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ON CONDITIONS OF THE CENTRAL LIMIT THEOREM FOR INHOMOGENEOUS MARKOV CHAINS

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This note is devoted to a further development of results presented in our note [1].*

With each operator of the transition probabilities $P(x, B)$ we associate the number

$$\beta = 1 - \sup |P(x, B) - P(y, B)|, \tag{1}$$

where the sup is taken over all $x \in \Omega$, $y \in \Omega$, $B \in \mathfrak{B}$.

Let us consider the linear space $L = \lambda$ of completely additive functions of the set $\lambda(B)$ given in $B \in \mathfrak{B}$ and such that $\lambda(\Omega) = 0$. Let us transform L into a Banach space by putting $\|\lambda\| = \sup_{B \in \mathfrak{B}} \lambda(B)$. By putting

$$P\lambda(B) = \int_\Omega P(x, B)\lambda(dx), \tag{2}$$

we obtain $P\lambda \in L$ and we can consider P as an operator acting in L. Then, as is easily shown, the number $1 - \beta$ coincides with the norm of the operator P. In the case of a denumerable chain given by the matrix (p_{ij})

$$\beta = \inf_{i,j} \sum_{k=1}^{\infty} \min(p_{ik}, p_{jk}), \tag{3}$$

where the inf is taken over all possible pairs of subscripts i and j and the minimum is the minimum of the two probabilities mentioned.

It is not difficult to show that $\beta \geq \rho$ always, where ρ is the coefficient of ergodicity introduced in [1]. In a broad class of cases $\rho = 0$ but $\beta > 0$. The quantity β, just as ρ does, characterizes the ergodic properties of the operator P. The qualitative distinction between them is, roughly speaking, that the conditions bounding the quantity β from below guarantee the presence of a common part in each pair of probability distributions $P(x, B)$, $P(y, B)$ at the same time as the conditions bounding ρ from below guarantee at once the presence of a common part in all the distributions $P(x, B)$, $x \in \Omega$.

Theorems 1, 2 and 3 of our note [1] remain true if the quantity ρ therein is replaced by the corresponding quantity β.

In this note, we shall formulate several criteria of the asymptotic normality of the sums ζ_n. In all the subsequent theorems, except Theorem 6, we shall assume that

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*A review of the question under consideration here and the notations used here is given in [1].

$$0 < c \leq D\xi_{in} \leq C < \infty. \tag{4}$$

Theorem 1. *Let the quantities* $|\xi_{in}| \leq c_n$ *with probability* 1. *Then, for the asymptotic normality of the normalized sums (i. e., for compliance with the relation* (1) *from* [1]), *it is sufficient that as* $n \to \infty$

$$c_n (\beta_n)^{-3/2} n^{-1/2} \to 0. \tag{5}$$

Conversely, if for certain given sequences of constants c_n *and quantities* β_n

$$\lim_{n\to\infty} c_n (\beta_n)^{-3/2} n^{-1/2} = a > 0, \tag{5'}$$

then an example can be constructed in which the quantities c_n *and* β_n *have prescribed values but the limit distribution for the normalized sums is not normal.*

Theorem 2. *Let a certain number* $\delta (0 < \delta < 1/3)$ *be given, and let the following condition be satisfied:*

$$\beta_n n^{\delta} \to \infty. \tag{6}$$

If for

$$m = 2\frac{1-\delta}{1-3\delta} \tag{7}$$

the following moments are bounded uniformly from above for all i *and* n

$$M|\xi_{in} - M\xi_{in}|^m \leq \gamma_m < \infty, \tag{8}$$

then the normal sums ζ_n *are asymptotically normal.*

Conversely, an example can be constructed in which (6) *is true for any* $\widetilde{m} < m$ *and all* i *and* n

$$M|\xi_{in} - M\xi_{in}|^{\widetilde{m}} \leq \gamma_{\widetilde{m}} < \infty, \tag{8'}$$

but the limit distribution for the normalized sums ζ_n *is not normal.*

Theorem 3. *Let all the quantities* ξ_{in} *have normal distributions. Then for the asymptotic normality of the sums* ζ_n, *it is sufficient that*

$$\lim_{n\to\infty} \beta_n n^{1/3} (\ln n)^{-1/3} = \infty. \tag{9}$$

Conversely, for any $a < \infty$ *an example can be presented where*

$$\lim_{n\to\infty} \beta_n n^{1/3} (\ln n)^{-1/3} = a \tag{9}$$

and the limit distribution is not normal.

The list of such criteria could certainly be continued. Let us now give a general condition of asymptotic normality which contains the sufficient conditions of asymptotic normality given by Theorems 1 and 3 of the note [1] and by Theorems 2 and 3 of the present note.

Theorem 4. *Let us put*

$$G_{in}(t) = P\{\xi_{in} - M\xi_{in} < t\}.$$

Then for the asymptotic normality of ζ_n *it is sufficient that for any* $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n(\beta_n)^2} \sum_{i=1}^{n} \int_{|t|\geq \epsilon n^{1/2}(\beta_n)^{3/2}} t^2 dG_{in}(t) = 0. \tag{10}$$

It is interesting to weaken somewhat the conditions imposed in Theorem 1 of [1] on the quantities $D\xi_{in}$ and β_{in} by replacing the requirement of nonconvergence to zero uniformly over all i by weaker conditions.

Theorem 5. *Let all the quantities* ξ_{in} *be bounded uniformly:* $|\xi_{in}| < c < \infty$. *Furthermore, let*

$$\lim_{n\to\infty} n^{-2/3} \sum_{i=1}^{n} \min\,(\beta_{in}, \beta_{i+1,n}) = \infty \tag{11}$$

and for all pairs k *and* l *such that* $l - k \geq n^{1/3}$ *and a certain function* $d(n) \longrightarrow \infty$ *as* $n \longrightarrow \infty$ *suppose*

$$\sum_{i=k}^{l} \beta_{in} \geq d(n). \tag{12}$$

Then the sums ζ_n *are asymptotically normal.*

The fact that the minimum of two adjacent quantities β_{in} enters in (11) is not chance. A simple example can be presented in which for all even i the quantities $\beta_{in} \geq c > 0$ but asymptotic normality does not occur.

Theorem 6. *Let the quantities* ξ_{in} *be bounded uniformly* $|\xi_{in}| \leq c < \infty$. *Then for asymptotic normality of the sums* ζ_n, *it is sufficient that*

$$\lim_{n\to\infty} n^{-2/3} \beta_n \left[\sum_{i=1}^{n} D\xi_{in} \right] = \infty. \tag{13}$$

Let us note that the method of proof described in [1] needs some revision, although the general plan of the proof remains the same. This is related to the fact that, in general, it can happen that $\sum_{\alpha=1}^{r_n} D\nu_\alpha^{(n)} \to \infty$ as $n \to \infty$. The limiting theorems for sums of quantities connected in a martingale, which are due to Lévy [2], must be used to prove Theorem 1 from [1] in its most general formulation with the quantities β.

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