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On the Innovation Problem for Gaussian Markov Random Fields

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1. Introduction.

The well-known innovation problem for random processes consists, roughly speaking, in the construction of a process with independent increments which contains all the information about the original process and which can be used for its reconstruction. The usual formulation of the innovation problem (see e.g. [5, 11]) is based on the notions of the "past" and the "future" of a process and so cannot be directly carried over to random fields. However, in connection with the recent progress in Markov random fields it appears that it is natural to replace the "past" by the exterior of a (bounded) set and the "future" by its interior when certain problems arising originally in the theory of random processes and employing these notions are discussed for random fields. Even with this convention about the "future" and the "past" there can be different formulations of the innovation problem depending on how we understand the "reconstruction of the original field from the innovating one". In this paper we discuss one of the possible variants to define the innovation problem for random fields and we will solve this problem for a wide class of Gaussian Markov fields. We want to point out that our definition of innovation when applied to the case of one-dimensional time differs a little from the 'classical' definition (of the innovation process) although in many interesting situations they coincide (see Proposition 1.1 below).

Let $(\Omega, \mathscr{F}, P)$ be a probability space, T be a Borel subset of the Euclidean space $\mathbb{R}^d$ $(d \geqq 1)$, $\mathscr{B}(T)$ the σ-algebra of Borel subsets of T. We assume in this paper that the basic σ-algebra $\mathscr{F}$ as well as any sub-σ-algebras of $\mathscr{F}$ discussed below are complete, i.e. include all P-null sets.

Let $(\mathscr{F}_V) = (\mathscr{F}_V,\ V \in \mathscr{B}(T))$ and $(\mathscr{O}_V,\ V \in \mathscr{B}(T))$ be two systems of sub-σ-algebras of $\mathscr{F}$ which are monotone in the sense that

$$\mathscr{F}_{V_1} \subseteq \mathscr{F}_{V_2}; \quad \mathscr{O}_{V_1} \subseteq \mathscr{O}_{V_2} \tag{1.1}$$

if $V_1 \subset V_2$ $(V_1, V_2 \in \mathscr{B}(T))$.

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Let there be given a system $\mathscr{U}\subseteq\mathscr{B}(T)$. We say that a system $(\mathscr{A}_V, V\in\mathscr{B}(T))$ of sub-σ-algebras of $\mathscr{F}$ is $\mathscr{U}$-independent if for any $U\in\mathscr{U}$, the σ-algebras $\mathscr{A}_U$ and $\mathscr{A}_{U^c}$ are independent. (Here and below $U^c=T\setminus U$). Given two σ-algebras $\mathscr{A},\mathscr{B}\subseteq\mathscr{F}$, we denote $\mathscr{A}\vee\mathscr{B}$ the σ-algebra generated by the sum of $\mathscr{A}$ and $\mathscr{B}$.

Finally, let there be given a system $\mathscr{V}_{1,2}$ of pairs (V_1, V_2) of Borel subsets of T. Set $\mathscr{V}_1=(V\in\mathscr{B}(T);\ (V, V_2)\in\mathscr{V}_{1,2}$ for some $V_2\in\mathscr{B}(T))$.

Definition 1.1. A system $(\mathscr{O}_V)$ is called the innovating system (I.S.) for $(\mathscr{F}_V)$ with respect to the system $\mathscr{V}_{1,2}$ (where $(\mathscr{O}_V)$, $(\mathscr{F}_V)$ and $\mathscr{V}_{1,2}$ are as above) if $(\mathscr{O}_V)$ is $\mathscr{V}_1$-independent and for any $(V_1, V_2)\in\mathscr{V}_{1,2}$

$$\mathscr{O}_{V_1} \text{ and } \mathscr{F}_{V_2} \text{ are independent;} \tag{1.2}$$

$$\mathscr{O}_{V_1}\vee\mathscr{F}_{V_2}=\mathscr{F}_{V_1\cup V_2}. \tag{1.3}$$

It should be noted that in general the I.S. need not exist (even in the case of a single sub-σ-algebra $\mathscr{F}'\subseteq\mathscr{F}$ there need not exist its orthogonal compliment [1], i.e. the σ-algebra $\mathscr{O}\subseteq\mathscr{F}'$ which is independent of $\mathscr{F}'$ and such that $\mathscr{O}\vee\mathscr{F}'=\mathscr{F}$). Some rather general sufficient conditions which guarantee the existence of the orthogonal compliment were found by Rohlin [10]. Let us remark too that in general the orthogonal compliment as well as the I.S. is not unique.

The first major problem (the 'innovation problem', or I.P.) which arises in connection with Definition 1.1 is the following.

I.P.1. Given a probability space $(\Omega,\mathscr{F},P)$, a system $\mathscr{V}_{1,2}$ of pairs of Borel subsets of T and a system of σ-algebras $(\mathscr{F}_V)=(\mathscr{F}_V, V\in\mathscr{B}(T))$, establish the existence of the I.S. $(\mathscr{O}_V)=(\mathscr{O}_V, V\in\mathscr{B}(T))$ for $(\mathscr{F}_V)$ with respect to $\mathscr{V}_{1,2}$ and describe it explicitly.

Apparently the most interesting situation where the concept of the I.S. can be discussed, and which we shall treat later on, is the case when the σ-algebras $\mathscr{F}_V$ are related to a random field on T. Let $X=(X(\varphi), \varphi\in\mathscr{X}(T))$, where $\mathscr{X}(T)$ is a linear topological space of functions on T, be a generalized random field (g.r.f.) [3], i.e. a system of real random variables $(X(\varphi), \varphi\in\mathscr{X}(T))$ on a probability space $(\Omega,\mathscr{F},P)$ such that $X(\varphi)$ is linear in φ and $X(\varphi)\to 0$ in probability as $\varphi\to 0$. (In the sequel (Sect. 2–4) $\mathscr{X}(T)=\mathscr{D}(T)$, where $\mathscr{D}(T)$ is the Schwartz space (see footnote 3 at p. 5).) We say that a system of σ-algebras $(\mathscr{A}_V, V\in\mathscr{B}(T))$ is related to the field[1] $(X(\varphi), \varphi\in\mathscr{X}(T))$ if $\mathscr{A}_V=\sigma(X(\varphi);\ \varphi\in\mathscr{X}(T), \operatorname{supp}\varphi\subseteq V)$ for open $V\subseteq T$, and $\mathscr{A}_V=\bigcap_{\varepsilon>0}\mathscr{A}_{V_\varepsilon}$ for non-open $V\in\mathscr{B}(T)$, where V_ε is the open ε-neighborhood of V, i.e. $V_\varepsilon=(t\in T\colon |t-s|<\varepsilon$ for some $s\in V)$.[2] If V is empty, let $\mathscr{A}_V$ be the trivial σ-algebra.

Definition 1.2. Let $(\mathscr{O}_V)=(\mathscr{O}_V, V\in\mathscr{B}(T))$ be the I.S. for a given system $(\mathscr{F}_V)=(\mathscr{F}_V, V\in\mathscr{B}(T))$, with respect to a given system $\mathscr{V}_{1,2}$. If $(\mathscr{O}_V)$ is related to a g.r.f.

[1] Given an ordinary random field $(X_t, t\in T)$ we shall consider instead a generalized one $(X(\varphi), \varphi\in\mathscr{X}(T))$ associated with X_t in the natural way by the relation $X(\varphi)=\int_T X_t\varphi(t)\,dt$ (this can be done in most of the interesting cases)

[2] Sometimes the σ-algebra $\mathscr{A}_V$ for non-open V is defined as $\bigcap(\mathscr{A}_{V'}, V'$ open, $V\subset V')$. These two definitions are equivalent if V is compact, but the first one is more convenient in the general case

$I=(I(\varphi), \varphi\in\mathscr{L}'(T))$ and $(\mathscr{F}_V)$ is related to a g.r.f. $X=(X(\varphi), \varphi\in\mathscr{L}(T))$ we call I the innovating field for X with respect to the system $\mathscr{V}_{1,2}$ (or shortly, the innovating field (I.F.)).

It is clear that if $\mathscr{V}_1$ is sufficiently big (to separate any nonintersecting supports of functions from $\mathscr{L}'(T)$), I is a g.r.f. with independent values [3].

In the case where the I.F. exists, the second part of I.P. is as follows:

I.P.2. For any $(V_1, V_2)\in\mathscr{V}_{1,2}$, describe the 'algorithm' to reconstruct the field X in the entire 'volume' $V_1\cup V_2$ given its values in V_2 and the I.F. I in V_1.

Although both broblems I.P.1 and I.P.2 are not strictly mathematically formulated, the intuitive meaning of them is rather clear and they can be successfully treated in some special cases. This applies in the first place to the 'classical' innovation process (i.p.) with one-dimensional time, and we now want to discuss briefly the relationship between the 'classical' definition and that of I.F. as given by Definition 1.2 in the case where $\mathscr{V}_{1,2}$ is the system of intervals corresponding to the usual 'future – past' division of time axis. We'll consider for simplicity only the case of scalar i.p. and an ordinary process X_t, $a<t<b$ satisfying in addition some regularity conditions.

Let $(X_t)=(X_t, a<t<b)$ and $(\zeta_t)=(\zeta_t, a<t<b)$, $-\infty\leqq a<b\leqq+\infty$ be two random processes such that $E[X_t^2]<+\infty$, $E[\zeta_t^2]<+\infty$, $a<t<b$. We assume that (X_t) and (ζ_t) are square mean continuous and have zero means. (We denote by $\mathfrak{X}(a,b)$ the class of all such processes.) Under these conditions there are defined the generalized random processes (g.r.p.)[3] $(X(\varphi))=(X(\varphi), \varphi\in C_0^\infty(a,b))$ and $(\zeta(\varphi))=(\zeta(\varphi), \varphi\in C_0^\infty(a,b))$ where $X(\varphi)=\int_a^b X_t\varphi(t)\,dt$ and $\zeta(\varphi)=-\int_a^b \zeta_t\varphi'(t)\,dt$ correspondingly. Set $F_t^X=\sigma(X_s, s\leqq t)$, $F_t^\zeta=\sigma(\zeta_s-\zeta_u,\ a<u<s\leqq t)$, and write $(\mathscr{F}_V^X, V\in\mathscr{B}(a,b))$, $(\mathscr{F}_V^\zeta)=(\mathscr{F}_V^\zeta, V\in\mathscr{B}(a,b))$ for the systems of σ-algebras related to g.r.p. $(X(\varphi))$, $(\zeta(\varphi))$ correspondingly.

Definition 1.3. (cf. [5, 11]). (ζ_t) is called the innovation process (i.p.) for (X_t) if

$$(\zeta_t) \text{ is the process with independent increments,} \tag{1.4}$$

$$\mathscr{F}_t=\mathscr{F}_t^\zeta, \quad a<t<b. \tag{1.5}$$

Let $\mathscr{V}_{1,2}$ be the system of pairs of 'past – future' intervals: $\mathscr{V}_{1,2}=(((t_1,t_2),(a,t_1]) : a\leqq t_1<t_2<b)$ (we set $(a,a]=\emptyset$).

Proposition 1.1. *If (X_t), $(\zeta_t)\in\mathfrak{X}(a,b)$, then (ζ_t) is the i.p. for (X_t) if and only if $(\zeta(\varphi))$ is the I.F. for $(X(\varphi))$ with respect to the system $\mathscr{V}_{1,2}$.*

The proof of this statement is given in the Appendix. Thus, the existence of the i.p. in the sense of Definition 1.3 under some regularity conditions implies the positive solution of the I.P.1 for the 'past-future intervals' system $\mathscr{V}_{1,2}$ (see e.g. [5, 11] for which classes of random processes this existence can be proved). Construction of Markov diffusion processes by means of K. Ito's stochastic differential equations provides us with the solution of I.P.1 and I.P.2 for this

[3] We assume that $C_0^\infty(a,b)$ is equipped with the usual inductive limit topology, i.e. in fact $C_0^\infty(a,b)=\mathscr{D}(a,b)$. where $\mathscr{D}$ is the Schwartz space (this applies also to the spaces $C_0^\infty(T)$, $T\subseteq\mathbb{R}^d$ below)

class of processes and the same system of 'past-future intervals'. In the case of Gaussian processes and the same system $\mathscr{V}_{1,2}$ the solution to these problems was obtained by Hida [4] (for purely nondeterministic stationary processes this solution reduces to the well-known Wold's representation).

Before abandoning the general discussion, we want to give a (new) definition of the linearly innovating field (L.I.F.) which is closely related to the notion of the i.p. in the theory of random processes in the wide sense (see e.g. [11]) and which is equivalent to Definition 1.2 in the case of (jointly) Gaussian X and I (see below).

If $Y=(Y(\varphi),\ \varphi\in\mathscr{L}(T))$ is a second order' g.r.f. (i.e. if $E[(Y(\varphi))^2]<+\infty$, $\varphi\in\mathscr{L}(T)$, and $E[(Y(\varphi))^2]\to 0$ $(\varphi\to 0)$), let $H(Y)$ denote the Hilbert space of random variables, with square mean norm, spanned by the set $(Y(\varphi),\ \varphi\in\mathscr{L}(T))$ (we write $H(Y)=\bigvee(Y(\varphi),\ \varphi\in\mathscr{L}(T))$). Set $H_V(Y)=\bigvee(Y(\varphi),\ \varphi\in\mathscr{L}(T),\ \operatorname{supp}\varphi\subseteq V)$ for V open, and $H_V(Y)=\bigcap_{\varepsilon>0} H_{V_\varepsilon}(Y)$ for any non-open $V\in\mathscr{B}(T)$, $H_\emptyset(Y)=\{0\}$. Given a system $\mathscr{U}\subseteq\mathscr{B}(T)$, we say that Y is $\mathscr{U}$-orthogonal if for any $U\in\mathscr{U}$, $H_U(Y)$ and $H_{U^c}(Y)$ are orthogonal subspaces of $H(Y)$.

Let $\mathscr{V}_{1,2}$ be again a system of pairs of Borel subsets of T and $\mathscr{V}_1$ be associated with it as above.

Definition 1.4. Let $X=(X(\varphi),\ \varphi\in\mathscr{L}(T))$ and $I=(I(\varphi),\ \varphi\in\mathscr{L}'(T))$ be 'second order' g.r.f.s, defined on the same probability space. Then I is called the L.I.F. for X with respect to the system $\mathscr{V}_{1,2}$ if I is $\mathscr{V}_1$-orthogonal and for any pair $(V_1,V_2)\in\mathscr{V}_{1,2}$

$$H_{V_1}(I) \text{ and } H_{V_2}(X) \text{ are orthogonal,} \tag{1.6}$$

$$H_{V_1}(I)\oplus H_{V_2}(X)=H_{V_1\cup V_2}(X) \tag{1.7}$$

(here $\oplus$ stands for the direct sum of Hilbert spaces).

The proof of the following statement is also carried over to Appendix.

Proposition 1.2. *Let $X=(X(\varphi),\ \varphi\in\mathscr{L}(T))$ and $I=(I(\varphi),\ \varphi\in\mathscr{L}'(T))$ be jointly Gaussian g.r.f.s, i.e. for any $\varphi_1,\dots,\varphi_m\in\mathscr{L}(T)$, $\varphi'_1,\dots,\varphi'_{m'}\in\mathscr{L}'(T)$ the joint probability distribution of $X(\varphi_1),\dots,X(\varphi_m)$, $I(\varphi'_1),\dots,I(\varphi'_{m'})$ is Gaussian. Then I is the I.F. for X if and only if is the L.I.F. for X (with respect to any given system $\mathscr{V}_{1,2}$ of pairs of Borel subsets of T).*

In what follows we shall consider only the case when $\mathscr{V}_1$ is the system of all open relatively compact subsets of T with a smooth[4] boundary, and $\mathscr{V}_{1,2}=((V_1,V_2):\ V_1\in\mathscr{V}_1,\ V_2=V_1^c)$ (actually, a slightly more general situation could be discussed along the same lines, namely the case when $\mathscr{V}_{1,2}=((V_1,V_2):\ V_1\in\mathscr{V}_1,\ V_2=U_1\setminus V_1$ for some $U_1\in\overline{\mathscr{V}_1}$ such that ∂V_1 and ∂U_1 do not intersect) (here $\overline{\mathscr{V}_1}=\mathscr{V}_1\cup\{T\}$, and ∂V is the boundary of V)). In this paper the I.P. are solved for a wide class of Gaussian Markov g.r.f.s in T which is a subclass of the so-called Markov fields of finite order (see [8, 12, 13]). In particular the solution of the I.P.2 is obtained in reducing it to some kind of (linear) 'stochastic' elliptic boundary value problem. It seems that there exist substantial difficulties for solving this innovation problem for non-Gaussian Markov fields, even in case $d=1$. For non-Gaussian diffusion processes to all appearance, the corresponding stochastic

[4] Everywhere below 'smooth' means 'of C^∞-class', i.e. infinitely differentiable

equations must be non-linear (quasilinear), and the question of the uniqueness of their solutions seems to be very difficult.

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2. The Class $\Phi_p(T)$ of Gaussian Markov Fields

Let T be an open domain in $\mathbb{R}^d$, $d \geqq 1$, $a_{\alpha\beta} \in C^\infty(T)$, $|\alpha|, |\beta| \leqq p$ be given real functions on T, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\beta = (\beta_1, \dots, \beta_d)$ multiindices, $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Here $C^\infty(T)$ stands for infinitely differentiable functions on T. We introduce also the following common notations:

$C_0^\infty(T)$ = the class of all $f \in C^\infty(T)$ with compact supports;

$$D^\alpha = \partial^{|\alpha|}/\partial t_1^{\alpha_1}, \dots, \partial t_d^{\alpha_d}$$

$$x^\alpha = x_1^{\alpha_1}, \dots, x_d^{\alpha_d} \qquad (x = (x_1, \dots, x_d) \in \mathbb{R}^d).$$

Let A be a symmetric differential operator given by

$$Af = \sum_{|\alpha|, |\beta| \leqq p} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta} D^\beta f) \tag{2.1}$$

with[5] $D(A) = C_0^\infty(T)$. We assume that A is uniformly strongly elliptic, i.e.

$$\Big| \sum_{|\alpha| = |\beta| = p} a_{\alpha\beta}(t)\, x^\alpha x^\beta \Big| \geqq c_1 |x|^{2p}, \qquad x \in \mathbb{R}^d,\ t \in T,\ c_1 > 0, \tag{2.2}$$

and strictly positive (here and below $\int f\,dt = \int_T f(t)\,dt$), i.e.

$$\int f A f\,dt \geqq c_2 \int f^2\,dt, \qquad f \in C_0^\infty(T),\ c_2 > 0. \tag{2.3}$$

Let $\bar{A}$ denote the Friedrichs' self-adjoint extension of A. We recall that (see e.g. [14], pp. 620–621)

$$R(\bar{A}) = L^2(T), \tag{2.4}$$

$$D(\bar{A}) \subset \mathscr{H}(T), \tag{2.5}$$

$\bar{A}^{-1}$ exists and is a bounded operator $L^2(T) \to L^2(T)$. (2.6)

Here $\mathscr{H}(T)$ denotes the Hilbert space completion of $C_0^\infty(T)$ in the norm

$$\|f\|_{\mathscr{H}} = \Big(\int f A f\,dt\Big)^{1/2}, \tag{2.7}$$

which according to (2.2), (2.3) and the well-known Gårding's inequality (e.g. [1], p. 78) is equivalent to the Sobolev norm

$$\|f\|_p = \Big(\sum_{|\alpha| \leqq p} \int (D^\alpha f)^2\,dt\Big)^{1/2}. \tag{2.8}$$

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5 We write $D(B)$ for the domain and $R(B)$ for range of an operator B

Hence $\mathscr{H}(T)$ can be identified (algebraically) with the Sobolev space $H_0^p(T)$ (= the completion of $C_0^\infty(T)$ in the norm $\|\cdot\|_p$) (c.f. [13], p. 378). Denote by $H^p(T)$ the Sobolev space of all $f \in L^2(T)$ whose distributional derivatives $D^\alpha f$, $|\alpha| \leqq p$ belong to $L^2(T)$, with the norm $\|\cdot\|_p$.

With every such operator A, we can associate a real Gaussian g.r.f. $X = (X(\varphi), \varphi \in C_0^\infty(T))$ on a probability space $(\Omega, \mathscr{F}, P)$ with mean zero and covariance

$$R(\varphi, \psi) = E[X(\varphi) X(\psi)] = \int \varphi \bar{A}^{-1} \psi \, dt, \qquad \varphi, \psi \in C_0^\infty(T). \tag{2.9}$$

We denote the class of all such fields by $\Phi_p(T)$. As it follows from Proposition 2.1 below, $\Phi_p(T)$ is contained in the class of Markov fields of finite order (of order p) which was considered by Pitt [13] in case of ordinary fields and by Molčan [8, 9] in the general case (see also Wong [16] for the isotropic case and $p=1$, and the recent work of Rozanov [12] on a related class of (Gaussian Markov) fields). The Markov property of order p of a g.r.f. $X = (X(\varphi), \varphi \in C_0^\infty(T))$ means, roughly speaking, that for any 'good' subset $V \subset T$, the 'future' $\mathscr{F}_{V^c} = \sigma(X(\varphi)\colon \operatorname{supp} \varphi \subset V^c)$ and the 'past' $\mathscr{F}_V = \sigma(X(\varphi)\colon \operatorname{supp} \varphi \subset V)$ are conditionaly independent given the 'present' – the 'germ' σ-algebra at V which is generated by p 'weak' normal derivatives of the field X on ∂V (see e.g. [13, 12]). (Here ∂V is the boundary of V).

Remark 2.1. If $X = (X(\varphi), \varphi \in C_0^\infty(\mathbb{R}^d))$ is a stationary Gaussian g.r.f. with the spectral density function $f(\lambda), \lambda \in \mathbb{R}^d$ then $X \in \Phi_p(\mathbb{R}^d)$ if and only if $f(\lambda) = (2\pi)^{-d} P(i\lambda)^{-1}$, where $P(i\lambda) = \sum_{|\alpha|, |\beta| \leqq p} (-1)^{|\alpha|} a_{\alpha\beta} (i\lambda)^\alpha (i\lambda)^\beta$ is an even elliptic positive polynomial of degree $2p$, i.e. if $P(i\lambda) = P(-i\lambda)$ and $P(i\lambda) \geqq c(1+|\lambda|^2)^p$, $\lambda \in \mathbb{R}^d$, $c>0$. The corresponding operator A is given by $Af = P(D) f = \sum_{|\alpha|, |\beta| \leqq p} (-1)^{|\alpha|} a_{\alpha\beta} D^\alpha (D^\beta f)$. An important special case is that of $f(\lambda) = (2\pi)^{-d} (1+|\lambda|^2)^{-1}$, or $A = 1 - \Delta$, corresponding to the so-called free Markov field which plays the basic role in many quantum field theories.

We recall below some definitions and properties of the reproducing kernel Hilbert spaces (RKHS) which appeared to be very useful in the study of Gaussian Markov fields.

If $X = (X(\varphi), \varphi \in C_0^\infty(T))$ is a Gaussian g.r.f. with mean zero and covariance $R(\varphi, \psi) = E[X(\varphi) X(\psi)]$, $\varphi, \psi \in C_0^\infty(T)$, then a Hilbert subspace $\mathscr{H}(X)$ of[6] $\mathscr{D}'(T)$ is called the RKHS of X if it is obtained by completion of the set of (generalized) functions $(R(\cdot, \varphi), \varphi \in C_0^\infty(T))$ in the norm $\|F\|_{\mathscr{H}(X)} = (R(\varphi, \varphi))^{1/2}$, $F(\cdot) = R(\cdot, \varphi)$ [4, 6, 12]. It follows from the definition that for any $F \in \mathscr{H}(X)$,

$$\langle F, R(\cdot, \varphi) \rangle_{\mathscr{H}(X)} = F(\varphi), \qquad \varphi \in C_0^\infty(T), \tag{2.10}$$

where $\langle \cdot, \cdot \rangle_{\mathscr{H}(X)}$ is the scalar product in $\mathscr{H}(X)$ (the 'reproducing property'). There exists an isometrical isomorphism $\imath$ from $H(X) = \bigvee (X(\varphi), \varphi \in C_0^\infty(T))$ onto $\mathscr{H}(X)$ which acts on $(X(\varphi), \varphi \in C_0^\infty(T))$ according to the rule

$$\imath[X(\varphi)] = R(\cdot, \varphi). \tag{2.11}$$

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[6] $\mathscr{D}'(T)$ (the space of 'generalized functions') = the dual of the Schwartz space $\mathscr{D}(T)$ (see footnote 3 at p. 3)

Proposition 2.1. *Let $X \in \Phi_p(T)$ and $\mathscr{H}(X)$ be the corresponding RKHS. Then*[7] $\mathscr{H}(X)=\mathscr{H}(T)$.

Proof. By (2.6), $R(\cdot,\varphi)$ $(=E[X(\cdot)X(\varphi)])$ is equivalent to an ordinary function $R_\varphi=\bar{A}^{-1}\varphi \in L^2(T)$. As $R_{A\varphi}=\varphi$, $\varphi \in C_0^\infty(T)$ and

$$\|R(\cdot, A\varphi)\|^2_{\mathscr{H}(X)}=R(A\varphi, A\varphi)=\int A\varphi\, \bar{A}^{-1} A\varphi\, dt = \int \varphi\, A\varphi\, dt=\|\varphi\|^2_{\mathscr{H}(T)},$$

so $\mathscr{H}(T)\subseteq\mathscr{H}(X)$. But $R_\varphi \in D(A)\subset \mathscr{H}(T)$ by (2.5), which implies $\mathscr{H}(T)=\mathscr{H}(X)$.

Let $V\in\mathscr{V}_1$ be an open relatively compact subset of T with a smooth boundary $\Gamma=\partial V$, and let $(\Gamma_h, -h_0<h<h_0)$, $\Gamma_0=\Gamma$ be a family of smooth surfaces 'parallel to Γ' and 'convergent to Γ as $h\to 0$' ([6], Ch. 2, 8.1). This means in particular that there exists a finite system of neighborhoods (patches) U_i, $i=1,\dots,N$, $U_i\subset T$ such that Γ_h $(|h|<h_0)$ are contained in $\bigcup_{i=1}^N U_i$ and on every U_i there can be introduced 'local coordinates' $t\to g_i t\equiv t'$ as smooth $1-1$ maps (with smooth inverses) from U_i onto the 'cylinder' $U'\equiv\left(t'=(t'_1,\dots,t'_d)\in\mathbb{R}^d: |t'_1|<h_0, \sum_{i=2}^d t_i'^2<1\right)$ such that $t\in\Gamma_h \Leftrightarrow t'\in\Gamma'_h\equiv(t'\in U': t'_1=h)$. If f is a given function on U_i, set $f'(t')=f(g_i^{-1}t')$, $t'\in U'$. Conversely, for any given f on U', set $'f(t)={}'f_i(t)=f(g_i t)$, $t\in U_i$. Finally, let us denote by $X'=X'_i=(X'(\varphi)=X'_i(\varphi),\ \varphi\in C_0^\infty(U'))$ the 'field X in local coordinates $t'=g_i t$', i.e.

$$X'(\varphi)=X({}'\varphi\cdot J), \qquad \varphi\in C_0^\infty(U'),$$

where J is the Jacobian: $J(t)=|\partial(g_i t)/\partial t|$, which is smooth and positive on U_i. If $\rho\in C_0^\infty(-h_0,h_0)$ and $\psi\in C_0^\infty(\Gamma')$ $(\Gamma'\equiv\Gamma'_0)$ we write $(\rho\otimes\psi)(t_1,\dots,t_d)=\rho(t_1)\,\psi(t_2,\dots,t_d)$.

Definition 2.1. Let $X=(X(\varphi),\ \varphi\in C_0^\infty(T))$ be a g.r.f. We say that X has the property of stochastic continuity of order $p-1$ $(p=0,1,\dots)$ on the surfaces parallel to Γ if, first, for any $j=0,1,\dots,p-1$, $\psi\in C_0^\infty(\Gamma')$ and any patch U_i, $i=1,\dots,N$ there exists a unique square mean continuous process $\partial_j^+ X'_\psi=\partial_j^+ X'_\psi(h)$, $h\in(-h_0,h_0)$, such that

$$(-1)^j \int_{-h_0}^{h_0} \rho(h)\,\partial_j^+ X'_\psi(h)\,dh = X'_i(\rho^{(j)}\otimes\psi) \tag{2.12}$$

for every $\rho\in C_0^\infty(-h_0,h_0)$, where $\rho^{(j)}=d^j\rho/dh^j$; and, second, for any function ψ_h: $(-h_0,h_0)\to C_0^\infty(\Gamma')$ which is $L^2(\Gamma')$-continuous[8] the process $\partial_j^+ X_{\psi_h}(h)$, $h\in(-h_0,h_0)$ is square mean continuous.

The following statement was in fact discussed by Pitt [13] in the case of ordinary fields (Lemma 5.3) and also by Molčan (unpublished) in the general case (although in a slightly different formulation and without the continuity of

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[7] We assume that $\mathscr{H}(T)$ is embedded into $\mathscr{D}'(T)$ by means of the usual relation $u(\varphi)=\int u\varphi\,dt$, $\varphi\in C_0^\infty(T)$, $u\in\mathscr{H}(T)$

[8] I.e. $\int_{\Gamma'}(\psi_{h'}-\psi_{h''})^2\,dt\to 0$ as $h''\to h'$, $h', h''\in(-h_0,h_0)$

'weak derivatives'). The proof presented below is different from that of Pit and Molčan, too.

Proposition 2.2. *Any (Gaussian) g.r.f.* $X \in \Phi_p(T)$ *has the property of stochastic continuity of order* $p-1$ *on surfaces parallel to the boundary* $\Gamma = \partial V$, *for any* $V \in \mathscr{V}_1$.

Remark 2.2. If $p - d/2 > 0$ and $X \in \Phi_p(T)$, then X is an ordinary random field [13] and $D^\alpha X(t)$ exists in the usual sense and is continuous in t for every α: $0 < |\alpha| < [p - d/2]$ [15]. In this case ($t_0 = (t_2, \ldots, t_d) \in \mathbb{R}^{d-1}$, $\{h, t_0\} = (h, t_2, \ldots, t_d) \in \mathbb{R}^d$):

$$\partial_j^+ X'_\psi(h) = \int_{\Gamma'} \partial^j X'(\{h, t_0\})/\partial h^j \psi(t_0)\, dt_0 \tag{2.13}$$

for j: $0 \leqq j < [p - d/2]$.

Proof. Let $Y \in H(X)$ and $u \in \mathscr{H}(X) = \mathscr{H}(T)$ be the corresponding element in the RKHS, i.e. $u = \imath[Y]$. Then

$$\begin{aligned} E[Y \cdot X'(\rho^{(j)} \otimes \psi)] &= E[Y \cdot X('(\rho^{(j)} \otimes \psi) \cdot J)] \\ &= \int_{U_i} u(t)\, J(t)\, '(\rho^{(j)} \otimes \psi)(t)\, dt = \int_{U'} u'(t)(\rho^{(j)} \otimes \psi)(t)\, dt \\ &= (-1)^j \int_{-h_0}^{h_0} \rho(h)\, dh \int_{\Gamma'} \frac{\partial^j u'(\{h, t_0\})}{\partial h^j} \psi(t_0)\, dt_0 \end{aligned} \tag{2.14}$$

as the transform $u \to u'$ is a continuous mapping from $H^p(U_i)$ onto $H^p(U')$ and the last equality which is obvious for smooth u' holds for $u' \in H^p(U')$ and $j \leqq p-1$ as well (c.f. [6], Chap. 1, Th. 8.3).

Consider the linear functional

$$u \to \int_{\Gamma'} \partial^j u'(\{h, t_0\})/\partial h^j \psi(t_0)\, dt_0 \equiv f_{h,\psi}(u),$$

which is continuous in $\mathscr{H}(T)$ and therefore defines an element $u_{h,\psi} \in \mathscr{H}(T)$ such that $\langle u_{h,\psi}, u \rangle_{\mathscr{H}} = f_{h,\psi}(u)$, $u \in \mathscr{H}(T)$. Write $\partial_j^+ X'_\psi(h)$ for the corresponding random variable in $H(X)$. Let ψ_h: $(-h_0, h_0) \to C_0^\infty(\Gamma')$ be $L^2(\Gamma')$-continuous (see above). If we show that the function u_{h,ψ_h}: $(-h_0, h_0) \to \mathscr{H}(T)$ is strongly continuous in h, then $\partial_j^+ X_{\psi_h}(h)$ is a (square mean) continuous process and, by using this fact for $\psi_h = \psi$,

$$\begin{aligned} E\left[Y \cdot \int_{-h_0}^{h_0} \partial_j^+ X'_\psi(h)\, \rho(h)\, dh\right] &= \int_{-h_0}^{h_0} E[Y \cdot \partial_j^+ X'_\psi(h)] \cdot \rho(h)\, dh \\ &= \int_{-h_0}^{h_0} \langle u, u_{h,\psi} \rangle_{\mathscr{H}}\, \rho(h)\, dh = \int_{-h_0}^{h_0} \left\{ \int_{\Gamma'} \partial^j u'(\{h, t_0\})/\partial h^j \psi(t_0)\, dt_0 \right\} \rho(h)\, dh \\ &= E[Y \cdot X'(\rho^{(j)} \otimes \psi)] \quad \text{by (2.14)}. \end{aligned}$$

Therefore (2.12) holds. Thus, it remains to prove the continuity of u_{h,ψ_h}.

By the equivalence of the norms $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_p$ and the continuity of the transform $H^p(U_i)\ni u\to u'\in H^p(U')$ it is sufficient to prove the continuity of the function u_h: $(-h_0, h_0)\to H^p(U')$ given by the relation

$$\langle u_h, u\rangle_p = \int_{\Gamma'} \partial^j u(\{h, t_0\})/\partial h^j \psi_h(t_0)\, dt_0, \qquad u\in H^p(U'). \tag{2.15}$$

We prove the continuity of u_h at $h=0$, as the general case is no more complicated. Let v_h: $(-h_0, h_0)\to H^p(U')$ be defined by the relation (2.15), with ψ_h replaced by ψ_0. Then $v_0=u_0$ and $\|u_h-u_0\|_p\leqq\|u_h-v_h\|_p+\|v_h-v_0\|_p$. Let $\tau\in C_0^\infty(-h_0, h_0)$ be monotone on the intervals $(-h_0, -h_0/2)$, $(h_0/2, h_0)$ and $\tau(h)=1$, $h\in(-h_0/2, h_0/2)$. Consider the 'τ-shift' T_δ ($|\delta|<h_0/2$) of $u\in H^p(U')$, given by the formula

$$(T_\delta u)(\{h, t_0\}) = u(\{h+\delta\,\tau(h), t_0\}).$$

It is easy to see that T_δ and T_δ^* (=the dual of T_δ) are continuous operators $H^p(U')\to H^p(U')$ and $T_\delta, T_\delta^*\to I$ (=the identity) strongly as $\delta\to 0$. Note that $\langle v_h, u\rangle_p=\langle v_0, T_h u\rangle_p=\langle T_h^* v_0, u\rangle_p$, $|h|<h_0/2$. Hence

$$\|v_h-v_0\|_p=\sup_{\substack{u\in H^p(U')\\ \|u\|_p\leqq 1}} |\langle v_h-v_0, u\rangle_p| = \|T_h^* v - v_0\|_p\to 0 \qquad (h\to 0).$$

Next,

$$\|u_h-v_h\|_p\leqq \sup_{\substack{u\in H^p(U')\\ \|u\|_p\leqq 1}} \Big(\int_{\Gamma'} (\partial^j T_h u/\partial t_1^j|_{t_1=0})^2\, dt_0\Big)^{1/2} \times\Big(\int_{\Gamma'} (\psi_h-\psi_0)^2\, dt_0\Big)^{1/2}\to 0 \qquad (h\to 0)$$

as $\int_{\Gamma'} (\partial^j u'/\partial t_1^j)^2\, dt_0\leqq\text{const}\,\|u'\|_p^2$ for $u'\in H^p(U')$ (and $j\leqq p-1$) by the trace theorem ([6], Chap. 1, Th. 8.3) and $\|T_h u\|_p\leqq c\|u\|_p$ with c independent of $u\in H^p(U')$ and h for h sufficiently small.

This ends the proof of Proposition 2.2 except for the uniqueness part. But the formula (2.12) easily implies that every such process $\partial_j^+ X'_\psi$ is unique in the sense that any two such processes satisfying (2.12) coincide a.s. for every $h\in(-h_0, h_0)$.

3. The Innovating Field I of a Random Field $X\in\Phi_p(T)$.

Now we turn to the I.P.1, in our case, to the existence and description of the I.F. (or the L.I.F., as the both notions mean the same in the case discussed here (see Proposition 1.2)) of a given field X from the class $\Phi_p(T)$. The solution of this problem is particularly simple in this case, namely, we have the following

Theorem 3.1. *The I.F. I of a given field $X\in\Phi_p(T)$ exists and can be defined by the formula*

$$I(\varphi)=X(A\,\varphi), \qquad \varphi\in C_0^\infty(T). \tag{3.1}$$

Its covariance functional is given by

$$E[I(\varphi)\,I(\psi)]=\int \varphi\, A\, \psi\, dt=\langle\varphi,\psi\rangle_{\mathscr{H}}, \qquad \varphi,\psi\in C_0^\infty(T). \tag{3.2}$$

Moreover,

$$H(I)=H(X). \tag{3.3}$$

Remark 3.1. Theorem 3.1 follows essentially from the results of [9] and, in some particular cases, also from [13, 12], the main difference being in the point of view. However, we present a short proof of it to make the discussion self-contained and easy to follow; the exposition in [9] applies to a much more general situation and is, in fact, rather brief. The reader familiar with the papers [9, 12, 8] will notice that the I.F. I defined by (3.1) is nothing else but the dual field of X as discussed in the cited references. We recall [12, 8] that a Gaussian g.r.f. $X^*=X^*(\varphi)$, $\varphi\in C_0^\infty(T)$) is called the dual field of a given Gaussian g.r.f. $X=(X(\varphi), \varphi\in C_0^\infty(T))$ if $H(X^*)=H(X)$, $E[X(\varphi)\,X^*(\varphi)]=\int\varphi\,\psi\,dt$, $\varphi,\psi\in C_0^\infty(T)$ and $H_U(X^*)\oplus H_{U^c}(X)=H(X)$ for all U from a 'sufficiently big' family $\mathscr{U}\subseteq\mathscr{B}(T)$. It is easy to see that all these conditions including the last one in the case $\mathscr{U}=\mathscr{V}_1$ are satisfied by the I.F. I defined by (3.1) as it follows from Theorem 3.1 and Definition 1.4.

Proof of Theorem 3.1. It follows straight from the definition (3.1) of I and (2.9) that

$$E[X(\varphi)\,I(\psi)]=\int\varphi\,\bar{A}^{-1}A\,\psi\,dt=\int\varphi\,\psi\,dt \tag{3.4}$$

and

$$E(I(\varphi)\,I(\psi)]=\int\varphi\,A\,\psi\,dt \tag{3.5}$$

for any $\varphi,\psi\in C_0^\infty(T)$ (notice that (3.5) implies that I is a Gaussian field with independent values ([3], p. 338), hence also $\mathscr{V}_1$-orthogonal). Turning now to the conditions (1.6)–(1.7) of Definition 1.4 (of the L.I.F.), (1.6) is clear by (3.4), and (1.7) follows easily from the standard argument by considering the corresponding subspaces in the RKHS $\mathscr{H}(T)$ of X (c.f. [13, 9, 12]). (We recall that here and below $\mathscr{V}_{1,2}=((V_1,V_2)\colon V_1\in\mathscr{V}_1,\ V_2=V_1^c)$.) Thus, the subspace $\mathscr{H}_V(I)\subset\mathscr{H}(T)$ corresponding to $H_V(I)$ $(V\in\mathscr{V}_1)$ under the isometrical isomorphism $\imath$ (2.11) turns out to be the subspace of all $u\in\mathscr{H}(T)$ which vanish on $(\bar{V})^c$ while the subspace $\mathscr{H}_{V^c}(X)\subset\mathscr{H}(T)$ corresponding to $H_{V^c}(X)$ consists of all $u\in\mathscr{H}(T)$ which solve $A\,u=0$ on V. Clearly $\mathscr{H}_V(I)\perp\mathscr{H}_{V^c}(X)$ and $\mathscr{H}_V(I)\oplus\mathscr{H}_{V^c}(X)=\mathscr{H}(T)$ by the existence and the uniqueness of the solution of the Dirichlet problem in $\mathscr{H}(T)$ ([6], Chap. 2, Th. 8.3).

To prove (3.3), let $Y\in H(X)$ be orthogonal to $H(I)$ and let u be the corresponding element in $\mathscr{H}(T)$. Then $0=E[Y\,I(\psi)]=E[Y\,X(A\,\psi)]=\int u\,A\,\psi\,dt$ (by (2.11)) $=\langle u,\psi\rangle_{\mathscr{H}}$ for every $\psi\in C_0^\infty(T)$ which implies $u=0$ as $C_0^\infty(T)$ is dense in $\mathscr{H}(T)$. Consequently $H(I)=H(X)$ which ends the proof of Theorem 3.1.

Remark 3.2. It follows easily from an argument similar to that given above that for every random field $X\in\Phi_p(T)$, the subspaces $H_{V^c}(X)$, $V\in\mathscr{V}_1$ are 'continuous at the boundary $\Gamma = \partial V$' in the sense that

$$H_{V^c}(X) = H_{(\bar{V})^c}(X), \tag{3.6}$$

where $\bar{V}$ denotes the closure of V. In fact, let $u \in \mathscr{H}_{V^c}(X)$ be orthogonal to $\mathscr{H}_{(\bar{V})^c}(X)$ $(= \imath[H_{(\bar{V})^c}(X)])$. This implies $\langle u, v\rangle_{\mathscr{H}} = 0$ for every v of the form $v = \bar{A}^{-1}\varphi$, with $\varphi \in C_0^\infty(T)$ and $\operatorname{supp}\varphi \subseteq (\bar{V})^c$. We know that for smooth φ, $v = \bar{A}^{-1}\varphi \in C^\infty(T)$ as A is elliptic (see e.g. [17], Chap. VI,9), hence $\langle u, v\rangle_{\mathscr{H}} = \int u\,A\,v\,dt = \int u\,\varphi\,dt = 0$ and consequently $u = 0$ on $(\bar{V})^c$. But this implies $u \in \mathscr{H}_{\bar{V}}(I) \perp \mathscr{H}_{V^c}(X)$ (see above) and so $u = 0$.

The relation (3.6) is important to define in a natural way the solution of the I.P.2, as discussed below.

4. The Dirichlet Problem for X in V

Let again V be a domain from $\mathscr{V}_1$, $\Gamma = \partial V$ be its boundary, and let there be given the 'field X in V^c' and the 'innovating field I in V'. More precisely, let $X_{V^c} = (X_{V^c}(\varphi),\ \varphi \in C_0^\infty(T),\ \operatorname{supp}\varphi \subset (V)^c)$ and $I_V = (I_V(\varphi),\ \varphi \in C_0^\infty(T),\ \operatorname{supp}\varphi \subset V)$ be independent Gaussian g.r.f.s. on a probability space $(\Omega, \mathscr{F}, P)$ with mean zero and covariance functionals $E[X_{V^c}(\varphi)\,X_{V^c}(\psi)] = \int \varphi\,\bar{A}^{-1}\psi\,dt$, $\operatorname{supp}\varphi$, $\psi \subset (\bar{V})^c$ and $E[I_V(\varphi)\,I_V(\psi)] = \int \varphi\,A\,\psi\,dt$, $\operatorname{supp}\varphi$, $\psi \subset V$ respectively, where A satisfies the assumptions of Sect. 2. Our next problem is what we called the I.P.2 – to describe an 'algorithm' to reconstruct the field X in T given its 'trajectory in V^c' and a 'trajectory of the I.F. in V'. Intuitively it is clear that the solution of this problem consists in reducing it to the Dirichlet problem[9]

$$A\tilde{X} = I_V \quad \text{in} \quad V, \tag{4.1}$$

$$\partial_j^+ \tilde{X} = \partial_j^+ X_{V^c} \quad \text{on } \Gamma, j = 0, \ldots, p-1 \tag{4.2}$$

and piecing together $\tilde{X}$ (in V) and X_{V^c} (in V^c) to obtain X in T. Now, the problem (4.1)–(4.2) should be treated with care as both the right hand sides of (4.1) and (4.2) are in fact generalized random functions, and as far as we know this Dirichlet problem with such 'bad' data was not considered in the literature (c.f. [6, 1]). Hence we start first with the definition of the solution of (4.1)–(4.2) which reflects the peculiarity of our 'stochastic problem'. Recall beforehand, that according to Minlos' theorem ([3], Th. 6 Chap. IV §2) we can (and shall) assume without loss of generality that the trajectories of all Gaussian g.r.f. $Y = (Y(\varphi),\ \varphi \in C_0^\infty(T))$ considered below are generalized functions on T a.s., i.e. $Y(\cdot)(\omega) \in \mathscr{D}'(T)$ a.s.

Definition 4.1. We call a g.r.f. $\tilde{X} = (\tilde{X}(\varphi),\ \varphi \in C_0^\infty(T))$ the solution of the Dirichlet problem (4.1)–(4.2) given g.r.f.s $I_V = (I_V(\varphi),\ \varphi \in C_0^\infty(T),\ \operatorname{supp}\varphi \subset V)$ and $X_{V^c} = (X_{V^c}(\varphi),\ \varphi \in C_0^\infty(T),\ \operatorname{supp}\varphi \subset (\bar{V})^c)$ if

(a) $(A\tilde{X})_V = I_V$ a.s. as elements of $\mathscr{D}'(V)$,

(b) $\tilde{X}_{V^c} = X_{V^c}$ a.s. as elements of $\mathscr{D}'((\bar{V})^c)$

[9] Here and below ∂_j^+ denotes the j-th inward pointing normal derivative

(here $(A\tilde{X})_V$ and $\tilde{X}_{V^c}$ are the restrictions of $A\tilde{X}$ and $\tilde{X}$ on V and $(\bar{V})^c$ correspondingly), and

(c) X has the property of stochastic continuity of order $p-1$ on surfaces parallel to Γ (see Definition 2.1 above).

Remark 4.1. If $\tilde{X}\in\Phi_p(T)$, $X_{V^c}=(X(\varphi),\ \operatorname{supp}\varphi\subset(\bar{V})^c)$, $I_V=(X(A\,\varphi),\ \operatorname{supp}\varphi\subset V)$ then X is the solution of (4.1)–(4.2), as it follows from Proposition 2.2.

Theorem 4.1. *Let A satisfy the assumptions of Sect. 2. Then the solution of Dirichlet problem* (4.1)–(4.2) *is unique.*

Proof. Let $\tilde{X}_1$, $\tilde{X}_2$ be solutions of (4.1)–(4.2), and set $\tilde{X}=\tilde{X}_1-\tilde{X}_2$. By the theorem about regularity of solutions of elliptic equations ([17], Chap. VI.9), $\tilde{X}$ is smooth in V, $\tilde{X}(t)=0$ in $(\bar{V})^c$ and $A\tilde{X}(t)=0$ in V. If we show that

$$\tilde{X}(t)=0 \qquad t\in V \tag{4.3}$$

then of course $\partial_j^+\tilde{X}'_\psi(h)=0$ ($|h|<h_0$) and the condition (c) of Definition 4.1 implies that $\tilde{X}'(\rho\otimes\psi)=0$ for any $\rho\in C_0^\infty(-h_0,h_0)$, $\psi\in C_0^\infty(\Gamma')$ and any 'local coordinate' $t'=g_i t$, $i=1,\dots,N$. This implies $\tilde{X}(\cdot)=0$ in the neighborhood of Γ, i.e. $\tilde{X}(\cdot)=0$ a.s. in $\mathscr{D}'(T)$. It remains to prove (4.3).

Let V_h ($|h|<h_0$) denote the domain inside Γ_h ($V_h\subset V$ if $h>0$ and $V\subset V_h$ if $h<0$). Let $\varphi\in C_0^\infty(T)$, $\operatorname{supp}\varphi\subset V$, and let f_h solve the Dirichlet problem

$$\begin{aligned} Af_h&=\varphi, && t\in V_h,\\ \partial_j^+ f_h&=0, && t\in\Gamma_h,\ j=0,\dots,p-1. \end{aligned} \tag{4.4}$$

If $h=0$, $f_0=f$ will denote the solution of (4.4) with V_h and Γ_h replaced by V and Γ, correspondingly. To prove the theorem, we shall need the following fact;

$$\sup_{t\in V_h\cap V}|D^\alpha f_h(t)-D^\alpha f(t)|\to 0 \qquad (h\to 0) \tag{4.5}$$

for any $\alpha=(\alpha_1,\dots,\alpha_d)$. Now, (4.5) can be proved by the following argument. As Γ_h is 'parallel' to Γ, we can define a 'shift' θ_h as a smooth homeomorphism with a smooth inverse which maps V_h onto V and Γ_h onto Γ, and such that all the derivatives of a 'shifted' function $(\theta_h\varphi)(t)$, $t\in V$, $\varphi\in C_0^\infty(T)$, where $(\theta_h f)(t)=f(\theta_h^{-1}t)$, $t\in V$, are bounded in V by a constant which is not dependent on h (but dependent on φ and on the order of derivative) (see e.g. [6], Chap. 2, 8.1), also the argument in the proof of Proposition 2.2). It is easy to see that $f'_h\equiv\theta_h f_h$ solves the Dirichlet problem

$$\begin{aligned} Af'_h&=\varphi_h, && t\in V,\\ \partial_j^+ f'_h&=\psi_{h,j}, && t\in\Gamma,\ j=0,\dots,p-1, \end{aligned} \tag{4.6}$$

and $\varphi_h-\varphi\to 0$, $\psi_{h,j}\to 0$, $j=0,\dots,p-1$ as $h\to 0$ with all their derivatives uniformly in V and Γ, respectively. This implies $f'_h\to f$ ($h\to 0$) with all derivatives uniformly in V (apply e.g. Th. 8.3, Chap. 2 [6] and Sobolev's lemma ([7], p. 280)), which yields (4.5).

Let $h>0$ be sufficiently small and let f_h denote the solution of (4.4), as above. By Green's formula (recall that A is symmetric):

$$\begin{aligned}\int_V \tilde{X}(t)\,\varphi(t)\,dt &= \int_{V_h} \tilde{X}(t)\,Af_h(t)\,dt \\ &= \sum_{j=0}^{p-1} \int_{\Gamma_h} \delta_j^h f_h(t)\,\partial_j^+ \tilde{X}(t)\,\sigma_h(dt),\end{aligned} \tag{4.7}$$

where $\delta_j^h = \sum_{|\alpha|\leqq 2p-j-1} b_{\alpha,j}^h(t)\,D^\alpha$, $j=0,\dots,p-1$ is a system of 'boundary' differential operators of order $2p-j-1$ respectively, and $\sigma_h(dt)$ is the surface measure on Γ_h (see e.g. [6], Chap. 2.2). Moreover, as it follows from the definition of δ_j^h, its coefficients $b_{\alpha,j}^h$ depend on h continuously, i.e.

$$\sup_{t\in\Gamma} |(\theta_h b_{\alpha,j}^h)(t) - b_{\alpha,j}^0(t)| \to 0 \qquad (h\to 0). \tag{4.8}$$

By means of a partition of unity in $C_0^\infty(\Gamma)$ as well as (4.5), (4.8), condition (c) in Definition 4.1 and the condition $\tilde{X}(t)=0$, $t\in(\bar{V})^c$ it can be shown that

$$E\Big[\Big(\int_{\Gamma_h} \delta_j^h f_h(t)\,\partial_j^+ \tilde{X}(t)\,\sigma_h(dt)\Big)^2\Big] \to 0 \qquad (h\to 0),$$

$j=0,\dots,p-1$. This implies $E[(\int_V \tilde{X}(t)\,\varphi(t)\,dt)^2]=0$ for $\varphi\in C_0^\infty(T)$, $\operatorname{supp}\varphi\subset V$ and hence (4.3) by the smoothness of $\tilde{X}$ in V (see above). Theorem 4.1 is proved.

Let's turn now to the existence of solution of the problem (4.1)–(4.2). Let again $(\Gamma_h, |h|<h_0)$ be a system of 'parallel' surfaces as in Sect. 2. We assume in addition that it satisfies the condition $2h \geqq \operatorname{dist}(\Gamma_h,\Gamma) \geqq h$ (such systems do exist), where $\operatorname{dist}(\Gamma_h,\Gamma)$ is the Euclidean distance between the sets Γ_h and Γ. Let $\omega = \omega(t)$, $t\in\mathbb{R}^d$ be a *regularizator*, i.e. a nonnegative smooth function with a compact support, say, in the unit ball of $\mathbb{R}^d$. We introduce a *regularization* of the field I_V in V_h (the inside of Γ_h) $(0<h<h_0)$ by[10]

$$I_V^h(t) = I_V(\omega_h(t-\cdot)), \qquad t\in V_h,$$

where $\omega_h(s)=\omega(s/h)(\int \omega(s/h)\,ds)^{-1}$. Denote $T_h=(t\in T\colon \operatorname{dist}(t,\partial T)>h)$, $\Lambda_h = T_h\cap(\bar{V}_{-h})^c$, $0<h<h_0$. Let $X_{V^c}^h$ be a regularization of X_{V^c} given by

$$X_{V^c}^h(t) = X_{V^c}(\omega_h(t-\cdot)), \qquad t\in\Lambda_h.$$

Finally, let $Z^h(t)$, $t\in V_{-h}$ be the solution of the Dirichlet problem in V_{-h} with the smooth data

$$\begin{aligned} AZ^h(t) &= \chi_h(t)\,I_V^h(t), && t\in V_{-h}, \\ \partial_j^+ Z^h(t) &= \partial_j^+ X_{V^c}^h(t), && t\in\Gamma_{-h},\ j=0,\dots,p-1 \end{aligned} \tag{4.9}$$

(we set $I_V^h(t)=0$, $t\in V_{-h}\setminus V_h$), where $\chi_h(t)$, $t\in V$ is a smooth function, $0\leqq\chi_h\leqq 1$, such that $\chi_h(t)=1$, $t\in V_{2h}$; $=0$, $t\in V_{-h}\setminus V_h$. Set

[10] Recall that $V_h\subset V$ and $V\subset V_{-h}$ if $0<h<h_0$

$$\tilde{X}^h(t)=\begin{cases} Z^h(t), & t\in V_{-h}, \\ X_V^h(t), & t\in \bar{\Lambda}_h = T_h\setminus V_{-h}, \\ 0, & t\in T\setminus T_h. \end{cases} \tag{4.10}$$

Then $\tilde{X}^h$ is an ordinary Gaussian field in T which is smooth in V_{-h} and Λ_h. The following theorem guarantees the existence of solution of the Dirichlet problem (4.1)–(4.2) in the sense of Definition 4.1 and also suggests a more 'constructive' method to obtain it (via approximations (4.9)–(4.10)).

Theorem 4.2. *Let A satisfy assumptions of Sect. 2. Then the solution $X=(X(\varphi), \varphi\in C_0^\infty(T))\in\Phi_p(T)$ of the problem* (4.1)–(4.2) *exists. Moreover, if $\tilde{X}^h$ is defined as above, then*

$$E[(X(\varphi)-\tilde{X}^h(\varphi))^2]\to 0 \qquad (h\to 0) \tag{4.11}$$

for any $\varphi\in C_0^\infty(T)$, where $\tilde{X}^h(\varphi)=\int \tilde{X}^h(t)\,\varphi(t)\,dt$.

Proof. To prove the existence of solution X, consider another probability space $(\Omega', \mathscr{F}', P')$ with a Gaussian g.r.f. $X'=(X'(\varphi), \varphi\in C_0^\infty(T))\in\Phi_p(T)$ on it, corresponding to the same operator A. Recall that $H(X')=H_{(\bar{V})^c}(X')\oplus H_V(I')$, where $I'(\varphi)=X'(A\varphi)$ (see the proof of Theorem 3.1, also Remark 3.2). Consider the operator S given by $SX'(\varphi)=X_{V^c}(\varphi)$, $\operatorname{supp}\varphi\subset(\bar{V})^c$; $SI'(\varphi)=I_V(\varphi)$, $\operatorname{supp}\varphi\subset V$. Then S can be extended to a unitary operator from $H(X')$ onto $H\equiv H(X_{V^c})\otimes H(I_V)$ which we denote by the same letter. For any $\varphi\in C_0^\infty(T)$, set

$$X(\varphi)=SX'(\varphi).$$

It follows from Remark 4.1 that $X=(X(\varphi), \varphi\in C_0^\infty(T))$ is solution of (4.1)–(4.2).

Consider now the regularization X^h of X:

$$X^h(t)=X(\omega_h(t-\cdot)), \qquad t\in T_h.$$

As $E[(X^h(\varphi)-X(\varphi))^2]\to 0$ ($h\to 0$ where $X^h(\varphi)=\int_{T_h} X^h(t)\,\varphi(t)\,dt$, to prove (4.11) is is enough to show that

$$E[(X^h(\varphi)-\tilde{X}^h(\varphi))^2]\to 0 \qquad (h\to 0),\ \varphi\in C_0^\infty(T).$$

Let $\varphi\in C_0^\infty(T)$ be given, and $f^{-h}=f^{-h}(t)$, $t\in V_{-h}$ $(0<h<h_0)$ be the solution of the Dirichlet problem (4.4). By Green's formula,

$$\begin{aligned}\tilde{X}^h(\varphi)&=\int_{\Lambda_h}\tilde{X}^h(t)\,\varphi(t)\,dt+\int_{V_{-h}}\tilde{X}^h(t)\,Af^{-h}(t)\,dt\\ &=\int_{\Lambda_h}\tilde{X}^h(t)\,\varphi(t)\,dt+\int_{V_{-h}}A\tilde{X}^h(t)\,f^{-h}(t)\,dt\\ &\quad+\sum_{j=0}^{p-1}\int_{\Gamma_{-h}}\delta_j^{-h}f^{-h}(t)\,\partial_j^+X^h(t)\,\sigma_{-h}(dt),\end{aligned}$$

and similary,

$$X^h(\varphi)=\int_{A_h} X^h(t)\,\varphi(t)\,dt+\int_{V_{-h}} AX^h(t)\,f^{-h}(t)\,dt$$
$$+\sum_{j=0}^{p-1}\int_{\Gamma_{-h}} \delta_j^{-h} f^{-h}(t)\,\partial_j^+ X^h(t)\,\sigma_{-h}(dt)$$

(see (4.7) for notations of δ_j^{-h}, and σ_{-h}). By definition,

$$\int_{A_h} (\tilde{X}^h(t)-X^h(t))\,\varphi(t)\,dt=0,$$

$$\int_{\Gamma_{-h}} \delta_j^{-h} f^{-h}(t)\,\partial_j^+ (\tilde{X}^h-X^h)(t)\,\sigma_{-h}(dt)=0, \qquad j=0,\dots,p-1$$

and

$$\int_{V_{2h}} (A\tilde{X}^h(t)-AX^h(t))\,f^{-h}(t)\,dt=0$$

as

$$A\tilde{X}^h(t)=I_V^h(t)=I_V(\omega_h(t-\cdot))=X(A\,\omega_h(t-\cdot))=AX^h(t)$$

for $t\in V_{2h}$ (note that $A_t\,\omega_h(t-s)=A_s\,\omega_h(t-s)$ as A is symmetric). Thus, it remains to prove that

$$E[(\int_{V_h\setminus V_{2h}} I_V^h(t)\,(1-\chi_h(t))\,f^{-h}(t)\,dt)^2]\to 0 \qquad (h\to 0) \tag{4.12}$$

and

$$E[(\int_{V_{-h}\setminus V_h} AX^h(t)\,f^{-h}(t)\,dt)^2]\to 0 \qquad (h\to 0). \tag{4.13}$$

Let us prove (4.13) as (4.12) can be treated in a similar way. Set $g_h(t)=f^{-h}(t)$ if $t\in V_{-h}\setminus V_h$, $=0$ if otherwise in T and write $\varepsilon(h)$ for the left hand side of (4.13). Then, as $AX^h(t)=X(A\,\omega_h(t-\cdot))$, $\varepsilon(h)=E[(X(A\,\omega_h^- * g_h))^2]=\|\omega_h^- * g_h\|_{\mathscr{H}}^2$ by (2.9), (2.7), where $*$ stands for the convolution in $\mathbb{R}^d$ and $\omega_h^-(s)=\omega_h(-s)$. By equivalence of the norms $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_p$, it suffices to verify that

$$\int_T |D^\alpha(\omega_h^- * g_h)(t)|^2\,dt\to 0 \qquad (h\to 0) \tag{4.14}$$

for any α: $|\alpha|\leqq p$. But

$$\begin{aligned}|D^\alpha(\omega_h^- * g_h)(t)|&=|D^\alpha \int_T \omega_h^-(t-s)\,g_h(s)\,ds|\\ &\leqq \int_{(s:\,\mathrm{dist}(s,\Gamma)\leqq 4h)} |(D^\alpha\,\omega_h^-)(t-s)|\,|g_h(s)|\,ds\end{aligned} \tag{4.15}$$

and

$$(D^\alpha\,\omega_h^-)(t)=\mathrm{const}\,h^{-d-|\alpha|}(D^\alpha\,\omega)(-t/h). \tag{4.16}$$

Now (4.4) implies that

$$\sup_{t\in T}|g_h(t)|\leqq \text{const}\, h^p \tag{4.17}$$

(this can be shown by an argument similar to that applied to prove relation (4.5) above). Therefore, by (4.15)-(4.17), $D^\alpha(\omega_h^- * g_h^-)(t)=0$ if $\operatorname{disr}(t,\Gamma)\geqq 5h$, and $|D^\alpha(\omega_h^- * g_h)(t)|\leqq \text{const}\, h^{p-d-|\alpha|}\int|(D^\alpha\omega)(s/h)|\,ds = \text{const}\, h^{p-|\alpha|}$ if otherwise. This implies (4.14) for $|\alpha|\leqq p$. Theorem 4.2 is proved.

5. Appendix. Proof of Proposition 1.1

Suppose that (ζ_t) is the i.p. for (X_t), and let be given a pair $(V_1, V_2)=((t_1,t_2), (a,t_1])\in\mathscr{V}_{1,2}$ $(a\leqq t_1<t_2<b)$. Write $\mathscr{A}\perp\!\!\!\perp\mathscr{B}$ whenever σ-algebras $\mathscr{A}$ and $\mathscr{B}$ are independent, $\mathscr{A},\mathscr{B}\subseteq\mathscr{F}$. By independence of increments of (ζ_t), $\mathscr{F}^\zeta_{(t_1+\varepsilon,t_2-\varepsilon)}\perp\!\!\!\perp\mathscr{F}^\zeta_{(t_1,t_2)^c}$ for ε small enough which implies $\mathscr{F}^\zeta_{(t_1,t_2)}=\bigvee_{\varepsilon>0}\mathscr{F}^\zeta_{(t_1+\varepsilon,t_2-\varepsilon)}\perp\!\!\!\perp\mathscr{F}^\zeta_{(t_1,t_2)^c}$, i.e. the system $(\mathscr{F}^\zeta_V)$ is $\mathscr{V}_1$-independent. Similarly, $\mathscr{F}^X_{(a,t_1]}=\bigcap_{\delta>0}\mathscr{F}^X_{t_1+\delta}\perp\!\!\!\perp\mathscr{F}^\zeta_{(t_1+\varepsilon,t_2)}$ for $\varepsilon>0$ and consequently $\mathscr{F}^X_{(a,t_1]}\perp\!\!\!\perp\mathscr{F}^\zeta_{(t_1,t_2)}$. It remains to verify (1.3), in our case the relation

$$\mathscr{F}^X_{(a,t_1]}\vee\mathscr{F}^\zeta_{(t_1,t_2)}=\mathscr{F}^X_{(a,t_2)}. \tag{5.1}$$

Note that stochastic continuity of (X_t) and (ζ_t) implies that

$$\mathscr{F}^X_t=\mathscr{F}^X_{(a,t)},\quad \mathscr{F}^\zeta_t=\mathscr{F}^\zeta_{(a,t)},\qquad t\in(a,b). \tag{5.2}$$

If $t_1=a$, $\mathscr{F}^X_{(a,t_1]}$ is trivial by the definition and (5.1) follows immediately from (1.5) and (5.2). If $a<t_1<t_2$, by (5.2), (1.5) and the definition of $(\zeta(\varphi))$ we have

$$\begin{aligned}\mathscr{F}^X_{(a,t_2)}&=\mathscr{F}^X_{t_2}=\mathscr{F}^\zeta_t=\mathscr{F}^\zeta_t\vee\mathscr{F}^\zeta_{(t_1,t_2)}\\&=\mathscr{F}^X_{t_1}\vee\mathscr{F}^\zeta_{(t_1,t_2)}=\mathscr{F}^X_{(a,t_1)}\vee\mathscr{F}^\zeta_{(t_1,t_2)},\end{aligned} \tag{5.3}$$

from which $\mathscr{F}^X_{(a,t_2)}\subseteq\mathscr{F}^X_{(a,t_1]}\vee\mathscr{F}^\zeta_{(t_1,t_2)}$. But (5.3) implies also $\mathscr{F}^\zeta_{(t_1,t_2)}\subseteq\mathscr{F}^X_{(a,t_2)}$ while $\mathscr{F}^X_{(a,t_1]}\subseteq\mathscr{F}^X_{(a,t_2)}$, i.e. $\mathscr{F}^X_{(a,t_1]}\vee\mathscr{F}^\zeta_{(t_1,t_2)}\subseteq\mathscr{F}^X_{(a,t_2)}$ which yields (5.1).

To prove the converse part, assume that $(\tilde\zeta(\varphi))$ is the I.F. for $(X(\varphi))$. This means in particular that $\mathscr{F}^{\tilde\zeta}_{(t_1,t_2)}\perp\!\!\!\perp\mathscr{F}^{\tilde\zeta}_{(a,t_1)}=\mathscr{F}^{\tilde\zeta}_{t_1}$ as $(\mathscr{F}^{\tilde\zeta}_V)$ is $\mathscr{V}_1$-independent and $\mathscr{F}^{\tilde\zeta}_{(a,t_1)}\subseteq\mathscr{F}^{\tilde\zeta}_{(t_1,t_2)^c}$, i.e. $(\tilde\zeta_t)$ is a process with independent increments. Now (5.1) and (5.2) with ζ replaced by $\tilde\zeta$ and $t_1=a$ results in $\mathscr{F}^X_{t_2}=\mathscr{F}^{\tilde\zeta}_{t_2}$, $a<t_2<b$, which completes the proof.

Proof of Proposition 1.2. Let $(\mathscr{O}_V)$ and $(\mathscr{F}_V)$ again be the systems related to g.r.f.s. I and X, correspondingly. It is not difficult to prove (see e.g. [2], Proposition 4.3.2) that[11]

$$\sigma(H_V(I))=\mathscr{O}_V,\qquad \sigma(H_V(X))=\mathscr{F}_V \tag{5.4}$$

[11] Erroneously the statement of Proposition 4.3.2 [2] was formulated for the version of definition of $\mathscr{A}_V$ as given in the footnote 2 on p. 276. In fact this result applies to the version which is used in this paper.

for any $V \in \mathscr{B}(T)$. Now the standard argument for Gaussian processes based on the equivalence between independence and orthogonallity in the Gaussian case implies the statement of Proposition 1.2.

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