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Fluctuations of the Phase Boundary in the 2D Ising Ferromagnet

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Abstract: We discuss some statistical properties of the phase boundary in the 2D low-temperature Ising ferromagnet in a box with the two-component boundary conditions. We prove the weak convergence in $\mathbf{C}[0,1]$ of measures describing the fluctuations of phase boundaries in the canonical ensemble of interfaces with fixed endpoints and area enclosed below them. The limiting Gaussian measure coincides with the conditional distribution of certain Gaussian process obtained by the integral transformation of the white noise.

1. Introduction

The large deviation probabilities for the total magnetization in the two-dimensional (2D) Ising ferromagnet are known to possess the non-classical asymptotics in the phase coexistence region. The exponential decay here is of the surface order [25, 14] reflecting the fact that the phase separation is the main mechanism responsible for this asymptotic behaviour. (Without being explicitly stated, this fact was essentially presented in the early papers by Minlos and Sinai [19, 20] where the case of d-dimensional ($d \geq 2$) Ising model was rigorously studied.) The rate function corresponds to the total surface tension of the phase boundary and the limiting shape of the latter can be described in the framework of the Wulff theory [7, 23]. Particularly, in the typical configurations, the immersed phase tends to form a unique macroscopic droplet with the shape and the area close to that of the Wulff droplet, i. e., the solution of the related variational problem. As a result, the optimal value of the Wulff functional provides the correct constant on the surface scale of the exponential decay of large deviations probabilities. Note the really remarkable fact that the last observation is actually true for all subcritical temperatures, i. e., in the whole phase coexistence region [15, 16].

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The results obtained in [7, 23, 15, 16] describe many interesting properties of the phase boundary as well as typical configurations in the considered situation. However, they are not sufficient to deliver the exact asymptotics of the probabilities of large deviations. To this end one needs more detailed information about the fluctuations of phase boundary with respect to the limiting Wulff shape, the information that is also of independent interest.

The present paper is an attempt on the way to fill this gap. Namely, we discuss statistical properties of phase boundary in the 2D low-temperature Ising ferromagnet with the two-component boundary conditions in the canonical ensemble of interfaces with fixed endpoints and fixed "area enclosed below them". We prove the weak convergence in $\mathbf{C}[0,1]$ of the probability distributions describing the fluctuations of such interfaces around the corresponding part of the Wulff shape to certain conditional Gaussian distribution. This limiting measure coincides with the conditional distribution of a Gaussian random process obtained by the integral transformation of the white noise.

As in the preceding paper [6], where a similar problem for general model of the SOS-type was investigated, we use extensively the large deviation principle in the strong form [8] combined with ideas further developed from the original book [7]. These results were announced in [13].

To our knowledge, there were only two mathematical papers [1] studying weak convergence of measures describing fluctuations of the phase boundary in the 2D Ising ferromagnet [12, 5]. Nevertheless, the methods used there were adjusted to the investigation of interfaces with fixed endpoints (even only horizontal ones in [12]) and are not applicable to the additional volume constraint discussed here.

The paper is organized as follows. Section 2 contains notions and known facts to be used later on. The main results are stated in Sect. 3. The basic polymer representation of the partition function is developed in Sect. 4. Then, in Sect. 5 we prove the analyticity of the corresponding free energy and discuss some its properties that are used in proofs of limit theorems in Sect. 6. Convergence of finite dimensional distributions of the considered conditional process is established in Sect. 7. The proof of the main result is completed in Sect. 8, where the tightness condition for the sequence of measures is checked. Finally, in the Appendix we present the geometric construction of the solution to the Wulff variational problem corresponding to the discussed situation.

Professor Roland Dobrushin left us forever when the work described in this paper was still in progress. But even this irreversible loss could not reduce his personal influence on the whole work – without any doubts, he is the main author of this result. In fact, this text is an attempt by the second author to realize some ideas of his Teacher. This paper is devoted to the memory of R. L. Dobrushin.

2. Preliminaries

To fix the notations let us recall briefly certain notions and facts from the theory of the 2D Ising model (for detailed discussion see, e. g., [7]).

Lattices. Let $\mathbb{Z}^2$ be the two-dimensional integer lattice and $\left(\mathbb{Z}^2\right)^*$ be its dual, $\left(\mathbb{Z}^2\right)^* = (\mathbb{Z}+1/2)^2$, both consisting of *sites*. These lattices are immersed into $\mathbb{R}^2$ equipped with the usual Euclidean distance $|\cdot|$, $|x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$, where $x=(x_1,x_2)$

[1] Many interesting ideas appeared already in the pioneering paper [9], where however only a particular one-dimensional distribution of the phase boundary was discussed.

and $y = (y_1, y_2)$. We call a *bond* any segment of unit length connecting two neighbouring sites of the dual lattice.

Let s, t be two neighbours in $\mathbb{Z}^2$ and f denote the unit segment connecting s and t. By definition, a bond e *separates* these sites if the segments f and e are orthogonal and meet at their midpoints.

Fix one of the two directions $(1, 1)$ and $(1, -1)$. Any straight line passing through a site in this fixed direction is called a *diagonal*. Thus, any site belongs to certain (uniquely determined) diagonal. By definition, a site $s \in \mathbb{Z}^2$ is *attached* to $s^* \in \left(\mathbb{Z}^2\right)^*$ provided they share the diagonal and $|s - s^*| = \sqrt{2}/2$. A site $s \in \mathbb{Z}^2$ is attached to a bond e if s is attached to one end of e.

Let e_1 and e_2 be two orthogonal bonds that share a site of the dual lattice. We say that e_1 and e_2 form a *linked pair* of bonds if they belong to the same half-plane in $\mathbb{R}^2$ determined by the diagonal passing through their common point.

For a set $V \subset \mathbb{Z}^2$, $|V|$ denotes its cardinality and ∂V is its outer boundary,

$$\partial V = \Big\{ s \in \mathbb{Z}^2 \setminus V : \exists t \in V \text{ with } |t - s| = 1 \Big\}.$$

A bond e is called a *boundary bond* of the set V if there exist $t \in V$ and $s \in \mathbb{Z}^2 \setminus V$ such that e separates t and s.

Configurations. For $V \subset \mathbb{Z}^2$ denote by $\Omega_V = \{-1, 1\}^V$ the set of all possible configurations $\sigma = \sigma_V$ in V. In the case $V = \{s\}$ the configuration σ_V is reduced to the *spin* at the site s and is denoted simply by σ_s. If V_N, $N > 1$, is the vertical strip in $\mathbb{Z}^2$ of the width N,

$$V_N = \Big\{ t = (t_1, t_2) \in \mathbb{Z}^2 : 0 < t_1 < N \Big\}, \tag{2.1}$$

we denote the corresponding set $\{-1, 1\}^{V_N}$ of configurations by Ω_N.

Fix any $V \subset \mathbb{Z}^2$. A configuration $\overline{\sigma} = \overline{\sigma}_{\mathbb{Z}^2 \setminus V}$ in the complement $\mathbb{Z}^2 \setminus V$ is called a *boundary condition* (for V). Two kinds of boundary conditions will be considered mainly in the following: the constant *plus* boundary condition $\overline{\sigma}^+$,

$$\overline{\sigma}^+_t = 1, \qquad \text{for all } t \in \mathbb{Z}^2 \setminus V, \tag{2.2}$$

and the two-component boundary condition $\overline{\sigma}^\varphi$, $\varphi \in (-\pi/2, \pi/2)$,

$$\overline{\sigma}^\varphi_t = \begin{cases} 1, & \text{if } t_2 > t_1 \tan \varphi, \\ -1, & \text{otherwise.} \end{cases} \tag{2.3}$$

Contours. Let σ be a configuration in a set $V \subset \mathbb{Z}^2$ and $\overline{\sigma}$ be a boundary condition. The *boundary* $\Gamma(\sigma, \overline{\sigma})$ of the configuration σ under the boundary condition $\overline{\sigma}$ is the collection of all bonds separating the sites in $\mathbb{Z}^2$ with different values of spins. Then any site s^* of the dual lattice is the meeting point of an even number of such bonds. If four bonds meet at a common vertex we split them up into two pairs of linked bonds. This procedure is actually a fixed choice of the so-called “rounding of corners” along the diagonal passing through the common vertex of these bonds. Apply this procedure at any dual site that is a meeting point of four bonds from $\Gamma(\sigma, \overline{\sigma})$. Then the boundary $\Gamma(\sigma, \overline{\sigma})$ splits up into connected components to be called *contours.*

Let V_{NM}, $M > 1$, be the set (cf. (2.1))

$$V_{NM} = \Big\{ t = (t_1, t_2) \in V_N : 1 - M < t_2 < M \Big\} \tag{2.4}$$

and $\overline{\sigma} \equiv \overline{\sigma}^{+}$. Then every contour of $\Gamma(\sigma, \overline{\sigma})$, $\sigma \in \Omega_{NM} = \{-1,1\}^{V_{NM}}$, is a closed polygon. For $\overline{\sigma} \equiv \overline{\sigma}^{\varphi}$ the boundary $\Gamma(\sigma, \overline{\sigma})$ contains one (infinite) open polygon S. In the case $M \geq [N \tan\varphi] + 1$ this open polygon passes through the points $(0, 1/2)$ and $(N, [N \tan\varphi] + 1/2)$.

Phase boundary. Let σ be a configuration in V_N (recall (2.1)) and $\overline{\sigma}^{\varphi}$ be the boundary condition defined in (2.3). As before, denote by $S \in \Gamma(\sigma, \overline{\sigma})$ the (infinite) open contour passing through the points $(0, 1/2)$ and $(N, [N \tan\varphi] + 1/2)$. Let $\Delta(S)$ be the set of all points from $\mathbb{Z}^2 \cap R_N$,

$$R_N = \Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0, N]\Big\},$$

that are attached to bonds of S. The restriction of S to the vertical strip R_N is called the *phase boundary* and is denoted also by S.

Let $\mathcal{T}_N^{\varphi}$ denote the set of all phase boundaries consistent with the boundary condition $\overline{\sigma}^{\varphi}$. Fix any $S \in \mathcal{T}_N^{\varphi}$. The point $(0, 1/2)$ is the *initial* point and $(N, [N \tan\varphi] + 1/2)$ is the *ending* point of the phase boundary S. By definition, the *height* $h(S)$ of S is the difference in the ordinates of the ending and the initial points of S. Thus, for $S \in \mathcal{T}_N^{\varphi}$ one has $h(S) = [N \tan\varphi]$.

Assume that $M = M(S) > 1$ is such that the contour S is covered by the rectangle $R_{NM} = [0, N] \times (1 - M, M)$. Then the polygon S splits up the rectangle R_{NM} into two parts, the "upper" and the "lower" ones, with the areas Q_N^{+} and Q_N^{-} respectively. The quantity

$$a(S) = a_N(S) = \frac{Q_N^{-} - Q_N^{+}}{2} \tag{2.5}$$

is called the *area* under the phase boundary S. Clearly, this definition does not depend upon M provided it is sufficiently large, $M \geq M_0(S)$. Observe also that for a "nice" contour S that intersects any vertical line $x = k$, $k = 1, 2, \dots, N-1$, at a unique point the quantity $a(S)$ gives the value of the integral of the piecewise constant function appearing after removing all vertical segments from S.

Gibbs measures. Let V be a finite subset of $\mathbb{Z}^2$ and $\overline{\sigma}$ be a boundary condition. The *Gibbs distribution* $\mathbb{P}_{V,\beta}(\cdot|\overline{\sigma})$ in V with the boundary condition $\overline{\sigma}$ is the probability measure in Ω_V given by

$$\mathbb{P}_{V,\beta}(\sigma|\overline{\sigma}) = Z(V, \beta, \overline{\sigma})^{-1} \exp\{-\beta \mathcal{H}(\sigma|\overline{\sigma})\}, \qquad \sigma \in \Omega_V, \tag{2.6}$$

where the hamiltonian $\mathcal{H}(\sigma|\overline{\sigma})$ is defined by

$$\mathcal{H}(\sigma|\overline{\sigma}) = -\sum_{\substack{s,t \in V,\\ |s-t|=1}} \sigma_s \sigma_t - \sum_{\substack{s \in V, t \in \partial V,\\ |s-t|=1}} \sigma_s \overline{\sigma}_t, \tag{2.7}$$

the partition function $Z(V, \beta, \overline{\sigma})$ is

$$Z(V, \beta, \overline{\sigma}) = \sum_{\sigma \in \Omega_V} \exp\{-\beta \mathcal{H}(\sigma|\overline{\sigma})\}, \tag{2.8}$$

and $\beta > 0$ denotes the inverse temperature. In what follows we will always assume that β is sufficiently large.

Ensembles of phase boundaries. Consider the box V_{NM} defined in (2.4) and let $\overline{\sigma}^{\varphi}$ be the boundary condition from (2.3). Let $\mathbb{P}_{N,M,\beta}(\cdot|\overline{\sigma}^{\varphi})$ be the Gibbs distribution in

$\Omega_{NM} = \{-1,1\}^{V_{NM}}$ defined as in (2.6)–(2.8). For $M > N\tan\varphi$ denote by $\mathcal{T}_{NM}^{\varphi}$ the set of all phase boundaries in V_{NM} consistent with the boundary condition $\overline{\sigma}^{\varphi}$. The Gibbs distribution $\mathbb{P}_{N,M,\beta}(\cdot|\overline{\sigma}^{\varphi})$ induces the probability distribution $\mathbf{P}_{N,M,\beta,\varphi}(\cdot)$ in $\mathcal{T}_{NM}^{\varphi}$ according to the following formula:

$$\mathbf{P}_{N,M,\beta,\varphi}(S) = \mathbf{P}_{N,M,\beta}\Big(\Big\{\sigma \in \Omega_{NM} : \Gamma(\sigma,\overline{\sigma}^{\varphi}) \ni S\Big\} \;\Big|\; \overline{\sigma}^{\varphi}\Big), \qquad S \in \mathcal{T}_{NM}^{\varphi}.$$

Another form of this distribution will be of importance in the following ([7, §4.3]). Namely, let $\Phi(\Lambda)$ be the function of finite subsets in $\mathbb{Z}^2$ determined from the cluster expansion of the partition function $Z(V_{NM},\beta,\overline{\sigma}^{+})$ ([7, §3.9]), $|S|$ denote the length [2] of the polygon S, and $\Delta(S)$ is the set of all sites attached to the phase boundary. Then, defining the weights $w_{NM}(S)$ via

$$w_{NM}(S) = \exp\Big\{-2\beta|S| - \sum_{\Lambda\subset V_{NM}:\Lambda\cap\Delta(S)\neq\emptyset}\Phi(\Lambda)\Big\}, \tag{2.9}$$

we rewrite

$$\mathbf{P}_{N,M,\beta,\varphi}(S) = \frac{w_{NM}(S)}{\Xi(N,M,\varphi)}, \tag{2.10}$$

where $\Xi(N,M,\varphi)$ is the corresponding partition function,

$$\Xi(N,M,\varphi) = \sum_{S\in\mathcal{T}_{NM}^{\varphi}} w_{NM}(S).$$

For future reference we recall here the following important properties of the function $\Phi(\Lambda)$ ([7, §3.9,§4.3]): $\Phi(\Lambda)$ is a translation invariant function vanishing on non-connected sets $\Lambda\subset\mathbb{Z}^2$; moreover, there exists $\beta_0<\infty$ such that for all $\beta\geq\beta_0$ one has

$$|\Phi(\Lambda)| \leq \exp\{-2(\beta-\beta_0)d(\Lambda)\}, \tag{2.11}$$

where the function $d(\Lambda)$ satisfies the inequality

$$d(\Lambda) > 2\mathrm{diam}(\Lambda) + 2 \tag{2.12}$$

with $\mathrm{diam}(\Lambda)$ denoting the diameter of the set Λ, $\mathrm{diam}(\Lambda) = \max\{|x-y| : x,y\in\Lambda\}$. According to Lemma 3.10 ([7]), estimate (2.11) implies the inequality

$$\sum_{\Lambda\subset\mathbb{Z}^2:\Lambda\cap\Delta(S)\neq\emptyset} |\Phi(\Lambda)| \leq K|S|, \tag{2.13}$$

where $K = K(\beta)$ is a constant such that $K\searrow 0$ as $\beta\nearrow+\infty$. Therefore, for all sufficiently large β the weights (cf. (2.9))

$$w(S) = \exp\Big\{-2\beta|S| - \sum_{\Lambda:\Lambda\cap\Delta(S)\neq\emptyset}\Phi(\Lambda)\Big\} \tag{2.14}$$

are well defined.

Let $\mathcal{T}_N^{\varphi} = \cup_M \mathcal{T}_{NM}^{\varphi}$ be the set of all phase boundaries in V_N consistent with the boundary condition $\overline{\sigma}^{\varphi}$ and $\mathcal{T}_N = \cup_{\varphi}\mathcal{T}_N^{\varphi}$ denote the set of all possible phase boundaries

[2] Observe that two external halfbonds of S did not contribute to $|S|$ in [7] but this does not affect the value on the right-hand side of (2.10).

in V_N (the union here is over all $\varphi \in (-\pi/2, \pi/2)$). Due to [7, Theorem 4.8] the quantities

$$\Xi(N,\varphi) = \sum_{S \in \mathcal{T}_N^{\varphi}} w(S), \qquad \Xi(N) = \sum_{S \in \mathcal{T}_N} w(S)$$

are finite (in fact, $\Xi(N)$ coincides with the partition function $\Xi(N, 0, \mathrm{restr})$, where $\Xi(N, H, \mathrm{restr})$ is the partition function for the restricted grand canonical ensemble of the phase boundaries (see definition (4.3.16) in [7])). As a result, one can define the probability distributions $\mathbf{P}_{N,\beta,\varphi}(\cdot) \equiv \mathbf{P}_{N,+\infty,\beta,\varphi}(\cdot)$ and $\mathbf{P}_{N,\beta}(\cdot)$ in $\mathcal{T}_N^{\varphi}$ and $\mathcal{T}_N$ respectively via the following formulas:

$$\mathbf{P}_{N,\beta,\varphi}(S) = \frac{w(S)}{\Xi(N,\varphi)}, \qquad S \in \mathcal{T}_N^{\varphi}, \tag{2.15}$$

and

$$\mathbf{P}_{N,\beta}(S) = \frac{w(S)}{\Xi(N)}, \qquad S \in \mathcal{T}_N. \tag{2.16}$$

Here again one has the condition $\beta \geq \beta_1 > \beta_{\mathrm{cr}}$ that is a consequence of application of the cluster expansions technique.

Surface tension, free energy, Legendre transformation. For any fixed $\varphi \in (-\pi/2, \pi/2)$ denote by $\mathbf{n} = \mathbf{n}(\varphi) = (-\sin\varphi, \cos\varphi)$ the unit orthogonal vector to the straight line $t_2 = t_1 \tan\varphi$ in $\mathbb{R}^2$. Let the box V_{NM}, $M > N \tan\varphi$, be as in (2.4) and $Z(V_{NM}, \beta, \overline{\sigma})$ denote the partition function in Ω_{NM} corresponding to the boundary condition $\overline{\sigma}$. By definition, the *surface tension* in the direction of $\mathbf{n}$ is given by

$$\tau_\beta(\mathbf{n}) = -\lim_{N\to\infty} \lim_{M\to\infty} \frac{\cos\varphi}{\beta N} \log \frac{Z(V_{NM}, \beta, \overline{\sigma}^{\varphi})}{Z(V_{NM}, \beta, \overline{\sigma}^{+})}, \tag{2.17}$$

where the boundary conditions $\overline{\sigma}^{\varphi}$ and $\overline{\sigma}^{+}$ are defined by (2.3) and (2.2) respectively.

The surface tension is closely related to another important function, the so-called free energy. To define it we fix any $\delta > 0$ and for any complex number H satisfying the condition

$$|\Re H| < 2 - \delta/\beta, \tag{2.18}$$

we introduce the partition function

$$\Xi(N, H) = \sum_{S \in \mathcal{T}_N} \exp\{\beta H h(S)\} w(S) \tag{2.19}$$

with $h(S)$ denoting the height of the phase boundary S. The limit

$$F(H) = \lim_{N\to\infty} \frac{\log \Xi(N,H)}{N} \tag{2.20}$$

is called the *free energy* corresponding to the height $h(S)$ of the phase boundary. According to Theorem 4.8 [7] this limit exists and is an analytical function of H in the domain (2.18).

The free energy $F(H)$ defined in (2.20) is dual to the surface tension $\tau_\beta(\cdot)$. Namely ([7, Theorem 4.12]), one has

$$\tau_\beta(\mathbf{n}) = \frac{1}{\beta} F^*(\beta \tan\varphi) \cos\varphi, \tag{2.21}$$

where $f^*(\cdot)$ denotes the *Legendre transformation* of the real convex function [3] $f : \mathbb{R} \to \mathbb{R}$,

$$f^*(p) = \sup_x \big(px - f(x)\big).$$

The following property of the Legendre transformation will be used below.

Property 2.1. *Let $f(\cdot)$ be a strictly convex twice continuously differentiable real function defined in a region $U \subset \mathbb{R}^m$, $m \geq 1$, and $f^*(p)$ be its Legendre transformation, $f^*(p) \equiv \sup_x \big((x,p) - f(x)\big)$, $p \in \mathbb{R}^m$. Assume that the values $x \in U$ and $p \in \mathbb{R}^m$ are related via $\nabla f(x) = p$. Then the following relations hold:*

$$\begin{aligned} &f^*(p) = (x,p) - f(x),\\ &\nabla f^*(p) = x,\\ &\mathbf{Hess}\, f^*(p) = \big(\mathbf{Hess}\, f(x)\big)^{-1}. \end{aligned} \tag{2.22}$$

Observe that in the considered case the matrix **Hess** $f(x)$ of the second derivatives $f(x)$ as a function of $x \in \mathbb{R}^m$ is strictly positive definite at x.

This duality property of the Legendre transformation can be verified directly or induced from the known facts ([24, Chap. 5]).

Wulff shape. Let $\tau_\beta(\varphi) = \tau_\beta(\mathbf{n})$ be the surface tension defined in (2.17). Using the symmetry properties of the lattice $\mathbb{Z}^2$ we easily have

$$\tau_\beta(\varphi) \equiv \tau_\beta(\pi/2 - \varphi), \qquad \tau_\beta(\varphi) \equiv \tau_\beta(-\varphi),$$

and thus $\tau_\beta(\mathbf{n})$ can be defined for all unit vectors $\mathbf{n} \in \mathbb{S}^1$.

Denote by $\mathcal{D}$ the set of all closed self-avoiding rectifiable curves $\gamma \subset \mathbb{R}^2$ that are boundaries of bounded regions (thus, boundary of any bounded convex region belongs to $\mathcal{D}$). Recall that any such rectifiable curve has finite length and has a tangent at its almost every point. To each $\gamma \in \mathcal{D}$ we assign the quantity

$$\mathcal{W}(\gamma) = \mathcal{W}_\beta(\gamma) = \int_\gamma \tau_\beta(\mathbf{n}_s)\, ds, \tag{2.23}$$

where ds denotes the length element and $\mathbf{n}_s$ is the unit outward normalvector to the curve γ at the point $s \in \gamma$. The functional (2.23) is called the *Wulff functional* corresponding to the surface tension $\tau_\beta(\cdot)$.

For any $\gamma \in \mathcal{D}$ denote by Vol(γ) the area of the enclosed region. By definition, the *Wulff shape* w_β is a solution to the variational problem

$$\mathcal{W}_\beta(\gamma) \to \inf : \qquad \gamma \in \mathcal{D}, \quad \mathrm{Vol}(\gamma) \geq 1.$$

Alternatively, one defines

$$W_{\beta,\lambda} = \cap_{\mathbf{n}\in\mathbb{S}^1} \left\{x \in \mathbb{R}^2 : (x, \mathbf{n}) \leq \lambda \tau_\beta(\mathbf{n})\right\},$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $\mathbb{R}^2$, $\mathbf{n}$ is a unit vector, and $\tau_\beta(\cdot)$ is the surface tension defined in (2.17). Then the Wulff shape w_β coincides with the boundary of the set W_{β,λ_0}, where λ_0 is determined from the condition Vol$(W_{\beta,\lambda_0}) = 1$. The Wulff

[3] Here and in the following we omit restrictions near the signs like upper bounds, sums, integrals, etc. when the appropriate operation is going over the whole set of possible values of parameters, summation indices, integration variables respectively.

shape is known to be unique up to translations in $\mathbb{R}^2$ [26, 27]. Due to positiveness of the stiffness, [4] $\tau_\beta(\varphi) + \frac{d^2}{d\varphi^2}\tau_\beta(\varphi)$, the Wulff shape is a smooth strictly convex closed curve in $\mathbb{R}^2$ and inherits the natural symmetries from $\mathbb{Z}^2$ [7, §2.20, §4.21].
Wulff profile. The main goal of the present paper is to study the statistical properties of phase boundaries of the 2D Ising ferromagnet in a bulk with the two-component boundary conditions $\overline{\sigma}^\varphi$. More precisely, we investigate the limiting behaviour of probability distributions $\mathbf{P}_{N,\beta,\varphi}(\cdot)$ ($\mathbf{P}_{N,M,\beta,\varphi}(\cdot)$ resp.) in the canonical ensemble of phase boundaries $S \in \mathcal{T}_N^\varphi$ ($\mathcal{T}_{NM}^\varphi$ resp.) with fixed value of the area (recall (2.5))

$$a_N(S) = N^2 q_N, \qquad q_N \to q \quad \text{as } N \to \infty,$$

enclosed below them. The phase boundary here is an open polygon; thus, its limiting behaviour is closely related to the corresponding piece of the Wulff shape to be called below the *Wulff profile*.

To construct the Wulff profile we use the following geometric algorithm. [5] Let l be a non-vertical straight line intersecting the Wulff shape at two different points O and A (we denote by A that of them that is to the left; see Fig. 1,a)). The segment OA splits up the interior of the Wulff shape into two parts, the "upper" one Q_l^+ and the "lower" one Q_l^- with the areas $|Q_l^+|$ and $|Q_l^-| = 1 - |Q_l^+|$ correspondingly. Clearly, Q_l^+ and Q_l^- are convex sets having tangents at all their boundary points except O and A.

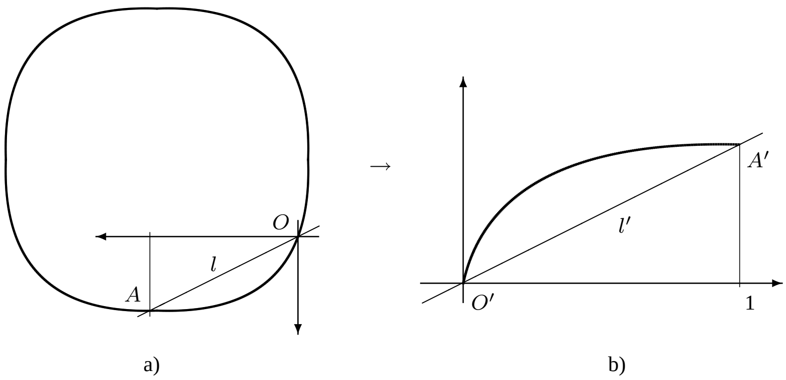

Fig. 1. Geometric construction of the Wulff profile

We say that the line l generates a (q, φ)-cutting of the Wulff shape if the following two conditions hold: a) the line l has the slope angle φ; b) the area $|Q_l^-|$ ($|Q_l^+|$ in the case $q < \frac{1}{2}\tan\varphi$) satisfies the equality

$$|Q_l^-| = |q - \frac{\tan\varphi}{2}| \cdot |OA|^2 \cos^2\varphi$$

with $|OA|$ denoting the length of the segment OA (and thus $|OA|\cos\varphi$ is its horizontal projection). Due to the strict convexity of the Wulff shape w_β, for any $q \in \mathbb{R}$ and

[4] Here we treat the surface tension $\tau_\beta(\cdot)$ as a function of φ (recall that $\mathbf{n} = (-\sin\varphi, \cos\varphi)$).
[5] The analytical expression for the Wulff profile in terms of the free energy $F(\cdot)$ from (2.20) is given in (3.14) below. See also the Appendix for more detailed discussion of the problem in a framework of a general 1D SOS model.

$\varphi \in (-\pi/2, \pi/2)$ there exists a unique (q, φ)-cutting of w_β (for $q = \frac{1}{2}\tan\varphi$ the points O and A coincide, and l becomes a tangent to the Wulff shape). If, in addition, the limiting value q is relatively small,

$$\left| q - \frac{1}{2}\tan\varphi \right| < Q_0(\varphi) \tag{2.24}$$

(with $Q_0(\varphi)$ easily identified in terms of the Wulff shape), all the tangents to Q_l^- at its boundary points (different from O and A) have uniformly bounded slope angles. Then the simple transformation (reflection + scaling; see Fig. 1,b)) of the arc OA gives the corresponding Wulff profile (in the degenerate case $q = \frac{1}{2}\tan\varphi$ the Wulff profile becomes a segment $O'A'$).

It what follows we will always assume the validity of condition (2.24) (which, in particular, will make possible the SOS approximation of phase boundaries for sufficiently large values of the inverse temperature β).

3. Results

Let $\mathcal{T}_N$ be the set of all possible phase boundaries in V_N and $\mathbf{P}(\cdot) \equiv \mathbf{P}_{N,\beta}(\cdot)$ denote the probability distribution from (2.16). Let $\mathbf{E}(\cdot) \equiv \mathbf{E}_{N,\beta}(\cdot)$ be the corresponding operator of mathematical expectation.

Fix any $S \in \mathcal{T}_N$ and for all $k = 0, 1, \ldots, N$ define

$$g_N^+(k) = \max\{t_2 : (k, t_2) \in S\}. \tag{3.1}$$

Let $g_N^+(x)$, $x \in [0, N]$, be the piecewise linear interpolation of the values $g_N^+(k)$. Denote by $\xi_N^+(t)$, $t \in [0, 1]$, the random polygonal function

$$\xi_N^+(t) = g_N^+(Nt) - g_N^+(0). \tag{3.2}$$

Our aim here is to describe the statistical properties of trajectories $\xi_N^+(t)$ conditioned by fixing the values of the area $a_N(S)$ and the height $h(S)$.

More precisely, let Λ_N be the random vector

$$\Lambda_N = (Y_N, h_N), \tag{3.3}$$

where $h_N = h_N(S)$ is the height of $S \in \mathcal{T}_N$ and

$$Y_N = \frac{1}{N} a_N(S) \tag{3.4}$$

is the normalized area under S (recall (2.5)). For $\mathbf{H} = (H_0, H_1)$, denote by $L_{\Lambda_N}(\mathbf{H})$ the logarithmic moment generating function of the random vector Λ_N (recall (2.16)),

$$L_{\Lambda_N}(\mathbf{H}) \equiv \log \mathbf{E} \exp\Big\{\beta\big(\mathbf{H}, \Lambda_N\big)\Big\} = \log \Xi(N, \Lambda, \mathbf{H}) - \log \Xi(N), \tag{3.5}$$

where the partition function $\Xi(N, \Lambda, \mathbf{H})$ is calculated via

$$\Xi(N, \Lambda, \mathbf{H}) = \sum_{S \in \mathcal{T}_N} \exp\Big\{-2\beta|S| + \beta H_0 Y_N + \beta H_1 h_N - \sum_{\Lambda : \Lambda \cap \Delta(S) \neq \emptyset} \Phi(\Lambda)\Big\}. \tag{3.6}$$

We will show below (see Remark 5.1.1) that the last expression is finite provided the real part $\Re\mathbf{H}$ of $\mathbf{H} = (H_0, H_1)$ belongs to the set

$$\mathcal{D}_\delta^2 = \Big\{(H_0, H_1) \in \mathbb{R}^2 : |H_1| < 2 - \delta/\beta, |H_1 + H_0| < 2 - \delta/\beta\Big\} \tag{3.7}$$

with some $\delta > 0$ and $\beta \geq \beta_0(\delta)$.

Consider any sequence of real vectors $A_N = (Nq_N, Nb_N)$ such that $2N^2 q_N$ and Nb_N are integer numbers and

$$N^{-1}A_N \to A = (q, b), \qquad 2q \neq b, \tag{3.8}$$

in such a way that

$$N^{-1}A_N - A = o\Big(\frac{1}{\sqrt{N}}\Big) \qquad \text{as } N \to \infty. \tag{3.9}$$

Definition 3.1. *Let δ be a positive number. Any sequence A_N satisfying (3.8)–(3.9) is called (Λ_N, δ)-regular if the following conditions hold:*

1) for any $N > 1$,

$$\mathbf{P}(\Lambda_N = A_N) > 0; \tag{3.10}$$

2) for all $N > 1$ there exists a solution $\mathbf{H}_N \in \mathcal{D}_\delta^2$ of the equation

$$\beta^{-1}\nabla_{\mathbf{H}} L_{\Lambda_N}(\mathbf{H}) \Big|_{\mathbf{H}=\mathbf{H}_N} = A_N; \tag{3.11}$$

3) there exists a solution $\widehat{\mathbf{H}} = (Q, H) \in \mathcal{D}_\delta^2$ of the equation

$$\mathcal{I}(\mathbf{H}) \equiv \beta^{-1}\nabla_{\mathbf{H}} \int_0^1 F(H_0 y + H_1)\, dy \Big|_{\mathbf{H}=\widehat{\mathbf{H}}} = A. \tag{3.12}$$

Here $\mathcal{D}_\delta^2$ is the set from (3.7), $\nabla_{\mathbf{H}}$ denotes the gradient with respect to $\mathbf{H} = (H_0, H_1)$ and $F(\cdot)$ is the free energy from (2.20).

Remark 3.1.1. It can be checked directly that (3.10) is true provided Nb_N and $2N^2q_N$ are integer numbers of the same parity.

Remark 3.1.2. The condition $\mathbf{H}_N \in \mathcal{D}_\delta^2$ for all $N > 1$ is a technical one; namely, we will show below (see the discussion after (7.5)) that the inclusion $\widehat{\mathbf{H}} \in \mathcal{D}_\delta^2$ implies $\mathbf{H}_N \in \mathcal{D}_\delta^2$ for all sufficiently large N.

Remark 3.1.3. Using the strict convexity of the function $F(\cdot)$ one can show that the relations $2q \neq b$ and $Q \neq 0$ are equivalent (see also the discussion the in Appendix below).

Fix any $(\Lambda_N, \bar{\delta})$-regular sequence A_N and consider the conditional random process

$$\theta_N^+(t) = \big(\xi_N^+(t) | \Lambda_N = A_N\big) \tag{3.13}$$

with $\xi_N^+(t)$ defined in (3.2). Applying arguments similar to those used in [7] one can prove the law of large numbers for the process $\theta_N^+(t)$. Namely, the distribution of the process tends weakly in the space $\mathbf{C}[0,1]$ of continuous function on the segment $[0,1]$ to the distribution concentrated on some deterministic function $\hat{e}(t)$, $t \in [0,1]$. The function $\hat{e}(t)$ presents the solution of the following variational problem (cf. (2.23), (2.21)):

$$\mathcal{W}(f) = \int_0^1 \beta^{-1} F^*(f'(t))\, dt \to \inf,$$
$$f \in \left\{ g \in \mathcal{AC}[0,1] : g(0) = 0, g(1) = b, \int_0^1 g(t)\, dt = q \right\}$$

(here $\mathcal{AC}[0,1]$ is the space of absolutely continuous functions on $[0,1]$) and can be computed explicitly,

$$\hat{e}(t) = \big(F(H+Q) - F(H+Q-Qt)\big)\big/\beta Q, \tag{3.14}$$

where (Q,H) is the solution of (3.12). Observe that due to Remark 3.1.3 one has $Q \neq 0$ and thus $\hat{e}(t)$ is well defined. Moreover, in view of the inclusion $(Q,H) \in \mathcal{D}_\delta^2$, the derivative of $\hat{e}(t)$ is uniformly bounded in $[0,1]$.

Consider the random process

$$\theta_N^*(t) = \frac{1}{\sqrt{N}}\big(\theta_N^+(t) - N\hat{e}(t)\big), \qquad t \in [0,1], \tag{3.15}$$

and denote the corresponding measure in $\mathbf{C}[0,1]$ by $\mu_N^* = \mu_N^{+,*}$. The following theorem formulates the main result of the present paper.

Theorem 3.2. *Let a $(\Lambda_N, \overline{\delta})$-regular sequence A_N be as described above. Then there exists $\beta_0 = \beta_0(\overline{\delta}) < \infty$ such that for all $\beta \geq \beta_0$ the sequence of measures μ_N^* converges weakly to some Gaussian measure μ^* in $\mathbf{C}[0,1]$. The limiting measure μ^* coincides with the conditional probability distribution of the random process $\hat{\xi}(t)$, $t \in [0,1]$, obtained by the integral transformation of the white noise dw_s,*

$$\hat{\xi}(t) \equiv \beta^{-1} \int_0^t \big(F''(H+Q-Qs)\big)^{1/2}\, dw_s,$$

conditioned by the conditions

$$\hat{\eta} \equiv \int_0^1 \hat{\xi}(t)\, dt = 0 \qquad \textit{and} \qquad \hat{\xi}(1) = 0.$$

Remark 3.2.1. The random vector Λ_N from (3.3) has zero mean and the variances of its components are of order N (see Lemma 6.1 below). Therefore, the condition $2q \neq b$ means that the events $\{\Lambda_N = A_N\}$ are in the large deviation region for the distribution $\mathbf{P}_{N,\beta}(\cdot)$.

Plan of the proof of Theorem 3.2. The proof of our main result follows the same scenario used in the case of random walks [6] with necessary modifications.

Namely, for any natural number k and a set $\mathcal{S}$ of real numbers s_i, $0 < s_1 < s_2 < \ldots < s_k < 1 = s_{k+1}$, consider the random vector

$$\Theta_N \equiv \big(Y_N, X_N(s_1), \ldots, X_N(s_k), X_N(1)\big) \in \mathbb{R}^{k+2}, \tag{3.16}$$

where Y_N was defined in (3.4), and $X_N(t)$, $t \in [0,1]$, are calculated via (cf. (3.2))

$$X_N(t) \equiv g_N^+([Nt]) - g_N^+(0), \tag{3.17}$$

with $[Nt]$ denoting the integral part of Nt. Let $\mathcal{M}_N^{k+2}$, $k = 0, 1, \ldots$, be the set

$$\mathcal{M}_N^{k+2} = \Big\{\mathbf{M} = (m_0, m_1, \ldots, m_{k+1}) : \{2Nm_0, m_1, \ldots, m_{k+1}\} \subset \mathbb{Z}^1\Big\}. \tag{3.18}$$

Then for any $\mathbf{M}_N \in \mathcal{M}_N^{k+2}$ of the kind $\mathbf{M}_N = (Nq_N, m_n^1, \ldots, m_N^k, Nb_N)$ one has the relation

$$\mathbf{P}\big(X_N(s_1) = m_N^1, \ldots, X_N(s_k) = m_N^k \mid \Lambda_N = A_N\big) = \frac{\mathbf{P}(\Theta_N = \mathbf{M}_N)}{\mathbf{P}(\Lambda_N = A_N)}. \tag{3.19}$$

Here $\Lambda_N = (Y_N, X_N(1))$ is the vector from (3.3) and $A_N = (Nq_N, Nb_N)$ is the $(\Lambda_N, \overline{\delta})$-regular sequence fixed above.

First, we investigate the asymptotical behaviour of the numerator and the denominator in (3.19) and obtain the central limit theorem for the finite dimensional distributions of the random process

$$\Theta_N(t) \equiv \big(X_N(t) \mid \Lambda_N = A_N\big). \tag{3.20}$$

Then, we prove that the difference between the conditional process $\theta_N^+(t)$ (recall (3.13)) and $\Theta_N(t)$ has uniformly bounded exponential moments in some neighbourhood of the origin. This observation implies immediately the same central limit theorem for the corresponding finite dimensional distributions of the process $\theta_N^+(t)$.

Finally, we check the following inequality:

$$\mathbf{E}|\theta_N^*(t) - \theta_N^*(s)|^4 \le C|t-s|^{7/4}$$

with some constant $C > 0$ uniformly in $s, t \in [0,1]$ and sufficiently large N. This implies the weak compactness of the sequence μ_N^* and finishes the proof by applying known results on weak convergence of measures in $\mathbf{C}[0,1]$ ([10]). □

A similar result holds also for the random process

$$\theta_N^-(t) = \big(\xi_N^-(t) | \Lambda_N = A_N), \qquad t \in [0,1],$$

induced by the lowest points of intersection (cf. (3.1)),

$$g_N^-(k) = \min\{t_2 : (k, t_2) \in S\},$$

via

$$\xi_N^-(t) = g_N^-(Nt) - g_N^-(0).$$

Let $\mu_N^{-,*}$ denote the probability distribution in $\mathbf{C}[0,1]$ corresponding to the process (recall (3.15))

$$\theta_N^{-,*}(t) = \frac{1}{\sqrt{N}}\big(\theta_N^-(t) - N\hat{e}(t)\big), \qquad t \in [0,1].$$

Theorem 3.3. *For the sequences of measures $\mu_N^{-,*}$ the statement of Theorem 3.2 holds true. Moreover, for any sequence of real numbers α_N, $\alpha_N \to 0$ as $N \to \infty$, one has the convergence*

$$\alpha_N\big(\theta_N^+(t) - \theta_N^-(t)\big) \to 0 \tag{3.21}$$

in probability as $N \to \infty$.

Clearly, the formulated results are valid also for the measures $\mu_{NM}^{\pm,*}$ describing the statistical properties of the phase boundaries $S \in \mathcal{T}_{NM}^{\varphi}$ in the box V_{NM} with the boundary condition $\overline{\sigma}^{\varphi}$, provided only $M > \big(\max_{t\in[0,1]} |\hat{e}(t)| + \varepsilon\big)N$ with any fixed $\varepsilon > 0$. This follows immediately from the observation that the events $\{\max_{t\in[0,1]}|N^{-1}\theta_N^{\pm}(t) - \hat{e}(t)| > \varepsilon\}$ belong to the large deviations region for the measures $\mu_N^{\pm}$ and thus have exponentially small probabilities as $N \to \infty$.

4. Basic Representation of the Partition Function

We start with discussing the statistical properties of the vector $\varTheta_N$ of joint distribution (recall (3.16)),

$$\varTheta_N \equiv \bigl(Y_N, X_N(s_1), \dots, X_N(s_k), X_N(s_{k+1})\bigr) \in \mathbb{R}^{k+2}, \tag{4.1}$$

where k is a natural number, the quantities s_i satisfy the condition $0 < s_1 < \dots < s_k < s_{k+1} = 1$, the normalized area Y_N is defined in (3.4), and the process $X_N(t)$, $t \in [0,1]$, is determined via (recall (3.17))

$$X_N(t) \equiv g_N^+([Nt]) - g_N^+(0). \tag{4.2}$$

For future reference we consider the more general situation. Namely, fix any natural number k and a collection $\mathcal{R} = \{r_1, \dots, r_{k+1}\}$ of natural numbers (they can depend on N, i. e., $r_i = r_{i,N}$) such that for all sufficiently large $N \ge N_0(\mathcal{R})$ one has the relation

$$0 < r_1 < \dots < r_k < r_{k+1} = N.$$

Denote (cf. (4.2))

$$X(r_i) \equiv g_N^+(r_i) - g_N^+(0), \tag{4.3}$$

and consider the random vector

$$\varTheta_{N,\mathcal{R}} = \bigl(Y_N, X(r_1), \dots, X(r_k), X(r_{k+1})\bigr) \in \mathbb{R}^{k+2}. \tag{4.4}$$

For any complex vector $\mathbf{H} = (H_0, H_1, \dots, H_{k+1}) \in \mathbb{C}^{k+2}$ we denote by $L_{N,\mathcal{R}}(\mathbf{H})$ the logarithmic moment generating function of the random vector $\varTheta_{N,\mathcal{R}}$,

$$L_{N,\mathcal{R}}(\mathbf{H}) \equiv \log \mathbf{E} \exp\{\beta\bigl(\mathbf{H}, \varTheta_{N,\mathcal{R}}\bigr)\}.$$

Observe that the last equality can be rewritten in the form (cf. (3.5))

$$L_{N,\mathcal{R}}(\mathbf{H}) = \log \varXi(N, \mathcal{R}, \mathbf{H}) - \log \varXi(N), \tag{4.5}$$

where

$$\varXi(N, \mathcal{R}, \mathbf{H}) = \sum_{S \in \mathcal{T}_N} \exp\Bigl\{-2\beta|S| + \beta\bigl(\mathbf{H}, \varTheta_{N,\mathcal{R}}\bigr) - \sum_{\Lambda:\Lambda\cap\Delta(S)\neq\emptyset} \Phi(\Lambda)\Bigr\}. \tag{4.6}$$

As we will show below (see Theorem 5.1), the last expression is finite provided $\Re\mathbf{H}$ belongs to the set

$$\begin{aligned}\widehat{\mathcal{D}}_\delta^{k+2} = \{\mathbf{H} = (H_0, H_1, \dots, H_{k+1}) \in \mathbb{R}^{k+2} : H_0 \in \bigl(-\tfrac{\delta}{4\beta(k+2)}, Q + \tfrac{\delta}{4\beta(k+2)}\bigr),\\ |H_i| < \tfrac{\delta}{4\beta(k+2)}, i = 1, \dots, k, |H_{k+1} - H| < \tfrac{\delta}{4\beta(k+2)}\},\end{aligned} \tag{4.7}$$

where (Q, H) is the solution of (3.12) and δ is the positive number fixed in Definition 3.1 above.

Since the partition function $\varXi(N, \mathcal{R}, \mathbf{H})$ contains all the information about the statistical properties of the random vector $\varTheta_{N,\mathcal{R}}$, we will study it carefully in the remaining part of this section. Following [7], we split up every phase boundary $S \in \mathcal{T}_N$ into pieces that are typical at low temperatures ("tame animals") and pieces to be interpreted as excitations appearing at non-vanishing temperatures ("wild animals").

Let us recall briefly the necessary considerations ([7, §4.4]). Denoting

$$\Psi(\Lambda) = \exp\{-\Phi(\Lambda)\} - 1,$$

we observe that there exists $\beta_0 < \infty$ such that

$$|\Psi(\Lambda)| \leq \exp\{-2(\beta - \beta_0) d(\Lambda)\} \tag{4.8}$$

for all $\beta \geq \beta_0$ and any finite set Λ (cf. (2.11)–(2.12)). In particular, $\Psi(\Lambda)$ vanishes on non-connected sets Λ.

Denote by $\mathcal{C}_N$ the set of all collections $\mathbf{C} = \{S, \Lambda_1, \dots, \Lambda_j\}$, where $S \in \mathcal{T}_N$, finite sets $\Lambda_i \subset \mathbb{Z}^2$ are connected and satisfy the condition $\Lambda_i \cap \Delta(S) \neq \emptyset$, $i = 1, \dots, j$; $j = 0, 1, \dots$; here $\Delta(S)$ is the set of all sites attached to the phase boundary S. Then the partition function $\Xi(N, \mathcal{R}, \mathbf{H})$ can be rewritten in the form

$$\begin{aligned} \Xi(N, \mathcal{R}, \mathbf{H}) &= \sum_{S \in \mathcal{T}_N} \exp\Big\{-2\beta|S| + \beta\big(\mathbf{H}, \Theta_{N,\mathcal{R}}\big)\Big\} \prod_{\Lambda : \Lambda \cap \Delta(S) \neq \emptyset} \big(\Psi(\Lambda) + 1\big) \\ &= \sum_{\mathbf{C} \in \mathcal{C}_N} \exp\Big\{-2\beta|S| + \beta\big(\mathbf{H}, \Theta_{N,\mathcal{R}}\big)\Big\} \prod_{l=1}^{j} \Psi(\Lambda_l). \end{aligned} \tag{4.9}$$

Fix any $\mathbf{C} = \{S, \Lambda_1, \dots, \Lambda_j\} \in \mathcal{C}_N$. We say that the collection $\mathbf{C}$ is *regular* in the column $m \in \mathbb{N}$ if the line $\{(x, y) \in \mathbb{R}^2 : x = m\}$ intersects the set $S \cup \Lambda_1 \cup \ldots \cup \Lambda_j$ at a unique point. Let $1 \leq m_1 < m_2 < \ldots < m_l \leq N - 1$, $l = l(\mathbf{C}) \in \{0, 1, \dots, N - 1\}$, be the set of all m, $1 \leq m \leq N - 1$, such that the collection $\mathbf{C}$ is regular in the column m. Denote

$$\begin{aligned} \triangle_1 &= \{(x, y) \in \mathbb{R}^2 : x \leq m_1\}, \\ \triangle_2 &= \{(x, y) \in \mathbb{R}^2 : m_1 \leq x \leq m_2\}, \\ &\dots \\ \triangle_l &= \{(x, y) \in \mathbb{R}^2 : m_{l-1} \leq x \leq m_l\}, \\ \triangle_{l+1} &= \{(x, y) \in \mathbb{R}^2 : m_l \leq x\} \end{aligned}$$

(in the case $l = 0$ we have $\triangle_1 = \mathbb{R}^2$). By definition, the *animal* ξ_i, $i = 1, \dots, l + 1$, is the collection

$$\xi_i = \{S_i, \Lambda_{j_1}, \dots, \Lambda_{j_s}\},$$

where

$$S_i = S \cap \triangle_i, \qquad \{\Lambda_{j_1}, \dots, \Lambda_{j_s}\} = \{\Lambda \in \mathbf{C} : \Lambda \subset \triangle_i\}.$$

Let $(m_i, y_i) = S \cap \{(x, y) \in \mathbb{R}^2 : x = m_i\}$, $i = 1, \dots, l$. We put also $(m_0, y_0) = (0, 1/2)$ and $(m_{l+1}, y_{l+1}) = (N, h(S) + 1/2)$. For any animal ξ_i we define the following quantities: the *length* $|\xi_i|$ that coincides with the length of the polygon S_i; the *base* $J(\xi_i) = (m_{i-1}, m_i]$; the *width* $|J(\xi_i)| = m_i - m_{i-1}$; the *height* $h(\xi_i) = y_i - y_{i-1}$ with (m_{i-1}, y_{i-1}) and (m_i, y_i) denoting the *beginning* and the *end* of the animal ξ_i. Then, we define the *area* $a(\xi_i)$ below ξ_i as

$$a(\xi_i) = \frac{1}{2}(a_i^- - a_i^+),$$

where a_i^- and a_i^+ denote the areas of the lower and the upper parts of the rectangle $[m_{i-1}, m_i] \times [y_{i-1} - M, y_{i-1} + M]$ that appear after cutting it along S_i (clearly, this definition is independent of M provided it is sufficiently large, $M \geq M_0(S)$; cf. (2.5)). Finally, for $r \in J(\xi_i) = (m_{i-1}, m_i]$ we denote by $h(r, \xi_i)$ the height of the animal ξ_i in the r^{th} column,

$$h(r,\xi_i) = g_N^+(r) - g_N^+(m_{i-1}).$$

Direct computations give us the following relations:

$$\begin{aligned} h(S) &= \sum_{i=1}^{l+1} h(\xi_i), \\ X(r) &= \sum_{i=1}^{j(r)-1} h(\xi_i) + h(r,\xi_j(r)), \\ a(S) &= \sum_{i=1}^{l+1} \big(a(\xi_i) + (N-m_i)h(\xi_i)\big) \end{aligned} \tag{4.10}$$

with $j(r)$ denoting j such that $r \in J(\xi_j)$. Define the *activity* of ξ_i via

$$\begin{aligned} \Psi_{N,\mathcal{R},\mathbf{H}}(\xi_i) = \exp\Big\{ &-2\beta|\xi_i| + \beta h(\xi_i)\Big(\Big(1-\frac{m_i}{N}\Big)H_0 + \sum_{n=1}^{k+1} \mathbf{1}_{\{i<j(r_n)\}} H_n\Big) \\ &+ \beta \sum_{n=1}^{k+1} \mathbf{1}_{\{i=j(r_n)\}} H_n h(r_n, \xi_{j(r_n)}) + \beta H_0 \frac{1}{N} a(\xi_i)\Big\} \prod_{\Lambda_s \in \xi_i} \Psi(\Lambda_s), \end{aligned} \tag{4.11}$$

where $\mathbf{1}_{\{i<j(r_n)\}}$ and $\mathbf{1}_{\{i=j(r_n)\}}$ denote the indicator functions of the relations $i < j(r_n)$ and $i = j(r_n)$ correspondingly. Then the partition function $\Xi(N,\mathcal{R},\mathbf{H})$ can be rewritten in the form

$$\Xi(N,\mathcal{R},\mathbf{H}) = \sum_{\mathbf{C}\in\mathcal{C}_N} \prod_{i=1}^{l(\mathbf{C})} \Psi_{N,\mathcal{R},\mathbf{H}}(\xi_i). \tag{4.12}$$

Fix any animal ξ. An animal ξ' is called *vertically congruent* to ξ iff it can be obtained by shifting all components of ξ on the same distance in the vertical direction. Let $\hat{\xi}$ denote the class of all animals that are vertically congruent to ξ. Clearly, all $\xi \in \hat{\xi}$ have the same length, base, height, etc. and thus have the same activity $\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})$. Observe that any collection $\mathbf{C} \in \mathcal{C}_N$ can be rewritten in the form $\{\hat{\xi}_1,\dots,\hat{\xi}_{l+1}\}$ such that the class $\hat{\xi}_i$ has the base $J(\hat{\xi}_i) = (m_{i-1}, m_i]$ and $0 = m_0 < m_1 < \ldots < m_{l+1} = N$. On the other hand, to any such collection $\{\hat{\xi}_1,\dots,\hat{\xi}_{l+1}\}$ corresponds a unique $\mathbf{C} \in \mathcal{C}_N$; therefore, there exists a one-to-one mapping between $\mathcal{C}_N$ and the set $\widehat{\mathcal{K}}_N$ of all ordered collections $\{\hat{\xi}_1,\dots,\hat{\xi}_{l+1}\}$ described above. As a result, (4.12) can be rewritten in the form

$$\Xi(N,\mathcal{R},\mathbf{H}) = \sum_{\{\hat{\xi}_1,\dots,\hat{\xi}_{l+1}\}\in\widehat{\mathcal{K}}_N} \prod_{i=1}^{l+1} \Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}_i). \tag{4.13}$$

In a similar way we consider the set $\widehat{\mathcal{K}}_{(a,b]}$, $(a,b] \subseteq [0,N]$, a, $b \in \mathbb{N}$, of ordered collections $\{\hat{\xi}_1,\dots,\hat{\xi}_l\}$ of the equivalence classes $\hat{\xi}_i$ such that $J(\hat{\xi}_i) = (m_{i-1}, m_i]$ and $a = m_0 < m_1 < \ldots < m_{l+1} = b$. Using the activities from (4.11) we introduce the partition function

$$\Xi((a,b],N,\mathcal{R},\mathbf{H}) = \sum_{\{\hat{\xi}_1,\dots,\hat{\xi}_{l+1}\}\in\widehat{\mathcal{K}}_{(a,b]}} \prod_{i=1}^{l+1} \Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}_i). \tag{4.14}$$

(In the case $a = b$ we put as usually $\Xi(\emptyset, N, \mathcal{R}, \mathbf{H}) = 1$.) Relations (4.13) and (4.14) will be the starting point of our considerations.

It follows from estimate (2.13) that the weights $w(S)$ from (2.14) coincide asymptotically as $\beta \to \infty$ with $\exp\{-2\beta|S|\}$. Therefore, the probability distribution (2.15) is "close" to the distribution concentrated on the polygons $S \in \mathcal{T}_N^{\varphi}$ of minimal length. It is convenient to consider a slightly larger set of phase boundaries

$$\mathcal{T}_{N,\infty} = \{S \in \mathcal{T}_N : |S \cap \{(x,y) : x = m\}| = 1, \forall m = 0, \dots, N\} \tag{4.15}$$

and the probability distribution

$$\mathbf{P}_{N,\beta,\infty}(S) = \frac{\exp\{-2\beta|S|\}}{\Xi(N,\beta,\infty)}, \qquad S \in \mathcal{T}_{N,\infty}, \tag{4.16}$$

with the partition function

$$\Xi(N,\beta,\infty) = \sum_{S \in \mathcal{T}_{N,\infty}} \exp\{-2\beta|S|\}. \tag{4.17}$$

Note that according to definition (4.15) every $S \in \mathcal{T}_{N,\infty}$ is regular in any column m, $m = 0, \dots, N$. Therefore, any animal ξ corresponding to $S \in \mathcal{T}_{N,\infty}$ has unit width and is called a *tame* animal. The probability distribution $\mathbf{P}_{N,\beta,\infty}(\cdot)$ from (4.16)–(4.17) is called the *ensemble of tame animals*. Any animal that is not tame is called *wild.*

For any $S \in \mathcal{T}_{N,\infty}$ one has $|S| = |\xi_1| + \dots + |\xi_N|$. Moreover, for any tame animal ξ one easily gets $|J(\xi)| = 1$, $|\xi| = |h(\xi)| + 1$, $a(\xi) = h(\xi)/2$, and therefore (cf. (4.10))

$$X(r) = \sum_{j=1}^{r} h(\xi_j), \qquad a(S) = \sum_{j=1}^{N} \bigl(N - j + 1/2\bigr) h(\xi_j).$$

As a result, the distribution (4.16)–(4.17) coincides with the distribution of the homogeneous random walk with the generating function $Z(H)$ of one step,

$$Z(H) \equiv \mathbf{E} \exp\{\beta H h(\xi)\} = Q(H)/Q(0),$$

where

$$Q(H) = \sum_{k=-\infty}^{+\infty} \exp\{-2\beta\bigl(|k| + 1\bigr) + \beta H k\} = e^{-2\beta} \frac{\sinh(2\beta)}{\cosh(2\beta) - \cosh(H\beta)}. \tag{4.18}$$

Thus, the limiting behaviour of the phase boundary S in the ensemble of tame animals with fixed values $X(N) = Nb_N$ and $a(S) = N^2 q_N$ can be described by Theorem 2.3 from [6], where such asymptotics for a general random walk was investigated. To extend that result to the case of the probability distribution $\mathbf{P}_{N,\beta}(\cdot)$ (recall (2.16)) in the ensemble of phase boundaries $S \in \mathcal{T}_N$ (i. e., to prove Theorem 3.2) is the main goal of the present paper.

In the ensemble of tame animals the partition function (4.6) is reduced to

$$\Xi(N, \mathcal{R}, \mathbf{H}, \infty) = \sum_{S \in \mathcal{T}_{N,\infty}} \exp\Bigl\{-2\beta|S| + \beta\bigl(\mathbf{H}, \Theta_{N,\mathcal{R}}\bigr)\Bigr\}. \tag{4.19}$$

We rewrite it in the form

$$\Xi(N,\mathcal{R},\mathbf{H},\infty) = \prod_{j=1}^{N} Q(H_{N,j}), \tag{4.20}$$

where $Q(\cdot)$ was defined in (4.18) and the quantities $H_{N,j}$, $j = 1,\dots,N$, are calculated via

$$H_{N,j} = \big(1-(j-1/2)/N\big)H_0 + \sum_{n=1}^{k+1} H_n \mathbf{1}_{\{j\le r_n\}}. \tag{4.21}$$

For future reference we define also the partition function (cf. (4.14))

$$\Xi((a,b],N,\mathcal{R},\mathbf{H},\infty) = \prod_{j=a+1}^{b} Q(H_{N,j}), \tag{4.22}$$

where $(a,b] \subseteq [0,N]$ is a segment with integer endpoints a, b. Here again $\Xi(\emptyset,N,\mathcal{R},\mathbf{H},\infty) = 1$.

Observe that the function $Q(\cdot)$ is finite for all H such that $|\Re H| < 2$. Moreover, if for some $\delta > 0$ and $\beta \ge \beta_0(\delta) > 0$ one has

$$|\Re H| < 2 - \delta/2\beta, \tag{4.23}$$

then

$$\frac{|\cosh(H\beta)|}{\cosh(2\beta)} \le \frac{\cosh(|\Re H|\beta)}{\cosh(2\beta)} \le \frac{\cosh(2\beta-\delta/2)}{\cosh(2\beta)} < e^{-\delta/4} \tag{4.24}$$

if only $\beta \ge \beta_0(\delta) > 0$ and therefore

$$\Big|\frac{\tanh(2\beta)}{e^{2\beta}Q(H)} - 1\Big| \le \frac{\cosh(2\beta-\delta/2)}{\cosh(2\beta)} < e^{-\delta/4}. \tag{4.25}$$

As a result, $\log Q(H)$ is well defined and uniformly bounded for all real H satisfying (4.23) with any fixed $\beta \ge \beta_0(\delta) > 0$.

Consider arbitrary $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ (recall (4.7)). Then any $H_{N,j}$ from (4.21) satisfies (4.23) and therefore the function $N^{-1}\log\Xi(N,\mathcal{R},\mathbf{H},\infty)$ is bounded uniformly in N and any such $\mathbf{H}$. Since the asymptotical properties of the partition function $\Xi(N,\mathcal{R},\mathbf{H},\infty)$ are well understood ([6]), we can reduce the investigation of the partition function $\Xi(N,\mathcal{R},\mathbf{H})$ from (4.6) to the study of the relative partition function

$$\widehat{\Xi}(N,\beta,\mathcal{R},\mathbf{H}) = \frac{\Xi(N,\mathcal{R},\mathbf{H})}{\Xi(N,\mathcal{R},\mathbf{H},\infty)}. \tag{4.26}$$

In the remaining part of this section we develop the so-called polymer representation of this partition function and obtain certain estimates for the polymer weights. All the considerations will be applicable also to the relative partition function

$$\widehat{\Xi}((a,b],N,\mathcal{R},\mathbf{H}) \equiv \frac{\Xi((a,b],N,\mathcal{R},\mathbf{H})}{\Xi((a,b],N,\mathcal{R},\mathbf{H},\infty)} \tag{4.27}$$

(recall (4.14), (4.22)) for any interval $(a,b] \subseteq (0,N]$ with integer endpoints.

Substituting (4.13) and (4.20) into (4.26) one easily obtains [6]

[6] Here and below j is always an integer number; therefore, $j \in J(\hat\xi)$ means $j \in J(\hat\xi) \cap \mathbb{Z}^1$. For any segment $I = (a,b] \subseteq [0,N]$ with integer endpoints we denote by $|I|$ its length, $|I| = b-a$.

$$\widehat{\Xi}(N,\beta,\mathcal{R},\mathbf{H}) = \sum_{\{\hat{\xi}_1,\dots,\hat{\xi}_{l+1}\}\in\widehat{\mathcal{K}}_N} \prod_{i=1}^{l+1}\Big(\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}_i)\prod_{j\in J(\hat{\xi}_i)} Q(H_{N,j})^{-1}\Big). \tag{4.28}$$

For any segment $I = (a,b] \subseteq [0,N]$ denote

$$\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I) = \Big(\prod_{j\in I} Q(H_{N,j})\Big)^{-1} \sum_{\hat{\xi}:J(\hat{\xi})=I} \Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}). \tag{4.29}$$

Then (4.28) can be rewritten in the form

$$\widehat{\Xi}(N,\beta,\mathcal{R},\mathbf{H}) = \sum_{\alpha=0}^{[N/2]} \sum_{\{I_1,I_2,\dots,I_\alpha\}} \prod_{i=1}^{\alpha} \widehat{X}_{N,\mathcal{R},\mathbf{H}}(I_i), \tag{4.30}$$

where the inner sum is taken over all families of mutually disjoint intervals $I_i = (a_i,b_i] \subseteq [0,N]$ such that $|I_i| \geq 2$. Observe that $|I| = 1$ implies $\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I) = 1$.

Formula (4.30) is a particular case of the polymer representation of the partition function ([17, 7]). To apply the cluster expansions technique we need the following estimate (cf. [7, Lemma 4.7]).

Lemma 4.1. *Let $\mathbf{H} \in \mathbb{C}^{k+2}$ be such that $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ and a real number γ satisfies the condition*

$$0 \leq \gamma \leq \delta/8.$$

For any interval $I \subset (0,N]$ with integer endpoints put

$$\widehat{\widehat{\mathrm{X}}}_{N,\mathcal{R},\mathbf{H}}(I) = \Big(\prod_{j\in I} Q(H_{N,j})\Big)^{-1} \sum_{\hat{\xi}:I(\hat{\xi})=I} \Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}) \exp\{\gamma|\hat{\xi}|\}.$$

Then there exists $\overline{\beta} > 0$ depending only upon the value β_0 from (4.8) and on the constant δ such that for all $\beta \geq \overline{\beta}$ and all intervals $I \subset (0,N]$ under consideration one has

$$|\widehat{\widehat{\mathrm{X}}}_{N,\mathcal{R},\mathbf{H}}(I)| \leq \exp\{-4(\beta-\overline{\beta})(|I|-1)\}. \tag{4.31}$$

The functions $\widehat{\widehat{\mathrm{X}}}_{N,\mathcal{R},\mathbf{H}}(I)$ depend analytically on such $\mathbf{H}$.

Remark 4.1.1. Putting $\gamma = 0$ we obtain estimate (4.31) for the polymer weights $\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I)$ from (4.29).

Proof. We start with the following observation. Let ξ be a wild animal with the base $J(\xi) = (m', m'']$ and let a natural number m satisfy the condition $m' < m < m''$. Since ξ is not regular in the column m at least one of the following two events occurs: 1) the vertical line $\{(x,y) \in \mathbb{R}^2 : x = m\}$ intersects the corresponding part $S = S_\xi$ of the phase boundary at least at three points; 2) a point from some set $\Lambda \in \xi$ belongs to the column m and thus at least two boundary bonds of the set Λ are intersected by this line. Therefore, for any wild animal $\xi = (S, \Lambda_1, \dots, \Lambda_k)$ one has the inequality

$$|J(S)| - 1 \leq \frac{1}{2}\Big(N_h(S) - \big(|J(S)|-1\big) + \sum_{\Lambda\in\xi} d(\Lambda)\Big),$$

where $N_h(S)$ denotes the number of full horizontal bonds in S, the function $d(\cdot)$ satisfies (2.11)–(2.12) and $J(S) \equiv J(\xi)$. As a result,

$$\sum_{\Lambda\in\xi} d(\Lambda) \geq 3\big(|J(S)| - 1\big) - N_h(S).$$

Denote

$$\widetilde{X}(S) \equiv \sum_{\substack{k,\Lambda_1,\ldots,\Lambda_k:\Lambda_i\cap\Delta(S)\neq\emptyset \\ \sum_{i=1}^k d(\Lambda)\geq 3(|J(S)|-1)-N_h(S)}} \exp\{-2(\beta-\beta_0)\sum_{i=1}^k d(\Lambda_i)\} \tag{4.32}$$

and fix any $\beta_1 > 0$. As it was shown in [7] (see Eq. (4.7.11)), there exists a function $\varepsilon = \varepsilon(\beta_1)$, $\varepsilon(\beta_1)\searrow 0$ as $\beta_1\nearrow\infty$, such that

$$\widetilde{X}(S) \leq \exp\{-6(\beta-\beta_2)\big(|J(S)| - 1\big) + 2(\beta-\beta_2)N_h(S)\}\exp\{\varepsilon|S|\} \tag{4.33}$$

with $\beta_2 = \beta_0 + \beta_1$. Define

$$X_{N,\mathcal{R},\mathbf{H}}(S) = \sum_{\xi:S_\xi=S} |\Psi_{N,\mathcal{R},\mathbf{H}}(\xi)|, \tag{4.34}$$

where the sum is taken over all wild animals ξ with fixed $S_\xi = S$. We prove below the following estimate:

$$X_{N,\mathcal{R},\mathbf{H}}(S) \leq \exp\{-2\beta|S| + (2\beta - \delta/2)N_v(S)\}\widetilde{X}(S) \tag{4.35}$$

with $N_v(S)$ denoting the number of vertical bonds in S. Then (4.31) follows directly.

Indeed, for any $\mathbf{H}$, $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, one has (recall (4.25))

$$|Q(H_{N,j})|^{-1} \leq e^{2\beta+2\beta_3}$$

with some $\beta_3 = \beta_3(\beta_0,\delta)$. Therefore the inequality

$$\left|\widehat{\mathrm{X}}_{N,\mathcal{R},\mathbf{H}}(I)\right| \leq \left|\prod_{j\in I} Q(H_{N,j})\right|^{-1} \sum_{S:I(S)=I,\ y_{\text{in}}(S)=0} X_{N,\mathcal{R},\mathbf{H}}(S)e^{\gamma|S|}$$

(here $y_{\text{in}}(S)$ denotes the y-coordinate of the initial point of S) can be rewritten in the form

$$\begin{aligned}\left|\widehat{\mathrm{X}}_{N,\mathcal{R},\mathbf{H}}(I)\right| \leq{}& e^{-(4\beta-6\beta_2+2\beta_3)(|I|-1)}e^{2\beta+2\beta_3}e^{-(2\beta-\varepsilon-\gamma)} \\ &\sum_{S:I(S)=I,\ y_{\text{in}}(S)=0} \exp\{\big(2\beta - 2\beta_2 - (2\beta-\varepsilon-\gamma)\big)N_h(S)\} \\ &\qquad \exp\{(-\delta/2 + \varepsilon + \gamma)N_v(S)\},\end{aligned}$$

where the identity $|S| = N_v(S) + N_h(S) + 1$ was used. Let β_1 be such that $\varepsilon = \varepsilon(\beta_1) < \delta/8$ and $\beta_2 = \beta_0 + \beta_1 \geq \beta_3$. Then

$$\left|\widehat{\mathrm{X}}_{N,\mathcal{R},\mathbf{H}}(I)\right| \leq e^{-4(\beta-2\beta_2)(|I|-1)} \sum_{\substack{S:I(S)=I, \\ y_{\text{in}}(S)=0}} e^{-\beta_4(N_h(S)+1)-\delta N_v(S)/4}, \tag{4.36}$$

where we used the obvious inequality $2\beta_2 + 2\beta_3 \le 2(\beta_2 + \beta_3)\bigl(|I| - 1\bigr)$ (recall that for any wild animal ξ one has $|J(\xi)| > 1$) and denoted $\beta_4 = 2\beta_2 - \varepsilon - \gamma$. It remains to observe that the last sum was shown to be bounded [7, p. 119],

$$\sum_{S:I(S)=I,\ y_{\text{in}}(S)=0} e^{-\beta_4(N_h(S)+1)-\delta N_v(S)/4} \le R(\beta_4,\delta)^{|I|}\bigl(1 - R(\beta_4,\delta)\bigr)^{-1}, \tag{4.37}$$

provided β_4 is large enough, $\beta_4 \ge \overline{\beta}_4(\delta)$, to guarantee the estimate

$$R(\beta_4,\delta) = 2e^{-\beta_4}\frac{1 + e^{-\delta/4}}{1 - e^{-\delta/4}} < 1.$$

As a result, (4.31) follows directly from (4.36) and (4.37).

It remains to establish (4.35). To do this we cut the polygon S into pieces by any vertical line $x = m$, $m \in \mathbb{N}$. Then S splits up into certain collection of zigzag fragments f_n consisting of two horizontal half-bonds and (possibly) a vertical segment of S. The ordering of f_n in S determines in a unique way the initial and ending points of f_n. Define the height $h(f_n)$ of f_n as the difference between ordinates of ending and initial points of f_n. Clearly,

$$h(\xi_i) = \sum_{f_n\in\xi_i} h(f_n), \qquad N_v(\xi_i) = \sum_{f_n\in\xi_i} |h(f_n)|. \tag{4.38}$$

Define the midpoint c_n of the fragment f_n as the midpoint of the vertical segment belonging to f_n (provided it is not empty) or as the midpoint of the fragment f_n itself (otherwise). Let d_n denote the distance from c_n to the vertical line $x = m_i$ passing through the ending point of the animal ξ_i (recall that $J(\xi_i) = (m_{i-1}, m_i]$). The direct geometric considerations give the equality

$$a(\xi_i) = \sum_{f_n\in\xi_i} d_n h(f_n). \tag{4.39}$$

Now, (4.38) and (4.39) imply the relation (cf. (4.11))

$$h(\xi_i)\Bigl(1 - \frac{m_i}{N}\Bigr) + \frac{1}{N}a(\xi_i) = \sum_{f_n\in\xi_i} h(f_n)\Bigl(1 - \frac{m_i - d_n}{N}\Bigr). \tag{4.40}$$

Then, the inclusion $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ and the inequality

$$1/2 \le d_n \le |I| - 1/2 < m_i \tag{4.41}$$

imply the estimate (recall (3.7), (4.7))

$$\begin{aligned}\Re\Bigl\{\beta h(\xi_i)\Bigl(\Bigl(1 - \frac{m_i}{N}\Bigr)H_0 + \sum_{l=1}^{k}\mathbf{1}_{\{i<j(r_l)\}}H_l + H_{k+1}\Bigr) + \beta H_0\frac{1}{N}a(\xi_i)\Bigr\}& \\ \le \bigl(2\beta - 3\delta/4\bigr)N_v(S).&\end{aligned} \tag{4.42}$$

On the other hand, from the inclusion $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ and the obvious inequality $|h(r_l, \xi_{j(r_l)})| \le N_v(S_{\xi_{j(r_l)}})$ one easily obtains

$$\Re\Big\{\beta\sum_{n=1}^{k}\mathbf{1}_{\{i=j(r_n)\}}H_n h(r_n,\xi_{j(r_n)})\Big\}\le\frac{\delta}{4}N_v(S_{\xi_i}).\tag{4.43}$$

Finally, (4.35) follows immediately from (4.34), (4.11), (4.7), (4.42) and (4.43). Estimate (4.31) is proved.

It remains to observe that the uniform estimates obtained above imply the analyticity of $\widehat{\widehat{\mathrm{X}}}_{N,\mathcal{R},\mathbf{H}}(I)$ as a function of $\mathbf{H}$, $\Re\mathbf{H}\in\widehat{\mathcal{D}}_{\delta}^{k+2}$. □

Corollary 4.2. *Let the polymer weights $\widetilde{X}_{N,\mathcal{R},\mathbf{H}}(I)$ be defined as in (4.29) with the activities $\Psi_{N,\mathcal{R},\mathbf{H}}(\xi_i)$ replaced by (cf. (4.11))*

$$\widetilde{\Psi}_{N,\mathcal{R},\mathbf{H}}(\xi_i)=\exp\Big\{-2\beta|\xi_i|+\beta h(\xi_i)\Big(H_0\Big(1-\frac{m_i}{N}\Big)+\sum_{n=1}^{k+1}H_n\mathbf{1}_{\{i<j(r_n)\}}\Big)+\beta\sum_{n=1}^{k+1}H_n h(r_n,\xi_{j(r_n)})\mathbf{1}_{\{i=j(r_n)\}}\Big\}\prod_{\Lambda_s\in\xi_i}\Psi(\Lambda_s).\tag{4.44}$$

Then there exist constants β_0 and $N_0=N_0(\beta_0)$ such that for all $\beta\ge\beta_0$, $N\ge N_0$ and all segments $I=(a,b]\subseteq[0,N]$, $b-a\le\log^2 N$, with integer endpoints one has the estimate

$$\Big|\widetilde{X}_{N,\mathcal{R},\mathbf{H}}(I)-\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I)\Big|\le 2\Big(e^{2\beta\log^4 N/N}-1\Big)\exp\{-4(\beta-\beta_0)(|I|-1)\}.\tag{4.45}$$

Proof. We start with the following simple observation. There exists $\bar{\beta}>0$ (probably different from β in (4.31)) such that for all $\alpha_N>0$ and all $\beta\ge\bar{\beta}$ one has

$$\Big|\Big(\prod_{j\in I}Q(H_{N,j})\Big)^{-1}\sum_{\substack{\hat{\xi}:I(\hat{\xi})=I\\ N_v(\hat{\xi})\ge\alpha_N}}\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big|\le e^{-\delta\alpha_N/8}e^{-4(\beta-\bar{\beta})(|I|-1)}.\tag{4.46}$$

Indeed, using the relation (cf. (4.37))

$$\begin{aligned}\sum_{\substack{S:J(S)=I,\ y_{\text{in}}(S)=0,\\ N_v(S)\ge\alpha_N}}&e^{-\beta_4(N_h(S)+1)-\delta N_v(S)/4}\\ &\le e^{-\delta\alpha_N/8}\sum_{S:J(S)=I,\ y_{\text{in}}(S)=0}e^{-\beta_4(N_h(S)+1)-\delta N_v(S)/8}\\ &\le e^{-\delta\alpha_N/8}R(\beta_4,\delta/2)^{|I|}\big(1-R(\beta_4,\delta/2)\big)^{-1}\end{aligned}$$

one easily deduces (4.46) from (4.36).

Now,

$$
\begin{aligned}
&\Big|\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I) - \widetilde{X}_{N,\mathcal{R},\mathbf{H}}(I)\Big| \prod_{j\in I} |Q(H_{N,j})| \\
&\quad \le \sum_{\hat{\xi}:J(\hat{\xi})=I,\ N_v(\hat{\xi})\le \log^2 N} \Big|\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}) - \widetilde{\Psi}_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big| \\
&\quad + \sum_{\substack{\hat{\xi}:J(\hat{\xi})=I,\\ N_v(\hat{\xi})>\log^2 N}} \Big|\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big| + \sum_{\substack{\hat{\xi}:J(\hat{\xi})=I,\\ N_v(\hat{\xi})>\log^2 N}} \Big|\widetilde{\Psi}_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big|.
\end{aligned}
\tag{4.47}
$$

Then, using the simple estimate $|a(\hat{\xi})| \le |I| N_v(\hat{\xi})$, definitions (4.11) and (4.44), one obtains

$$
\begin{aligned}
\Big|\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}) - \widetilde{\Psi}_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big| &\le \Big(e^{2\beta|I|\frac{N_v(S)}{N}} - 1\Big)\Big|\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big| \\
&\le \Big(e^{2\beta \log^4 N/N} - 1\Big)\Big|\Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi})\Big|,
\end{aligned}
\tag{4.48}
$$

provided $|I| \le \log^2 N$ and $N_v(S) \le \log^2 N$. Finally, substituting (4.48) into (4.47) and using (4.46) to evaluate the last two sums in (4.47), one easily deduces (4.45) from Lemma 4.1 for all sufficiently large N. $\square$

5. Cluster Expansion and Limiting Properties of the Partition Function

We establish here the cluster expansion for the relative partition function $\widehat{\Xi}(N, \beta, \mathcal{R}, \mathbf{H})$ and investigate some asymptotical properties of the corresponding free energy to be used later. The following statement presents the main result of this section.

Theorem 5.1. *There exists a constant β_0 depending only on δ and the constant β_0 from (2.11) such that for all $\beta \ge \beta_0$, N, and $\mathbf{H}$, $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ (recall (4.7)), the partition function $\Xi(N, \mathcal{R}, \mathbf{H})$ is finite (i. e., the defining series is absolutely convergent) non-vanishing analytical function of $\mathbf{H}$ satisfying the bound*

$$
\begin{aligned}
|\log \widehat{\Xi}(N, \beta, \mathcal{R}, \mathbf{H})| &= \Big|\log \Xi(N, \mathcal{R}, \mathbf{H}) - \sum_{j=1}^{N} \log Q(H_{N,j})\Big| \\
&\le N \exp\{-4(\beta - \beta_0)\}.
\end{aligned}
\tag{5.1}
$$

There exist functions $\Phi_{N,\mathcal{R},\mathbf{H}}(I)$ of intervals $I = (a, b] \subseteq (0, N]$ with integer end-points such that

$$
\Big|\Phi_{N,\mathcal{R},\mathbf{H}}(I)\Big| \le \exp\{-4(\beta - \beta_0)\big(|I| - 1\big)\},
\tag{5.2}
$$

and

$$
\log \widehat{\Xi}(N, \beta, \mathcal{R}, \mathbf{H}) = \sum_{I\subset[0,N]} \Phi_{N,\mathcal{R},\mathbf{H}}(I).
\tag{5.3}
$$

Finally, the functions $\Phi_{N,\mathcal{R},\mathbf{H}}(I)$ depend analytically on polymer weights $X_{N,\mathcal{R},\mathbf{H}}(I')$, $I' \subseteq I$, and the following inequality holds

$$
\left|\frac{\partial \Phi_{N,\mathcal{R},\mathbf{H}}(I)}{\partial X_{N,\mathcal{R},\mathbf{H}}(I')}\right| \le \big(|I| - |I'| + 1\big)^2 \exp\{|I'| \exp\{-4(\beta - \beta_0)\}\}.
\tag{5.4}
$$

Remark 5.1.1. For $k = 0$ one has $\Xi(N, \mathcal{R}, \mathbf{H}) \equiv \Xi(N, \Lambda, \mathbf{H})$ (recall (3.6)) and therefore this partition function is finite for all $\mathbf{H}$, $\Re\mathbf{H} \in \mathcal{D}_\delta^2$ (recall (3.7)).

Proof. In view of the polymer representation (4.30) and Lemma 4.1, expansion (5.3) and estimates (5.2) follow from any of the numerous versions of cluster expansions for polymer models (see, e. g., [18, 17]).

Then, (5.1) follows directly from (5.2) and the inequality

$$\Big|\sum_{I:I_0\subseteq I} \Phi_{N,\mathcal{R},\mathbf{H}}(I)\Big| \le \sum_{i=1}^{\infty}(i+1)\exp\{-4(\beta-\beta_0)i\} \le \exp\{-4(\beta-\bar{\beta}_0)\}, \tag{5.5}$$

that is valid for some $\bar{\beta}_0 < \infty$ and arbitrary $I_0 = (a, a+1] \subset [0, N]$, $a \in \mathbb{Z}$.

It remains to check (5.4). Due to the Möbius inversion formula (see, e. g., [18, §2.6], [7, §3.8], [8, §3.3]) the cluster weights $\Phi_{N,\mathcal{R},\mathbf{H}}(I)$ can be calculated from (recall (4.27))

$$\Phi_{N,\mathcal{R},\mathbf{H}}(I) = \sum_{I^*:\,\emptyset\neq I^*\subseteq I} (-1)^{|I\setminus I^*|}\log\widehat{\Xi}(I^*, N, \mathcal{R}, \mathbf{H}), \tag{5.6}$$

where again I^* are intervals with integer endpoints. According to Proposition 3.6 ([8]) the functions $\log\widehat{\Xi}(I^*, N, \mathcal{R}, \mathbf{H})$ depend analytically on the polymer weights $\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I')$, $I' \subseteq I^*$. Moreover, using (4.30) and (5.3) one has[7]

$$\frac{\partial\log\widehat{\Xi}(I^*, N, \mathcal{R}, \mathbf{H})}{\partial\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I')} = \frac{\widehat{\Xi}(I^*\setminus I', N, \mathcal{R}, \mathbf{H})}{\widehat{\Xi}(I^*, N, \mathcal{R}, \mathbf{H})} = \exp\Big\{-\sum_{\substack{\tilde{I}=(a,b]:\tilde{I}\subseteq I^*,\\ \tilde{I}\cap I'\neq\emptyset}} \Phi_{N,\mathcal{R},\mathbf{H}}(\tilde{I})\Big\}.$$

As a result, (5.5) implies directly that

$$\left|\frac{\partial\log\widehat{\Xi}(I^*, N, \mathcal{R}, \mathbf{H})}{\partial\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I')}\right| \le \exp\{|I'|e^{-4(\beta-\tilde{\beta}_0)}\}, \tag{5.7}$$

with some $\tilde{\beta}_0$ depending only on β_0. It remains to observe that for any pair I, I', $I' \subseteq I$, of intervals with integer endpoints there exists no more than $\big(|I| - |I'| + 1\big)^2$ intervals $\tilde{I}$ satisfying the condition $I' \subseteq \tilde{I} \subseteq I$. Finally, (5.4) follows immediately from (5.6), (5.7) and the last observation. □

Remark 5.1.2. We have proved (5.4) using only the polymer representation (4.30) of the partition function $\widehat{\Xi}(N, \beta, \mathcal{R}, \mathbf{H})$ and the estimate (4.31) of polymer weights $\widehat{X}_{N,\mathcal{R},\mathbf{H}}(I)$ (recall Remark 4.1.1). Since the explicit form of these polymer weights was not used, our result is valid for any partition function defined via (4.30) with any collection of polymer weights satisfying (4.31).

---

[7] In the case $I^*\setminus I = I_1 \cup I_2$ with disjoint intervals I_1 and I_2 we denote

$$\widehat{\Xi}(I_1\cup I_2, N, \mathcal{R}, \mathbf{H}) \equiv \widehat{\Xi}(I_1, N, \mathcal{R}, \mathbf{H})\widehat{\Xi}(I_2, N, \mathcal{R}, \mathbf{H}).$$

In the remaining part of the present section we obtain some corollaries of Theorem 5.1 to be used later on.

Let first $k = 0$, $\mathcal{R} = \{r_1\}$, $r_1 = N$ and $\mathbf{H} = (0, H) \in \mathbb{C}^2$. Then the partition function $\Xi(N, \mathcal{R}, \mathbf{H})$ from (4.6) coincides with the partition function $\Xi(N, H)$ (recall (2.19)) for the height $h(S)$ of the phase boundary S. Define $\widehat{\Xi}(N, H)$ similarly to (4.26). The following result was obtained in [7].

Corollary 5.2 ([7], Theorem 4.8). *Let H satisfy the condition*

$$|\Re H| < 2 - \delta/2\beta. \tag{5.8}$$

Then all the statements of Theorem 5.1 are valid for the partition function $\widehat{\Xi}(N, H)$. Moreover, the functions $\Phi_{N,\mathcal{R},\mathbf{H}}(I)$ do not depend on N,

$$\Phi_{N,\mathcal{R},\mathbf{H}}(I) \equiv \Phi_H(|I|),$$

where $|I|$ denotes the length of the interval I, and there exists a limit

$$\widehat{F}(H) = \lim_{n\to\infty} \frac{\log \widehat{\Xi}(N, H)}{N}, \tag{5.9}$$

that presents an analytical function of H in the region (5.8). Finally, one has the expansion

$$\widehat{F}(H) = \sum_{i=2}^{\infty} \Phi_H(i) \tag{5.10}$$

and the estimate

$$|\widehat{F}(H)| \le \exp\{-4(\beta - \beta_0)\}, \tag{5.11}$$

where $\beta \ge \beta_0$ with sufficiently large β_0.

Remark 5.2.1. Due to (4.27) one has $\widehat{\Xi}(I, N, \mathcal{R}, \mathbf{H}) = 1$ for any $I \subset [0, N]$ such that $|I| = 1$. Thus, (5.6) implies $\Phi_H(1) = 0$ that explains the absence of $i = 1$ in (5.10). The expansion from (5.10) plays the important role in the following considerations.

Remark 5.2.2. It follows from (5.9), definitions (4.26) and (4.20) that the limit (recall (2.20))

$$F(H) = \lim_{N\to\infty} \frac{\log \Xi(N, H)}{N}$$

exists, is an analytical function of H in the region (5.8), and satisfies there the following identity ([7, p. 120])

$$F(H) \equiv \widehat{F}(H) + \log Q(H).$$

To study the asymptotical properties of the area $a_N(S)$ below the phase boundary S we put $k + 1 = 0$ in (4.1). Denote the corresponding partition function by $\Xi(N, H, \text{area})$ and define the relative partition function $\widehat{\Xi}(N, H, \text{area})$ as in (4.26).

Corollary 5.3. *Assume that H satisfies (5.8). Then $\widehat{\Xi}(N, H, area)$ is a non-vanishing analytical function of such H. Moreover, there exists the limit*

$$\widehat{F}_{area}(H) \equiv \lim_{N\to\infty} \frac{\log \widehat{\Xi}(N, H, area)}{N} = \int_0^1 \widehat{F}\big((1-x)H\big)\, dx, \tag{5.12}$$

where $\widehat{F}(\cdot)$ is the free energy from (5.9) corresponding to the height $h(S)$ of the phase boundary S. Finally, there exist constants β_0 and N_0 such that for all $N \geq N_0$ and $\beta \geq \beta_0$,

$$\left|\log \widehat{\Xi}(N, H, area) - N \int_0^1 \widehat{F}\big((1-x)H\big)\, dx\right| \leq \exp\{-3(\beta - \beta_0)\} \log^{10} N. \tag{5.13}$$

Remark 5.3.1. Due to the integral representation in (5.12), the function $\widehat{F}_{\text{area}}(\cdot)$ is an analytical function of H in the region (5.8).

Remark 5.3.2. The derivatives of $N^{-1} \log \widehat{\Xi}(N, H, \text{area})$ with respect to H converge to the corresponding derivatives of $\widehat{F}_{\text{area}}(H)$. In this case estimate (5.13) is also true with possibly another constant β_0.

The following simple property of real functions will be used below.

Property 5.4. *Let $f(\cdot)$ be a smooth real function, $f : U \to \mathbb{R}^1$, where U is some open convex set in $\mathbb{R}^k$. Assume that for any $i = 1, \dots, k$ one has*

$$\left|\frac{\partial f(x)}{\partial x_i}\right| \Big|_{x=y} \leq a_i \tag{5.14}$$

uniformly in $y \in U$. Then for all $y, z \in U$,

$$|f(y) - f(z)| \leq \sum_{i=1}^k a_i |y_i - z_i|. \tag{5.15}$$

Proof. Define $g(t) = f\big(z + t(y - z)\big)$. Then

$$g'(t) = \sum_{i=1}^k \frac{\partial f}{\partial x_i}\big(z + t(y-z)\big) \cdot (y_i - z_i),$$

and therefore (recall (5.14))

$$|f(y) - f(z)| \equiv |g(1) - g(0)| \leq \int_0^1 |g'(t)|\, dt \leq \sum_{i=1}^k a_i |y_i - z_i|.$$

□

Proof of Corollary 5.3. The analyticity of $\log \widehat{\Xi}(N, H, \text{area})$ with respect to H in the region (5.8), the cluster expansion

$$\log \widehat{\Xi}(N, H, \text{area}) = \sum_{I \subset [0,N]} \Phi_{N,H,\text{area}}(I) \tag{5.16}$$

and the estimates for $\Phi_{N,H,\text{area}}(I)$ of the type (5.2) and (5.4) follow directly from Theorem 5.1. It remains to establish (5.13).

We will check below that there exists a constant $C_1 = C_1(\delta, \beta_0)$ such that for all $\beta \geq \beta_0$ and all intervals $I = (m', m''] \subset [0, N]$, $|m' - m''| \leq \log^2 N$, with integer endpoints the following inequality holds:

$$\Big|\Phi_{N,H,\text{area}}(I) - \Phi_{(1-m''/N)H}(|I|)\Big| \leq C_1 \frac{\log^8 N}{N} \exp\{-3(\beta - \beta_0)\}, \tag{5.17}$$

where the quantities $\Phi_H(k)$ coincide with the elements of expansion (5.10). Then (5.13) will follow directly from (5.17).

Indeed, using (5.16) we obtain

$$\begin{aligned} &\Big|\log \widehat{\Xi}(N, H, \text{area}) - N \int_0^1 \widehat{F}\big((1-x)H\big)\, dx\Big| \\ &\leq \sum_{m''=1}^{N} \Big|\widehat{F}\big((1 - m''/N)H\big) - \sum_{I=(m,m'']:I \subseteq (0,m'']} \Phi_{N,H,\text{area}}(I)\Big| \\ &+ \Big|N \int_0^1 \widehat{F}\big((1-x)H\big)\, dx - \sum_{j=1}^{N} \widehat{F}\big((1 - j/N)H\big)\Big|, \end{aligned} \tag{5.18}$$

where in view of (5.10),

$$\widehat{F}\big((1 - j/N)H\big) = \sum_{k=2}^{\infty} \Phi_{(1-j/N)H}(k). \tag{5.19}$$

Let us estimate every term on the right-hand side of (5.18). First of all, due to analyticity of $\widehat{F}(\cdot)$ there exists a constant $C_2 = C_2(\delta, \beta) > 0$ such that for all $\beta \geq \beta_0$ and H in the region (5.8) one has

$$\Big|N \int_0^1 \widehat{F}\big((1-x)H\big)\, dx - \sum_{j=1}^{N} \widehat{F}\Big(\big(1 - \frac{j}{N}\big)H\Big)\Big| \leq C_2.$$

Then, (5.11) and analog of (5.2) imply

$$\begin{aligned} R_N &= \Big|\widehat{F}\Big(\big(1 - m''/N\big)H\Big) - \sum_{I=(m,m'']:I \subseteq (0,m'']} \Phi_{N,H,\text{area}}(I)\Big| \\ &\leq \exp\{-4(\beta - \beta_0)\} + \exp\{-4(\beta - \overline{\beta}_0)\} = C_3 < \infty. \end{aligned} \tag{5.20}$$

Finally, for any $m'' \geq \log^2 N$ we rewrite (recall (5.19))

$$\begin{aligned} R_N &= \Big|\widehat{F}\Big(\big(1-m''/N\big)H\Big) - \sum_{I=(m,m'']:I\subseteq(0,m'']} \Phi_{N,H,\text{area}}(I)\Big| \\ &\le \sum_{\substack{I=(m,m'']:I\subseteq(0,m''] \\ |m''-m|\le\log^2 N}} \big|\Phi_{N,H,\text{area}}(I) - \Phi_{(1-m''/N)H}(I)\big| \\ &+ \sum_{\substack{I=(m,m'']:I\subseteq(0,m''] \\ |m''-m|>\log^2 N}} \big(|\Phi_{N,H,\text{area}}(I)| + \big|\Phi_{(1-m''/N)H}(I)\big|\big)\,. \end{aligned}$$

Applying (5.17) to every term in the first sum and using (5.2) for all other terms we obtain

$$\begin{aligned} R_N \le \log^2 N\cdot C_1\frac{\log^8 N}{N}\exp\{-3(\beta-\beta_0)\} \\ +2\sum_{k\ge\log^2 N}\exp\{-4(\beta-\beta_0)k\} \\ \le C_4\frac{\log^{10} N}{N}\exp\{-3(\beta-\beta_0)\}. \end{aligned} \tag{5.21}$$

for all sufficiently large N. Finally, applying (5.20) for $m''\le\log^2 N$ and (5.21) in the opposite case, $\log^2 N < m'' \le N$, we obtain

$$\begin{aligned} &\Big|\log\widehat{\Xi}(N,H,\text{area}) - N\int_0^1 \widehat{F}\big((1-x)H\big)\,dx\Big| \\ &\le C_3\log^2 N + C_4(N-\log^2 N)\frac{\log^{10} N}{N}\exp\{-3(\beta-\beta_0)\} + C_2 \\ &\le C_5\exp\{-3(\beta-\beta_0)\}\log^{10} N \end{aligned}$$

(with some constant $C_5 > 0$) for all sufficiently large N.

Thus, it remains to prove (5.17). Fix any $I = (m', m''] \subset [0, N]$, $|m''-m'| \le \log^2 N$, with integer endpoints. Recall that the partition function $\widehat{\Xi}(N,H,\text{area})$ corresponding to the normalized area Y_N is expressed in terms of activities

$$\Psi_{N,H,\text{area}}(\xi_i) = \exp\Big\{-2\beta|\xi_i| + \beta H\Big(1-\frac{m_i}{N}\Big)h(\xi_i) + \beta H\frac{1}{N}a(\xi_i)\Big\}\prod_{\Lambda_s\in\xi_i}\Psi(\Lambda_s),$$

where the animal ξ_i has the base $J(\xi_i) = (m_{i-1}, m_i]$. Define (cf. (4.29))

$$\widehat{X}_{N,H,\text{area}}(I') = \Big(\prod_{j\in I'} Q\Big(\big(1-j/N\big)H\Big)\Big)^{-1}\sum_{\hat{\xi}:I(\hat{\xi})=I'}\Psi_{N,H,\text{area}}(\hat{\xi}), \qquad I'\subseteq I.$$

For all $\hat{\xi}$ with $J(\hat{\xi})\subset(m', m'']$ consider also new activities

$$\overline{\Psi}_{N,H,\text{area}}(\hat{\xi}) = \exp\Big\{-2\beta|\hat{\xi}| + \beta H\Big(1-\frac{m''}{N}\Big)h(\hat{\xi})\Big\}\prod_{\Lambda_s\in\xi_i}\Psi(\Lambda_s)$$

(with the same value m'' for all such animals $\hat{\xi}$) and polymer weights

$$\widehat{\widehat{\mathrm{X}}}_{N,H,\text{area}}(I') = \Big(\prod_{j\in I'} Q\Big(\big(1-j/N\big)H\Big)\Big)^{-1} \sum_{\hat\xi : I(\hat\xi)=I'} \overline{\Psi}_{N,H,\text{area}}(\hat\xi), \qquad I' \subseteq I.$$

Clearly, the polymer weights $\widehat{\widehat{\mathrm{X}}}_{N,H,\text{area}}(\cdot)$ satisfy (4.31). Moreover, for all I', $I' \subseteq I$, one has

$$\begin{aligned} &\Big|\widehat{X}_{N,H,\text{area}}(I') - \widehat{\widehat{\mathrm{X}}}_{N,H,\text{area}}(I')\Big| \\ &\qquad \le 2\Big(e^{2\beta \log^4 N/N} - 1\Big) \exp\{-4(\beta-\overline{\beta})(|I|-1)\} \\ &\qquad \le 4\beta \frac{\log^4 N}{N} e^{2\beta \log^4 N/N} \exp\{-4(\beta-\overline{\beta})(|I|-1)\}, \end{aligned} \tag{5.22}$$

provided N and β are sufficiently large, $\beta \ge \overline{\beta}$ and $N \ge N_0$. In the second inequality above we have used the simple inequality $e^x - 1 \le xe^x$ that is true for all $x \ge 0$.

Let $\Phi_{N,H,\text{area}}(I)$ and $\overline{\Phi}_{N,H,\text{area}}(I)$ be the cluster weights generated by $\widehat{X}_{N,H,\text{area}}(I')$ and $\widehat{\widehat{\mathrm{X}}}_{N,H,\text{area}}(I')$, $I' \subseteq I$, correspondingly. In view of Remark 5.1.2 we apply (5.4) and (5.15) to obtain

$$\begin{aligned} &\Big|\Phi_{N,H,\text{area}}(I) - \overline{\Phi}_{N,H,\text{area}}(I)\Big| \\ &\qquad \le \sum_{I' : I' \subset I} e^{|I'| e^{-4(\beta-\beta_0)}} \cdot \log^4 N \cdot \Big|\widehat{X}_{N,H,\text{area}}(I') - \widehat{\widehat{\mathrm{X}}}_{N,H,\text{area}}(I')\Big|. \end{aligned}$$

Then, using (5.22) one gets

$$\begin{aligned} &\Big|\Phi_{N,H,\text{area}}(I) - \overline{\Phi}_{N,H,\text{area}}(I)\Big| \\ &\qquad \le 4\beta \frac{\log^8 N}{N} e^{2\beta \log^4 N/N} \sum_{I' : I' \subset I} e^{-4(\beta-\beta_0)(|I'|-1)} e^{|I'| e^{-4(\beta-\beta_0)}}, \\ &\qquad \le \frac{\log^8 N}{N} e^{-3(\beta-\beta')} \end{aligned}$$

provided N is sufficiently large and $\beta \ge \beta' > 0$. It remains to observe that due to its definition $\overline{\Phi}_{N,H,\text{area}}(I)$ coincides with $\Phi_{(1-m''/N)H}(I)$. □

Finally, consider the random vector Θ_N from (4.1)–(4.2),

$$\Theta_N \equiv \big(Y_N, X_N(s_1), \dots, X_N(s_k), X_N(1)\big) \in \mathbb{R}^{k+2},$$

where the collection $\mathcal{S} = \{s_1, \dots, s_{k+1}\}$ is such that $0 < s_1 < \dots < s_{k+1} = 1$. Denote

$$\mathcal{R}(\mathcal{S}) = \Big\{[Ns_1], \dots, [Ns_k], N\Big\}.$$

Then the corresponding partition function $\widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H})$ is given by (4.6) with $\mathcal{R}$ replaced by $\mathcal{R}(\mathcal{S})$. For any $\mathbf{H}$, $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, define

$$\widetilde{H}(x) = (1-x)H_0 + \sum_{l=1}^{k+1} H_l \mathbf{1}_{\{x<s_l\}}. \tag{5.23}$$

Corollary 5.5. *The partition function* $\widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H})$ *is a non-vanishing analytical function of* $\mathbf{H}$, $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$. *There exist the limit*

$$\widehat{F}_{\mathcal{R}(\mathcal{S})}(\mathbf{H}) \equiv \lim_{N\to\infty} \frac{\log \widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H})}{N} = \int_0^1 \widehat{F}\big(\widetilde{H}(x)\big)\,dx, \tag{5.24}$$

where $\widehat{F}(\cdot)$ *is the free energy from (5.9) and* $\widetilde{H}(x)$ *was defined in (5.23). Finally, there exist constants* N_0 *and* $\tilde{\beta}_0$ *such that for all* $N \geq N_0$ *and* $\beta \geq \tilde{\beta}_0$,

$$\left|\log \widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H}) - N\int_0^1 \widehat{F}\big(\widetilde{H}(x)\big)\,dx\right| \leq \log^{10} N \exp\{-3(\beta - \beta_0)\}. \tag{5.25}$$

Remark 5.5.1. Due to the integral representation in (5.24), the free energy $\widehat{F}_{\mathcal{R}(\mathcal{S})}(\mathbf{H})$ is an analytical function of $\mathbf{H}$, $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$.

Remark 5.5.2. The analog of (5.25) holds for any partial derivative of the function $\log \widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H})$ as a function of $\mathbf{H}$, $\Re\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, with possibly different constant $\tilde{\beta}_0$.

Proof. Arguments similar to those used in the proof of Corollary 5.3 imply the following estimate

$$\begin{aligned}\left|N\int_{s_i}^{s_{i+1}} \widehat{F}\big(\widetilde{H}(x)\big)\,dx - \log \widehat{\Xi}\big((r_i, r_{i+1}], N, \mathcal{R}, \mathbf{H}\big)\right| \\ \leq \log^{10} N \exp\{-3(\beta - \beta_0)\}\end{aligned} \tag{5.26}$$

for any $i = 0, 1, \ldots, k$ and $N \geq N_0$ with $s_0 = r_0 = 0$, $r_i = [Ns_i]$, $i = 1, \ldots, k+1$. On the other hand,

$$\begin{aligned}&\left|\log \widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H}) - N\int_0^1 \widehat{F}\big(\widetilde{H}(x)\big)\,dx\right| \\ &\quad\leq \sum_{i=0}^{k}\left|N\int_{s_i}^{s_{i+1}} \widehat{F}\big(\widetilde{H}(x)\big)\,dx - \log \widehat{\Xi}\big((r_i, r_{i+1}], N, \mathcal{R}, \mathbf{H}\big)\right| \\ &\quad+ \sum_{i=1}^{k}\ \sum_{I:\,(r_i, r_i+1]\subseteq I\subset[0,N]}\left|\Phi_{N,\mathcal{R}(\mathcal{S}),\mathbf{H}}(I)\right|.\end{aligned}$$

Therefore, (5.26) and (5.5) imply the inequality

$$\begin{aligned}&\left|\log \widehat{\Xi}(N, \mathcal{R}(\mathcal{S}), \mathbf{H}) - N\int_0^1 \widehat{F}\big(\widetilde{H}(x)\big)\,dx\right| \\ &\quad\leq (k+1)\log^{10} N\ \exp\{-3(\beta - \beta_0)\} + k\exp\{-4(\beta - \beta_0)\} \\ &\quad\leq \log^{10} N\ \exp\{-3(\beta - \tilde{\beta}_0)\}\end{aligned}$$

for all sufficiently large N and $\beta \geq \tilde{\beta}_0$. $\square$

6. Limit Theorems for the Joint Distribution

We study here the asymptotical behaviour of the probabilities $\mathbf{P}(\Theta_N = \mathbf{M}_N)$ and $\mathbf{P}(\Lambda_N = A_N)$ entering the right-hand side of (3.19).

Let an integer number $k \geq 0$ and a set $\mathcal{S}$ of real numbers s_i, $\{0 < s_1 < \ldots < s_k < 1 = s_{k+1}$ be fixed. Denote

$$\mathcal{R} = \{r_i : r_i = [Ns_i], i = 1, \ldots, k+1\},$$

and for $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ consider the logarithmic moment generating function $L_{N,\mathcal{R}}(\mathbf{H})$ corresponding to the random vector $\Theta_{N,\mathcal{R}} \equiv \Theta_N$ from (4.1)–(4.2),

$$L_{N,\mathcal{R}}(\mathbf{H}) = \log \mathbf{E} \exp\{\beta\big(\mathbf{H}, \Theta_{N,\mathcal{R}}\big)\}. \tag{6.1}$$

For any $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ we introduce also the random vector $\Theta_{N,\mathcal{R},\mathbf{H}}$ with $\mathbf{H}$-tilted distribution,

$$\mathbf{P}(\Theta_{N,\mathcal{R},\mathbf{H}} = \mathbf{M}) = \exp\{\beta(\mathbf{M},\mathbf{H}) - L_{N,\mathcal{R}}(\mathbf{H})\}\mathbf{P}(\Theta_{N,\mathcal{R}} = \mathbf{M}), \tag{6.2}$$

where $\mathbf{M} \in \mathcal{M}_N^{k+2}$ (recall (3.18)). Observe that the mean vector $\mathbf{E}\Theta_{N,\mathcal{R},\mathbf{H}}$ and the covariance matrix $\mathbf{Cov}\Theta_{N,\mathcal{R},\mathbf{H}}$ of $\Theta_{N,\mathcal{R},\mathbf{H}}$ can be calculated via

$$\beta\,\mathbf{E}\Theta_{N,\mathcal{R},\mathbf{H}} = \nabla_{\mathbf{H}} L_{N,\mathcal{R}}(\mathbf{H}), \qquad \beta^2 \mathbf{Cov}\Theta_{N,\mathcal{R},\mathbf{H}} = \mathbf{Hess} L_{N,\mathcal{R}}(\mathbf{H}), \tag{6.3}$$

where $\nabla_{\mathbf{H}}$ denotes the gradient and $\mathbf{Hess}L_{N,\mathcal{R}}(\mathbf{H})$ is the Hessian (the matrix of the second derivatives) of $L_{N,\mathcal{R}}(\mathbf{H})$ as the function of $\mathbf{H} = (H_0, H_1, \ldots, H_{k+1})$.

Assuming that $\mathbf{H}$ and $\mathbf{M}$ are related via

$$\beta\mathbf{M} = \nabla_{\mathbf{H}} L_{N,\mathcal{R}}(\mathbf{H}),$$

one easily obtains (recall (6.2), (4.5))

$$\begin{aligned} \mathbf{P}\left(\Theta_{N,\mathcal{R}} = \mathbf{M}\right) &= \exp\{-\beta(\mathbf{M},\mathbf{H})\} \frac{\Xi(N,\mathcal{R},\mathbf{H})}{\Xi(N)} \mathbf{P}\big(\Theta_{N,\mathcal{R},\mathbf{H}} = \mathbf{M}\big) \\ &= \exp\{-L^*_{N,\mathcal{R}}(\mathbf{M})\}\, \mathbf{P}\big(\Theta_{N,\mathcal{R},\mathbf{H}} = \mathbf{M}\big) \end{aligned} \tag{6.4}$$

with $L^*_{N,\mathcal{R}}(\cdot)$ denoting the Legendre transformation

$$L^*_{N,\mathcal{R}}(\mathbf{M}) \equiv \sup_{\mathbf{H}} \big(\beta(\mathbf{M},\mathbf{H}) - L_{N,\mathcal{R}}(\mathbf{H})\big).$$

In view of (6.4) the problem is reduced to the investigation of the asymptotical behaviour of the probability $\mathbf{P}\big(\Theta_{N,\mathcal{R},\mathbf{H}} = \mathbf{M}\big)$.

For any $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ define the matrix

$$B_{N,\mathcal{R}}(\mathbf{H}) \equiv \frac{1}{\beta^2 N} \mathbf{Hess} L_{N,\mathcal{R}}(\mathbf{H}) \tag{6.5}$$

and introduce the quadratic form $\mathcal{B}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$, $\mathbf{T} = (t_0, t_1, \ldots, t_{k+1}) \in \mathbb{R}^{k+2}$,

$$\mathcal{B}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) \equiv \big(B_{N,\mathcal{R}}(\mathbf{H})\mathbf{T}, \mathbf{T}\big).$$

Consider also the quadratic form

$$\mathcal{B}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) \equiv \big(B_{\mathcal{R}}(\mathbf{H})\mathbf{T}, \mathbf{T}\big)$$

corresponding to the matrix (recall (5.24))

$$B_{\mathcal{R}}(\mathbf{H}) \equiv \frac{1}{\beta^2}\mathbf{Hess}\int_0^1 \left(\log Q + \widehat{F}\right)\left(\widetilde{H}(x)\right)dx, \tag{6.6}$$

where $Q(\cdot)$, $\widehat{F}(\cdot)$, and $\widetilde{H}(x)$ were defined in (4.18), (5.9), and (5.23) respectively.

Lemma 6.1. *Let $\beta \geq \beta_0$ with β_0 fixed in (5.11). Then uniformly in $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ and $\mathbf{T} \in \mathbb{R}^{k+2}$, $|\mathbf{T}| = 1$, one has*

$$\mathcal{B}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) \to \mathcal{B}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) \quad \textit{as } N \to \infty. \tag{6.7}$$

Moreover, there exist positive constants b, N_0, and $\overline{\beta}$ depending only on β_0 from (5.11) and δ such that uniformly in $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, $N \geq N_0$, and $\beta \geq \overline{\beta}$ one has

$$\mathcal{B}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) \geq b|\mathbf{T}|^2, \qquad \mathcal{B}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) \geq b|\mathbf{T}|^2. \tag{6.8}$$

Proof. In view of (6.5), (4.5), and (4.26) one easily obtains

$$B_{N,\mathcal{R}}(\mathbf{H}) = \frac{1}{\beta^2 N}\mathbf{Hess}\log \Xi(N,\mathcal{R},\mathbf{H},\infty) + \frac{1}{\beta^2 N}\mathbf{Hess}\log \widehat{\Xi}(N,\beta,\mathcal{R},\mathbf{H}). \tag{6.9}$$

The first term on the right-hand side of (6.9) presents the normalized covariance matrix for the ensemble of tame animals. Due to (4.20) the corresponding quadratic form $\mathcal{Q}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$ satisfies the relation

$$\mathcal{Q}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) = \mathcal{Q}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) + O(N^{-1})|\mathbf{T}|^2 \quad \text{as } N \to \infty, \tag{6.10}$$

where the limiting quadratic form $\mathcal{Q}_{\mathcal{R},\mathbf{H}}(\mathbf{T})$ is calculated via

$$\mathcal{Q}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) = \frac{1}{\beta^2}\int_0^1 (\log Q)''\left(\widetilde{H}(x)\right)\left((1-x)t_0 + \sum_{l=1}^{k+1}\mathbf{1}_{\{x<s_l\}}t_l\right)^2 dx. \tag{6.11}$$

Let $\widehat{\mathcal{F}}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$ be the quadratic form corresponding to the second term on the right-hand side of (6.9). According to Remark 5.5.2 one has

$$\widehat{\mathcal{F}}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) = \widehat{\mathcal{F}}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) + O\Big(\frac{\log^{10} N}{N}\exp\{-3(\beta-\beta_0)\}\Big)|\mathbf{T}|^2 \quad \text{as } N \to \infty \tag{6.12}$$

with the limiting quadratic form (cf. (6.11))

$$\widehat{\mathcal{F}}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) = \frac{1}{\beta^2}\int_0^1 (\widehat{F})''\left(\widetilde{H}(x)\right)\left((1-x)t_0 + \sum_{l=1}^{k+1}\mathbf{1}_{\{x<s_l\}}t_l\right)^2 dx.$$

As a result, (6.7) follows immediately from (6.10) and (6.12).

It remains to prove the inequalities in (6.8). First, observe that

$$\mathcal{B}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) \equiv \mathcal{Q}_{\mathcal{R},\mathbf{H}}(\mathbf{T}) + \widehat{\mathcal{F}}_{\mathcal{R},\mathbf{H}}(\mathbf{T}).$$

We will show later that the function $\beta^{-2}\left(\log Q + \widehat{F}\right)''(H)$ is uniformly bounded from below (and above) by two positive constants uniformly in H, $|H| < 2-3\delta/4\beta$, provided

β is sufficiently large, $\beta \geq \overline{\beta}_0$. Then the first inequality in (6.8) follows from the observation that the quadratic form

$$\int_0^1 \Big((1-x)t_0 + \sum_{l=1}^{k+1} \mathbf{1}_{\{x<s_l\}} t_l\Big)^2 \, dx$$

is a positive continuous function of $\mathbf{T} = (t_0, \dots, t_{k+1})$ on the unit sphere $|\mathbf{T}| = 1$, and thus is bounded from below by some positive constant C_1.

To prove that the function $\beta^{-2}\big(\log Q + \widehat{F}\big)''(H)$ is bounded uniformly in H, $|H| < 2 - 3\delta/4\beta$, we observe that due to (4.18),

$$\frac{\partial^2}{\beta^2 \partial H^2} \log Q(H) = \frac{\cosh(2\beta)\cosh(H\beta) - 1}{\big(\cosh(2\beta) - \cosh(H\beta)\big)^2}$$

and thus (recall (4.24))

$$e^{-2\beta} \frac{\cosh(2\beta_0) - 1}{\cosh(2\beta_0)} \leq \frac{\cosh(2\beta_0) - 1}{\cosh(2\beta_0)} \cdot \frac{\cosh(H\beta)}{\cosh(2\beta)} \leq \frac{\partial^2}{\beta^2 \partial H^2} \log Q(H) \leq \frac{e^{\delta/4}}{\big(e^{\delta/4} - 1\big)^2}$$

if only $\beta \geq \beta_0$ and $H \in \mathbb{R}^1$ satisfies (4.23). On the other hand, due to Corollary 5.2 for any fixed H_0, $|H_0| < 2 - 3\delta/4\beta$, the function $\widehat{F}(H)$ is analytic in the disk of radius $\delta/4\beta$ with the center at H_0. Applying the Cauchy formula and estimate (5.11) one obtains

$$\Big| \frac{\partial^2}{\beta^2 \partial H^2} \widehat{F}(H) \Big| \leq C(\delta) \exp\{-4(\beta - \beta_0)\},$$

where $C(\delta) > 0$ is a constant depending only on δ. The needed inequality follows immediately provided β is such that

$$C(\delta) \exp\{-4(\beta - \beta_0)\} \leq \frac{1}{2} e^{-2\beta} \frac{\cosh(2\beta_0) - 1}{\cosh(2\beta_0)} = q_1.$$

Put $b = q_1 C_1/2$. Since the convergence in (6.7) is uniform in $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, the last inequality in (6.8) follows for all sufficiently large N, $N \geq N_0$. □

Let $\overline{\Theta}$ be the Gaussian random vector with zero mean and the covariance matrix $B_{\mathcal{R}}(\mathbf{H})$ (recall (6.6)). Denote its characteristic function by $\overline{\chi}_{\mathbf{H}}(\mathbf{T})$,

$$\overline{\chi}_{\mathbf{H}}(\mathbf{T}) \equiv \exp\Big\{-\frac{1}{2} \mathcal{B}_{\mathcal{R},\mathbf{H}}(\mathbf{T})\Big\}, \qquad \mathbf{T} \in \mathbb{R}^{k+2}. \tag{6.13}$$

Since the matrix $B_{\mathcal{R}}(\mathbf{H})$ is positively definite, the distribution of $\overline{\Theta}$ is non-degenerate and has the density $\overline{p}_{\mathbf{H}}(\mathbf{X})$, $\mathbf{X} \in \mathbb{R}^{k+2}$.

Theorem 6.2. *Let a sequence of vectors $\mathbf{H}_N \in \widehat{\mathcal{D}}_\delta^{k+2}$ satisfy the condition $\mathbf{H}_N \to \mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ as $N \to \infty$. Consider the random vector*

$$\Theta_N^* \equiv \frac{1}{\sqrt{N}}\big(\Theta_{N,\mathcal{R},\mathbf{H}_N} - \mathbf{E}\Theta_{N,\mathcal{R},\mathbf{H}_N}\big). \tag{6.14}$$

Then for all $\beta \geq \beta_0$ with sufficiently large β_0 the distribution of Θ_N^* converges weakly as $N \to \infty$ to the distribution of the random vector $\overline{\Theta}$ with the characteristic function $\overline{\chi}_{\mathbf{H}}(\mathbf{T})$.

Proof. Let $\chi_N(\mathbf{T})$ be the characteristic function of the random vector $\Theta_{N,\mathcal{R},\mathbf{H}_N}$,

$$\chi_N(\mathbf{T}) \equiv \mathbf{E}\exp\{i(\mathbf{T}, \Theta_{N,\mathcal{R},\mathbf{H}_N})\} = \frac{\Xi(N,\mathcal{R},\mathbf{H}_N + i\beta^{-1}\mathbf{T})}{\Xi(N,\mathcal{R},\mathbf{H}_N)}. \tag{6.15}$$

Then the characteristic function $\chi_N^*(\mathbf{T})$ of the random vector Θ_N^* equals

$$\log \chi_N^*(\mathbf{T}) = -\frac{1}{2}\mathcal{B}_{N,\mathcal{R},\mathbf{H}_N}(\mathbf{T}) - \frac{i}{6N^{3/2}}R_N, \tag{6.16}$$

where

$$R_N = \frac{1}{\beta^3}\sum_{l,m,p=0}^{k+1} t_l t_m t_p \frac{\partial^3}{\partial H_l \partial H_m \partial H_p}\log \Xi(N,\mathcal{R},\mathbf{H})\bigg|_{\mathbf{H}=\mathbf{H}_N+\frac{i\omega}{\beta\sqrt{N}}\mathbf{T}} \tag{6.17}$$

with some $\omega = \omega(\mathbf{H}_N, \mathbf{T})$, $0 \leq \omega \leq 1$. Since the convergence in (6.7) is valid for $\mathbf{T}$ belonging to any compact set in $\mathbb{R}^{k+2}$ (uniformly in $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$), it remains to prove that

$$R_N = o\big(N^{3/2}\big) \quad \text{as } N \to \infty. \tag{6.18}$$

Let $\chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$ be the characteristic function of the random vector $\Theta_{N,\mathcal{R},\mathbf{H}}$, $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, (cf. (6.15))

$$\chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) = \frac{\Xi(N,\mathcal{R},\mathbf{H} + i\beta^{-1}\mathbf{T})}{\Xi(N,\mathcal{R},\mathbf{H})}. \tag{6.19}$$

We will show below that the function $\log \chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$ can be extended to an analytical function of $\mathbf{T}$ in the region $\{\mathbf{T} \in \mathbb{C}^{k+2}, \sum_{i=0}^{k+1} |\Im t_i| < \delta/4\}$. Then, applying the Cauchy formula one obtains

$$\left|\frac{\partial^3}{\partial t_l \partial t_m \partial t_p}\log \chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})\right| \leq C(\delta)\sup_{(\mathbf{H},\mathbf{T})\in G(\delta)}|\log \chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})| \tag{6.20}$$

for all such $\mathbf{T}$, where (recall (4.7))

$$G(\delta) = \Big\{(\mathbf{H},\mathbf{T}) : \mathbf{H} \in \mathcal{D}_{\delta/2}^{k+2},\ \ \mathbf{T} \in \mathbb{C}^{k+2}, \sum_{i=0}^{k+1}|\Im t_i| < \delta/4\Big\},$$

and the constant $C(\delta)$ depends only on δ. This will give us the needed estimate for the remainder R_N.

Using (4.26) we rewrite (6.19) in the form

$$\chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) = \chi^{\infty}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) \frac{\widehat{\Xi}(N,\mathcal{R},\mathbf{H}+i\beta^{-1}\mathbf{T})}{\widehat{\Xi}(N,\beta,\mathcal{R},\mathbf{H})}, \tag{6.21}$$

where $\chi^{\infty}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$ denotes the corresponding characteristic function in the ensemble of tame animals (recall (4.20), (4.18)),

$$\chi^{\infty}_{N,\mathcal{R},\mathbf{H}}(\mathbf{T}) = \prod_{j=1}^{N} \frac{Q(H_{N,j}+i\beta^{-1}t_{N,j})}{Q(H_{N,j})}, \tag{6.22}$$

and the quantities $t_{N,j}$ are calculated via (cf. (4.21))

$$t_{N,j} = \big(1-(j-1/2)/N\big)t_0 + \sum_{n=1}^{k+1} t_n \mathbf{1}_{\{j\le r_n\}}.$$

It follows from (5.1) that

$$\left|\log \frac{\widehat{\Xi}(N,\mathcal{R},\mathbf{H}+i\beta^{-1}\mathbf{T})}{\widehat{\Xi}(N,\beta,\mathcal{R},\mathbf{H})}\right| \le 2N\exp\{-4(\beta-\beta_0)\} \tag{6.23}$$

uniformly in $(\mathbf{H},\mathbf{T}) \in G(\delta)$ provided $\beta \ge \beta_0$ with $\beta_0 = \beta_0(2\delta/3) > 0$. On the other hand (see Eq. (4.10.18) in [7]), the inequality

$$|\log Q(H_{N,j}+i\beta^{-1}t_{N,j}) - \log Q(H_{N,j})| \le \bar{C}(\delta)e^{-\beta(2-|H_{N,j}|)} \le \bar{C}(\delta)e^{-3\beta/4} \tag{6.24}$$

holds uniformly in N, $j=1,\dots,N$ and $(\mathbf{H},\mathbf{T}) \in G(\delta)$. Then, (6.21), (6.22), (6.23), and (6.24) imply the estimate

$$|\log \chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})| \le \widetilde{C}(\delta)Ne^{-3\delta/4} \tag{6.25}$$

for all N, $(\mathbf{H},\mathbf{T}) \in G(\delta)$, provided $\beta \ge \beta_0(2\delta/3) > 0$. Finally, the analyticity of $\log \chi_{N,\mathcal{R},\mathbf{H}}(\mathbf{T})$ follows directly from (6.25), definitions (6.21), (6.22), (4.18), and Theorem 5.1.

Since (6.18) follows directly from (6.17), (6.20), and (6.25), one has the convergence

$$\chi^{*}_{N}(\mathbf{T}) \to \overline{\chi}_{\mathbf{H}}(\mathbf{T}), \qquad \text{as } N\to\infty \tag{6.26}$$

that is uniform in $\mathbf{T}$ belonging to any compact set in $\mathbb{R}^{k+2}$ provided β is sufficiently large. □

Let $\mathbf{H}_N$, $\mathbf{H}_N \to \mathbf{H} \in \widehat{\mathcal{D}}^{k+2}_{\delta}$ be the sequence of vectors from Theorem 6.2. For any N define

$$\mathbf{E}_N \equiv \mathbf{E}\Theta_{N,\mathcal{R},\mathbf{H}_N} = \frac{1}{\beta}\nabla_H L_{N,\mathcal{R}}(\mathbf{H})\Big|_{\mathbf{H}=\mathbf{H}_N},$$

and for any $\mathbf{M}_N \in \mathcal{M}^{k+2}_N$ (recall (3.18)) put

$$\mathbf{X}_N = \frac{1}{\sqrt{N}}\big(\mathbf{M}_N - \mathbf{E}_N\big).$$

Theorem 6.3. *Uniformly in* $\mathbf{M}_N \in \mathcal{M}_N^{k+2}$ *and* $\mathbf{H}_N \in \widehat{\mathcal{D}}_\delta^{k+2}$, $\mathbf{H}_N \to \mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, *one has*

$$2N^{\frac{k+4}{2}}\mathbf{P}(\Theta_{N,\mathcal{R},\mathbf{H}_N} = \mathbf{M}_N) - \overline{p}_{\mathbf{H}}(\mathbf{X}_N) \to 0 \qquad \text{as } N \to \infty,$$

where $\overline{p}_{\mathbf{H}}(\cdot)$ *denotes the density of the random vector* $\overline{\Theta}$ *from Theorem 6.2, provided* $\beta \geq \beta_0$ *with sufficiently large* $\beta_0 > 0$.

Proof. Using the well-known inversion formula for the Fourier transformation we rewrite the difference

$$\rho_N = 2N^{\frac{k+4}{2}}\mathbf{P}(\Theta_{N,\mathcal{R},\mathbf{H}_N} = \mathbf{M}_N) - \overline{p}_{\mathbf{H}}(\mathbf{X}_N)$$

in the form

$$\begin{aligned}\rho_N = \frac{1}{(2\pi)^{k+2}} \int_A \chi_N^*(\mathbf{T}) e^{-i(\mathbf{T},\mathbf{H}_N)}\, d\mathbf{T} \\ - \frac{1}{(2\pi)^{k+2}} \int_{\mathbb{R}^{k+2}} \overline{\chi}_{\mathbf{H}}(\mathbf{T}) e^{-i(\mathbf{T},\mathbf{H}_N)}\, d\mathbf{T},\end{aligned} \tag{6.27}$$

where

$$A = \{\mathbf{T} = (t_0, \dots, t_{k+1}) \in \mathbb{R}^{k+2} : |t_0| \leq 2\pi N^{3/2}, |t_l| \leq \pi\sqrt{N}, l = 1, 2, \dots, k+1\}.$$

Following the standard proof of the local limit theorem (see, e. g., [11, §43]) we evaluate the right-hand side of (6.27) by the sum of four terms,

$$(2\pi)^{-(k+2)}\,(J_1 + J_2 + J_3 + J_4)\,,$$

where for some positive constants A and Δ,

$$\begin{aligned} J_1 &= \int_{A_1} |\chi_N^*(\mathbf{T}) - \overline{\chi}_{\mathbf{H}}(\mathbf{T})|\, d\mathbf{T}, \qquad A_1 = [-A, A]^{k+2}, \\ J_2 &= \int_{A_2} \overline{\chi}_{\mathbf{H}}(\mathbf{T})\, d\mathbf{T}, \qquad A_2 = \mathbb{R}^{k+2} \setminus A_1, \\ J_p &= \int_{A_p} |\chi_N^*(\mathbf{T})|\, d\mathbf{T}, \qquad p = 3, 4, \end{aligned}$$

with

$$\begin{aligned} A_3 &= \{\mathbf{T} \in \mathbb{R}^{k+2} : |t_l| \leq \Delta\sqrt{N}, l = 0, 1, \dots, k+1\} \setminus A_1, \\ A_4 &= A \setminus (A_1 \cup A_3)\,. \end{aligned}$$

Fix any $\varepsilon > 0$. We will show in the following that the constants A and Δ can be chosen in such a way to imply $J_p < \varepsilon/4$, $p = 1, \dots, 4$, if only $\beta \geq \beta_0$ (and $N \geq N_0$) with sufficiently large $\beta_0 > 0$ (and $N_0 > 1$).

First, due to (6.26) one has $J_1 \to 0$ as $N \to \infty$ for any fixed $A > 0$ and all $\beta \geq \beta_0$, provided β_0 is sufficiently large.

Then, since the distribution of the random vector $\overline{\Theta}$ is non-degenerate, one has $J_2 \to 0$ as $A \to \infty$ for all $\beta \geq \beta_0$ with sufficiently large β_0.

To estimate J_3 fix any $\mathbf{T} \in A_3$. Then $|\mathbf{T}| \leq \Delta\sqrt{N(k+2)}$ and for any N one gets (recall (6.17), (6.20), and (6.25))

$$|R_N| \le C_1(\delta) N e^{-3\delta/4} \Bigl(\sum_{l=0}^{k+1} |t_l| \Bigr)^3 \le C_1(\delta) N \exp\{-3\delta/4\} (k+2)^{3/2} |\mathbf{T}|^3$$
$$\le C_1(\delta) N^{3/2} \exp\{-3\delta/4\} (k+2)^2 \Delta |\mathbf{T}|^2.$$

Consequently (recall (6.16)),

$$\Bigl| \log \chi_N^*(\mathbf{T}) + \frac{1}{2} \mathcal{B}_{N,\mathcal{R},\mathbf{H}_N}(\mathbf{T}) \Bigr| = \Bigl| \frac{i}{6N^{3/2}} R_N \Bigr| \le \frac{C_1(\delta)(k+2)^2}{6} e^{-3\delta/4} \Delta |\mathbf{T}|^2.$$

Let $\Delta > 0$ be such that

$$\frac{C_1(\delta)(k+2)^2}{6} \exp\{-3\delta/4\} \Delta \le \frac{b}{4}$$

with the constant b from (6.8). Then

$$\Re \log \chi_N^*(\mathbf{T}) \le -\frac{1}{2} \mathcal{B}_{N,\mathcal{R},\mathbf{H}_N}(\mathbf{T}) + \frac{b}{4} |\mathbf{T}|^2 \le -\frac{b}{4} |\mathbf{T}|^2$$

and therefore

$$|\chi_N^*(\mathbf{T})| \le \exp\{\Re \log \chi_N^*(\mathbf{T})\} \le \exp\{-b|\mathbf{T}|^2/4\}$$

for all $\mathbf{T} \in A_3$ uniformly in $N \ge N_0$ and $\beta \ge \overline{\beta}$ (with N_0 and $\overline{\beta}$ from Lemma 6.1). As a result,

$$J_3 = \int_{A_3} |\chi_N^*(\mathbf{T})| \, d\mathbf{T} \le \int_{A_2} e^{-b|\mathbf{T}|^2/4} \, d\mathbf{T} \searrow 0 \qquad \text{as} \qquad A \nearrow \infty.$$

Finally, fix any $\mathbf{T} \in A_4$ and rewrite $|\chi_N^*(\mathbf{T})|$ in the form (recall (6.21), (6.14))

$$|\chi_N^*(\mathbf{T})| = |\chi_{N,\mathcal{R},\mathbf{H}_N}^{\infty}(N^{-1/2}\mathbf{T})| \frac{|\widehat{\Xi}(N, \mathcal{R}, \mathbf{H}_N + i\mathbf{T}/\beta\sqrt{N})|}{|\widehat{\Xi}(N, \mathcal{R}, \mathbf{H}_N)|}. \tag{6.28}$$

The arguments, similar to those used in the proof of Theorem 4.2 from [6], imply the existence of a constant $C = C(\mathcal{R}, \delta, \beta_0) > 0$ such that for all $\mathbf{T} \in A_4$, $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, $\beta \ge \beta_0$, and N sufficiently large one has

$$|\chi_{N,\mathcal{R},\mathbf{H}}^{\infty}(N^{-1/2}\mathbf{T})| \le \exp\{-CN\}.$$

Then, applying (5.1) to estimate the partition functions on the right-hand side of (6.28) one immediately gets

$$|\chi_N^*(\mathbf{T})| \le \exp\{-N(C - 2\exp\{-4(\beta - \beta_0)\})\}.$$

Therefore, for all sufficiently large β, $\beta \ge \overline{\beta}_0$, one obtains

$$J_4 = \int_{A_4} |\chi_N^*(\mathbf{T})| \, d\mathbf{T} \le \int_A e^{-CN/2} \, d\mathbf{T} = (2\pi)^{k+2} N^{\frac{k+4}{2}} \exp\{-CN/2\} \searrow 0$$

as $N \to \infty$ that finishes the proof of the theorem. $\square$

In the arguments above the Gaussian density $\overline{p}_{\mathbf{H}}(\cdot)$ can be replaced by the density of zero-mean Gaussian distribution with the covariance matrix $B_{N,\mathcal{R}}(\mathbf{H}_N)$ (recall (6.5), (6.16), and (6.26)). In particular, one has

Corollary 6.4. *There exist positive constants N_0, β_0, c_0, and C_0 such that for all $N \geq N_0$ and $\beta \geq \beta_0$,*

$$\frac{c_0}{N}\beta^2 \leq \Big(\det \mathbf{Hess}\, L_{\Lambda_N}(\mathbf{H}_N)\Big)^{1/2} \mathbf{P}(\Lambda_{N,\mathbf{H}_N} = A_N) \leq \frac{C_0}{N}\beta^2, \tag{6.29}$$

where $L_{\Lambda_N}(\cdot)$ was determined in (3.5) and $\mathbf{H}_N$ – in (3.11).

For future reference we formulate also the following simple statement.

Corollary 6.5. *Let all $\mathbf{X}_N$ be uniformly bounded. Then under the conditions of Theorem 6.3 one has*

$$\mathbf{P}(\Theta_{N,\mathcal{R},\mathbf{H}_N} = \mathbf{M}_N) = \frac{1}{2} N^{-\frac{k+4}{2}} \overline{p}_{\mathbf{H}}(\mathbf{X}_N) \cdot (1 + o(1)),$$

where the estimate $o(1)$ is uniform with respect to the considered sequences $\mathbf{H}_N \in \widehat{\mathcal{D}}_\delta^{k+2}$ and $\mathbf{X}_N$, provided only β is sufficiently large.

Moreover, there exist positive constants β_0, c_i, C_i, $i = 1, 2$, and a number N_0 such that

$$c_2\beta^{k+2} \leq c_1 \overline{p}_{\mathbf{H}}(\mathbf{X}_N) \leq N^{\frac{k+4}{2}} \mathbf{P}\left(\Theta_{N,\mathcal{R},\mathbf{H}_N} = \mathbf{M}_N\right) \leq C_1 \overline{p}_{\mathbf{H}}(\mathbf{X}_N) \leq C_2 \beta^{k+2} \tag{6.30}$$

uniformly in $N \geq N_0$ and the sequences $\mathbf{H}_N$, $\mathbf{X}_N$ under consideration, provided only $\beta \geq \beta_0$.

7. Convergence of Finite Dimensional Distributions

We prove here the convergence of finite dimensional distributions of the conditional random process (recall (3.15))

$$\theta_N^*(t) = \frac{1}{\sqrt{N}}\big(\theta_N^+(t) - N\hat{e}(t)\big), \qquad t \in [0, 1],$$

to the corresponding distributions of the Gaussian measure μ^* from Theorem 3.2.

Consider first the vector Λ_N of conditions (3.3) with the logarithmic moment generating function $L_{\Lambda_N}(\mathbf{H})$ from (3.5). Assume that $\mathbf{H}$ belongs to the set $\mathcal{D}_\delta^2$ defined in (3.7). Then

$$\frac{1}{N\beta} \nabla_{\mathbf{H}} L_{\Lambda_N}(\mathbf{H}) = \mathcal{I}(\mathbf{H}) + O\Big(\frac{\log^{10} N}{N}\Big), \tag{7.1}$$

where $\mathcal{I}(\mathbf{H})$ was defined in (3.12) and the estimate $O(\cdot)$ is uniform in $\mathbf{H} \in \mathcal{D}_\delta^2$. Indeed, it follows from (3.5) and (4.26) that

$$\beta^{-1}\nabla_{\mathbf{H}} L_{\Lambda_N}(\mathbf{H}) = \beta^{-1}\nabla_{\mathbf{H}} \left(\log \widehat{\Xi}(N, \Lambda, \mathbf{H}) + \log \Xi(N, \Lambda, \mathbf{H}, \infty)\right). \tag{7.2}$$

Then, due to Remark 5.5.2 one has

$$\begin{aligned} \Big|\frac{1}{N\beta}\nabla_{\mathbf{H}} \log \widehat{\Xi}(N, \Lambda, \mathbf{H}) - \frac{1}{\beta}\nabla_{\mathbf{H}} \int_0^1 \widehat{F}\big((1-x)H_0 + H_1\big)\, dx\Big| \\ \leq e^{-3(\beta - \tilde{\beta}_0)} \frac{\log^{10} N}{N}. \end{aligned} \tag{7.3}$$

On the other hand, the analyticity and uniform boundedness of $\log Q(\cdot)$ in the region (4.23) imply the estimate

$$\frac{1}{N\beta}\nabla_{\mathbf{H}}\log \Xi(N,\Lambda,\mathbf{H},\infty)-\frac{1}{\beta}\nabla_{\mathbf{H}}\int_0^1 \log Q\bigl((1-x)H_0+H_1\bigr)\,dx = O\bigl(N^{-1}\bigr). \quad (7.4)$$

Finally, (7.1) follows directly from (7.2)–(7.4) and definition (3.12).

Let $\mathbf{H}_N$ and $\widehat{\mathbf{H}}$ be the solutions of (3.11) and (3.12) respectively. Applying the implicit function theorem to $\mathcal{I}(\cdot)$ and taking into account estimate (7.1) one easily obtains

$$|\mathbf{H}_N-\widehat{\mathbf{H}}| = \beta^{-1}O\Bigl(\frac{\log^{10} N}{N}\Bigr) + \beta^{-1}O(N^{-1}A_N - A), \quad (7.5)$$

where the estimates $O(\cdot)$ are uniform in $\mathbf{H}_N \in \mathcal{D}_\delta^2$, and $N^{-1}A_N \in \mathcal{I}\bigl(\mathcal{D}_{\delta'}^2\bigr)$ respectively (here $\delta' > 0$ is any fixed number and $\mathcal{I}\bigl(\mathcal{D}_{\delta'}^2\bigr)$ denotes the image of the region $\mathcal{D}_{\delta'}^2$). Thus $\mathbf{H}_N \to \widehat{\mathbf{H}}$ as $N\to\infty$ and therefore all $\mathbf{H}_N$ with sufficiently large N belong to the region $\mathcal{D}_\delta^2$ from (3.7) (recall Remark 3.1.2).

Let Θ_N be the random vector from (3.16),

$$\Theta_N \equiv \bigl(Y_N, X_N(s_1),\dots,X_N(s_k),X_N(s_{k+1})\bigr) \in \mathbb{R}^{k+2}.$$

For $\mathbf{H}_N = \bigl(H_N^0, H_N^1\bigr)$ determined from (3.11) we introduce the vector

$$\mathbf{H}_N^0 \equiv (H_N^0,0,\dots,0,H_N^1) \in \mathbb{R}^{k+2}.$$

Clearly, the sequence $\mathbf{H}_N^0$ converges to

$$\mathbf{H}^0 = (Q,0,\dots,0,H) \in \mathbb{R}^{k+2},$$

where $\widehat{\mathbf{H}} = (Q,H)$ denotes the solution of (3.12); thus, all $\mathbf{H}_N^0$ with sufficiently large N belong to the region $\widehat{\mathcal{D}}_\delta^{k+2}$ from (4.7). Denote (recall (6.1), (6.3))

$$\mathbf{E}_N^0 \equiv \mathbf{E}\Theta_{N,\mathcal{R},\mathbf{H}_N^0} = \bigl(Nq_N, e_N^1,\dots,e_N^k, Nb_N\bigr)$$

with

$$e_N^i = e_N(s_i) = \frac{1}{\beta}\frac{\partial}{\partial H_i}L_{N,\mathcal{R}}(\mathbf{H})\Big|_{\mathbf{H}=\mathbf{H}_N^0}.$$

Similarly to (7.1) one easily obtains the relation (recall (7.5))

$$\frac{1}{N}e_N(s) = \hat{e}(s) + sO\Bigl(\frac{\log^{10} N}{N}\Bigr) + sO\bigl(N^{-1}A_N - A\bigr), \quad (7.6)$$

where (cf. (3.14))

$$\hat{e}(s) = \frac{1}{\beta}\int_0^s F'\bigl((1-x)Q+H\bigr)\,dx = \bigl(F(H+Q)-F(H+Q-Qs)\bigr)\big/\beta Q, \quad (7.7)$$

and the estimates $O(\cdot)$ are uniform in $s\in[0,1]$, provided β is sufficiently large.

For any $\mathbf{M}_N \in \mathcal{M}_N^{k+2}$ (see (3.18)) of the kind

$$\mathbf{M}_N = (Nq_N, m_N^1,\dots,m_N^k, Nb_N),$$

we put

$$x_N^i = \frac{1}{\sqrt{N}}(m_N^i - e_N^i), \qquad i = 1, \dots, k.$$

Let $\overline{p}_k(\cdot)$ denote the probability distribution of the Gaussian random vector $\overline{\Theta} = (\overline{\eta}, \overline{\xi}_1, \dots, \overline{\xi}_{k+1})$ with the characteristic function $\overline{\chi}_{\mathbf{H}^0}(\mathbf{T})$ from (6.13). Then

$$\tilde{p}_k(x^1, \dots, x^k|\mathbf{0}) \equiv \frac{\overline{p}_k(\mathbf{X}^0)}{\overline{p}_0(\mathbf{0})}, \qquad \mathbf{X}^0 = (0, x^1, \dots, x^k, 0) \in \mathbb{R}^{k+2},$$

presents the density of the conditional distribution $(\overline{\xi}_1, \dots, \overline{\xi}_k | \overline{\eta} = 0, \overline{\xi}_{k+1} = 0)$. Finally, define the random process (recall (3.20), (7.7))

$$\Theta_N^*(t) = \frac{1}{\sqrt{N}}\big(\Theta_N(t) - N\hat{e}(t)\big). \tag{7.8}$$

Theorem 7.1. *Let a natural number k and a collection of real numbers t_i, $0 < t_1 < \dots < t_k < 1$, be fixed. Then for all $\beta \geq \beta_0$ with sufficiently large β_0 the distribution of the random vector $\big(\Theta_N^*(t_1), \dots, \Theta_N^*(t_k)\big)$ converges weakly to the Gaussian distribution with the density $\tilde{p}_k(\cdot|\mathbf{0})$. This limiting distribution coincides with the corresponding distribution of the measure μ^* from Theorem 3.2.*

The proof of Theorem 7.1 can be obtained by literal repetition of that of Theorem 5.2 in [6]. It is based on the following simple observation that follows immediately from Theorem 6.3 (cf. Lemma 5.1 in [6]).

Lemma 7.2. *Let all x_N^i be uniformly bounded. Then*

$$\mathbf{P}\left(\Theta_N(s_1) = m_N^1, \dots, \Theta_N(s_k) = m_N^k\right) = N^{-\frac{k}{2}}\tilde{p}_k(x_N^1, \dots, x_N^k|\mathbf{0})\,(1 + o(1))$$

as $N \to \infty$ if only β is sufficiently large, $\beta \geq \beta_0 > 0$; the estimate $o(\cdot)$ is uniform in such x_N^i.

Denote (cf. (4.3))

$$\Delta_j X = \Delta_j X(S) = g_N^+(j) - g_N^+(j-1)$$

and choose any ρ,

$$0 < \rho < \overline{\delta}/12 \tag{7.9}$$

with $\overline{\delta}$ fixed in Theorem 3.2.

Lemma 7.3. *There exist positive constants C, β_0, and N_0 such that for all $\beta \geq \beta_0$, $N \geq N_0$, and all $j = 1, \dots, N$ one has*

$$\mathbf{E}\big(\exp\{\rho|\Delta_j X|\} \mid \Lambda_N = A_N\big) \leq C. \tag{7.10}$$

Proof. Fix any $j \in \{1, 2, \dots, N\}$ and a phase boundary $S \in \mathcal{T}_N$. Applying to S the animal decomposition described in Sect. 4 we observe that $\Delta_j X$ is uniquely determined by the animal ξ satisfying the condition $J(\hat{\xi}) \supseteq (j-1, j]$. Denote by $\{\hat{\xi}\}$ the event

$$\{\hat{\xi}\} = \{S \in \mathcal{T}_N : \text{ the animal decomposition of } S \text{ contains } \hat{\xi}\}.$$

Then one has

$$\mathbf{E}\big(e^{\rho|\Delta_j X|}\,|\,\Lambda_N = A_N\big) = \sum_{\hat{\xi}} \exp\{\rho|\Delta_j X(\hat{\xi})|\}\mathbf{P}(\{\hat{\xi}\}\,|\,\Lambda_N = A_N\big), \tag{7.11}$$

where the summation is going over the whole set of disjoint events $\{\hat{\xi}\}$ such that $J(\hat{\xi}) = (m_{i-1}, m_i] \supseteq (j-1, j]$. Relation (7.11) will be the initial point of our reasoning.

We start with the following simple observation. Let $\varXi(N, \mathcal{R}, \mathbf{H})$, $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$, be the partition function from (4.13) and $\hat{\xi} \in \widehat{\mathcal{K}}_N$ be the animal fixed above. Denote by $\widehat{\mathcal{K}}_N(\hat{\xi}) \subset \widehat{\mathcal{K}}_N$ the set of all collections from $\widehat{\mathcal{K}}_N$ that contain $\hat{\xi}$,

$$\widehat{\mathcal{K}}_N(\hat{\xi}) = \{\{\hat{\xi}_1, \dots, \hat{\xi}_{l+1}\} \in \widehat{\mathcal{K}}_N : \hat{\xi} \in \{\hat{\xi}_1, \dots, \hat{\xi}_{l+1}\}\}.$$

Clearly, the sets $\widehat{\mathcal{K}}_N(\hat{\xi})$ form the partition of $\widehat{\mathcal{K}}_N$ labeled by $\hat{\xi}$ under consideration. Define (cf. (4.13))

$$\begin{aligned}\varXi\big(N, \mathcal{R}, \mathbf{H}; \hat{\xi}\big) &\equiv \sum_{\{\hat{\xi}_1, \dots, \hat{\xi}_{l+1}\} \in \widehat{\mathcal{K}}_N(\hat{\xi})} \prod_{i=1}^{l+1} \Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}_i) \\ &= \varXi\big(N, \mathcal{R}, \mathbf{H}\,|\,\hat{\xi}\big) \cdot \Psi_{N,\mathcal{R},\mathbf{H}}(\hat{\xi}).\end{aligned} \tag{7.12}$$

Then for all $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ and sufficiently large β one has

$$\left| \frac{\varXi\big(N, \mathcal{R}, \mathbf{H}\,|\,\hat{\xi}\big)}{\varXi(N, \mathcal{R}, \mathbf{H})} \right| \le \exp\{(2\beta + Q_\delta + \rho)|J(\hat{\xi})|\}, \tag{7.13}$$

where $J(\hat{\xi})$ is the base of the animal $\hat{\xi}$ and (recall (4.25))

$$Q_\delta \equiv \max_{H:|H|<2-\delta/2\beta} |\log Q(H) + 2\beta|. \tag{7.14}$$

To check (7.13) observe that the cluster expansion of $\log \varXi\big(N, \mathcal{R}, \mathbf{H}\,|\,\hat{\xi}\big)$ contains only the cluster weights depending on $I = (a, b] \subseteq (0, N] \setminus J(\hat{\xi})$. Since the same weights appear in the expansion for $\log \varXi(N, \mathcal{R}, \mathbf{H})$ one easily obtains (recall (4.26), (4.20), (4.21), and (5.5))

$$\left| \log \varXi\big(N, \mathcal{R}, \mathbf{H}\,|\,\hat{\xi}\big) - \log \varXi(N, \mathcal{R}, \mathbf{H}) + \sum_{j \in J(\hat{\xi})} \log Q(H_{N,j}) \right| \le K|J(\hat{\xi})| \tag{7.15}$$

for all $\mathbf{H} \in \widehat{\mathcal{D}}_\delta^{k+2}$ and sufficiently large β, where the constant $K = K(\beta) \searrow 0$ as $\beta \nearrow \infty$. Thus, (7.13) follows directly from (7.15), (7.14), and the inequality $|H_{N,j}| < 2 - \delta/2\beta$ (cf. (4.7), (4.23)).

We will show below that for some constant $\bar{C} > 0$ and all sufficiently large β one has

$$\begin{aligned}\mathbf{P}(\{\hat{\xi}\}\,|\,\Lambda_N = A_N\big) &= \frac{\mathbf{P}(\Lambda_N = A_N; \{\hat{\xi}\}\big)}{\mathbf{P}(\Lambda_N = A_N\big)} \\ &\le \bar{C} \exp\{(2\beta + Q_\delta + 2\rho)|J(\hat{\xi})| + \rho|\Lambda[\hat{\xi}]|\}\Psi_{N,\Lambda,\mathbf{H}_N}\big(\hat{\xi}\big),\end{aligned} \tag{7.16}$$

where

$$\Lambda[\hat{\xi}] = \big(a[\hat{\xi}], h[\hat{\xi}]\big) = \Big(\frac{1}{N} a_N(\hat{\xi}) + \Big(1 - \frac{m_i}{N}\Big) h(\hat{\xi}), h(\hat{\xi})\Big)$$

(recall that $J(\hat\xi) = (m_{i-1}, m_i]$), $\mathbf{H}_N$ is the solution to (3.11), and (cf. (4.11))

$$\begin{aligned}\Psi_{N,\Lambda,\mathbf{H}}\big(\hat\xi\big) &\equiv \exp\Big\{-2\beta|\hat\xi| + \beta h(\hat\xi)\Big(\Big(1-\frac{m_i}{N}\Big)H_0 + H_1\Big)\\ &\qquad\qquad + \beta H_0\frac{1}{N}a(\hat\xi)\Big\}\prod_{\Lambda_s\in\hat\xi}\Psi(\Lambda_s)\\ &= \exp\Big\{-2\beta|\hat\xi| + \beta\big(\Lambda[\hat\xi],\mathbf{H}\big)\Big\}\prod_{\Lambda_s\in\hat\xi}\Psi(\Lambda_s)\end{aligned}$$

with $\mathbf{H}\in\mathcal{D}_\delta^2$. Then (7.10) follows directly.

Indeed, according to (4.42),

$$\beta\big(\Lambda[\hat\xi],\mathbf{H}\big) \le \big(2\beta - 3\delta/4\big)\,N_v(\hat\xi) \tag{7.17}$$

for any $\mathbf{H}\in\mathcal{D}_\delta^2$. Then, the inequalities

$$|h(\hat\xi)| \le N_v(\hat\xi), \qquad |a(\hat\xi)| \le |J(\hat\xi)|\cdot N_v(\hat\xi), \qquad |J(\hat\xi)| \le m_i \tag{7.18}$$

imply $|a[\hat\xi]| \le N_v(\hat\xi)$ and therefore

$$|\Lambda[\hat\xi]| \le |a[\hat\xi]| + |h[\hat\xi]| \le 2N_v(\hat\xi). \tag{7.19}$$

As a result, using the simple observation

$$|\Delta_j X(\hat\xi)| \le N_v(\hat\xi), \tag{7.20}$$

one obtains (recall (7.9))

$$\begin{aligned}&\mathbf{E}\big(\exp\{\rho|\Delta_j X|\}\mid \Lambda_N = A_N\big)\\ &\quad\le \bar C\sum_{\hat\xi} e^{-2\beta|\hat\xi|+(2\beta+Q_\delta+2\rho)|J(\hat\xi)|+(2\beta-\delta/2)N_v(\hat\xi)}\prod_{\Lambda_s\in\hat\xi}\Psi(\Lambda_s)\\ &\quad= \bar C\sum_I e^{(2\beta+Q_\delta+2\rho)|I|}\sum_{\hat\xi:J(\hat\xi)=I} e^{-2\beta|\hat\xi|+(2\beta-\delta/2)N_v(\hat\xi)}\prod_{\Lambda_s\in\hat\xi}\Psi(\Lambda_s),\end{aligned} \tag{7.21}$$

where $\sum_I$ denotes the summation over all $I = (a,b] \subseteq (0,N]$ such that $I \supseteq (j-1,j]$. As in Sect. 4 (recall (4.32)–(4.35)) we estimate the inner sum via

$$\begin{aligned}&\sum_{S:I(S)=I,\ y_{\text{in}}(S)=0} e^{-2\beta|\hat\xi|+(2\beta-\delta/2)N_v(\hat\xi)}\widetilde X(S)\\ &\qquad\le e^{-6(\beta-\beta_2)(|I|-1)+2(\beta_2-\beta)}\sum_{S:I(S)=I,\ y_{\text{in}}(S)=0} e^{-\delta N_v(S)/4-\beta_3(N_h(S)+1)},\end{aligned}$$

where $\beta_3 = 2\beta_2 - \varepsilon$ and β_2 in (4.33) is sufficiently large to imply $\varepsilon(\beta_2) < \delta/4$. Evaluating the last sum with the help of (4.37) one easily obtains (7.10),

$$\mathbf{E}\big(e^{\rho|\Delta_j X|}\mid \Lambda_N = A_N\big) \le \tilde C\sum_{n=1}^{\infty}(n+1)e^{-4(\beta-\beta_4)n}\frac{R(\beta_3,\delta)^n}{1-R(\beta_3,\delta)} \le C, \tag{7.22}$$

for $\beta \geq \beta_4$ and some constant C, where we set $\beta_4 = (6\beta_2 + Q_\delta + 2\rho)/4$ and $\tilde{C} = C\exp\{2\beta_2 + Q_\delta + 2\rho\}$.

It remains to establish (7.16). First, we apply the analog of (6.4) to rewrite

$$\frac{\mathbf{P}\big(\Lambda_N = A_N; \{\hat{\xi}\}\big)}{\mathbf{P}\big(\Lambda_N = A_N\big)} = \frac{\Xi(N, \mathbf{H}_N, \Lambda; \hat{\xi})}{\Xi(N, \mathbf{H}_N, \Lambda)} \cdot \frac{\mathbf{P}\big(\Lambda_{N;\hat{\xi},\mathbf{H}_N} = A_N\big)}{\mathbf{P}\big(\Lambda_{N,\mathbf{H}_N} = A_N\big)} \tag{7.23}$$

with $\mathbf{H}_N$ denoting the solution of (3.11). The first fraction on the right-hand side of (7.23) can be estimated with the help of (7.12)–(7.13),

$$\begin{aligned} \frac{\Xi(N, \mathbf{H}_N, \Lambda; \hat{\xi})}{\Xi(N, \mathbf{H}_N, \Lambda)} &= \frac{\Xi(N, \mathbf{H}_N, \Lambda \mid \hat{\xi})}{\Xi(N, \mathbf{H}_N, \Lambda)} \Psi_{N,\Lambda,\mathbf{H}_N}\big(\hat{\xi}\big) \\ &\leq \exp\{(2\beta_2 + Q_\delta + \rho)|J(\hat{\xi})|\}\Psi_{N,\Lambda,\mathbf{H}_N}\big(\hat{\xi}\big). \end{aligned} \tag{7.24}$$

On the other hand, similarly to (7.13) one obtains

$$\Big|\frac{1}{\beta}\nabla_{\mathbf{H}}\Big(\log \Xi(N, \mathbf{H}_N, \Lambda \mid \hat{\xi}) - \log \Xi(N, \mathbf{H}_N, \Lambda)\Big)\Big| \leq (Q_{\delta,1} + \rho)|J(\hat{\xi})| \tag{7.25}$$

with (recall (4.24))

$$\begin{aligned} Q_{\delta,1} &\equiv \max_{H:|H|<2-\delta/2\beta}\Big|\frac{\partial}{\beta\partial}\log Q(H)\Big| = \max_{H:|H|<2-\delta/2\beta}\frac{\sinh H\beta}{\cosh 2\beta - \cosh H\beta} \\ &\leq \frac{\sinh(2\beta - \delta/2)}{\cosh 2\beta - \cosh(2\beta - \delta/2)} \leq \frac{e^{-\delta/2}}{1 - e^{-\delta/4}} = \frac{e^{-\delta/4}}{e^{\delta/4} - 1} \end{aligned}$$

for all $\beta \geq \beta_0(\delta)$. Thus, taking into account the simple identity

$$\frac{1}{\beta}\nabla_{\mathbf{H}}\log\Psi_{N,\Lambda,\mathbf{H}}\big(\hat{\xi}\big) = \Lambda[\hat{\xi}]$$

(that can be obtained by direct computation) one deduces immediately that

$$\mathbf{E}\Lambda_{N;\hat{\xi},\mathbf{H}_N} \equiv \frac{1}{\beta}\nabla_{\mathbf{H}}\log\Xi(N, \mathbf{H}, \Lambda; \hat{\xi})\,\Big|_{\mathbf{H}=\mathbf{H}_N}$$

satisfies the estimate

$$\Big|\mathbf{E}\Lambda_{N;\hat{\xi},\mathbf{H}_N} - A_N - \Lambda[\hat{\xi}]\Big| \leq (Q_{\delta,1} + \rho)|J(\hat{\xi})|. \tag{7.26}$$

It remains to evaluate the last fraction in (7.23). Let first $|J(\hat{\xi})| \leq A\sqrt{N}$ with some fixed constant $A > 0$. Observe that the analog of (7.25) for the second derivatives can be obtained in a similar way; therefore, the analog of (5.24) for our special case $\mathcal{R} = \{N\}$ imply the convergence

$$\frac{1}{\beta^2 N}\mathbf{Hess}\log\Xi(N, \mathbf{H}, \Lambda; \hat{\xi}) \to \frac{1}{\beta^2}\mathbf{Hess}\int_0^1 F\big((1-x)H_0 + H_1\big)\,dx$$

for any $\beta \geq \beta_0(\delta)$ uniformly in $\mathbf{H} = (H_0, H_1) \in \widehat{\mathcal{D}}_\delta^2$. Thus, the limiting properties of the random vector $\big(\Lambda_N; \{\hat{\xi}\}\big)$ are the same as that of Λ_N. In particular, if $|\Lambda[\hat{\xi}]| \leq B\sqrt{N}$ with any fixed constant $B > 0$, one can apply Corollary 6.4 to obtain (recall (7.26))

$$\frac{\mathbf{P}\big(\Lambda_{N;\hat{\xi},\mathbf{H}_N} = A_N\big)}{\mathbf{P}\big(\Lambda_{N,\mathbf{H}_N} = A_N\big)} \le \bar{C}_1 \tag{7.27}$$

provided β is sufficiently large. In the opposite case, $|\Lambda[\hat{\xi}]| > B\sqrt{N}$, one has (recall (6.30))

$$\frac{\mathbf{P}\big(\Lambda_{N;\hat{\xi},\mathbf{H}_N} = A_N\big)}{\mathbf{P}\big(\Lambda_{N,\mathbf{H}_N} = A_N\big)} \le \frac{1}{\mathbf{P}\big(\Lambda_{N,\mathbf{H}_N} = A_N\big)} \le \frac{\bar{C}_2 N^2}{\beta^2} \le \bar{C}_3 e^{\rho|\Lambda[\hat{\xi}]|}. \tag{7.28}$$

Finally, for $|J(\hat{\xi})| > A\sqrt{N}$, one gets

$$\frac{\mathbf{P}\big(\Lambda_{N;\hat{\xi},\mathbf{H}_N} = A_N\big)}{\mathbf{P}\big(\Lambda_{N,\mathbf{H}_N} = A_N\big)} \le \frac{1}{\mathbf{P}\big(\Lambda_{N,\mathbf{H}_N} = A_N\big)} \le \frac{\bar{C}_2 N^2}{\beta^2} \le \bar{C}_4 e^{\rho|J(\hat{\xi})|}. \tag{7.29}$$

Now, (7.16) follows immediately from (7.23), (7.24), and (7.27)–(7.29). □

Observe that this proof can be applied to any local variable that satisfies the analog of (7.20) with the right-hand side of the kind $CN_v(\hat{\xi})$, where $C > 0$ is any fixed constant; then (7.9) should be replaced by

$$0 < \rho < \bar{\delta}/12C.$$

In particular, one has

Corollary 7.4. *Let the constants C, β_0, and N_0 be as determined in Lemma 7.3. Then*

$$\mathbf{E}\big(\exp\{\rho|g_N^+(j) - g_N^-(j)|\} \mid \Lambda_N = A_N\big) \le C$$

for all $j = 1, 2, \ldots, N$, provided only $N \ge N_0$ and $\beta \ge \beta_0$.

For future reference we formulate here the following corollary of Lemma 7.3 that could be obtained directly from (7.16) using calculations similar to those in (7.21)–(7.22).

Corollary 7.5. *Fix a number $j \in \{1, 2, \ldots, N\}$. For any phase boundary $S \in \mathcal{T}_N$ apply the animal decomposition and denote by $\xi(j)$ the animal satisfying $J(\xi(j)) \supseteq (j-1, j]$. Then there exists $\bar{\beta} < \infty$ such that for all $\beta \ge \bar{\beta}$ and all $l \ge 1$ one has*

$$\mathbf{P}\big(|J(\xi(j))| \ge l + 1 \mid \Lambda_N = A_N\big) \le \exp\{-4(\beta - \bar{\beta})l\}.$$

Another consequence of Lemma 7.3 is the following

Theorem 7.6. *For all $\beta \ge \beta_0$ with β_0 determined in Theorem 7.1 the finite dimensional distributions of the random process $\theta_N^*(t)$, $t \in [0, 1]$, have the same limiting behaviour as that of $\Theta_N^*(t)$.*

Proof. In view of the observation (recall (3.15), (3.20), (7.8), (3.17), and (3.2))

$$\theta_N^*(t) - \Theta_N^*(t) = \frac{\{Nt\}}{\sqrt{N}} \big(g_N^+([Nt] + 1) - g_N^+([Nt]) \mid \Lambda_N = A_N\big)$$

the statement of the theorem follows immediately from (7.10). For details see [6, Theorem 5.4]. □

8. Proof of Main Theorems

To complete the proof of our main result we need to check the weak compactness of the sequence of measures μ_N^*. We obtain it here as an implication of Theorem 2.2 from [10, Chap. 9] which provides the sufficient condition for the weak compactness of measures in $\mathbf{C}[0,1]$. The following statement verifies the assumption of the above mentioned theorem.

Theorem 8.1. *There exist positive numbers C, β_0, and N_0 such that*

$$\mathbf{E}\left|\theta_N^*(t)-\theta_N^*(s)\right|^4 \le C|t-s|^{7/4}$$

uniformly in $N \ge N_0$ and all segments $[s,t] \subseteq [0,1]$, $s<t$, provided only $\beta \ge \beta_0$.

As in [6] we consider two cases, $\Delta = \Delta_N \equiv |t-s| \le N^{-8/9}$ and $\Delta > N^{-8/9}$, separately.

Lemma 8.2. *There exist positive numbers C_1 and N_1 such that*

$$\mathbf{E}\left|\theta_N^*(t)-\theta_N^*(s)\right|^4 \le C_1|t-s|^{7/4}$$

uniformly in $[s,t] \subset [0,1]$, $\Delta \le N^{-8/9}$, if only $N \ge N_1$ and $\beta \ge \beta_0$ with β_0 determined in Lemma 7.3.

The proof is based on estimate (7.10) and can be obtained by literal repetition of that of Lemma 6.2 from [6].

Lemma 8.3. *There exist positive numbers C_2, β_2, and N_2 such that*

$$\mathbf{E}\left|\theta_N^*(t)-\theta_N^*(s)\right|^4 \le C_2|t-s|^2 \tag{8.1}$$

uniformly in $[s,t] \subset [0,1]$, $\Delta > N^{-8/9}$, if only $N \ge N_2$ and β is sufficiently large, $\beta \ge \beta_2$.

Proof. Denote (recall (3.2))

$$\zeta_N \equiv \xi_N^+(t) - \xi_N^+(s) = g_N^+(Nt) - g_N^+(Ns)$$

and introduce the random vector (cf. (3.3))

$$\widetilde{\Lambda}_N = \left(Y_N, h_N, \zeta_N/\sqrt{\Delta}\right)$$

with the logarithmic moment generating function $\widetilde{L}_{\Lambda_N}(\mathbf{H})$, $\mathbf{H} \in \mathbb{R}^3$, (cf. (3.5))

$$\widetilde{L}_{\Lambda_N}(\mathbf{H}) \equiv \log \mathbf{E} \exp\Big\{\beta\big(\mathbf{H}, \widetilde{\Lambda}_N\big)\Big\} = \log \widetilde{\Xi}(N,\Lambda,\mathbf{H}) - \log \Xi(N). \tag{8.2}$$

For $\mathbf{H}_N = (H_N^0, H_N^1)$ determined from (3.11) we define

$$\mathbf{H}_N^0 = (H_N^0, H_N^1, 0)$$

and

$$\widetilde{\mathbf{E}}_N \equiv \frac{1}{\beta}\nabla_{\mathbf{H}} \log \widetilde{\Xi}(N,\Lambda,\mathbf{H})\Big|_{\mathbf{H}=\mathbf{H}_N^0} = (Nq_N, Nb_N, \tilde{e}_N), \tag{8.3}$$

where similarly to (7.6) one obtains the relation (recall (3.9))

$$\tilde{e}_N = N(\hat{e}(t) - \hat{e}(s)) + \Delta o(\sqrt{N}).$$

As a result, for all sufficiently large N one has

$$\mathbf{E}\Big(\frac{\theta_n^*(t) - \theta_n^*(s)}{\sqrt{\Delta}}\Big)^4 \leq 2\sum_{k\geq 0}(k+1)^4 \mathbf{P}\Big(\frac{|\zeta_N - \tilde{e}_N\sqrt{\Delta}|}{\sqrt{N\Delta}} > k \,|\, \Lambda_N = A_N\Big). \tag{8.4}$$

We will show below that for all $N \geq N_2$ and $\beta \geq \beta_2$ with sufficiently large $N_2 > 0$ and $\beta_2 > 0$ one gets the estimate

$$\mathbf{P}\big(|\zeta_N - \tilde{e}_N\sqrt{\Delta}| > k\sqrt{N\Delta} \,|\, \Lambda_N = A_N\big) \leq f_N(k), \tag{8.5}$$

where

$$f_N(k) = \begin{cases} D_1 \exp\{-\alpha_1 k^2\}, & \text{if} \quad |k| \leq \varepsilon\sqrt{N\Delta}, \\ D_2 \exp\{-\alpha_2 N^{1/18}|k|\}, & \text{if} \quad |k| > \varepsilon\sqrt{N\Delta}, \end{cases} \tag{8.6}$$

and D_1, D_2, α_1, α_2, ε are some fixed positive constants. Thus, the series in (8.4) is convergent and (8.1) follows immediately.

It remains to establish estimates (8.5)–(8.6). To do this we introduce the vector (recall (8.3))

$$\widetilde{\mathbf{Z}}_N \equiv (Nq_N, Nb_N, \tilde{e}_N + k\sqrt{N}) = \widetilde{\mathbf{E}}_N + (0, 0, k\sqrt{N}) \tag{8.7}$$

and determine $\widetilde{\mathbf{H}}_N = \widetilde{\mathbf{H}}_N(k) = (\widetilde{H}_N^0(k), \widetilde{H}_N^1(k), \widetilde{H}_N^2(k))$ from the equation

$$\frac{1}{\beta}\nabla_{\mathbf{H}} \log \widetilde{\Xi}(N, \Lambda, \mathbf{H}) \Big|_{\mathbf{H}=\widetilde{\mathbf{H}}_N} = \widetilde{\mathbf{Z}}_N. \tag{8.8}$$

It follows from (8.2) and the implicit function theorem that provided k in (8.7) is of order $\sqrt{N\Delta}$ the quantities $\widetilde{H}_N^0(k) - H_N^0$, $\widetilde{H}_N^1(k) - H_N^1$, and $\widetilde{H}_N^2(k)\sqrt{\Delta}$ are of order Δ. Therefore, there exist $\varepsilon = \varepsilon(\rho) > 0$, $N_3 > 0$, and $\beta_3 > 0$ such that for all k, $|k| \leq \varepsilon\sqrt{N\Delta}$, all $\beta \geq \beta_3$, and all $N \geq N_3$ the following inequalities hold true

$$|\widetilde{H}_N^0(k) - H_N^0| < \rho\Delta, \qquad |\widetilde{H}_N^1(k) - H_N^1| < \rho\Delta, \qquad |\widetilde{H}_N^2(k)| < \rho\sqrt{\Delta}.$$

Thus, applying arguments similar to those used in the proof of Lemma 6.1 one obtains the inequality (cf. (6.8))

$$\big(\mathbf{Hess}\, \widetilde{L}_{\Lambda_N}(\mathbf{H})\mathbf{T}, \mathbf{T}\big) \Big|_{\mathbf{H}=\widetilde{\mathbf{H}}_N(k)} \geq C\beta^2 N|\mathbf{T}|^2 \tag{8.9}$$

for all k, $|k| \leq \varepsilon\sqrt{N\Delta}$, all $\mathbf{T} \in \mathbb{R}^3$, $\beta \geq \beta_4$, $N \geq N_4$, where C, β_4, and N_4 are some positive constants depending only on ε and β_0 from (5.11). For future reference we fix such a value of $\varepsilon > 0$.

Assuming that $\zeta_N - \tilde{e}_N\sqrt{N\Delta} \geq 0$ (in the opposite case the estimates are similar) we rewrite

$$\begin{aligned} &\mathbf{P}\big(\zeta_N > \tilde{e}_N\sqrt{\Delta} + k\sqrt{N\Delta} \,|\, \Lambda_N = A_N\big) \\ &\quad = \frac{\mathbf{P}\big(\Lambda_N = A_N, \zeta_N > \tilde{e}_N\sqrt{\Delta} + k\sqrt{N\Delta}\big)}{\mathbf{P}(\Lambda_N = A_N)} \\ &\quad = \frac{e^{-\widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{Z}}_N)}}{e^{-L^*_{\Lambda_N}(A_N)}} \frac{\mathbf{P}_{\widetilde{\mathbf{H}}_N}\big(\Lambda_N = A_N, \zeta_N > \tilde{e}_N\sqrt{\Delta} + k\sqrt{N\Delta}\big)}{\mathbf{P}_{\mathbf{H}_N}(\Lambda_N = A_N)}, \end{aligned} \tag{8.10}$$

where $\widetilde{L}^*_{\Lambda_N}(\cdot)$ and $L^*_{\Lambda_N}(\cdot)$ denote the Legendre transformations of the functions $\widetilde{L}_{\Lambda_N}(\cdot)$ and $L_{\Lambda_N}(\cdot)$ correspondingly, $\widetilde{\mathbf{H}}_N$ was determined in (8.8), $\mathbf{H}_N$ – in (3.11), and $\mathbf{P}_{\widetilde{\mathbf{H}}_N}(\cdot)$, $\mathbf{P}_{\mathbf{H}_N}(\cdot)$ denote the tilted distributions of the random vectors $\widetilde{\Lambda}_N$ and Λ_N with parameters $\widetilde{\mathbf{H}}_N$ and $\mathbf{H}_N$ respectively.

Let us evaluate first the difference $\widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{Z}}_N) - L^*_{\Lambda_N}(A_N)$. It follows from (8.2), (8.3), (3.11) and the duality relations (2.22) for the Legendre transformation that

$$\widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}_N) = L^*_{\Lambda_N}(A_N) \qquad \text{and} \qquad \partial_2 \widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}_N) = 0,$$

where $\partial_2 \widetilde{L}^*_{\Lambda_N}(\cdot)$ denotes the derivative of the function $\widetilde{L}^*_{\Lambda_N}(x_0, x_1, x_2)$ with respect to x_2. Consequently (cf. relation (A.5) in [6]),

$$\widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{Z}}_N) - \widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}_N) = \int_0^{k\sqrt{N}} (k\sqrt{N} - y)(\partial_2)^2 \widetilde{L}^*_{\Lambda_N}(Nq_N, Nb_N, \tilde{e}_N + y)\, dy, \quad (8.11)$$

and one needs to evaluate $(\partial_2)^2 \widetilde{L}^*_{\Lambda_N}(\cdot)$ from below. Denote

$$\widetilde{\mathbf{E}}^y_N = \widetilde{\mathbf{E}}_N + (0, 0, y) = (Nq_N, Nb_N, \tilde{e}_N + y). \quad (8.12)$$

We will show below that in the case $|k| \le \varepsilon\sqrt{N\Delta}$ there exist positive constants $\alpha_1 = \alpha_1(\varepsilon)$ and β_5 such that for all y, $|y| \le k\sqrt{N}$, one has

$$(\partial_2)^2 \widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}^y_N) \ge \alpha_1 / N. \quad (8.13)$$

Then (8.11) implies

$$\widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{Z}}_N) - \widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}_N) \ge \alpha_1 k^2, \quad (8.14)$$

provided $|k| \le \varepsilon\sqrt{N\Delta}$ and due to the convexity of $\widetilde{L}^*_{\Lambda_N}(\cdot)$ (see also Property A.2 in [6])

$$\widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{Z}}_N) - \widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}_N) \ge 2\alpha_2 N^{1/18} |k| \quad (8.15)$$

in the opposite case, $|k| > \varepsilon\sqrt{N\Delta}$. Thus, it remains to prove (8.13). To do this determine $\widetilde{\mathbf{H}}^y_N = (\widetilde{H}^0_N(y), \widetilde{H}^1_N(y), \widetilde{H}^2_N(y))$ from the condition (recall (8.12))

$$\frac{1}{\beta} \nabla_{\mathbf{H}} \widetilde{L}^*_{\Lambda_N}(\mathbf{H}) \Big|_{\mathbf{H}=\widetilde{\mathbf{H}}^y_N} = \widetilde{\mathbf{E}}^y_N,$$

and consider the matrix $\mathbf{Hess}\, \widetilde{L}_{\Lambda_N}(\widetilde{\mathbf{H}}^y_N)$. Since it is positive definite (recall inequality (8.9)) there exists $C_5 = C_5(\varepsilon) > 0$ such that for all y, $|y| \le k\sqrt{N} \le \varepsilon N\sqrt{\Delta}$, one has

$$\left[\Big(\frac{\partial}{\partial H_0}\Big)^2 \widetilde{L}_{\Lambda_N}(\mathbf{H}) \Big(\frac{\partial}{\partial H_1}\Big)^2 \widetilde{L}_{\Lambda_N}(\mathbf{H}) - \Big(\frac{\partial^2}{\partial H_1 \partial H_0} \widetilde{L}_{\Lambda_N}(\mathbf{H})\Big)^2 \right] \Bigg|_{\mathbf{H}=\widetilde{\mathbf{H}}^y_N} \ge C_5 N^2. \quad (8.16)$$

On the other hand,

$$\det \mathbf{Hess}\, \widetilde{L}_{\Lambda_N}(\mathbf{H}) \Big|_{\mathbf{H}=\widetilde{\mathbf{H}}^y_N} \le C_6 N^3 \quad (8.17)$$

uniformly in such y with some fixed constant $C_6 > 0$. Since due to the duality relations (2.22) the value of the derivative $(\partial_2)^2 \widetilde{L}^*_{\Lambda_N}(\widetilde{\mathbf{E}}^y_N)$ coincides with the ratio of the left-hand sides in (8.16) and (8.17), one immediately obtains (8.13).

It remains to evaluate the last fraction in (8.10). Consider first the case $|k| \leq \varepsilon\sqrt{N\Delta}$. Let $\Lambda_{N,\widetilde{\mathbf{H}}_N}$ be the random vector with the distribution induced by $\mathbf{P}_{\widetilde{\mathbf{H}}_N}(\cdot)$ and $L_{\Lambda_N,\widetilde{\mathbf{H}}_N}(\mathbf{H})$, $\mathbf{H} = (H_0, H_1)$, be its logarithmic moment generating function,

$$L_{\Lambda_N,\widetilde{\mathbf{H}}_N}(\mathbf{H}) \equiv \log\Big(\sum_{\mathbf{M}\in\mathcal{M}_N^2} e^{\beta(\mathbf{H},\mathbf{M})}\mathbf{P}_{\widetilde{\mathbf{H}}_N}(\Lambda_N = \mathbf{M})\Big)$$
$$= \widetilde{L}_{\Lambda_N}(\widetilde{\mathbf{H}}_N + (H_0, H_1, 0)) - \widetilde{L}_{\Lambda_N}(\widetilde{\mathbf{H}}_N).$$

Note that this function is strictly convex and satisfies the condition

$$\det \mathbf{Hess}\, L_{\Lambda_N,\widetilde{\mathbf{H}}_N}(\mathbf{H}) \Big|_{\mathbf{H}=(0,0)} \geq C_5 N^2 \tag{8.18}$$

(since the expression on the left-hand side of (8.18) coincides with the left-hand side of (8.16) with $y = k\sqrt{N}$). As a result, applying the analog of (6.29) one gets

$$\mathbf{P}_{\widetilde{\mathbf{H}}_N}\big(\Lambda_N = A_N\big) \leq \frac{C_0}{N^2}\beta^2.$$

On the other hand, the denominator $\mathbf{P}_{\mathbf{H}_N}(\Lambda_N = A_N)$ can be evaluated from below via the analog of (6.30). Thus, there exist positive constants C_7, β_7, and N_7 such that for all $N \geq N_7$, $\beta \geq \beta_7$, and $|k| \leq \varepsilon\sqrt{N\Delta}$ one has

$$\frac{\mathbf{P}_{\widetilde{\mathbf{H}}_N}\big(\Lambda_N = A_N, \zeta_N > \tilde{e}_N\sqrt{\Delta} + k\sqrt{N\Delta}\big)}{\mathbf{P}_{\mathbf{H}_N}(\Lambda_N = A_N)} \leq \frac{\mathbf{P}_{\widetilde{\mathbf{H}}_N}(\Lambda_N = A_N)}{\mathbf{P}_{\mathbf{H}_N}(\Lambda_N = A_N)} \leq C_7. \tag{8.19}$$

In the opposite case, $|k| > \varepsilon\sqrt{N\Delta}$, one easily gets (recall (6.30))

$$\begin{aligned}\frac{\mathbf{P}_{\widetilde{\mathbf{H}}_N}\big(\Lambda_N = A_N, \zeta_N > \tilde{e}_N\sqrt{\Delta} + k\sqrt{N\Delta}\big)}{\mathbf{P}_{\mathbf{H}_N}(\Lambda_N = A_N)} &\leq \frac{1}{\mathbf{P}_{\mathbf{H}_N}(\Lambda_N = A_N)} \\ &\leq \frac{N^2}{c_2\beta^2} \leq C_8 \exp\{\alpha_2 N^{1/18}|k|\}.\end{aligned} \tag{8.20}$$

It remains to observe that (8.5)–(8.6) follow immediately from (8.10), (8.14), (8.15), (8.19), and (8.20). □

Proof of Theorem 3.2. The statement of the theorem follows directly from Theorems 7.1, 7.6, 8.1, and Theorem 2.2 from [10]. □

Proof of Theorem 3.3. The first part of the theorem can be obtained in the same way as Theorem 3.2. The convergence in (3.21) follows from Corollary 7.4. □

A. Wulff Construction in 1D Models of SOS Type

The 1D SOS model is the simplest interface model. In view of its simplicity it is very popular in the physical literature and is used mainly as a "toy model" for discussing the statistical properties of interfaces. In particular, the Wulff construction for this model is well understood ([1, 21]).

On the other hand, the interfaces appearing in the 1D SOS model present sample paths of the 1D random walk of the special type (see, e. g., [6, Sect. 3]) and therefore the Wulff construction here follows immediately from the known facts of the sample paths large deviations theory ([3, Chap. 5], [22]). Using the probabilistic interpretation one can investigate a much more general case of random walks than those usually appearing in the physical literature in the context of 1D model of SOS type (see, e. g., [2] for a list of typical examples). In this sense, the random walks provide the most general model of SOS type and for this reason we will use the probabilistic language in the present section. We will restrict ourselves to the discrete case, though the generalization to the continuous one is straightforward [6, Sect. 2].

Let ξ_i be a sequence of independent integer valued random variables having the same non-degenerate distribution that is concentrated on the lattice $\mathbb{Z}^1$. Then the interface is described by the sequence of partial sums, $S_0 = 0$, $S_k = \sum_{i=1}^k \xi_i$, of the corresponding random walk. Denote by

$$L(h) \equiv \log \mathbf{E} \exp\{h\xi\}$$

the logarithmic moment generating function (the free energy) of a single step of this random walk. Assume in addition that $L(\cdot)$ is a finite function (and thus analytical) in some open neighbourhood of the origin. [8] Finally, for any $n \geq 1$ and $t \in [0,1]$ define a random polygonal function (a piece-wise linearly interpolated interface)

$$x_n(t) = S_{[nt]} + \{nt\}\xi_{[nt]+1} = \sum_{i=1}^{[nt]} \xi_i + \{nt\}\xi_{[nt]+1}$$

with $[nt]$ and $\{nt\}$ denoting the integral and the fractional parts of nt correspondingly.

Then the distribution of $n^{-1}x_n(t)$ satisfies the large deviations principle with the rate function ([22, 4, 3])

$$\mathcal{J}(f) = \begin{cases} \displaystyle\int_0^1 L^*\bigl(f'(t)\bigr)\,dt, & \text{if } f \in \mathcal{AC}[0,1],\ f(0) = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{AC}[0,1]$ is the space of absolutely continuous functions on $[0,1]$ and $L^*(\cdot)$ is the Legendre transformation of $L(\cdot)$,

$$L^*(x) = \sup_h \bigl(xh - L(h)\bigr),$$

that is well defined due to the strict convexity of $L(\cdot)$. In particular, for any admissible pair (q,b) (i. e., satisfying condition (A.4) below) one has

$$\lim_{\varepsilon \searrow 0} \lim_{n\to\infty} \frac{\log \mathbf{P}\bigl(x_n(1) \in (b, b+\varepsilon), \int_0^1 x_n(t)\,dt \in (q, q+\varepsilon)\bigr)}{n} = -\mathcal{J}(\bar{f}),$$

[8] This is a usual conjecture in applications; moreover, typically one demands the existence of *all* exponential moments for ξ (see, e. g., [2]).

where $\bar{f}(\cdot)$ presents the solution of the variational problem:

$$\mathcal{J}(f) \to \inf : \qquad f(0) = 0, \quad f(1) = b, \quad \int_0^1 f(t)\,dt = q. \tag{A.1}$$

Note that the functional $\mathcal{J}(\cdot)$ is closely related to the Wulff functional with naturally defined surface tension (see, e.g., [6, Sect. 3]), and therefore the function $\bar{f}(\cdot)$ is the Wulff profile in the considered situation.

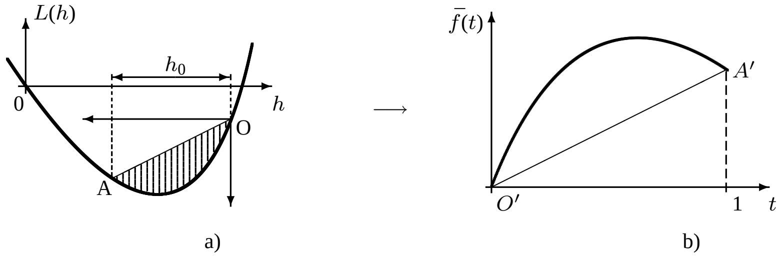

Fig. 2. Wulff construction in a general 1D model of SOS type

It turns out that the variational problem (A.1) can be solved explicitly. Namely, define the quantities $\hat{h}_0 = \hat{h}_0(q, b)$ and $\hat{h}_1 = \hat{h}_1(q, b)$ from the equations

$$\begin{cases} \displaystyle\int_0^1 L'\big(\hat{h}_1 + y\hat{h}_0\big)\,dy = b, \\ \displaystyle\int_0^1 y\,L'\big(\hat{h}_1 + y\hat{h}_0\big)\,dy = q. \end{cases} \tag{A.2}$$

Then the Wulff profile $\bar{f}(\cdot)$ is defined via ([6, Sect. 2])

$$\bar{f}(t) = \begin{cases} \Big(L\big(\hat{h}_1 + \hat{h}_0\big) - L\big(\hat{h}_1 + (1-t)\hat{h}_0\big)\Big)\,/\hat{h}_0, & \text{if } \hat{h}_0 \not\equiv 0, \\ L'(\hat{h}_1)t \equiv bt, & \text{otherwise.} \end{cases} \tag{A.3}$$

Relations (A.2)–(A.3) have a simple geometric interpretation. Namely, rewriting (A.2) in the form (cf. [21, Theorem 3])

$$\begin{cases} \Big(L\big(\hat{h}_1 + \hat{h}_0\big) - L\big(\hat{h}_1\big)\Big)\,/\hat{h}_0 = b, \\ \displaystyle\frac{1}{\hat{h}_0^2}\int_0^{\hat{h}_0} \left(\frac{L(\hat{h}_1 + \hat{h}_0) + L(\hat{h}_1)}{2} - L(\hat{h}_1 + y)\right) dy = q - b/2 \end{cases}$$

we infer that these conditions prescribe to find two points $A(\hat{h}_1, L(\hat{h}_1))$ and $O(\hat{h}_1 + \hat{h}_0, L(\hat{h}_1 + \hat{h}_0))$ on the graph of the function $L(\cdot)$ such that (see Fig. 2,a)): 1) the straight line passing through O and A has the slope coefficient b; 2) the area $Q_b(h_0)$ of the figure bounded by the segment OA and the arc of the graph of $L(\cdot)$ with the endpoints A and O equals $(q - b/2){h_0}^2$, where h_0 denotes the horizontal separation of the points A and O (in the case $q < b/2$ one should interchange these points). Then the Wulff proflie $\bar{f}(\cdot)$

is obtained by simple transformation (reflection + scaling) of the arc OA (see Fig. 2,b)). In the critical case $2q = b$ the points O and A coincide and due to the second line in (A.3) the corresponding Wulff profile is reduced to the segment $O'A'$ (Fig. 2,b)).

Due to the strict convexity and analyticity of the function $L(\cdot)$, the normalized area $Q_b(h_0)/{h_0}^2$ is an increasing function of h_0 and $Q_b(h_0)/{h_0}^2 \to 0$ as $h_0 \to 0$. In particular, the conditions $\hat{h}_0 = 0$ and $2q = b$ are equivalent (recall (A.3)). As a result, equations (A.2) have at most one solution. Such solution clearly exists for every pair (q, b) satisfying the condition [9]

$$|q - b/2| < \sup_h Q_b(h)/h^2. \tag{A.4}$$

Here the supremum corresponds to the most "upper" limiting position of the secant OA; thus, (A.4) means that the real secant should be below the limiting one (if such exists).

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[9] In some cases a solution of (A.2) exists also when both expressions in (A.4) are equal. But this depends on the distribution of the single step ξ, and we will not discuss this question here.

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