

“Non-Gibbsian” States and their Gibbs Description

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Abstract: The driving principle behind this paper is the following thesis: “Every physically reasonable random field has to be a Gibbs random field”. In this paper the so-called “non-Gibbsian” random fields are considered. The usual definition of the Gibbs field is generalized in such a way so as to include some of the discovered “non-Gibbsian” fields. The new definition is then used to show that the projection of the two-dimensional Ising model onto the one-dimensional sublattice \mathbb{Z}^1 falls into the class of the generalized Gibbs fields.

A foreword by S. Shlosman

I had the sad duty to complete one of the papers by R. L. Dobrushin, on which he was working during the last several months of his life. He got the idea of this work right after the paper of Roberto Schonmann [S] appeared in 1989. He discussed the project of bringing the “non-Gibbsian” fields back to the Gibbsian fold with me several times.

There are two versions of Dobrushin’s manuscript. The first one deals with the projection of the (+)-phase of the 2D Ising model onto an arbitrary countable subset of the lattice \mathbb{Z}^2 . Because of the wish to have a preliminary version [D95] of this result in time for the conference in The Netherlands (the conference in August–September, 1995, in Renkum, the last one in which R. L. Dobrushin participated), he started to write down the special case of projecting onto the sublattice $\mathbb{Z}^1 \subset \mathbb{Z}^2$. Both manuscripts were left unfinished.

The following is the result of my attempt to finish the second manuscript. I had to change some of the initial statements, and I also added some new ones. I think that,

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if completed by R. L. Dobrushin, the final version would also differ from his original manuscript, though for sure the changes would be different.

1. Introduction

In the last years a following surprising possibility was discovered and intensively discussed in the literature (see the excellent review paper of van Enter, Fernandez, and Sokal [EFS], the recent informative paper of van Enter, Fernandez and Kotecky [EFK] and the references there). Natural functionals of values of Gibbs random fields of the type used in the renormalization-group theory can (in some sense) be non-Gibbsian.

More exactly, it means the following. Let T be a countable set and X be a finite set. For any subsets $V' \subset V \subseteq T$ and any configuration $\sigma_V \in X^V$ we denote the restriction $\sigma_V|_{V'} \in X^{V'}$ of σ_V to V' by $\sigma_{V'}$. For any two mutually disjoint subsets $V_1, V_2 \subseteq T$ and any configurations $\sigma_{V_1} \in X^{V_1}$ and $\sigma_{V_2} \in X^{V_2}$ the configuration $\tilde{\sigma} \in X^{V_1 \cup V_2}$, such that its restriction $\tilde{\sigma}|_{V_1}$ on V_1 coincides with σ_{V_1} and its restriction $\tilde{\sigma}|_{V_2}$ on V_2 coincides with σ_{V_2} , is denoted by $\sigma_{V_1} \cup \sigma_{V_2}$. A *Gibbsian potential* $\mathcal{U} = (U_A(\sigma_A), A \subset T, 0 < |A| < \infty)$ is a system of real-valued functions $U_A(\sigma_A)$ of $\sigma_A \in X^A$, labelled by the system of all finite nonempty subsets $A \subset T$. It is assumed that for any $t \in T$ the following series absolutely converges:

$$\sum_{A \subset T: t \in A, |A| < \infty} \max_{\sigma_A \in X^A} |U_A(\sigma_A)| < \infty. \quad (1.1)$$

For any finite $V \subset T$, any configuration $\bar{\sigma}_{T \setminus V} \in X^{T \setminus V}$, called a *boundary condition*, and any $\sigma_V \in X^V$ consider the *relative energy*

$$\begin{aligned} E_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) &= \sum_{A \subseteq V, A \neq \emptyset} U_A(\sigma_A) \\ &+ \sum_{A \subset T: A \cap V \neq \emptyset, A \cap (T \setminus V) \neq \emptyset, |A| < \infty} U_A(\sigma_{A \cap V} \cup \bar{\sigma}_{A \cap (T \setminus V)}). \end{aligned} \quad (1.2)$$

The condition (1.1) guarantees the convergence of the series in (1.2) for all boundary conditions $\bar{\sigma}_{T \setminus V}$ and configurations σ_V . Let

$$p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) = \frac{\exp\{-E_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})\}}{Z_V^{\mathcal{U}}(\bar{\sigma}_{T \setminus V})}, \quad (1.3)$$

where the *partition function*

$$Z_V^{\mathcal{U}}(\bar{\sigma}_{T \setminus V}) = \sum_{\sigma_V \in X^V} \exp\{-E_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})\}. \quad (1.4)$$

The *transition function* $p_V(\sigma_V / \bar{\sigma}_{T \setminus V})$ is defined for all σ_V and all boundary conditions $\bar{\sigma}_{T \setminus V}$. The system $p^{\mathcal{U}} = \{p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}), \sigma_V \in X^V, \bar{\sigma}_{T \setminus V} \in X^{T \setminus V}, |V| < \infty\}$ of transition functions is called the *Gibbs specification with the potential* \mathcal{U} .

For any subset $W \subseteq T$ we consider the smallest σ -algebra of subsets of the space X^T with respect to which all the coordinate functions $\sigma_t, t \in W$, are measurable. This σ -algebra is denoted by \mathfrak{B}_W . A probability measure \mathbb{P} on the measurable space (X^T, \mathfrak{B}_T)

is called *consistent with the Gibbs specification* $p^{\mathcal{U}}$, if for any finite $V \subset T$, any function $\phi(\sigma_V)$ of $\sigma_V \in X^V$ and any subset $B \in \mathfrak{B}_{T \setminus V}$,

$$\int_B \phi(\sigma_V) \mathbb{P}(d\sigma) = \int_B \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) \mathbb{P}_{T \setminus V}(d\bar{\sigma}_{T \setminus V}), \quad (1.5)$$

where $\mathbb{P}_{T \setminus V}$ is the restriction of the measure \mathbb{P} to the σ -algebra $\mathfrak{B}_{T \setminus V}$. The probability measure \mathbb{P} is called “*non-Gibbsian*”, if there exists no potential \mathcal{U} satisfying the condition (1.1), such that the measure \mathbb{P} is consistent with the Gibbs specification $p^{\mathcal{U}}$.

To be concrete we restrict ourselves to functionals of the type of decimation, applied to the two-dimensional ferromagnetic symmetric Ising random field with large values of the inverse temperature $\beta > 0$ and (+)-boundary condition. Recall the corresponding definitions. Let $T = \mathbb{Z}^2$ and $X = \{1, -1\}$. Consider the probability measure $\mathbb{P}^{\beta,+}$ on the measurable space $(X^{\mathbb{Z}^2}, \mathfrak{B}_{\mathbb{Z}^2})$ defined by the following usual construction. Let a subset $V \subset \mathbb{Z}^2$ be finite and $P_V^{\beta,+}(\sigma_V)$, $\sigma_V \in X^V$, be the restriction of the measure $\mathbb{P}^{\beta,+}$ to the σ -algebra \mathfrak{B}_V . Let $W_N, N = 1, 2, \dots$ be the sequence of the lattice squares $W_N = \{t = (t_1, t_2) \in \mathbb{Z}^2 : |t_1| \leq N, |t_2| \leq N\}$. Then

$$P_V^{\beta,+}(\sigma_V) = \lim_{N \rightarrow \infty: V \subseteq W_N} \sum_{\sigma_{W_N} \in X^{W_N}: \sigma_{W_N}|_V = \sigma_V} p_{W_N}^{\beta,+}(\sigma_{W_N}), \quad \sigma_V \in X^V. \quad (1.6)$$

Here $p_W^{\beta,+}, W \subset \mathbb{Z}^2, |W| < \infty$, are the probability distributions defined by the Ising specification for the case of the plus-boundary conditions, i.e.

$$p_W^{\beta,+}(\sigma_W) = \frac{\exp\{-\beta E_W^{\text{Ising}}(\sigma_W /+)\}}{Z_W^{\beta,+}}, \quad (1.7)$$

where (for $W^c = \mathbb{Z}^2 \setminus W$)

$$E_W^{\text{Ising}}(\sigma_W /+) = - \sum_{(s,t) \subseteq W: |s-t|=1} \sigma_s \sigma_t - \sum_{(s,t): s \in W, t \in W^c, |s-t|=1} \sigma_s \quad (1.8)$$

and

$$Z_W^{\beta,+} = \sum_{\sigma_W \in X^W} \exp\{-\beta E_W^{\text{Ising}}(\sigma_W /+)\}. \quad (1.9)$$

It is well-known that the limits (1.6) exist. By the Kolmogorov theorem the system of the probability distributions $P_V^{\beta,+}, V \subset \mathbb{Z}^2, |V| < \infty$, defines the measure $\mathbb{P}^{\beta,+}$ in a unique way.

Let now a countable set $T \subseteq \mathbb{Z}^2$. Consider the projection $\Pi_T : X^{\mathbb{Z}^2} \rightarrow X^T$, defined by $\sigma = (\sigma_t, t \in \mathbb{Z}^2) \rightarrow \Pi_T \sigma = (\sigma'_t, t \in T)$, where $\sigma'_t = \sigma_t$ for all $t \in T$. Via this projection, the measure $\mathbb{P}^{\beta,+}$ induces a measure $\mathbb{P}_T^{\beta,+}$ on the space (X^T, \mathfrak{B}_T) , defined by

$$\mathbb{P}_T^{\beta,+}(B) = \mathbb{P}^{\beta,+}(\Pi_T^{-1} B), \quad B \in \mathfrak{B}_T, \quad (1.10)$$

which is called the *projection on T of the Ising measure* $\mathbb{P}^{\beta,+}$. In case of the sublattice $T = b\mathbb{Z}^2$ (then b should be an integer), the transformation described above is called

decimation with spacing b . It is one of the transformations used in renormalization-group theory. An important result of van Enter, Fernandez, and Sokal [EFS], extending earlier results of Griffiths and Pearce [GP78, GP79] and Israel [I], states that for decimation with any spacing $b \geq 2$ the measure $\mathbb{P}_T^{\beta,+}$ is "non-Gibbsian". The other important result is due to Schonmann [S]. He considers the case of $T = \{t = (t_1, t_2) \in \mathbb{Z}^2 : t_2 = 0\}$ and proved that the measure $\mathbb{P}_T^{\beta,+}$ is "non-Gibbsian" in this case also. The flow of witty examples of "non-Gibbsian" measures is on the increase. Sometimes these examples are treated as pathological ones and even as an insult to physical intuition, since the belief that any reasonable transformation of a Gibbs measure leads to another Gibbs measure is a foundation-stone of the generally accepted, but mathematically non-rigorous, renormalization-group theory.

The mathematical results discussed above are very interesting, but the used terminology seems to be misleading in some sense. The property (1.1) of the potential is very convenient, if it is valid, but is not intrinsic to the notion of Gibbs distribution. Moreover, it is not strictly necessary to assume that Gibbs transition functions (1.3) are defined for all boundary conditions $\bar{\sigma}_{T \setminus V}$; it is enough to assume that they are defined only almost everywhere with respect to the corresponding random field. This point of view is evidently unavoidable, if the space of values $X = \mathbb{R}^1$, and the interaction is infinite range. For example, Gaussian translation-invariant random fields are described as Gibbsian ones with the potential $\mathcal{U} = (U_A(\sigma_A), A \subset \mathbb{Z}^d, |A| < \infty)$, such that

$$U_A(\sigma_X) = \begin{cases} c_{s-t}\sigma_s\sigma_t & \text{if } A = \{s, t\}, \\ 0 & \text{in other cases,} \end{cases} \quad (1.11)$$

where $c_{s-t} = c_{t-s}$ are some nonnegative constants (see [D80]). It is clear that if all these constants do not vanish, the series in the definition (1.2) for the relative energy diverges for some boundary conditions $\bar{\sigma}_{V^c}$, and so the Gibbs specification can not be defined in a reasonable way for all boundary conditions. Of course, in the case of finite X similar possibilities appear exotic, but, if we meet the real difficulties in these cases, we can try to find a way out by using partly defined specifications. At least, physical intuition of renormalization-group theory is vague enough to be not in contradiction with such interpretation of Gibbs measures.

More exactly, let again T be a countable set and X be a finite set. We define the potential $\mathcal{U} = (U_A(\sigma_A), A \subset T, |A| < \infty)$ as above, but we will not introduce the condition (1.1). Instead, for any finite $V \subset T$ we introduce the set $\bar{X}_V^{\mathcal{U}} \in \mathfrak{B}_{T \setminus V}$ of all boundary conditions $\bar{\sigma}_{T \setminus V}$ such that the series (1.2) absolutely converges for all $\sigma_V \in X^V$. (This set $\bar{X}_V^{\mathcal{U}}$ can in principle be empty.) Then for $\bar{\sigma}_{T \setminus V} \in \bar{X}_V^{\mathcal{U}}$ we can still define the probability distribution $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})$ by the formula (1.3). The system $p^{\mathcal{U}} = \{p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}), \sigma_V \in X^V, \bar{\sigma}_{T \setminus V} \in \bar{X}_V^{\mathcal{U}}, V \subset T, |V| < \infty\}$ of transition functions will be called the *partly defined Gibbs specification with the potential \mathcal{U}* . A probability measure \mathbb{P} on the measurable space (X^T, \mathfrak{B}_T) is called *consistent with the partly defined Gibbs specification $p^{\mathcal{U}}$* , if for any finite $V \subset T$ the probability

$$\mathbb{P}_{T \setminus V}(\bar{X}_V^{\mathcal{U}}) = 1, \quad (1.12)$$

and the identity (1.5) holds for any function $\phi(\sigma_V)$ of $\sigma_V \in X^V$ and any set $B \in \mathfrak{B}_{T \setminus V}$ such that $B \subseteq \bar{X}_V^{\mathcal{U}}$.

Now we can formulate the main result of the paper.

Theorem 1.1. *There exists a value $\beta_0 < \infty$ such that for all $\beta > \beta_0$ and $T = \{t = (t_1, t_2) \in \mathbb{Z}^2 : t_2 = 0\} \subset \mathbb{Z}^2$ there exists a Gibbs potential $\mathcal{U}_T^\beta = \{U_A^\beta(\sigma_A), A \subset T, |A| < \infty, \sigma_A \in X^A\}$, such that the projection $\mathbb{P}_T^{\beta,+}$ of the Ising measure $\mathbb{P}^{\beta,+}$ is consistent with the partly defined Gibbs specification $p^{\mathcal{U}_T^\beta}$.*

Remark 1. A similar construction can be applied to the case of minus-boundary condition. As a result we obtain that the projection of the Ising measure with minus-boundary condition is consistent with the partly defined Gibbs specification with the potential $\bar{\mathcal{U}}_T^\beta$, which is obtained from the potential \mathcal{U}_T^β by the change of variables $\sigma_t \leftrightarrow -\sigma_t, t \in T$. For large enough β these two potentials do not coincide. We believe that there is no unique potential $\tilde{\mathcal{U}}_T^\beta$, such that both projections $\mathbb{P}_T^{\beta,+}$ and $\mathbb{P}_T^{\beta,-}$ are consistent with the partly defined Gibbs specification $p^{\tilde{\mathcal{U}}_T^\beta}$.

Remark 2. We consider the two-dimensional Ising model and the sublattice $T = \mathbb{Z}^1 \subset \mathbb{Z}^2$ only to simplify the following construction. In fact, the only essential feature of our approach is that in the considered case the contour method and the cluster expansion for the contour system are applicable. Therefore, similar results can be proved for other examples of the subsets $T \subset \mathbb{Z}^2$, for the Ising model in dimension $d > 2$ and for many other Gibbs models.

Remark 3. There is a hope that the partly defined Gibbs specifications exist also in other “non-Gibbsian” situations, but the investigation of this problem requires a lot of additional work.

Remark 4. There is the following doubt, formulated in connection with the approach developed in this paper: the class of partly defined potentials is too wide and we know nothing about corresponding “almost Gibbs” fields. One possible response is that we also know almost nothing about the usual Gibbs fields with potentials of the class (1.1). We have only a general theorem of existence, a theorem of uniqueness for high temperatures and some examples of phase transitions for low temperatures. It seems that similar results can be obtained under some appropriate restrictions for Gibbs fields with partly defined potentials.

1.1. Some heuristics. We finish this section by an informal description of some properties of the projected low-temperature Ising measure, which is meant to explain the origins of its “non-Gibbsianity” (in the standard meaning) and also to make the main results of this paper (see Propositions 2.2 and 2.3) look natural. Consider the segment $[-n, n] \subset \mathbb{Z}^1$. We want to fix the configuration σ everywhere on $[-n, n]$ except at the origin, and we want to estimate the influence of the rest of the configuration σ outside this segment on the value σ_0 . For the ordinary Gibbs field this influence vanishes as $n \rightarrow \infty$. As we will see soon, this is generally not the case in our example. To be specific, suppose that the configuration σ is antisymmetric on $[-n, n]$, that is $\sigma_s = -\sigma_{-s}$ for $1 \leq s \leq n$, and that $\sigma_{[1,n]}$ is somewhat neutral, that is for some fixed $k \ll n$ all the averages $\frac{1}{k} |\sigma_{s+1} + \dots + \sigma_{s+k}| < \frac{1}{2}$, say, for all $s, 0 \leq s \leq n - k$. The reader can consider, for example, the configuration $\sigma_t = (-1)^t, 0 < t \leq n$. Let us first consider the case when the configuration outside $[-n, n]$ is identically +1. Then the typical configuration of the (+)-phase of the 2D model under such condition in the vicinity of the origin looks as follows: the segments of minuses of the configuration σ are enclosed by “thin” contours

which stay very close to these segments, while the rest of the plane is filled by the (+)-phase. Because of the symmetry of the configuration $\sigma_{[-n,n]\setminus 0}$, its influence on the spin at the origin vanishes, and the influence left is that of the (+)-phase outside the thin contours, so the spin σ_0 behaves basically as if being in the (+)-phase.

Let us now add two more (-)-segments, $\pm[n+1, n+N]$. If N is moderate, then nothing new happens, and we have just two more thin contours, surrounding the new (-)-segments. However, when N becomes larger than a certain threshold $N(n, \beta)$, then the picture becomes qualitatively different: the two new contours prefer to merge into one big contour, containing the whole segment $[-N, N]$ inside. Energetically this is a loss, because the length of the new system of contours exceeds the minimal possible one by $\sim 2n$. The reason why the system is willing to pay this price is that the arrangement described gives more possibilities for fluctuations. Roughly speaking, the second picture corresponds to the pinning of the external contour of the third one to the segment $[-n, n]$. However, the probability of the event that this contour is located at some given height h above the origin, is of the order of $\frac{C(\beta)}{\sqrt{N}}$ for $h \sim \sqrt{N}$. So when $\sqrt{N} \gg \exp\{2\beta n\}$, the entropy considerations win over the energy considerations, and the system prefers the freedom of fluctuations to the energy gain. When the unifying external contour is formed (which happens with probability close to one, if $\sqrt{N} \gg \exp\{2\beta n\}$), the picture in the vicinity of the origin is reversed: inside the big contour we see thin contours surrounding (+)-segments of the configuration σ . Outside them what we see is basically the (-)-phase, and so the spin σ_0 behaves basically as if being in the (-)-phase.

This is how the “non-Gibbsian” nature of the projected field is manifested: no matter how large the segments $[-n, -1], [1, n]$ are, where the configuration is fixed, we still can influence the behavior at the origin – by choosing the segments $\pm[n+1, n+N]$ even bigger. That is a clear indication that if we still want to describe the field by a Gibbs potential, we have to be prepared to allow extremely long-range interactions, which might not even decay. Another, more quantitative information, we can get from this example is that the total energy of two strings of (-)-particles, $H(\sigma^{[a_1, b_1] \cup [a_2, b_2]})$, where $a_1 < b_1 < a_2 < b_2$, can be (approximately) equal to the sum of the energies of the two strings, $H(\sigma^{[a_1, b_1]}) + H(\sigma^{[a_2, b_2]})$, only if the strings are well separated. Here and in the following for $A \subset \subset \mathbb{Z}^1$ we denote by σ^A the configuration

$$\sigma^A(s) = \begin{cases} -1 & \text{for } s \in A, \\ +1 & \text{for } s \notin A. \end{cases} \quad (1.13)$$

The reason for that is that the approximate additivity of the energy is equivalent to the approximate independence. But when the distance between the segments, $|a_2 - b_1|$ is $o(\min(|b_1 - a_1|, |b_2 - a_2|))$, then we have to expect the two contours to coalesce, which rules out any hope for independence.

1.2. Organization of the paper. In the next section we write down the expression for the potential we are looking for, and formulate its main decay properties.

In Sect. 3 we study one of the main important objects we need: the meniscus. By this we mean the following. Suppose a box contains a system which is a mixture of two phases, with a phase separation surface between them. In case both phases are touching the walls of the box, the separation surface has its boundary on the wall. What we need is the information on how the phase separation surface approaches the wall of the box in question. We find in particular that the number of pure states of the Ising model in the half space equals the continuum!

In Sects. 4 and 5 we remind the reader about the cluster expansion technique.

Sections 6 and 7 contain the main technical results of the paper. There we are obtaining the estimates on the interaction, which are enough to prove the Gibbsianity of the projected field. This is done in Sect. 8.

A preliminary formulation of the results of the present paper appeared in [DS].

2. Construction of the Potential

The proof of Theorem 1.1 is based on an explicit construction of the potential \mathcal{U}_T^β . In this section we explain this construction.

In the case of the two-point space $X = \{1, -1\}$ we will use the following notations for the configurations $\sigma_V \in X^V$ in a volume V . For any $\sigma_V = (\sigma_t, t \in V) \in X^V$ we let

$$\sigma_V^+ = \{t \in V : \sigma_t = 1\}, \quad \sigma_V^- = \{t \in V : \sigma_t = -1\}. \quad (2.1)$$

As it is well-known (see [G] or [EFS]), distinct Gibbs potentials can define the same Gibbs specification, but there is a subclass of potentials, which are defined by the specification in a unique and even, in some sense, constructive way. It is the subclass of the so-called lattice gas potentials. These potentials are especially natural for the fields with two values ± 1 , when a typical configuration takes mainly one of these two values. In the considered case of the Ising measure $\mathbb{P}^{\beta,+}$ with β large, this dominant value is $+1$. We can treat the sites $t \in \sigma_V^-$ as the sites where interacting particles are situated, and the sites $t \in \sigma_V^+$ as empty (vacuum) sites.

We will use the following definitions. In the case $X = \{1, -1\}$ a Gibbs potential $\mathcal{U} = (U_A(\sigma_A), A \subset T, 0 < |A| < \infty)$ is called a *lattice gas potential* if

$$U_A(\sigma_A) = \begin{cases} U_A, & \text{if } \sigma_A^- = A, \\ 0 & \text{for all other } \sigma_A \in X^A, \end{cases} \quad (2.2)$$

where $\{U_A, A \subset T, 0 < |A| < \infty\}$ is a system of real numbers. We can treat U_A as the energy of mutual interaction between particles which occupy the set A . We show below that the potentials \mathcal{U}_T^β , the existence of which is stated in Theorem 1.1, can be constructed as lattice gas potentials.

There is a constructive way to recover the values U_A from the probability distribution of a Gibbs field with a lattice gas potential. Let W be a finite volume, $X = \{1, -1\}$, and p_W be a probability distribution on the set X^W . Assume that p_W is a Gibbs distribution with a lattice gas potential $U = \{U_A, A \subseteq W\}$. It means that

$$p_W(\sigma_W) = \frac{\exp\{-\sum_{A:A \subseteq \sigma_W^-} U_A\}}{Z_W^U}, \quad \sigma_W \in X^W, \quad (2.3)$$

where the partition function

$$Z_W^U = \sum_{\sigma_W \in X^W} \exp\left\{-\sum_{A:A \subseteq \sigma_W^-} U_A\right\}. \quad (2.4)$$

Let a set $D \subset W$, and σ_W^D is the configuration defined by (1.13), i.e. $(\sigma_W^D)^- = D$. It follows from (2.3) that

$$p_W(\sigma_W^D) = \frac{\exp\{-\sum_{A:A \subseteq D} U_A\}}{Z_W^U}. \quad (2.5)$$

In particular, for the case of the ‘‘vacuum’’ configuration σ_W^\emptyset on W (i.e. $(\sigma_W^\emptyset)^+ = W$) we have

$$p_W(\sigma_W^\emptyset) = (Z_W^U)^{-1}. \quad (2.6)$$

It follows from the relations (2.5) and (2.6) that

$$\sum_{A:A \subseteq D} U_A = -\ln \frac{p_W(\sigma_W^D)}{p_W(\sigma_W^\emptyset)}. \quad (2.7)$$

Recall now a well-known (and checked in an elementary way) Mobius inversion formula. Let $\Phi(\cdot), \Psi(\cdot)$ be a pair of functions defined on the set of all finite subsets $A \subseteq R$, where R is a finite or countable set. Then

$$\Psi(A) = \sum_{D \subseteq A} \Phi(D), \quad A \subseteq R, |A| < \infty, \quad (2.8)$$

if and only if

$$\Phi(A) = \sum_{D \subseteq A} (-1)^{|A \setminus D|} \Psi(D), \quad A \subseteq R, |A| < \infty. \quad (2.9)$$

It follows from the Mobius formula and the relation (2.7) that for any finite $A \subseteq W$,

$$U_A = -\sum_{D \subseteq A} (-1)^{|A \setminus D|} \ln \frac{p_W(\sigma_W^D)}{p_W(\sigma_W^\emptyset)}. \quad (2.10)$$

Of course, we do not know yet that the measure $\mathbb{P}_T^{\beta,+}$ is consistent with a partly defined Gibbs specification with a lattice gas potential \mathcal{U} . But the previous discussion suggests the following definitions, which will be justified below. We will treat the measure $\mathbb{P}_T^{\beta,+}$ as the limit, as $N \rightarrow \infty$, of the projections of the Ising distributions $p_{W_N}^{\beta,+}$ (recall (1.7)) to the sets $T \cap W_N$ and write the analog of the formula (2.10) for these projections. For any N and $D \subseteq (T \cap W_N)$ we let

$$Q_{T,D}^{\beta,+N} = -\ln \frac{p_{W_N}^{\beta,+}(\sigma_{W_N} : (\sigma_{W_N}|_{T \cap W_N})^- = D)}{p_{W_N}^{\beta,+}(\sigma_{W_N} : (\sigma_{W_N}|_{T \cap W_N})^- = \emptyset)}. \quad (2.11)$$

For any finite $D \subset T$ we let

$$Q_{T,D}^{\beta,+} = \lim_{N \rightarrow \infty} Q_{T,D}^{\beta,+N}, \quad (2.12)$$

if, of course, this limit exists. Let

$$U_{T,A}^{\beta,+N} = \sum_{D \subseteq A} (-1)^{|A \setminus D|} Q_{T,D}^{\beta,+N}, \quad (2.13)$$

$$U_{T,A}^{\beta,+} = \lim_{N \rightarrow \infty} U_{T,A}^{\beta,+N} = \sum_{D \subseteq A} (-1)^{|A \setminus D|} Q_{T,D}^{\beta,+}. \quad (2.14)$$

Theorem 2.1. *In the situation of Theorem 1.1 all the limits (2.12), (2.14) exist. The measure $\mathbb{P}_T^{\beta,+}$ is consistent with the partly defined Gibbs specification with the lattice gas potential $\mathcal{U}_T^{\beta,+}$ which is defined by the relation (2.2), where the values $U_A = U_{T,A}^{\beta,+}$.*

It is clear that the main Theorem 1.1 follows from Theorem 2.1. The main difficulty of its proof is a derivation of the appropriate estimates on the absolute values of the potentials $U_{T,A}^{\beta,+}$. Once obtained, they would imply easily the consistency of the measure $\mathbb{P}_T^{\beta,+}$ with the potential $\mathcal{U}_T^{\beta,+}$.

2.1. Bounds on $\mathcal{U}_T^{\beta,+}$. Here we formulate these estimates, the derivation of which is the main content of the paper.

Proposition 2.2. A rough estimate. *Let $A \subset T$ be any finite set. Then*

$$|U_A| \leq 4(2\beta + C) |A| 2^{|A|}. \tag{2.15}$$

The estimate (2.15) does not look very promising, and, what is worse, it seems that for some sets it cannot be improved significantly. These sets include segments and other sets of high density. Fortunately, the probability of the event that the segment A is occupied by the $(-)$ -particles decays as $\exp\{-4\beta|A|\}$, and so such a weak estimate as (2.15) by itself does not destroy our argument. The next statement supplements the estimate (2.15) with the bound needed to enable the proof of our main result.

Proposition 2.3. A refined estimate. *Let $\varepsilon > 0$ be fixed. Then there exists a value $\beta = \beta(\varepsilon)$, such that for all $\beta > \beta(\varepsilon)$ and for all finite sets $A \subset T$ with density $\rho(A)$ below $1 - \varepsilon$:*

$$\rho(A) \equiv \frac{|A|}{\text{diam}(A) + 1} < 1 - \varepsilon \tag{2.16}$$

we have

$$|U_A| \leq \exp\{-\beta' \text{diam}(A)\},$$

where $\beta' = \beta'(\beta) \rightarrow \infty$ as $\beta \rightarrow \infty$.

3. The Meniscus

As the reader will see in the following, the case of projecting a two-dimensional field onto the one-dimensional sublattice has the best chances to be non-Gibbsian. The reason, roughly speaking, is this: the question about weak dependence of the spatially separated boxes B_1, B_2 of the sublattice $\mathbb{Z}^{d'} \subset \mathbb{Z}^d$ according to the projection of the random field \mathcal{P} boils down to the following question:

Consider two closed contours $\Gamma_1, \Gamma_2 \subset \mathbb{Z}^d$, enclosing the boxes B_1, B_2 , respectively. Let the contours be distributed independently, each one governed by the field \mathcal{P} . The question is about the behavior of the probability of the event that their intersection is nonempty, as a function of $\text{dist}(B_1, B_2)$. It is more or less clear that the lower the dimension of the contour is the bigger are its fluctuations. In other words, the lower the dimension of the contour Γ the higher are chances to observe it at some distance from the enclosed set B . That is the reason why we think that the 2D case might be the most difficult one. To obtain our result we have therefore to control the fluctuations of

the contour surrounding a given set B , which is a segment, or punctured segment, in our case. We will do it by using the following meniscus theorem, which seems to be interesting on its own.

The meniscus theorem describes the phases of the semiinfinite Ising model, which correspond to various possible slopes of the interface as it approaches the wall. Namely, we consider the 2D Ising model on the half-infinite lattice $(\mathbb{Z}^2)^+ = \{(x, y) \in \mathbb{Z}^2 : y > 0\}$, and we are interested in the set of Gibbs states corresponding to the following boundary condition on the x -axis:

$$\bar{\sigma}_x = \begin{cases} +1 & \text{for } x \geq 0, \\ -1 & \text{for } x < 0. \end{cases}$$

To remind the reader of the definition of the Gibbs field on $(\mathbb{Z}^2)^+$, corresponding to the Ising model with a given boundary condition, we need some notation. Let $\Lambda \subset (\mathbb{Z}^2)^+$ be a finite subset. By $\partial\Lambda$ we denote the usual boundary of $\Lambda \subset \mathbb{Z}^2$, and by $\partial^+\Lambda \subset \partial\Lambda$ – the intersection $\partial\Lambda \cap (\mathbb{Z}^2)^+$. A random field on the set of all configurations on $(\mathbb{Z}^2)^+$ is called the Gibbs field with Ising interaction and boundary conditions $\bar{\sigma}$, iff its conditional distribution in any finite Λ given the configuration σ^c outside Λ is given by the usual Gibbs formula for the Ising interaction in the case $\partial^+\Lambda = \partial\Lambda$, while in the remaining case $\partial\Lambda \setminus \partial^+\Lambda \neq \emptyset$ we also use the same formula where we supplement the configuration $\sigma^c|_{\partial^+\Lambda}$ by the restriction $\bar{\sigma}|_{\partial\Lambda \setminus \partial^+\Lambda}$.

To formulate the meniscus theorem we introduce the following configurations, which will be used as boundary conditions. Let $\mathbf{n} = (n_x, n_y)$ be any unit vector in \mathbb{R}^2 with $n_x > 0$. We define the configuration $\sigma^{\mathbf{n}}$ by

$$\sigma^{\mathbf{n}}(\mathbf{x}) = \begin{cases} +1 & \text{for } \mathbf{n} \cdot \mathbf{x} \geq 0, \\ -1 & \text{for } \mathbf{n} \cdot \mathbf{x} < 0. \end{cases}$$

Denote by V_N the box

$$V_N = \{(x, y) \in \mathbb{Z}^2 : -N < x < N, 0 < y < N\}, \quad (3.1)$$

and let $\langle \cdot \rangle_{N, \mathbf{n}}^\beta$ be the Gibbs state in V_N , corresponding to the Ising model at inverse temperature β with boundary condition $\sigma^{\mathbf{n}}$. We also introduce the notation $h(N, \mathbf{n})$ for the integer point on the line $y = N$ closest to the line $\mathbf{n} \cdot \mathbf{x} = 0$.

Theorem 3.1 (The meniscus theorem). *If β is large enough, then*

1. *The thermodynamic limit $\langle \cdot \rangle_{\mathbf{n}}^\beta = \lim_{N \rightarrow \infty} \langle \cdot \rangle_{N, \mathbf{n}}^\beta$ exists for every \mathbf{n} .*
2. *The states $\langle \cdot \rangle_{\mathbf{n}}^\beta$ are mutually different Gibbs states of the Ising model on the half-lattice $(\mathbb{Z}^2)^+$, corresponding to the boundary condition $\bar{\sigma}$.*

For applications of this result to the problem of Gibbsianity we need a certain property of the phases $\langle \cdot \rangle_{\mathbf{n}}^\beta$. This property is formulated in terms of the contours of configurations. So in the next subsection we remind the reader of the relevant definitions.

3.1. Peierls contours. Let \mathbb{Z}^2 be the two-dimensional integer lattice. Assume that $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let \mathbb{Z}^{2*} be the *conjugate lattice* with vertices $(n_1 + 1/2, n_2 + 1/2)$, $n_1, n_2 \in \mathbb{Z}^1$. Let \mathbb{E} be the set of *edges of the conjugate lattice*, i.e. the set of all closed intervals of the length 1 connecting the adjacent points of this lattice. For each edge $e \in \mathbb{E}$ there are two vertices of the original lattice \mathbb{Z}^2 with the distance 1/2 from e . We say that they are *vertices adjacent to the edge e* . Let $W \subset \mathbb{Z}^2$ be a finite set. The set of edges $e \in \mathbb{E}$ such

that at least one of two points to which the edge e is adjacent belongs to W is called the set of *edges in the volume W* and is denoted by $\mathbb{E}(W)$.

Return now to the Ising distribution with the plus-boundary condition. The boundary $B(\sigma_W)$ of the configuration $\sigma_W = (\sigma_t, t \in W) \in X^W$, where $X = \{1, -1\}$, is defined as the set of all edges $e \in \mathbb{E}(W)$ such that either both points $t, t' \in \mathbb{Z}^2$ adjacent to e belong to W and $\sigma_t \neq \sigma_{t'}$ or one of these two points $t \in W$, the other one $t' \notin W$ and $\sigma_t = -1$. As usual, we represent the boundary $B(\sigma_W)$ as a sum of contours, but to do it in a unique way it is necessary to be careful in the definition of contours. Let $e_1, e_2 \in \mathbb{E}$ be two distinct edges containing a common vertex $t = (t_1, t_2) \in \mathbb{Z}^{2*}$. We say that these edges make a *legitimate turn*, if either one of these edges connects the vertex t with the vertex $(t_1 + 1, t_2)$ and the other vertex connects it with the vertex $(t_1, t_2 + 1)$ or if one of these edges connects the vertex t with the vertex $(t_1 - 1, t_2)$ and the other vertex connects it with the vertex $(t_1, t_2 - 1)$. A contour is defined as a sequence e_1, e_2, \dots, e_k of mutually distinct edges such that the edges $e_i, e_{i+1}, i = 1, 2, \dots, k$ (here $k + 1 = 1$) have a common vertex and in case there is another pair of edges $e_{i'}, e_{i'+1}$ of this contour having the same common vertex, the edges e_i, e_{i+1} make a legitimate turn. The set of all contours is denoted by \mathbb{G} . We say that a contour $\Gamma \in \mathbb{G}$ is a *contour in the volume W* , if all its edges belong to $\mathbb{E}(W)$. The set of all such contours is denoted by $\mathbb{G}(W)$. The number of edges in a contour $\Gamma \in \mathbb{G}$ is denoted $|\Gamma|$ and is called the *length* of this contour. The set of all points $t \in \mathbb{Z}^2$ such that there is no continuous curve in \mathbb{R}^2 which does not intersect a contour Γ and connect the point $t \in \mathbb{Z}^2 \subset \mathbb{R}^2$ to “infinity” will be called the *interior of the contour Γ* and will be denoted by $\text{Int } \Gamma$. We say that the contours Γ_1 and Γ_2 are *compatible* if they have no common edges and if at any vertex which is contained in both of contours these contours make legitimate turns.

We say that a finite system of contours $\pi \subset \mathbb{G}$ is a *compatible system of contours*, if any two different contours in π are compatible. Let H be the set of all systems of compatible contours and $H(W) \subseteq H$ be the set of all compatible systems of contours $\pi \subseteq \mathbb{G}(W)$ of contours in a volume W (the set $H(W)$ includes the empty system of contours). It is easy to understand that for any finite volume W and any configuration $\sigma_W \in X^W$ there exists a unique system of contours $\pi(\sigma_W) \in H(W)$ such that

$$B(\sigma_W) = \cup_{\Gamma \in \pi(\sigma_W)} \Gamma. \tag{3.2}$$

Further, for any system of contours $\pi \in H(W)$ there is a unique configuration $\sigma_W(\pi) \in X^W$ such that

$$\pi(\sigma_W(\pi)) = \pi. \tag{3.3}$$

For any point $t \in \mathbb{Z}^2$ denote by $O(t)$ the set of all contours $\Gamma \in \mathbb{G}$ such that $t \in \text{Int } \Gamma$. The configuration $\sigma_W = (\sigma_t, t \in W) \in X^W$ can be reconstructed from the contour system $\pi(\sigma_W)$ by the help of the relation

$$\sigma_t = \begin{cases} +1, & \text{if } |O(t) \cap \pi(\sigma_W)| \text{ is even,} \\ -1, & \text{if } |O(t) \cap \pi(\sigma_W)| \text{ is odd.} \end{cases}$$

Let Γ be a fixed contour. Define the subset $\Delta(\Gamma) \subset \mathbb{Z}^2$ by the relation:

$$\Delta(\Gamma) = \mathbb{Z}^2 \setminus \left(\bigcup_{\gamma: \gamma \text{ is compatible with } \Gamma} \text{Int } \gamma \right). \tag{3.4}$$

It is a subset of all sites of the lattice which are at the distance not greater than $\frac{\sqrt{2}}{2}$ from Γ . The set $\Delta(\pi)$ is defined in the obvious way.

3.2. Localization of the meniscus. We return now to the meniscus. Note first that if β is large enough, then with $\langle \cdot \rangle_{\mathbf{n}}^{\beta}$ -probability 1 every configuration σ on $(\mathbb{Z}^2)^+$ contains exactly one infinite contour, and it has an endpoint at the point $(-\frac{1}{2}, 0)$. Let us denote this contour by $\Gamma = \Gamma(\sigma)$, and let $[\gamma_1(\sigma), \gamma_2(\sigma)] \subseteq \mathbb{R}^1$ be the (random) segment of the x -axis, obtained by projecting the contour Γ onto \mathbb{R}^1 (the cases of the projection to be semiinfinite or infinite are not excluded, of course).

Theorem 3.2 (The theorem on the localization of the meniscus). *Suppose additionally to the conditions of the Meniscus Theorem that $n_y > 0$. (That means that the contours Γ go typically to the north-west.) Then the $\langle \cdot \rangle_{\mathbf{n}}^{\beta}$ -probability of the event $\{\sigma : \gamma_2(\sigma) \geq l, l > 0\}$ is bounded from above by*

$$\exp\{-c_{\mathbf{n}}\beta l\} \text{ with } c_{\mathbf{n}} > 0. \quad (3.5)$$

(The positivity of $c_{\mathbf{n}}$ is not uniform in \mathbf{n} , of course).

The above theorem estimates the probability of the event that the contour Γ crosses the vertical line and deviates from it into the positive quadrant by the distance l . A similar result holds for the probabilities of crossing and deviating by a distance l from other straight lines, passing through the origin. However, if the line has equation $y = kx$ with the slope $k < 0$, $k \neq -\frac{n_x}{n_y}$, then the corresponding estimate is of the form $\exp\{-c_k l\}$, and the positive exponent c_k does not diverge as $\beta \rightarrow \infty$, unlike (3.5). The reason is that such deviations are not suppressed even at zero temperature.

The following corollary of the above result is crucial for our purposes. Consider again the box V_N (see (3.1)) and endow it with the boundary condition $\bar{\sigma}$, which is +1 on the left, top and right border, as well as on the segment $[0, N]$, and is different from being identically +1 on the remaining part of the boundary. Then every configuration $\tilde{\sigma}$ in V_N has a certain amount of open contours $\Gamma_i(\tilde{\sigma})$ attached to the boundary of V_N . Denote by $[\tilde{\gamma}_1(\tilde{\sigma}), \tilde{\gamma}_2(\tilde{\sigma})]$ the smallest segment of the x -axis, containing the projections of all the contours $\Gamma_i(\tilde{\sigma})$. Let $\langle \cdot \rangle_{\bar{\sigma}, N}^{\beta}$ be the corresponding Gibbs state in V_N . Then the following statement holds:

Corollary 3.3. *The $\langle \cdot \rangle_{\bar{\sigma}, N}^{\beta}$ -probability of the event $\{\tilde{\sigma} : \tilde{\gamma}_2(\tilde{\sigma}) \geq l, l > 0\}$ is bounded from above by*

$$\exp\{-c\beta l\} \quad (3.6)$$

with $c > 0$, uniformly in $\bar{\sigma}, N$.

Proof. The state $\langle \cdot \rangle_{\bar{\sigma}, N}^{\beta}$ can be coupled with any of the states $\langle \cdot \rangle_{\mathbf{n}}^{\beta}$ in such a way that the former is higher than the latter in the FKG sense. (That means that with probability one, according to the coupling measure, $\tilde{\sigma}(t) \neq \sigma(t)$ for every $t \in V_N$.) In particular, the region between the contour $\Gamma(\sigma)$ and the ray $(-\infty, -\frac{1}{2})$ of the x -axis contains all the contours $\Gamma_i(\tilde{\sigma})$, and so $\gamma_2(\sigma) \geq \tilde{\gamma}_2(\tilde{\sigma})$. So the corollary holds with $c = c_{\mathbf{n}}$ for arbitrary \mathbf{n} with positive coordinates n_x, n_y . \square

3.3. *Properties of the phase separation line.* The reader who is familiar with the book [DKS] would note that the above statements are quite close to those in its Chapter 4, where the question about large deviations of the phase separation line is discussed. There the deviations were studied for two ensembles: one was the canonical ensemble of the phase separation lines $S \in \mathcal{I}_{N,\mathbf{n}}$, connecting two points $(\frac{1}{2}, \frac{1}{2})$ and $(h(N, \mathbf{n}), N - \frac{1}{2})$, while the other was the grand canonical ensemble of the phase separation lines, which were starting at $(\frac{1}{2}, \frac{1}{2})$, but which were terminating on the line $y = N - \frac{1}{2}$, with the position $h(S)$ of the endpoint on this line randomly distributed in such a way that its mean value was $h(N, \mathbf{n})$, and the distribution was asymptotically normal with the variance of the order of \sqrt{N} . Before discussing these results further, we recall some notions introduced in [DKS].

3.3.1. *The ensemble of tame animals.* We start by considering the canonical ensemble of tame animals, which is defined as a measure on the set

$$\mathcal{I}_{N,\mathbf{n}}^\infty = \{S \in \mathcal{I}_{N,\mathbf{n}} : |S \cap \{(x, y) : y = m\}| = 1 \text{ for all } m = 0, 1, \dots, N\} \quad (3.7)$$

of all SOS-trajectories S , starting at $(\frac{1}{2}, \frac{1}{2})$ and terminating at $(h(N, \mathbf{n}), N - \frac{1}{2})$. We denote by $k(r, S)$ the abscissa of the trajectory S at the level r :

$$k(r, S) = S \cap \{(x, y) : y = r\}. \quad (3.8)$$

We introduce the distribution

$$P_{N,\mathbf{n}}^\infty(S) = (\Xi(N, \mathbf{n}, \infty))^{-1} \exp\{-2\beta|S|\}, \quad S \in \mathcal{I}_{N,\mathbf{n}}^\infty, \quad (3.9)$$

with the partition function

$$\Xi(N, \mathbf{n}, \infty) = \sum_{S \in \mathcal{I}_{N,\mathbf{n}}^\infty} \exp\{-2\beta|S|\}. \quad (3.10)$$

The main tool in investigating this ensemble is the passage to the grand canonical ensemble, which is defined as a family of measures on the union

$$\mathcal{I}_N^\infty = \bigcup_{\mathbf{n}} \mathcal{I}_{N,\mathbf{n}}^\infty, \quad (3.11)$$

indexed by the real parameter H :

$$P_{N,H}^\infty(S) = (\Xi(N, H, \infty))^{-1} \exp\{-2\beta|S| + \beta H h(S)\}, \quad S \in \mathcal{I}_N^\infty, \quad (3.12)$$

with the partition function

$$\Xi(N, H, \infty) = \sum_{S \in \mathcal{I}_N^\infty} \exp\{-2\beta|S| + \beta H h(S)\}. \quad (3.13)$$

It is not difficult to calculate explicitly the partition function (3.13) of the ensemble of tame animals. Supposing that $-2 < H < 2$ we have

$$\Xi(N, H, \infty) = e^{2\beta(Q_H)^N} \quad (3.14)$$

with

$$Q_H = \sum_{k=-\infty}^{+\infty} \exp\{-2\beta(|k|+1) + \beta Hk\} = e^{-2\beta} \frac{\sinh(2\beta)}{\cosh(2\beta) - \cosh(H\beta)}. \quad (3.15)$$

The position $h(S)$ of the endpoint of the polygon S on the line $y = N - \frac{1}{2}$ is a random variable in the ensemble (3.12). It equals the sum of N identically distributed random variables with the probability distribution

$$P_H^\infty(k) = Q_H^{-1} \exp\{-2\beta(|k|+1) + \beta Hk\}, \quad (3.16)$$

the mean value

$$M_H^\infty = \beta^{-1} \frac{\partial \log Q_H}{\partial H} = \frac{\sinh(H\beta)}{\cosh(2\beta) - \cosh(H\beta)}, \quad (3.17)$$

and the variance

$$D_H^\infty = \beta^{-2} \frac{\partial^2 \log Q_H}{\partial H^2} = \frac{\cosh(2\beta) \cosh(H\beta) - 1}{(\cosh(2\beta) - \cosh(H\beta))^2}. \quad (3.18)$$

According to the standard local limit theorem for sums of independent random variables (see for example [Gn]), as $N \rightarrow \infty$ one has

$$P_{N,H}^\infty(\{S : h(S) = b_N\}) \sim \frac{1}{\sqrt{2\pi N D_H^\infty}} \exp\left\{-\frac{1}{2N D_H^\infty} (b_N - N M_H^\infty)^2\right\}, \quad (3.19)$$

whenever a sequence of integers b_N is chosen in such a way that the quantity

$$\frac{|b_N - M_H^\infty N|}{N^{1/2}}$$

is bounded uniformly in N .

3.3.2. The ensemble of wild animals. The “true” phase separation line of the Ising model is a small perturbation of the ensemble, introduced above, see [DKS]. The corresponding ensemble $\mathcal{I}_{N,\mathbf{n}}$ of the separation lines is again formed by the polygons S of the dual lattice, starting at $(\frac{1}{2}, \frac{1}{2})$ and terminating at $(h(N, \mathbf{n}), N - \frac{1}{2})$, but this time it is not required that the intersections $S \cap \{(x, y) : y = r\}$ are singletons. Therefore, instead of the random variable $k(r, S)$ (see (3.8)) we introduce two new variables:

$$\begin{aligned} \underline{k}(r, S) &= \min \{x : S \cap \{(x, y) : y = r\} \neq \emptyset\}, \\ \bar{k}(r, S) &= \max \{x : S \cap \{(x, y) : y = r\} \neq \emptyset\}. \end{aligned}$$

The probability distribution $P_{N,\mathbf{n}}(S)$ (compare with (3.9)) is the one induced on the set $\mathcal{I}_{N,\mathbf{n}}$ of separation lines by the Ising model random field $\langle \cdot \rangle_{N,\mathbf{n}}^\beta$, while the probability distribution $P_{N,H}(S)$ on $\mathcal{I}_N = \bigcup_{\mathbf{n}} \mathcal{I}_{N,\mathbf{n}}$ is obtained as a normalized mixture of the distributions $P_{N,\mathbf{n}}(S)$ with the weights $\exp\{\beta H h(S)\}$, compare with (3.12).

Turning back to the meniscus problem, the event $dev(r, c_k)$ we are interested in, is the following one: the random line deviates at the level r from the ray of its expected values by an amount linear in the distance from the starting point, i.e. by $c_k r$. In the ensemble \mathcal{I}_N its probability is given by

$$P_{N,H}(dev(r, c_k)) \equiv P_{N,H}(r, c_k) \equiv \sum_{\substack{S \in \mathcal{I}_N: \bar{k}(r,S) - rM_H \geq c_k r \text{ or} \\ rM_H - \underline{k}(r,S) \geq c_k r}} P_{N,H}(S). \quad (3.20)$$

The result of [DKS] is that this probability is exponentially small in r :

$$P_{N,H}(r, c_k) \leq A \exp\{-r\Psi(c_k)\}, \quad (3.21)$$

with $\Psi(\cdot)$ a positive function, defined on a positive semiaxis. This statement is almost what we need, though the estimator does not include the dependence on β . (To have such dependence one has to put restrictions on the slopes c_k and M_H .) However when one looks at the situation in the canonical ensemble, it is much less satisfactory; the corresponding estimate of [DKS] reads:

$$P_{N,\mathbf{n}}(r, c_k) \equiv \sum_{\substack{S \in \mathcal{I}_{N,\mathbf{n}}: \bar{k}(r,S) - rM_H \geq c_k r \text{ or} \\ rM_H - \underline{k}(r,S) \geq c_k r}} P_{N,\mathbf{n}}(S) \leq A\sqrt{N} \exp\{-r\Psi(c_k)\}. \quad (3.22)$$

This estimate was enough for the purposes of [DKS] of studying the surface tension. To obtain (3.22) from (3.21) is very easy:

$$\begin{aligned} P_{N,H}(r, c_k) &\geq P_{N,H}(r, c_k; h(S) = h(N, \mathbf{n})) \\ &\equiv P_{N,H}(r, c_k | h(S) = h(N, \mathbf{n})) P_{N,H}(h(S) = h(N, \mathbf{n})) \\ &\equiv P_{N,\mathbf{n}}(r, c_k) P_{N,H}(h(S) = h(N, \mathbf{n})), \end{aligned}$$

and we get the desired bound by using the statement that the random variable $h(S)$ has the standard local limit behavior with the variance of the order of N . It is clear however that the true estimate in (3.22) should be of the same order as in (3.21), and the probability $P_{N,H}(r, c_k)$ should in fact be asymptotically equal to the conditional probability $P_{N,H}(r, c_k | h(S) = h(N, \mathbf{n}))$, since the conditioning is done by fixing the random variable $h(S)$ to be equal to its mean value (which belongs to the region of its typical values). As we will see in the following, the corresponding improvement can indeed be done.

In this paper we will not prove the Meniscus Theorem, relegating it to the forthcoming publication. What we will establish is the Meniscus Localization Theorem for the particular choice of the vector $\mathbf{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Since the only thing we need for the purposes of the present paper is the above mentioned Corollary 3.3, it will be enough. The relevant events we should study are that the observables $k(r, S)$ reach some threshold values $l > 0$. Note however, that the probability of the event $k(r, S) \geq l$ only increases when the size N of the box V_N increases. So for any fixed r we can choose N as large as it is convenient for us.

3.3.3. Calculations in the grand canonical ensemble of tame animals. We start by estimating the probability of the deviation we are interested in, in the grand canonical ensemble $P_{N,H}^\infty$ of tame animals. In accordance with our choice of the direction \mathbf{n} , the field H has to satisfy the equation

$$M_H^\infty \equiv \beta^{-1} \frac{\partial \log Q_H}{\partial H} = \frac{\sinh(H\beta)}{\cosh(2\beta) - \cosh(H\beta)} = -1. \quad (3.23)$$

The function M_H^∞ is analytic and increasing, so the inverse function $H^\infty(M)$ is also analytic and increasing. The solution to Eq. (3.23) is $H = H^\infty(-1) = -2 + \frac{\ln 2}{\beta} + o(\frac{1}{\beta})$. The event we are occupied with is that the observable $k(r, S)$ reaches some threshold $l > 0$, $0 < r < N$. Denoting this probability by $P_{N,H}^\infty(r, l)$, we have for every choice of the auxiliary field K that

$$\begin{aligned} P_{N,H}^\infty(r, l) &\equiv \sum_{S \in \mathcal{I}_N^\infty: k(r, S)=l} \frac{\exp\{-2\beta|S| + \beta H h(S)\}}{\Xi(N, H, \infty)} \\ &\equiv e^{-\beta K l} \sum_{S \in \mathcal{I}_N^\infty: k(r, S)=l} \frac{\exp\{-2\beta|S| + \beta H h(S) + \beta K k(r, S)\}}{\Xi(N, H, \infty)} \\ &\leq e^{-\beta K l} \frac{\Xi(N, H, r, K, \infty)}{\Xi(N, H, \infty)}, \end{aligned} \quad (3.24)$$

where $\Xi(N, H, r, K, \infty) = \sum_{S \in \mathcal{I}_N^\infty} \exp\{-2\beta|S| + \beta H h(S) + \beta K k(r, S)\}$.

(The estimate (3.24) is the standard Cramer tilt.) The straightforward calculations (3.14), (3.15) imply that

$$P_{N,H}^\infty(r, l) \leq e^{-\beta K l} \left[\frac{\cosh(2\beta) - \cosh(H\beta)}{\cosh(2\beta) - \cosh((H+K)\beta)} \right]^r. \quad (3.25)$$

(Note that this estimate does not depend on N .) The choice of K is up to us. The approximate optimization in (3.25) suggests the following choice for K :

$$K + H = \begin{cases} 2 - \frac{1}{\beta} \ln(1 + \frac{r}{l}) & \text{for } \frac{r}{l} < \frac{e^{2\beta}}{2}, \\ \frac{1}{\beta} \frac{l}{r} e^{2\beta} & \text{for } \frac{r}{l} \geq \frac{e^{2\beta}}{2}. \end{cases}$$

The choice which we will use and which is simpler to handle is the following:

$$K + H = \begin{cases} -H & \text{for } r \leq l, \\ 2 - \frac{1}{\beta} \ln(1 + \frac{r}{l}) & \text{for } l < r < \frac{e^{2\beta} l}{2}, \\ \frac{1}{\beta} \frac{l}{r} e^{2\beta} & \text{for } r \geq \frac{e^{2\beta} l}{2}. \end{cases} \quad (3.26)$$

Hence we have the estimate

$$\begin{aligned} P_{N,H}^\infty(l) &\equiv \sum_r P_{N,H}^\infty(r, l) \\ &\leq e^{2\beta H l} \sum_{r < \frac{e^{2\beta} l}{2}} 1 + e^{\beta H l} \left[\sum_{l < r \leq 2l} \left(\frac{l+r}{2r}\right)^r + \sum_{2l < r \leq \frac{e^{2\beta} l}{2}} \left(\frac{l+r}{2r}\right)^r + \sum_{r > \frac{e^{2\beta} l}{2}} \left(\frac{1}{2}\right)^r \right] \\ &\leq e^{2\beta H l} \frac{e^{2\beta} l}{2} + e^{\beta H l} \left[l + \sum_{r > 2l} \left(\frac{3}{4}\right)^r + \sum_{r > \frac{e^{2\beta} l}{2}} \left(\frac{1}{2}\right)^r \right] \\ &< e^{-\beta l}, \end{aligned} \quad (3.27)$$

provided β is large enough.

3.3.4. *Calculations in the canonical ensemble of tame animals.* In complete analogy with (3.24) we obtain that in the canonical ensemble

$$P_{N,\mathbf{n}}^\infty(r, l) \leq e^{-\beta Kl} \frac{\Xi(N, \mathbf{n}, r, K, \infty)}{\Xi(N, \mathbf{n}, \infty)}, \quad (3.28)$$

where

$$\begin{aligned} \Xi(N, \mathbf{n}, \infty) &= \sum_{S \in \mathcal{I}_N^\infty : h(S)=h(N, \mathbf{n})} \exp\{-2\beta|S|\}, \\ \Xi(N, \mathbf{n}, r, K, \infty) &= \sum_{S \in \mathcal{I}_N^\infty : h(S)=h(N, \mathbf{n})} \exp\{-2\beta|S| + \beta Kk(r, S)\}. \end{aligned}$$

To proceed, we use the following relations between the canonical and grand canonical partition functions: for every H :

$$\begin{aligned} \Xi(N, \mathbf{n}, \infty) &= P_{N,H}^\infty\{S : h(S) = h(N, \mathbf{n})\} \\ &\quad \times \Xi(N, H, \infty) \exp\{-\beta Hh(N, \mathbf{n})\}, \\ \Xi(N, \mathbf{n}, r, K, \infty) &= P_{N,H,r,K}^\infty\{S : h(S) = h(N, \mathbf{n})\} \\ &\quad \times \Xi(N, H, r, K, \infty) \exp\{-\beta Hh(N, \mathbf{n})\}. \end{aligned}$$

Of course, the distribution $P_{N,H,r,K}^\infty$ is defined on \mathcal{I}_N^∞ by

$$P_{N,H,r,K}^\infty\{S\} = \frac{\exp\{-2\beta|S| + \beta Hh(S) + \beta Kk(r, S)\}}{\Xi(N, H, r, K, \infty)}.$$

The substitution to (3.28) of the last two relations (with different values H, \tilde{H}) gives

$$\begin{aligned} P_{N,\mathbf{n}}^\infty(r, l) &\leq e^{-\beta Kl} \times \\ &\times \frac{P_{N,\tilde{H},r,K}^\infty\{S : h(S) = h(N, \mathbf{n})\} \Xi(N, \tilde{H}, r, K, \infty) \exp\{-\beta \tilde{H}h(N, \mathbf{n})\}}{P_{N,H}^\infty\{S : h(S) = h(N, \mathbf{n})\} \Xi(N, H, \infty) \exp\{-\beta Hh(N, \mathbf{n})\}}. \end{aligned} \quad (3.29)$$

(Here, in contrast with (3.25), we have the N -dependence.) We obtain the best possible bound by choosing the magnetic fields H, \tilde{H} in such a way that the expectations of the random variable $h(S)$ are equal to the same value $h(N, \mathbf{n})$ under both $P_{N,H}^\infty$ and $P_{N,\tilde{H},r,K}^\infty$. Under that choice the first ratio in (3.29) is less than 1, provided N is large enough. To see this we note first that the distributions of $h(S)$ under both $P_{N,H}^\infty$ and $P_{N,\tilde{H},r,K}^\infty$ are asymptotically normal, and that the standard local limit theorem holds for them. Lemma 3.4, proven below, tells us that the variance of the random variable $h(S)$ under $P_{N,H}^\infty$ is not bigger than the one under $P_{N,\tilde{H},r,K}^\infty$, so that we can use the knowledge of the limit behavior of $h(S)$ and apply the relation (3.19). Also, the difference $|\tilde{H} - H| \rightarrow 0$, as $N \rightarrow \infty$, while the product $(\tilde{H} - H)h(N, \mathbf{n})$ tends to the derivative $\frac{d}{dM} H^\infty(-1)$. This derivative is finite, and that takes care of the last factor. (In the present situation also a different choice is possible: H is chosen in the way prescribed, while \tilde{H} is taken to be equal to H . In such a case the first ratio in (3.29) goes to 1 as $N \rightarrow \infty$, since the random variables $k(r, S)$ and $h(S) - k(r, S)$ are independent under the distribution $P_{N,H,r,K}^\infty$. Hence the random variables $h(S)$ under both $P_{N,H,r,K}^\infty$ and $P_{N,H}^\infty$ have the local limit behavior with the divergent variances, and these variances are asymptotically equal.) After these remarks the estimate (3.29) is reduced to (3.24). Of course, the values of N for which this reduction is valid, depend on (r, l) , but this is

irrelevant for our purposes. Thus, in the limit $N \rightarrow \infty$ we have the following analog of (3.27):

$$P_{N=\infty, \mathbf{n}}^\infty(l) \equiv \sum_r P_{N=\infty, \mathbf{n}}^\infty(r, l) < e^{-\beta l}.$$

3.3.5. Properties of the variance. Here we will study the properties of the variance of the random variable $h(S)$ when the trajectory S is subject to a varying “field”. To be specific, we suppose that a given fraction λ of the total life-span N of the trajectory is under the influence of the “field” K , while after that time the field has a different value L . If we denote by $h_\lambda(S)$ the location of the polygon S at the “time” $[\lambda N]$, then the probability distribution we are interested in is given by

$$P_{N, \lambda, K, L}^\infty(S) = e^{-2\beta(Q_K)^{-[\lambda N]}(Q_L)^{-(N-[\lambda N])}} \times \exp\{-2\beta|S| + \beta K h_\lambda(S) + \beta L(h(S) - h_\lambda(S))\}, \quad S \in \mathcal{I}_N^\infty. \quad (3.30)$$

To simplify the notations we will consider hereafter only the case when the total time length is $2N$, while $\lambda = \frac{1}{2}$. The corresponding measure (3.30) will be denoted by $P_{2N, K, L}^\infty(S)$.

Let $2NM_{K, L}^\infty$ be the mean value of the random variable $h(S)$ according to the distribution $P_{2N, K, L}^\infty(S)$. Clearly,

$$2NM_{K, L}^\infty = NM_K^\infty + NM_L^\infty. \quad (3.31)$$

We are interested in the one-parameter families of the distributions $P_{2N, K, L}^\infty(S)$, for which

$$M_{K, L}^\infty = \text{const} \equiv C. \quad (3.32)$$

(This restriction is natural, since our main interest is in the canonical ensemble, when the endpoint of the polygon S is fixed; the restriction (3.32) means that the endpoint is fixed “in the mean”.) The relation between K and L is then the following, according to (3.17) and (3.31):

$$\frac{\sinh(K\beta)}{\cosh(2\beta) - \cosh(K\beta)} + \frac{\sinh(L\beta)}{\cosh(2\beta) - \cosh(L\beta)} = 2\beta C. \quad (3.33)$$

What we are interested in is the behavior of the variance $D_{K, L}^\infty$, which is defined naturally by:

$$2ND_{K, L}^\infty = ND_K^\infty + ND_L^\infty \quad (3.34)$$

(see (3.18)).

Lemma 3.4. *Let C be any real number, and consider the function $D_{K, L}^\infty$ restricted to the curve (3.33). Let H_C be the value of magnetic field such that*

$$M_{H_C, H_C}^\infty = C. \quad (3.35)$$

Then for any K, L , satisfying $M_{K, L}^\infty = C$, we have

$$D_{K, L}^\infty \geq D_{H_C, H_C}^\infty. \quad (3.36)$$

Proof. The proof of this lemma consists in calculation of the derivative of the function $D_{K,L}^\infty$ along the curve $M_{K,L}^\infty = C$. As we will see, this function has on any such curve exactly one minimum at H_C for every C .

The gradient of the function $M_{K,L}^\infty$ is clearly equal to

$$\left(\frac{\cosh(2\beta)\cosh(K\beta) - 1}{(\cosh(2\beta) - \cosh(K\beta))^2}, \frac{\cosh(2\beta)\cosh(L\beta) - 1}{(\cosh(2\beta) - \cosh(L\beta))^2} \right),$$

hence the tangent direction (k, l) to the curve (3.33) satisfies the relation:

$$k \frac{\cosh(2\beta)\cosh(K\beta) - 1}{(\cosh(2\beta) - \cosh(K\beta))^2} + l \frac{\cosh(2\beta)\cosh(L\beta) - 1}{(\cosh(2\beta) - \cosh(L\beta))^2} = 0. \quad (3.37)$$

Suppose that $K > L$. Let us show that in this case the derivative of $D_{K,L}^\infty$ along the vector (k, l) , satisfying (3.37) with $k > 0$ is positive. Let us take $(k, l) = \left(\frac{\cosh(2\beta)\cosh(L\beta) - 1}{(\cosh(2\beta) - \cosh(L\beta))^2}, -\frac{\cosh(2\beta)\cosh(K\beta) - 1}{(\cosh(2\beta) - \cosh(K\beta))^2} \right)$. The derivative in question is then equal to

$$\frac{\sinh(K\beta)[\cosh(2\beta)(\cosh(2\beta) + \cosh(K\beta)) - 1][\cosh(2\beta)\cosh(L\beta) - 1]}{[\cosh(2\beta) - \cosh(L\beta)]^2[\cosh(2\beta) - \cosh(K\beta)]^3} - \frac{\sinh(L\beta)[\cosh(2\beta)(\cosh(2\beta) + \cosh(L\beta)) - 1][\cosh(2\beta)\cosh(K\beta) - 1]}{[\cosh(2\beta) - \cosh(K\beta)]^2[\cosh(2\beta) - \cosh(L\beta)]^3}.$$

Positivity of this expression is equivalent to the statement that the function

$$\frac{\sinh(H\beta)[\cosh(2\beta)(\cosh(2\beta) + \cosh(H\beta)) - 1]}{[\cosh(2\beta) - \cosh(H\beta)][\cosh(2\beta)\cosh(H\beta) - 1]}$$

is increasing. But this follows from direct calculation of its derivative, which is the sum of four manifestly nonnegative terms, one of which is even manifestly positive. \square

3.3.6. Wild animals. What should be done next is the same construction for the case of the real Ising model, which has to take into account the real behavior of the separation line, which has overhangs, and which thus has to be described by the ensemble of wild animals (in the terminology of [DKS]). But all the necessary constructions, involving the cluster expansion in the ensemble of wild animals, are presented in Chapter 4 of [DKS], and they need not be repeated here.

4. Contour Representation of the Partition Function

Now we recall the main definitions of the contour method in a variant convenient for our aims and introduce the notation used below. The definition (1.8) implies that the energy

$$E_W^{\text{Ising}}(\sigma_W / +) = 2|\pi(\sigma_W)| - |\mathbb{E}(W)|, \quad \sigma_W \in X^W, \quad (4.1)$$

where we let

$$|\pi| = \sum_{\Gamma \in \pi} |\Gamma|, \quad \pi \in H(W). \quad (4.2)$$

So recalling the definition (1.7) we find that

$$p_W^{\beta,+}(\sigma_W) = \tilde{p}_W(\pi(\sigma_W)), \quad \sigma_W \in X^W, \quad (4.3)$$

where *the contour probability distribution*

$$\tilde{p}_W(\pi) = (\tilde{Z}(W))^{-1} \exp\{-2\beta|\pi|\}, \quad \pi \in H(W), \quad (4.4)$$

and *the contour partition function*

$$\tilde{Z}(W) = \sum_{\pi \in H(W)} \exp\{-2\beta|\pi|\}. \quad (4.5)$$

We can also rewrite in contour terms the quantities $Q_{T,D}^{\beta,+N}$ introduced by relation (2.11) and used in the formulation of the main theorem. It follows from the relations (3.4) and (4.4) that for any finite set $D \subseteq T \cap W_N$,

$$p_{W_N}^{\beta,+}(\sigma_{W_N} : (\sigma_{W_N})^- = D) = (\tilde{Z}(W_N))^{-1} \tilde{Z}_T(D, W_N), \quad (4.6)$$

where

$$\tilde{Z}_T(D, W_N) = \sum_{\pi \in K_T(D, W_N)} \exp\{-2\beta|\pi|\}, \quad (4.7)$$

and the set $K_T(D, W_N) \subseteq H(W_N)$ of systems of contours is defined by the relation

$$K_T(D, W_N) = \{\pi \in H(W_N) : |O(t) \cap \pi| \text{ is odd, if } t \in D, \\ |O(t) \cap \pi| \text{ is even, if } t \in (T \setminus D)\}. \quad (4.8)$$

In a similar way

$$p_{W_N}^{\beta,+}(\sigma_{W_N} : (\sigma_{W_N})^- = \emptyset) = (\tilde{Z}(W_N))^{-1} \tilde{Z}_T(\emptyset, W_N). \quad (4.9)$$

Thus it follows from the definition (2.11)

$$Q_{T,D}^{\beta,+N} = -\ln \frac{\tilde{Z}_T(D, W_N)}{\tilde{Z}_T(\emptyset, W_N)}. \quad (4.10)$$

This formula is the starting point of the following estimates.

5. Cluster Expansions

The following estimates use the cluster expansion method which exists in many versions. We choose the general Kotecky-Preiss model ([KP]) with simplifications introduced in the paper [D 94], though some other variants of the cluster expansion method can be also applied to the derivation of the same estimates. In this section we formulate the definitions and the results from [D 94] which will be used below.

Let us describe the Kotecky-Preiss model. Let Θ be a finite or a countable set. Its elements will be called *animals* and so we call this model the *animal model*. (The ensembles of wild and tame animals of Sect. 3 were particular examples of the animal models.) Assume that a subset $\mathcal{S} \subseteq \Theta \times \Theta$ of the set of pairs (θ_1, θ_2) of animals is fixed, which is symmetric, i.e. a pair $(\theta_1, \theta_2) \in \mathcal{S}$ if and only if the pair $(\theta_2, \theta_1) \in \mathcal{S}$, and reflexive, i.e. the diagonal pairs $(\theta, \theta) \in \mathcal{S}$. If the pair of animals $(\theta_1, \theta_2) \in \mathcal{S}$, we say that they are *compatible* and write $\theta_1 \leftrightarrow \theta_2$. If the pair of animals $(\theta_1, \theta_2) \in (\Theta \times \Theta) \setminus \mathcal{S}$,

we say that they are *incompatible* and write $\theta_1 \leftrightarrow \theta_2$. The pair (Θ, \mathcal{S}) is called the *animal model*. Sometimes it is convenient to consider the undirected graph without loops and multiple edges with the set Θ as the set of its vertices, such that the vertices $\theta_1, \theta_2 \in \Theta$ are connected by an edge of this graph if and only if $\theta_1 \leftrightarrow \theta_2$. It is evident that this graph describes the animal model (Θ, \mathcal{S}) in a unique way.

Fix an animal model (Θ, \mathcal{S}) . A finite subset $\tau \subseteq \Theta$ is called a *herd*, if any two animals $\theta_1, \theta_2 \in \tau$ are compatible. For any finite $\Lambda \subseteq \Theta$ the set of all herds τ such that $\tau \subseteq \Lambda$ is denoted by $H(\Lambda)$ and is called the set of all *herds in Λ* . (The set $H(\Lambda)$ includes the empty herd which is denoted by \emptyset .)

Assume that a complex-valued function $w(\theta), \theta \in \Theta$, is given. The number $w(\theta)$ is called *the weight of the animal θ* . Let a finite set $\Lambda \subseteq \Theta$. The number

$$Z_w(\Lambda) = \sum_{\tau \in H(\Lambda)} \prod_{\theta \in \tau} w(\theta) \tag{5.1}$$

is called *the partition function in Λ defined by the weights $w = (w(\theta), \theta \in \Theta)$* . (For $\tau = \emptyset$ the product in (5.1) equals 1 by definition. So if Λ is an empty set, the partition function $Z_w(\Lambda) = 1$.)

A good control over the logarithm of the partition function of the animal models is possible only if the absolute values $|w(\theta)|$ are in some sense small enough. It leads to the main restrictions on the domain of applicability of the discussed approach. To be more definite we describe a condition on the weights which is used below and was introduced by Kotecky and Preiss [KP]. Assume that a positive-valued function $b(\theta), \theta \in \Theta$, is given. The value $b(\theta)$ will be called *the might of the animal θ* . A choice of the function $b(\theta)$ for a concrete animal model with given weights is determined simply by a wish to satisfy the needed conditions. Roughly speaking, the might $b(\theta)$ has to be large if the animal θ is incompatible with many other animals.

Definition 5.1. *We say that a weight function $w(\theta), \theta \in \Theta$, satisfies the KP-condition, if there exists a non-negative weight function $w_0(\theta) \geq 0, \theta \in \Theta$, such that for any $\theta \in \Theta$,*

$$\exp \left\{ \sum_{\tilde{\theta} \in \Theta: \tilde{\theta} \leftrightarrow \theta} w_0(\tilde{\theta})b(\tilde{\theta}) + w_0(\theta)b(\theta) \right\} \leq b(\theta) \tag{5.2}$$

and

$$|w(\theta)| \leq w_0(\theta), \quad \theta \in \Theta. \tag{5.3}$$

Proposition 5.2. *Fix a positive weight function $w_0(\theta), \theta \in \Theta$ and mights $b(\theta), \theta \in \Theta$, such that the condition (5.2) is satisfied. Let \mathcal{W}_0 be the set of all (complex-valued) weight functions $w = w(\theta)$ of $\theta \in \Theta$ such that*

$$|w(\theta)| \leq w_0(\theta), \quad \theta \in \Theta. \tag{5.4}$$

Consider a weight function $w \in \mathcal{W}_0$. Then for any finite set $\Lambda \subseteq \Theta$ the partition function $Z_w(\Lambda) \neq 0$ and

$$|\ln |Z_w(\Lambda)|| \leq \sum_{\theta \in \Lambda} w_0(\theta)b(\theta). \tag{5.5}$$

Proposition 5.2 is proved in [D 94] by simple induction in $|\Lambda|$. It follows from this proposition that the function $\ln Z_w(\Lambda)$ is an analytic function of the arguments $w(\theta), \theta \in \Theta$. It turns out (see [D 94]) that the cluster expansion is nothing else but the Taylor expansion of this function at the point $w(\theta) \equiv 0, \theta \in \Theta$. Its coefficients do not depend on Λ , the majority of them vanish and the rest can be estimated by the help of the usual Cauchy formula.

For any finite $\Lambda \subseteq \Theta$ the set of all pairs $\rho = (\bar{\rho}, \alpha)$, such that $\bar{\rho} \subseteq \Lambda$ is a subset and $\alpha = \alpha(\theta) \geq 1, \theta \in \bar{\rho}$, is an integer-valued function of $\theta \in \bar{\rho}$, will be denoted by $D(\Lambda)$ and will be called a *group of animals in Λ* . The set $\bar{\rho}$ will be called the *support of the group* and the value $\alpha(\theta)$ will be interpreted as the multiplicity of animals of the kind θ in the group ρ . We say that a group $\rho = (\bar{\rho}, \alpha)$ is a *sum of groups* $\rho_i = (\bar{\rho}_i, \alpha_i), i = 1, 2, \dots, k$, if $\bar{\rho}_i \subseteq \bar{\rho}, i = 1, 2, \dots, k$, and

$$\alpha(\theta) = \sum_{i=1,2,\dots,k;\theta \in \bar{\rho}_i} \alpha_i(\theta), \quad \theta \in \bar{\rho}. \quad (5.6)$$

A *gang of animals in Λ* is an non-empty group of animals $\rho = (\bar{\rho}, \alpha) \in D(\Lambda)$ such that for any two animals $\theta, \theta' \in \bar{\rho}$ there is a sequence $\theta = \theta_1, \theta_2, \dots, \theta_n = \theta'$ of animals in $\bar{\rho}$ such that the animals θ_i and θ_{i+1} are incompatible for all $i = 1, 2, \dots, n-1$, i.e. $\bar{\rho}$ is a connected subset of the graph Θ . The set of all gangs in Λ will be denoted by $G(\Lambda)$. The subsets $\bar{\rho}$, which are connected subsets of the graph Θ , will be called the *supports of gangs*. The set of all supports of gangs $\bar{\rho} \subseteq \Lambda$ is denoted by $\bar{G}(\Lambda)$.

Proposition 5.3. *Let the conditions of Proposition 5.2 be fulfilled and a finite set $\Lambda \subseteq \Theta$ be fixed. Consider a polydisk $\mathcal{W}_0(\Lambda) = \{w = (w(\theta), \theta \in \Lambda) : |w(\theta)| \leq w_0(\theta), \theta \in \Lambda\} \subset \mathbb{C}^\Lambda$ and the set $\mathcal{W}_0^{in}(\Lambda)$ of all inner points of the polydisk $\mathcal{W}_0(\Lambda)$. The partition function $Z_w(\Lambda)$ will be treated as a function of $w \in \mathbb{C}^\Lambda$. For any $w \in \mathcal{W}_0^{in}(\Lambda)$ a convergent expansion*

$$\ln Z_w(\Lambda) = \sum_{\rho \in G(\Lambda)} q_w(\rho) = \sum_{\rho \in G(\Lambda)} r(\rho) \prod_{\theta \in \bar{\rho}} w(\theta)^{\alpha(\theta)} \quad (5.7)$$

holds. The coefficients $r(\rho)$ are real numbers depending only on the restriction of the graph structure on Θ to $\bar{\rho}$. For any gang $\rho = (\bar{\rho}, \alpha)$,

$$|q_w(\rho)| = |r(\rho) \prod_{\theta \in \bar{\rho}} w(\theta)^{\alpha(\theta)}| \leq \left(\sum_{\theta \in \bar{\rho}} w_0(\theta) b(\theta) \right) \left(\prod_{\theta \in \bar{\rho}} \left(\frac{|w(\theta)|}{w_0(\theta)} \right)^{\alpha(\theta)} \right). \quad (5.8)$$

In the applications of the animal model considered below the animals will be contours Γ or some combinations of contours and their weights will be $e^{-\beta|\Gamma|}$ or something like it. Since we are interested in the case of large β , the following strong hypothesis is painless for us:

$$|w(\theta)| \leq \frac{1}{2} w_0(\theta), \quad \theta \in \Theta. \quad (5.9)$$

Then we have the following simplification of the cluster expansion (5.7).

Corollary 5.4. *Let*

$$\bar{q}_w(\bar{\rho}) = \sum_{\rho = (\bar{\rho}', \alpha) \in G(\Lambda); \bar{\rho}' = \bar{\rho}} q_w(\rho). \quad (5.10)$$

Then for any finite set $\Lambda \subseteq \Theta$,

$$\ln Z_w(\Lambda) = \sum_{\bar{\rho} \in \bar{G}(\Lambda)} \bar{q}_w(\bar{\rho}). \quad (5.11)$$

If the condition (5.9) is fulfilled, then

$$|\bar{q}_w(\bar{\rho})| \leq 2^{|\bar{\rho}|} \left(\sum_{\theta \in \bar{\rho}} w_0(\theta) b(\theta) \right) \prod_{\theta \in \bar{\rho}} \left(\frac{|w(\theta)|}{w_0(\theta)} \right), \quad \bar{\rho} \in \bar{G}(\Lambda). \quad (5.12)$$

Proof. The expansion (5.11) follows immediately from the expansion (5.7). We find using the condition (5.9) that for any $\bar{\rho} \in \bar{G}(\Lambda)$,

$$\begin{aligned} \sum_{\rho=(\bar{\rho}', \alpha) \in G(\Lambda); \bar{\rho}'=\bar{\rho}} \left(\prod_{\theta \in \bar{\rho}} \left(\frac{|w(\theta)|}{w_0(\theta)} \right)^{\alpha(\theta)} \right) &= \prod_{\theta \in \bar{\rho}} \left(\frac{|w(\theta)|}{w_0(\theta)} \right) \left(\sum_{a=0}^{\infty} \left(\frac{|w(\theta)|}{w_0(\theta)} \right)^a \right) \\ &= \prod_{\theta \in \bar{\rho}} \left(\frac{|w(\theta)|}{w_0(\theta)} \right) \left(\sum_{a=0}^{\infty} \left(\frac{1}{2} \right)^a \right) = \prod_{\theta \in \bar{\rho}} \left(2 \left(\frac{|w(\theta)|}{w_0(\theta)} \right) \right). \end{aligned} \quad (5.13)$$

The desired estimate (5.12) follows now from the relations (5.8) and (5.13). \square

6. A Rough Estimate on the Potential $U_{T,A}^{\beta,+}$

In this section we will obtain a rough estimate on the main quantity

$$Q_{T,D}^{\beta,+N} = - \ln \frac{\tilde{Z}_T(D, W_N)}{\tilde{Z}_T(\emptyset, W_N)}, \quad (6.1)$$

introduced in (4.10). A much more precise estimate will be obtained in the next section, at the price of restricting the range of the sets D allowed (and with much more labor). Here we will treat the case of general finite sets $D \subset T$.

We first rewrite $Q_{T,D}^{\beta,+N}$ in the following way:

$$Q_{T,D}^{\beta,+N} = - \ln \left(\sum_{\pi \in K_T(D, W_N)} E_N(\pi) \right), \quad (6.2)$$

where

$$E_N(\pi) = \frac{\hat{Z}_T(\pi, W_N)}{\hat{Z}_T(\emptyset, W_N)}, \quad (6.3)$$

and

$$\hat{Z}_T(\pi_0, W_N) = \sum_{\pi \in \mathcal{R}(\pi_0, W_N)} \exp\{-2\beta|\pi|\}. \quad (6.4)$$

We use the notation $\mathcal{R}(\pi_0)$ (correspondingly $\mathcal{R}(\pi_0, W_N)$) for the collection of all admissible systems of contours (correspondingly in W_N), containing π_0 , such that all contours

except these in π_0 do not intersect the axis $T = \mathbb{Z}^1$. Note, that $\tilde{Z}_T(D = \emptyset, W_N) = \hat{Z}_T(\pi_0 = \emptyset, W_N)$.

We will use the following representation for this ratio of the partition functions:

$$\frac{\hat{Z}_T(\pi, W_N)}{\hat{Z}_T(\emptyset, W_N)} = \exp \left\{ -2\beta|\pi| - \sum_{M: M \subset W_N, M \cap \Delta(\pi) \neq \emptyset, M \cap T = \emptyset} \Phi(M) \right\}, \quad (6.5)$$

see (3.4). The function Φ is defined for all finite subsets $M \subset \mathbb{Z}^2$, vanishes for all subsets which are disconnected, and satisfies the following estimate:

for some $\beta_0 < \infty$ and all $\beta \geq \beta_0$ one has

$$|\Phi(M)| \leq \exp\{-2(\beta - \beta_0)d(M)\}, \quad (6.6)$$

with $d(M)$ denoting the minimal cardinality of connected sets of bonds belonging to \overline{M} and containing all boundary bonds of the set M .

Also, the function Φ is invariant with respect to shifts:

$$\Phi(M) = \Phi(M + t), \quad t \in \mathbb{Z}^2. \quad (6.7)$$

Such a representation follows directly from the cluster expansion of the previous section for the partition functions $\hat{Z}_T(\pi_0, W_N)$ and $\hat{Z}_T(\emptyset, W_N)$. The animals here are the contours. Our choice of the weights w_0, w and the mights b is the following: $w_0(\Gamma) = \exp\{-\beta_0|\Gamma|\}$, $w(\Gamma) = \exp\{-\beta|\Gamma|\}$ with $\beta \geq 2\beta_0$, $b(\Gamma) = \exp\{|\Gamma|\}$. The quantity $\Phi(M)$ is given by the sum over all herds of contours, “covering” the subset M :

$$\Phi(M) = \sum_{\rho: \cup_{\Gamma \in \rho} \text{Int } \Gamma = M} q(\rho).$$

A connected set M such that $M \cap \Delta(\pi) \neq \emptyset$ will be called a *blob* (on π). For future use we fix for each M a connected set $\delta M \subset \overline{M}$ of $d(M)$ bonds containing all boundary bonds of it. We will call it a *c-boundary* of M .

It is easy to see from the representation (6.5), that the quantities $E_N(\pi)$ approach their limits $E(\pi)$, while $Q_{T,D}^{\beta,+N}$ go to a limit, $Q_{T,D}^{\beta,+}$, for every D , as $N \rightarrow \infty$ (though not uniformly in D). The study of the quantities $E_N(\pi)$ and $E(\pi)$, $Q_{T,D}^{\beta,+N}$ and $Q_{T,D}^{\beta,+}$ is done in the identical manner, so below we will treat only the latter cases of the infinite volume quantities (to save on notation). Their finite volume analogs will be needed only for Proposition 8.2 concerning the comparison of the interactions U_A^N and U_A .

We start with the following simple estimate.

Lemma 6.1. *If β is large enough, then there exists a value $C = C(\beta) \rightarrow 0$ as $\beta \rightarrow \infty$, such that for all π ,*

$$(-2\beta - C)|\pi| \leq \ln E(\pi) \leq (-2\beta + C)|\pi|. \quad (6.8)$$

Proof. Note that every blob M is defined by its c-boundary δM . So to sum over all possible blob assignments – which is what we have to do according to (6.5) – is the same as to sum over all possible c-boundaries, intersecting π . This summation is a standard combinatorics, which is done using the estimate (6.6) and the remark that for every blob M one has $d(M) \geq 4$. \square

Now we can estimate the inner sum in (6.2).

Lemma 6.2. *For some value $C = C(\beta)$, uniformly bounded as $\beta \rightarrow \infty$,*

$$\exp \{(-2\beta - C)(4|D|)\} \leq \sum_{\pi \in K_T(D, W_N)} E(\pi) \leq \exp \{(-2\beta + C)(2|D|)\}. \quad (6.9)$$

Proof. Since all the weights $E(\pi)$ are positive, the lower estimate follows by taking the shortest system π , isolating D . Such a system can have at most $4|D|$ bonds. It is easy to see that the number of systems of contours $\pi \in K_T(D, W_N)$ of the total length l does not exceed 3^l . Since for every π we have $|\pi| \geq 2|D|$, the result follows from (6.8). \square

Proof of Proposition 2.2. Since the number of summands in the definition (2.14) is $2^{|A|}$, the relation (2.15) follows immediately from (6.9). \square

7. The Estimate of the Potential $U_{T,A}^{\beta,+}$ for Punctured Sets A

In this section we will establish the asymptotic splitting property of the main quantity

$$Q_{T,D}^{\beta,+N} = -\ln \frac{\tilde{Z}_T(D, W_N)}{\tilde{Z}_T(\emptyset, W_N)},$$

introduced in (4.10), in the situation when the finite set D is *essentially disconnected*. By this we mean the following: Let $D_i \subset D, i \in \mathcal{I}$ be a fixed partition of D into disjoint subsets, such that each D_i is a union of connected components of D . Our goal is to show that under the condition that the elements of the partition $D_i \subset D, i \in \mathcal{I}$ are sufficiently separated, the main contribution to (6.1) comes from the terms corresponding to subsets D_i :

$$Q_{T,D}^{\beta,+N} = \sum_i Q_{T,D_i}^{\beta,+N} + \text{higher order terms} . \quad (7.1)$$

The precise meaning of (7.1) is given by Proposition 7.4 below. Before explaining the notion of essential disconnectedness, we will introduce the notion of $(1 - \varepsilon)$ -connected sets, which is crucial for the present section.

7.1. $(1 - \varepsilon)$ -connected sets. A finite set $A \subset \mathbb{Z}^1$ will be called *connected*, iff it is a segment, $A = [a, b] = \{n \in \mathbb{Z}^1, a \leq n \leq b\}, a, b \in \mathbb{Z}^1$. We are going to define $(1 - \varepsilon)$ -connected sets, $1 > \varepsilon \geq 0$. (The connected sets would then coincide with 1-connected ones.) To do this, we consider all segments $[a, b]$ with $a, b \in A$, inside which the set A has density above $1 - \varepsilon$, which means that

$$\frac{|[a, b] \cap A|}{|[a, b]|} \geq 1 - \varepsilon.$$

(Here $|B|$ is the number of elements in the subset $B \subset \mathbb{Z}^1$; in particular, $|[a, b]| = b - a + 1$.) Consider the union of all such segments $[a, b]$ of A -density above $1 - \varepsilon$. This is a finite subset of \mathbb{Z}^1 , and as such is a disjoint union of segments I_1, I_2, \dots, I_k , with $\text{dist}(I_i, I_j) \geq 2$. If it consists of just one segment, then A will be called $(1 - \varepsilon)$ -connected. Otherwise it will be called $(1 - \varepsilon)$ -disconnected, and the intersections $A_i = A \cap I_i$ will be called $(1 - \varepsilon)$ -connected components of A .

For a finite set $A \subset \mathbb{Z}^1$ we define the segment $[l_A, r_A] \subset \mathbb{Z}^1$, $l_A \leq r_A \in \mathbb{Z}^1$, as the shortest one containing the set A . If the set A is $(1 - \varepsilon)$ -disconnected, then the complement $[l_A, r_A] \setminus (I_1 \cup I_2 \cup \dots \cup I_k)$ is nonempty, and consists of segments J_1, \dots, J_{k-1} . These $(k - 1)$ segments would be called *lacunas of A* . The quantity

$$l(A) = \sum_{i=1}^{k-1} |J_i| \quad (7.2)$$

is then the total number of points in the lacunas. We introduce also the density $\rho(A)$ of a finite set A as

$$\rho(A) = \frac{|A|}{\text{diam}(A) + 1} \equiv \frac{|A|}{|[l_A, r_A]|}. \quad (7.3)$$

The next three statements contain the main properties of $(1 - \varepsilon)$ -connected sets.

7.1.1. Properties of $(1 - \varepsilon)$ -connected sets.

Lemma 7.1. *Each $(1 - \varepsilon)$ -connected set A (and, in particular, each $(1 - \varepsilon)$ -connected component of any set) has density $\rho(A) \geq 1 - 2\varepsilon$.*

Proof. To show this suppose that the set A is $(1 - \varepsilon)$ -connected, and let $[a_1, b_1], [a_2, b_2], \dots, [a_i, b_i] \in A$ be the collection of all segments of A -density above $1 - \varepsilon$, which are ordered in the lexicographic order. This order will be denoted by \prec . The segments $[a_1, b_1], [a_2, b_2], \dots$ form a covering of A . We want to construct a subcovering of that covering, which is “minimal”. We define the segments of this minimal covering inductively. Denoting by l_J (r_J) the left (right) endpoint of the segment J , define the segment J_1 as the (\prec)-last one among these segments in the collection, for which $l_J = a_1$. Suppose that the segments $J_k, k = 1, \dots, n - 1$ are already defined, and their union does not yet contain A . We define J_n to be the last segment J among these from our collection, which have the following two properties:

- i) The intersections $J \cap (\cup_{k=1}^{n-1} J_k) \neq \emptyset$, $J \cap (A \setminus \cup_{k=1}^{n-1} J_k) \neq \emptyset$. (Because A is $(1 - \varepsilon)$ -connected, the set of such J -s has to be nonempty.)
- ii) $r_{J_n} \geq r_J$ for all J satisfying i)

This process clearly terminates, and let N be the number of segments in the minimal covering thus constructed.

We claim now that $l_{J_n} > r_{J_{n-2}}$ for all $n \geq 3$. Indeed, otherwise the segment J_n would have been added to the minimal covering at the previous step. So every point of A belongs to at most two consecutive segments from our minimal covering. Define the segments K_i to be $J_i \cap J_{i+1}, i = 1, \dots, N - 1, K_0 = K_N = \emptyset, L_i = J_i \setminus \cup_{j \neq i} J_j, i = 1, \dots, N$. We have now that for every $i = 1, \dots, N$,

$$\begin{aligned} |A \cap K_{i-1}| + |A \cap L_i| + |A \cap K_i| &\equiv |A \cap J_i| \geq \\ &\geq (1 - \varepsilon) |J_i| \equiv (1 - \varepsilon)(|K_{i-1}| + |L_i| + |K_i|), \end{aligned}$$

since all the segments J_i have A -density above $1 - \varepsilon$. Adding all these N inequalities results in the following one:

$$\begin{aligned}
 & |A \cap L_1| + |A \cap K_1| + |A \cap L_2| + |A \cap K_2| + \cdots + |A \cap K_{N-1}| + |A \cap L_N| \\
 & \geq (1 - \varepsilon)(|L_1| + |K_1| + |L_2| + |K_2| + \cdots + |K_{N-1}| + |L_N|) \\
 & \quad + (1 - \varepsilon)(|K_1| + |K_2| + \cdots + |K_{N-1}|) - (|A \cap K_1| + |A \cap K_2| + \cdots + |A \cap K_{N-1}|) \\
 & \geq (1 - \varepsilon)(|L_1| + |K_1| + |L_2| + |K_2| + \cdots + |K_{N-1}| + |L_N|) - \\
 & \quad - \varepsilon(|K_1| + |K_2| + \cdots + |K_{N-1}|) \\
 & \geq (1 - 2\varepsilon)(|L_1| + |K_1| + |L_2| + |K_2| + \cdots + |K_{N-1}| + |L_N|),
 \end{aligned}$$

which proves our statement. \square

Lemma 7.2. *Suppose the set A is $(1 - \varepsilon)$ -disconnected, and the subset $B \subset A$ has nonempty intersections with at least two different $(1 - \varepsilon)$ -connected components of A . Then B is also $(1 - \varepsilon)$ -disconnected.*

Proof. The proof follows immediately from the fact that, by definition, the union of two intersecting $(1 - \varepsilon)$ -connected sets is again $(1 - \varepsilon)$ -connected. \square

As we know from Lemma 7.1, every finite set B with density $\rho(B) < (1 - 2\varepsilon)$ is $(1 - \varepsilon)$ -disconnected, and so has lacunas. We will show that if the density of B is even smaller, then the total length of lacunas of B is comparable to its size.

Lemma 7.3. *Suppose ε is small enough and the finite set B has density $\rho(B) \leq (1 - 3\varepsilon)$. Then*

$$l(B) \geq \frac{\varepsilon}{2} \text{diam}(B).$$

Proof. Let $B = B_1 \cup \cdots \cup B_k$ be the decomposition of B into its $(1 - \varepsilon)$ -connected components. Since each B_i is $(1 - \varepsilon)$ -connected,

$$\rho(B_i) = \frac{|B_i|}{\text{diam}(B_i) + 1} \geq (1 - 2\varepsilon).$$

On the other hand,

$$\rho(B) = \frac{\sum_{i=1}^k |B_i|}{\sum_{i=1}^k (\text{diam}(B_i) + 1) + l(B)}.$$

Hence

$$\frac{l(B)}{\sum_{i=1}^k |B_i|} \geq \frac{1}{1 - 3\varepsilon} - \frac{1}{1 - 2\varepsilon} > \varepsilon,$$

provided ε is small enough. Therefore

$$l(B) > \varepsilon \sum_{i=1}^k |B_i| \geq \varepsilon(1 - 2\varepsilon) \sum_{i=1}^k (\text{diam}(B_i) + 1),$$

hence

$$\begin{aligned}
 & l(B) [1 + \varepsilon(1 - 2\varepsilon)] > \\
 & > \varepsilon(1 - 2\varepsilon) \left[\sum_{i=1}^k (\text{diam}(B_i) + 1) + l(B) \right] \equiv \varepsilon(1 - 2\varepsilon) [\text{diam}(B) + 1],
 \end{aligned}$$

and the statement follows. \square

We now are in the position to formulate the condition of sufficient separation of the subsets D_i .

- *Condition of sufficient separation.* We suppose that the partition $D_i \subset D, i \in \{1, \dots, n\} \equiv \mathcal{I}$ is obtained in the following way: for some finite set \tilde{D} and its $(1 - \varepsilon)$ -connected components $\tilde{D}_i \subset \tilde{D}, i \in \mathcal{I}$ we have $D \subset \tilde{D}$ and $D_i = \tilde{D}_i \cap D$ for all $i \in \mathcal{I}$.

In particular, the partition of D into $(1 - \varepsilon)$ -connected components will go. To save on notation we will present below the proofs only for this specific case; the generalization to the general case will be obvious. For a finite set D endowed with its partition into sufficiently separated subsets $D_i, i \in \mathcal{I}$ we define, similarly to (7.2),

$$l(D) = l(D, \{D_i, i \in \mathcal{I}\}) = \sum_{i=1}^{n-1} (l_{D_{i+1}} - r_{D_i} - 1). \quad (7.4)$$

(We suppose here and in the following that the enumeration is such that $l_{D_1} < l_{D_2} < \dots < l_{D_n}$.)

Proposition 7.4. *Suppose that the finite set $D \subset T \cap W_N$ together with its partition into sufficiently separated subsets $D_i, i \in \mathcal{I}$ is fixed. Then if β is large enough, the following expansion holds:*

$$Q_{T,D}^{\beta,+N} = \sum_i Q_{T,D_i}^{\beta,+N} + \sum_{i < j \in \mathcal{I}} G_\beta(D_{[i,j]}), \quad (7.5)$$

where

$$D_{[i,j]} = \bigcup_{k \in [i,j]} D_k,$$

while the function $G_\beta(D_{[i,j]})$ depends only on the sets $D_k, k \in [i, j]$, and does not depend on the set D otherwise. We have also the following estimate:

$$|G_\beta(D_{[i,j]})| \leq \exp \left\{ -\beta' l(D_{[i,j]}, \{D_k, k \in [i, j]\}) \right\}, \quad (7.6)$$

with $\beta' \rightarrow \infty$ as $\beta \rightarrow \infty$.

The expansion (7.5) is somewhat similar to the standard cluster expansion of the logarithm of the partition function. The difference here lies in the fact that the usual low temperature expansion is made around a single ground state configuration, which has no contours at all. So in the usual case the leading term of the partition function corresponds to the vacuum and is 1, while here the situation is more complicated and is different from the standard one.

The next three subsections contain the proof of the above proposition.

7.2. Dressed system representation. As we have seen above, one of the main object we have to study is the following ratio of the partition functions:

$$E(\pi) = \frac{\widehat{Z}_T(\pi, W_N)}{\widehat{Z}_T(\emptyset, W_N)}, \quad (7.7)$$

while

$$Q_{T,D}^{\beta,+;N} = -\ln \left(\sum_{\pi \in K_T(D,W_N)} E(\pi) \right).$$

Recall that (see (6.4))

$$\widehat{Z}_T(\pi_0, W_N) = \sum_{\pi \in \mathcal{R}(\pi_0)} \exp\{-2\beta|\pi|\},$$

where $\mathcal{R}(\pi_0)$ is the collection of all admissible systems π of contours containing π_0 , such that all contours in π except those in π_0 do not intersect the axis $T = \mathbb{Z}^1$.

We will use again the representation (6.5), but this time we want to modify the weights Φ , in such a way that the new weights Φ' will be nonpositive. This can be achieved, but for the price that these new weights are not translation invariant anymore, and moreover, depend not only on M , but also on the set $\overline{M} \cap \pi$. This is the price we can afford. The construction is the following:

Let b be an arbitrary bond of \mathbb{Z}^{2*} , and $\tau(\beta) > 0$ be the sum

$$\tau(\beta) = \sum_{M \subset \mathbb{Z}^2, b \in \overline{M}} \exp\{-2(\beta - \beta_0)d(M)\}.$$

For every bond $b \in \mathbb{Z}^{2*} \setminus \mathbb{Z}^{1*}$ define

$$\tau(\beta, b) = \sum_{M \subset \mathbb{Z}^2, b \in \overline{M}, M \cap T \neq \emptyset} \exp\{-2(\beta - \beta_0)d(M)\}.$$

For the set M we put

$$\Phi'(M, \pi) = \begin{cases} \Phi(M) - |\overline{M} \cap \pi| \exp\{-2(\beta - \beta_0)d(M)\} & \text{for } |M| > 1, \\ \Phi(M) - |\overline{M} \cap \pi| \exp\{-2(\beta - \beta_0)d(M)\} - \\ - \sum_{b \in \overline{M} \cap \pi} n(b) \tau(\beta, b) & \text{for } |M| = 1, \end{cases} \quad (7.8)$$

where $n(b) = \frac{1}{\text{number of sites of } (\mathbb{Z}^2)^+, \text{ adjacent to } b}$. The idea behind the last definition is simple: we add a small fraction – namely, $\tau(\beta)$ – to the contribution of every bond b of the family π into the total weight $2\beta|\pi|$, and distribute the negative of it over all M 's, adjacent to b , according to (7.8). Each M gets as many contributions as there are bonds in π to which it is adjacent, while even one contribution is enough to make it negative. The only exceptions are the one-point M 's; we put all the addition which is not claimed by other M 's to these; this is the source of the non-translation invariance. (Note, however, that the translation invariance with respect to the horizontal shifts is retained.) The analog of (6.6) holds evidently.

We therefore have the following representation for $E(\pi)$:

$$E(\pi) = \exp \left\{ -(2\beta + \tau(\beta))|\pi| - \sum_{M: M \cap \Delta(\pi) \neq \emptyset, M \cap T = \emptyset} \Phi'(M, \pi) \right\}. \quad (7.9)$$

Next come the usual trick of the theory of cluster expansions, which starts by defining the function

$$\Psi(M, \pi) = e^{-\Phi'(M, \pi)} - 1, \quad (7.10)$$

which in our case satisfies

$$0 \leq \Psi(M, \pi) \leq \exp\{-2(\beta - \beta_0)d(M)\}. \quad (7.11)$$

Next, by a slight abuse of notation, we define a *dressed system* $\hat{\pi}$ (or a *system* π with a *dressing*) to be the following object: $\hat{\pi} = \{\pi, M_1, \dots, M_k\}$ consists of the system π itself plus a finite collection of blobs sitting on it, i.e. a finite collection of distinct finite connected sets $M_1, \dots, M_k \subset \mathbb{Z}^2, M_i \cap T = \emptyset$, such that $m_i \equiv M_i \cap \Delta(\pi) \neq \emptyset, i = 1, \dots, k; k = 0, 1, \dots$. A full notation for a dressed system should include also the sets m_i , i.e. it is $\hat{\pi} = \{\pi, M_1, \dots, M_k; m_1, \dots, m_k\}$. The reason for this inclusion lies in the fact that the natural weight of the dressed systems we are going to define next, depends on these intersections. In particular, the union, $\hat{\pi}' \cup \hat{\pi}'' = \{\pi' \cup \pi'', M'_1, \dots, M'_{k'}, M''_1, \dots, M''_{k''}; m'_1, \dots, m'_{k'}, m''_1, \dots, m''_{k''}\}$ of two systems $\hat{\pi}' = \{\pi', M'_1, \dots, M'_{k'}; m'_1, \dots, m'_{k'}\}$ and $\hat{\pi}'' = \{\pi'', M''_1, \dots, M''_{k''}; m''_1, \dots, m''_{k''}\}$ would be a dressed system only if, first, the systems π' and π'' are compatible, and, second, all the intersections $M'_i \cap \Delta(\pi'')$ and $M''_j \cap \Delta(\pi')$ are empty.

The expression (7.9) may be rewritten now in the form

$$\begin{aligned} E(\pi) &= \exp\{-(2\beta + \tau(\beta))|\pi|\} \prod_{M: M \cap \Delta(\pi) \neq \emptyset, M \cap T = \emptyset} (\Psi(M, \pi) + 1) \\ &= \sum_{k=1}^{\infty} \exp\{-(2\beta + \tau(\beta))|\pi|\} \prod_{i=1}^k \Psi(M_i, \pi) \equiv \sum_{\hat{\pi}} E(\hat{\pi}), \end{aligned} \quad (7.12)$$

where the weight $E(\hat{\pi})$ is, of course, nothing else but

$$E(\hat{\pi}) = \exp\{-(2\beta + \tau(\beta))|\pi|\} \prod_{i=1}^k \Psi(M_i, \pi). \quad (7.13)$$

The advantage of the positivity in (7.11) is that it allows us to interpret the last expression as the statistical weight and introduce the corresponding probability distributions on various ensembles of dressed systems of contours.

7.3. Structures. Fix now a finite set $D \subseteq T \cap W_N$. Let $D_i \subset D, i \in \mathcal{I}$ be a fixed partition of D into sufficiently separated components. It is convenient for us to suppose that the index set \mathcal{I} is a segment of integers, $\mathcal{I} = [1, n] \subset \mathbb{N}$. We denote by $G(\mathcal{I})$ the set of all possible graph structures on the set of vertices \mathcal{I} without loops and multiple edges; in other words, $G(\mathcal{I})$ is the set of all subsets of pairs of distinct elements of \mathcal{I} .

Let $\pi \in K_T(D, W_N)$ be a system of compatible contours, isolating D , and $\hat{\pi} = \{\pi, M_1, \dots, M_k\}$ be some dressing of it. (It might be an empty dressing, that is, it might contain no M 's at all.) Define a graph $g(\hat{\pi}) \in G(\mathcal{I})$ in the following manner: a bond $\{i, j\}, i, j \in \mathcal{I}$ belongs to $g(\hat{\pi})$ provided that either

- i) there is a contour Γ in π such that both subsets D_i, D_j belong to $\text{Int}(\Gamma)$, or
- ii) there are two contours Γ', Γ'' in π and an element M of the dressing $\hat{\pi}$, such that D_i belongs to $\text{Int}(\Gamma')$, D_j belongs to $\text{Int}(\Gamma'')$, while $\Delta(\Gamma') \cap M \neq \emptyset \neq \Delta(\Gamma'') \cap M$.

We call two dressed systems π and π' of compatible contours, isolating the set D , *equivalent*, iff $g(\hat{\pi}) = g(\hat{\pi}') = g \in G(\mathcal{I})$. We denote by $\mathbb{L} \equiv \mathbb{L}(D, \mathcal{I}, \{D_i, i \in \mathcal{I}\}, g)$ such a class of equivalence, and will call it a *structure (on D)*. If a system $\hat{\pi}$ belongs to the structure \mathbb{L} , then we call $\mathbb{L} \equiv \mathbb{L}(\hat{\pi})$ the *structure of the dressed system* $\hat{\pi}$. In

the case when $|\mathcal{I}| = 1$ and the partition $\{D_i, i \in \mathcal{I}\}$ containing just one element, the corresponding structure is unique and contains all systems of contours, isolating D . Such a structure will be called sometime *an elementary structure*, and will be denoted simply by $\mathbb{L}(D)$.

We would like to compare different D 's. So we equip every D with some partition $\{D_i\}$ into sufficiently separated components, and we define the set $\mathcal{L}_T(W_N)$ to be the union of all $\mathbb{L}(D, \mathcal{I}_D, \{D_i, i \in \mathcal{I}_D\}, g)$ over all various $D \in T$, all $g \in G(\mathcal{I}_D)$.

For any $\beta > 0$, any finite $D \subset T$, endowed with its partition $\{D_i\}$, and any structure $\mathbb{L} = \mathbb{L}(D, \{D_i\}, g)$ on it we let

$$S_\beta(\mathbb{L}) = \sum_{\hat{\pi} \in \mathbb{L}} E(\hat{\pi}). \quad (7.14)$$

Let now the system $\pi \in K_T(D, W_N)$ of compatible essential contours isolate D , and $\hat{\pi}$ is its dressing. Clearly, if a dressed system $\hat{\pi}'$ has the structure $\mathbb{L}(\hat{\pi})$, then π' also belongs to $K_T(D, W_N)$. Let $\mathcal{L}_T(D, \{D_i\}, W_N)$ be the set of all structures \mathbb{L} on D ;

$$\mathcal{L}_T(D, \{D_i\}, W_N) = \bigcup_{g \in G(\mathcal{I})} \mathbb{L}(D, \{D_i\}, g).$$

Then for any $D \subseteq T \cap W_N$,

$$\sum_{\pi \in K_T(D, W_N)} \frac{\widehat{Z}_T(\pi, W_N)}{\widehat{Z}_T(\emptyset, W_N)} = \sum_{\mathbb{L} \in \mathcal{L}_T(D, \{D_i\}, W_N)} S_\beta(\mathbb{L}) \equiv \sum_{g \in G(\mathcal{I})} S_\beta(\mathbb{L}(D, \{D_i\}, g)). \quad (7.15)$$

We now rewrite (7.1) and reformulate our goal. We want to show that under the condition that the elements of the partition $D_i \subset D, i \in \mathcal{I}$ are sufficiently separated, the main contribution to (7.15) comes from the term corresponding to just one graph $g_\emptyset \in G(\mathcal{I})$, where by g_\emptyset we denote the graph with \mathcal{I} as its set of vertices, which has no bonds. Moreover, under the same condition the term in question equals in leading order the product of the contributions of the vertices of g_\emptyset , which means that

$$\sum_{g \in G(\mathcal{I})} S_\beta(\mathbb{L}(D, \mathcal{I}, \{D_i\}, g)) = \prod_{j \in \mathcal{I}} S_\beta(\mathbb{L}(D_j)) + \text{higher order terms} . \quad (7.16)$$

According to our conventions the full notation for the structure $\mathbb{L}(D_j)$ should be $\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset)$; it corresponds to the only structure on D_j , with the partition $\{D_j\}$ containing the set D_j itself as its only element; the graph $(g_j)_\emptyset$ is the only element of the set $G(\{j\})$. The rest of this subsection and the next one is devoted to the proof of (7.16).

We say that a pair of systems of compatible contours $\pi, \pi' \in H(W_N)$ is *mutually compatible*, if these systems are disjoint and their sum $\pi \cup \pi'$ is again a compatible system of contours. Likewise, we say that a pair of dressed systems $\hat{\pi} = \{\pi, M_1, \dots, M_k\}$ and $\hat{\pi}' = \{\pi', M'_1, \dots, M'_{k'}\}$ is *mutually compatible*, if the pair π, π' is mutually compatible, while all the intersections $\overline{M}_i \cap \pi', \overline{M}'_{i'} \cap \pi$ are empty. (The intersections of M 's and M' 's are allowed.) Let $\mathbb{L}_1, \mathbb{L}_2 \in \mathcal{L}_T(W_N)$ be two different structures of contours, and $R(\mathbb{L}_1, \mathbb{L}_2) \subseteq \mathbb{L}_1 \times \mathbb{L}_2$ be the set of all pairs of dressed systems of contours $(\hat{\pi}_1, \hat{\pi}_2) \in \mathbb{L}_1 \times \mathbb{L}_2$ which are not mutually compatible.

Later we will need the following construction, which associates with every graph $g \in G(\mathcal{I})$ the partition $\Pi(g)$ of \mathcal{I} . It is defined in the following manner: we consider

the set \mathcal{I} to be a subset of the segment $[1, n] \subset \mathbb{R}^1$, and we associate with every bond $b = \{i_1, i_2\}$ of g the segment $u_b = [i_1, i_2] \subset [1, n]$. The union of all these segments is a closed subset of \mathbb{R}^1 , which is a collection of disjoint segments $v_j \subset \mathbb{R}^1$. The intersections $v_j \cap \mathcal{I}$ are by definition all the non-trivial elements of the partition $\Pi(g)$. The trivial elements of it are the singletons, corresponding to single points of \mathcal{I} , which do not belong to the above union of segments. For a given structure $\mathbb{L} = \mathbb{L}(D, \mathcal{I}, \{D_i\}, g)$ we define the partition $\Pi(\mathbb{L})$ to be the partition $\Pi(g)$. In case a dressed system $\hat{\pi}$ has g for its graph $g(\hat{\pi})$, we likewise define $\Pi(\hat{\pi})$ to be the partition $\Pi(g(\hat{\pi}))$. Note that $\Pi(g)$ is a partition of \mathcal{I} into consecutive segments, some of which might be degenerate. Let $\mathcal{J}(g)$ be the index set for these segments.

Let $\Pi \equiv \Pi_{\mathcal{I}} = (\Lambda_1, \Lambda_2, \dots, \Lambda_j, \dots, \Lambda_{|\mathcal{J}|}), j \in \mathcal{J}$ be a partition of the index set \mathcal{I} into consecutive segments. (The finest such partition $(\Lambda_1^*, \Lambda_2^*, \dots, \Lambda_j^*, \dots, \Lambda_n^*)$ of \mathcal{I} into singletons, $\Lambda_j^* = \{j\} \in \mathcal{I}$, will be denoted by $\Pi_{\mathcal{I}}^*$.) We then define the subsets $D(\Lambda_j) = \cup_{i \in \Lambda_j} D_i \subset D$, equipped with partitions into corresponding $D_i, i \in \Lambda_j$; let us also fix graphs $g_j \in G(\Lambda_j), j \in \mathcal{J}$. Consider the product

$$\mathcal{P}(\Pi, \{g_j, j \in \mathcal{J}\}) = \mathbb{L}_1 \times \mathbb{L}_2 \times \dots \times \mathbb{L}_{|\mathcal{J}|},$$

where $\mathbb{L}_j \equiv L(D(\Lambda_j), \Lambda_j, \{D_i, i \in \Lambda_j\}, g_j)$. This product consists of all collections of dressed systems $P = (\hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_{|\mathcal{J}|}), \hat{\pi}_j \in \mathbb{L}_j, j = 1, \dots, |\mathcal{J}|$, which, however, need not be compatible. So we consider the set of all possible unoriented graphs $G(\mathcal{J})$ on vertices \mathcal{J} , and for any graph $h \in G(\mathcal{J})$ we let

$$R(\Pi, \{g_j\}, h) = \bigcap_{\{j_1, j_2\} \in h} \{P \in \mathcal{P}(\Pi, \{g_j, j \in \mathcal{I}\}) : (\hat{\pi}_{j_1}, \hat{\pi}_{j_2}) \in R(\mathbb{L}_{j_1}, \mathbb{L}_{j_2})\}, \quad (7.17)$$

where $j_1, j_2 \in \mathcal{J}$, and the intersection is taken over all bonds of the graph h . In words, $R(\Pi, \{g_j\}, h)$ is the set of all collections P such that all the pairs $(\hat{\pi}_{j_1}, \hat{\pi}_{j_2}), \{j_1, j_2\} \in h$ of dressed system of contours are not mutually compatible. For a system of contours $P \in \mathcal{P}(\Pi, \{g_j, j \in \mathcal{I}\})$ we define the graph $h(P) \in G(\mathcal{J})$ to be the maximal one among these h 's, for which the inclusion $P \in R(\Pi, \{g_j\}, h)$ holds. For any $h \in G(\mathcal{J})$ let $\bar{h} \subseteq \mathcal{J}$ be the set of all vertices of the graph h which are adjacent to at least one bond of h .

Let

$$\bar{R}(\Pi_{\mathcal{I}}, \{g_j\}) = \bigcup_{i, k \in \mathcal{J}} \{P \in \mathcal{P}(\Pi, \{g_j\}) : (\hat{\pi}_i, \hat{\pi}_k) \in R(\mathbb{L}_i, \mathbb{L}_k)\} \quad (7.18)$$

be the set of all collections P such that some pair $(\hat{\pi}_i, \hat{\pi}_k)$ of systems of contours is not compatible. It is clear that for any collection of systems of contours $P = (\hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_{|\mathcal{J}|}) \in \mathcal{P}(\Pi, \{g_j\})$ the system of contours

$$\hat{\pi} = \bigcup_{j=1}^{|\mathcal{J}|} \hat{\pi}_j \quad (7.19)$$

belongs to some structure \mathbb{L} on D if and only if $P \notin \bar{R}(\Pi)$. On the other hand, if $\hat{\pi} \in \mathbb{L}$ and $\Pi(\hat{\pi}) = (\Lambda_1, \Lambda_2, \dots, \Lambda_k)$, then there exists a unique collection of graphs $g_j \in G(\Lambda_j), j = 1, \dots, k$ with $\Pi(g_j) = \Lambda_j$ and a unique collection of k dressed systems $\hat{\pi}_j \in \mathbb{L}(D(\Lambda_j), \Lambda_j, \{D_i, i \in \Lambda_j\}, g_j)$ such that (7.19) holds.

So we have:

$$\begin{aligned}
 & \sum_{g \in G(\mathcal{I})} S_\beta(\mathbb{L}(D, \mathcal{I}, \{D_i\}, g)) = \\
 & = \sum_{k=1}^n \sum_{\Pi=\Lambda_1, \dots, \Lambda_k} \sum_{g_1 \in G(\Lambda_1), \dots, g_k \in G(\Lambda_k): \Pi(g_j)=\Lambda_j} \\
 & \left(\prod_{j=1}^k S_\beta(\mathbb{L}(D(\Lambda_j), \Lambda_j, \{D_i, i \in \Lambda_j\}, g_j)) - \sum_{P \in \overline{\mathbb{R}}(\Pi, \{g_j\})} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j) \right).
 \end{aligned} \tag{7.20}$$

Since always $0 < S_\beta(\mathbb{L}) < \infty$, we can rewrite (7.20) as

$$\ln \left(\sum_{g \in G(\mathcal{I})} S_\beta(\mathbb{L}(D, \mathcal{I}, \{D_i\}, g)) \right) = \sum_{j=1}^n \ln (S_\beta(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset)) + F_\beta(D), \tag{7.21}$$

where

$$F_\beta(D) = \ln Z_\beta(D),$$

while

$$\begin{aligned}
 & Z_\beta(D) = 1 + \\
 & + \left(\sum_{k=1}^{n-1} \sum_{\Pi=\Lambda_1, \dots, \Lambda_k} \sum_{g_1 \in G(\Lambda_1), \dots, g_k \in G(\Lambda_k): \Pi(g_j)=\Lambda_j} \prod_{j=1}^k S_\beta(\mathbb{L}(D(\Lambda_j), \Lambda_j, \{D_i, i \in \Lambda_j\}, g_j)) \right) \times \\
 & \times \left(\prod_{j=1}^n S_\beta(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset)) \right)^{-1} \\
 & - \left(\sum_{k=1}^n \sum_{\Pi=\Lambda_1, \dots, \Lambda_k} \sum_{g_1 \in G(\Lambda_1), \dots, g_k \in G(\Lambda_k): \Pi(g_j)=\Lambda_j} \sum_{P \in \overline{\mathbb{R}}(\Pi, \{g_j\})} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j) \right) \times \\
 & \times \left(\prod_{j=1}^n S_\beta(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset)) \right)^{-1}.
 \end{aligned} \tag{7.22}$$

(The only term from (7.20) which is not present in (7.22) is the leading one, which equals the denominators in (7.22), and its absence is reflected in the fact that the first summation is up to $n - 1$.)

7.4. A polymer representation for $Z_\beta(D)$. We will study now various terms in (7.22), and we will show that many cancellations happen and that the terms are in fact small enough due either to the fact that the contours in the structures $\mathbb{L}(D(\Lambda_j))$ with nontrivial Λ_j 's are much longer than the contours from the structures $\mathbb{L}(D_i)$, or because we have incompatibility condition entering into the last term of (7.22), which again forces the presence of extra long terms.

Our immediate goal is to represent the sum $Z_\beta(D)$ in (7.22) as a partition function of an animal model. The animals turn out to be just segments s of the set \mathcal{I} .

Proposition 7.5. *There exists a function w defined for all finite $(1 - \varepsilon)$ -disconnected subsets of \mathbb{Z}^1 , such that the following holds. Let $\tilde{D} \subset \mathbb{Z}^1$ be a finite subset, which is $(1 - \varepsilon)$ -disconnected, and let \tilde{D}_i be its $(1 - \varepsilon)$ -connected components, $i \in \{1, \dots, n\} \equiv \mathcal{I}$. Let $D \subset \tilde{D}$ be a subset, such that all the components $D_i = D \cap \tilde{D}_i$ are nonempty. Then*

$$Z_\beta(D) = 1 + \sum_{\Pi} \prod_{\Lambda \in \Pi} w(D_\Lambda). \quad (7.23)$$

Here the summation goes over all partitions of the set \mathcal{I} into (disjoint) segments Π except the partition Π^* into singletons,

$$\Pi = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_k, \beta_k : 1 \leq k < n, \alpha_1 = 1 \\ \leq \beta_1 < \alpha_2 \leq \beta_2 < \dots \leq \beta_k = n\},$$

the product is taken over all segments $\Lambda \in \Pi$ of positive length (i.e. all Λ 's which are singletons, are excluded from (7.23)), and

$$D_\Lambda = \cup_{i \in \Lambda} D_i.$$

The function w satisfies the estimate:

$$|w(D_\Lambda)| \leq \exp \left\{ -\beta' l(D_\Lambda, \{D_i, i \in \Lambda\}) \right\}, \quad (7.24)$$

(see (7.4)), where $\beta' = \beta'(\beta) \rightarrow \infty$ as $\beta \rightarrow \infty$.

Once this proposition is proven, Proposition 7.4 follows from the estimates (7.23), (7.24) and the relations (7.20), (7.21) by applying the cluster expansion (5.11) to the logarithm of the partition function $Z_\beta(D)$. The animals here are the sets D_Λ ; they can be identified with the corresponding nontrivial segments $\Lambda \subset \mathcal{I}$. Our choice of the weights w is obvious; the weights $w_0(D_\Lambda)$ are given by the rhs of (7.24), and the mights $b(D_\Lambda)$ can be taken to be equal to $\exp\{|\Lambda|\}$. (Here $|\Lambda|$ is the number of points in the index set Λ .)

As the reader of this hard technical section sees, the generic term in the representation (7.22) corresponds to a pair of graphs. One is a graph $g \in G(\mathcal{I})$ (which is not necessarily connected), while another is a graph $h \in G(\mathcal{J}(g))$, where the set of indices $\mathcal{J}(g)$ enumerates the elements of the partition $\Pi(g)$. The first graph describes the connections between different components D_i , $i \in \mathcal{I}$, which occur due to the corresponding structures \mathbb{L} , while the second graph describes the incompatibility pattern between different \mathbb{L} 's. In the next subsection we will treat the case when the graph g is trivial (\equiv has no bonds), then we will treat the case of trivial h 's, and finally we will treat the general case.

7.4.1. A simple term: \mathcal{R}^1 . As a warm-up, we begin the necessary estimates with the term \mathcal{R}^1 of (7.22), corresponding to the case $k = n$ in the last term of (7.22). In that case the partition Π is the partition into singletons, $\Pi = \Pi_{\mathcal{I}}^*$, all the graphs g_j are trivial, so we will omit sometimes Π 's and/or g_j 's from our notations. So we have to consider the product $\mathcal{P}(\Pi_{\mathcal{I}}^*) = \mathbb{L}_1 \times \mathbb{L}_2 \times \dots \times \mathbb{L}_n$ of elementary structures $\mathbb{L}_j \equiv \mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset)$, consisting of all collections of dressed systems $P = (\hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_n), \hat{\pi}_j \in \mathbb{L}_j, j = 1, \dots, n$. The ratio which we are going to estimate is

$$\mathcal{R}^1(D) = - \frac{\sum_{P \in \overline{\mathbb{R}}(\Pi_{\mathcal{I}}^*)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j=1}^n S_{\beta}(\mathbb{L}_j)}.$$

Proposition 7.6. *In the notation of Proposition 7.5 there exists a function w^1 defined for all finite $(1 - \varepsilon)$ -disconnected subsets of \mathbb{Z}^1 , such that the following holds:*

$$\mathcal{R}^1(D) = \sum_{\Pi} \prod_{\Lambda \in \Pi} w^1(D_{\Lambda}). \tag{7.25}$$

The function w^1 satisfies the estimate:

$$|w^1(D_{\Lambda})| \leq \exp\{-\beta' l(D_{\Lambda})\},$$

where $\beta' = \beta'(\beta) \rightarrow \infty$ as $\beta \rightarrow \infty$.

Proof. We first rewrite the function \mathcal{R}^1 . Using a well-known formula for the union of events via their intersections, we find that

$$\mathcal{R}^1 = - \sum_{P \in \overline{\mathbb{R}}} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j=1}^n S_{\beta}(\mathbb{L}_j)} = \sum_{h \in G(\mathcal{I}), |h| > 0} (-1)^{|h|} \sum_{P \in R(h)} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j=1}^n S_{\beta}(\mathbb{L}_j)} \tag{7.26}$$

(recall (7.17)), where we denote by $|h|$ the number of bonds of the graph h . We now rewrite the inner sums in the rhs of (7.26) by singling out the factors corresponding to sites $j \in \mathcal{I} \setminus \bar{h}$. We find that

$$\sum_{P \in R(h)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j) = \left(\prod_{j \in \mathcal{I} \setminus \bar{h}} S_{\beta}(\mathbb{L}_j) \right) \sum_{P = \{\hat{\pi}_j : j \in \bar{h}\} \in R(\Pi_{\bar{h}}^*, h)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j). \tag{7.27}$$

Here $\Pi_{\bar{h}}^* = \{\Lambda_j, j \in \bar{h}\}$, and we use a slight abuse of notation in the expression $R(\Pi_{\bar{h}}^*, h)$ by treating the graph $h \in G(\mathcal{I})$ as a graph on the vertices \bar{h} , i.e. as an element of $G(\bar{h}) \subset G(\mathcal{I})$. The key observation now is that the last sum factors. Namely, if we introduce the set $\mathcal{C}(h)$ of all maximal connected subgraphs of h , *excluding isolated vertices*, then

$$\sum_{P = \{\hat{\pi}_j : j \in \bar{h}\} \in R(\Pi_{\bar{h}}^*, h)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j) = \prod_{k \in \mathcal{C}(h)} \sum_{P = \{\hat{\pi}_j : j \in \bar{k}\} \in R(\Pi_{\bar{k}}^*, k)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j). \tag{7.28}$$

So if we define the weight $Q_{\beta}^1(k)$ of a connected graph k with vertices among the points of the index set \mathcal{I} by

$$Q_{\beta}^1(k) = (-1)^{|k|} \frac{\sum_{P = \{\hat{\pi}_j : j \in \bar{k}\} \in R(\Pi_{\bar{k}}^*, k)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j \in \bar{k}} S_{\beta}(\mathbb{L}_j)}, \tag{7.29}$$

then it follows from the relations (7.22), (7.26), (7.27) and (7.28) that

$$\mathcal{R}^1 = \sum_{h \in G(\mathcal{I}), |h| > 0} \prod_{k \in \mathcal{C}(h)} Q_\beta^1(k). \quad (7.30)$$

One can estimate from above the weights of each graph k , contributing to (7.30), directly. However, the estimate, in general, would not be better than $\exp\{-c\beta l(k)\}$, where

$$l(k) = \sum_r \text{dist}(D_{i_r}, D_{i_{r+1}}) \quad (7.31)$$

and where the summation goes over all components $D_i \in \bar{k}$, ordered by the natural linear order inherited from \mathbb{Z}^1 . The more optimistic estimate with

$$L(k) = \sum_{\{i,j\} \in k} \text{dist}(D_i, D_j)$$

instead of $l(k)$ does not hold in general. The estimate available is not enough for our purposes, since the number of connected graphs on vertices \bar{k} is of the order $2^{(|\bar{k}|^2)}$, while $l(k)$ can well be of order $|\bar{k}|$. So we have too many animals with a given weight, and the straightforward application of the cluster expansion machinery would fail.

To cope with this problem we will introduce the partition of the set of all graphs $h \in G(\mathcal{I})$ into families, according to what the partition $\Pi(h)$ is. Namely, for every nondegenerate segment $s \subset \mathcal{I}$ denote by $H(s) \subset G(\mathcal{I})$ the subset of all graphs which have the segment s as the only nontrivial element of the partition $\Pi(h)$. Introduce now the weight $Q_\beta^1(s)$ by

$$Q_\beta^1(s) = \sum_{h \in H(s)} \prod_{k \in \mathcal{C}(h)} Q_\beta^1(k). \quad (7.32)$$

That definition clearly allows us to rewrite (7.30) as

$$\mathcal{R}^1 = \sum_{S \in S(\mathcal{I})} \prod_{s \in S} Q_\beta^1(s), \quad (7.33)$$

where the summation goes over all collections $S \in S(\mathcal{I})$ of disjoint segments of \mathcal{I} of positive lengths.

To estimate $Q_\beta^1(s)$, we first rewrite this weight in terms of the families P , contributing to it. Clearly,

$$\begin{aligned} Q_\beta^1(s) &= \sum_{h \in H(s)} (-1)^{|h|} \frac{\sum_{P=\{\hat{\pi}_j: j \in s\} \in R(\Pi_s^*, h)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j \in s} S_\beta(\mathbb{L}_j)} \\ &= \sum_{\substack{P=\{\hat{\pi}_j: j \in s\} \in \bar{R}(\Pi_s^*): \\ g(P) \in H(s)}} \left(\sum_{h \subseteq g(P): h \in H(s)} (-1)^{|h|} \right) \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j \in s} S_\beta(\mathbb{L}_j)}. \end{aligned}$$

We need an estimate on the coefficient $K(P) \equiv \left(\sum_{h \subseteq g(P): h \in H(s)} (-1)^{|h|} \right)$; being large, it can destroy our strategy. Happily, as we will show at the end of this subsection (see Lemma 7.8),

$$|K(P)| = \left| \sum_{h \subseteq g(P): h \in H(s)} (-1)^{|h|} \right| \leq 1, \quad (7.34)$$

so

$$|Q_\beta^1(s)| \leq \sum_{\substack{P = \{\hat{\pi}_j : j \in s\} \in \overline{\mathbb{K}}(\Pi_s^*) \\ g(P) \in H(s)}} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j \in s} S_\beta(\mathbb{L}_j)}. \quad (7.35)$$

The rhs of (7.35) can be interpreted as a probability of an event in a certain ensemble. Namely, introduce the following product distribution on $\mathcal{P}(\Pi_s^*)$:

$$q_s(P) \equiv q_s(\{\hat{\pi}_j, j \in s\}) = \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j \in s} S_\beta(\mathbb{L}_j)}.$$

This is just the ensemble of independent dressed families of contours, each family belonging to the corresponding structure \mathbb{L}_j , and being distributed according to

$$q_j(\hat{\pi}_j) = \frac{E(\hat{\pi}_j)}{S_\beta(\mathbb{L}_j)}. \quad (7.36)$$

Then we can rewrite (7.35) as

$$|Q_\beta^1(s)| \leq q_s(\{P \in \mathcal{P}(\Pi_s^*) : g(P) \in H(s)\}). \quad (7.37)$$

To estimate the last probability, we first define for every graph $g (= g(P))$, contributing to (7.37), a *spanning subgraph* $sp(g) \subseteq g$. The definition is inductive. Suppose the set of all vertices \bar{g} is enumerated in increasing order: $\bar{g} = \{i_1 < i_2 < \dots < i_{|\bar{g}|}\} \subset s$. The first bond $\{c_1, d_1\}$ of $sp(g)$ is the longest one of g among those incident to the vertex i_1 . Suppose inductively that the bonds $\{c_r, d_r\}$ are already constructed, $r = 1, 2, \dots, k$, and $c_1 (= i_1) < c_2 < \dots < c_k, d_1 < d_2 < \dots < d_k$. If $d_k = i_{|\bar{g}|}$, the process terminates. Otherwise the set of bonds $\{c, d\} \in g$, such that $c \leq d_k < d$, is nonempty (since $S(g) = s$), and we take for the bond $\{c_{k+1}, d_{k+1}\}$ the one from this set with the rightmost endpoint d . Let K be the total number of bonds in $sp(g)$. By construction, $c_1 < d_1 < c_3 < d_3 < \dots, c_2 < d_2 < c_4 < d_4 < \dots$, and so every vertex of the graph $sp(g)$ belongs to at most two bonds of it. Hence $K \leq |s|$. Moreover, every vertex of g belongs to at most one “even” bond and to at most one “odd” bond of $sp(g)$. Denote by $sp_o(g) \subset sp(g)$ (resp. $sp_e(g) \subset sp(g)$) the subgraph composed by only “odd” (resp. “even”) bonds, and let K_o (K_e) be their number. Since $\sum_{k=1}^K (d_k - c_k) \geq l(g)$ (see (7.31)), we have that at least one of the estimates holds:

$$\sum_{k=1}^{K_o} (d_{2k-1} - c_{2k-1}) \geq \frac{l(g)}{2}, \text{ or } \sum_{k=1}^{K_e} (d_{2k} - c_{2k}) \geq \frac{l(g)}{2}.$$

That implies the inclusion

$$\begin{aligned} & \{P \in \mathcal{P}(\Pi_s^*) : g(P) \in H(s)\} \\ & \subset \bigcup_{k=1}^{\lfloor \frac{|s|}{2} \rfloor + 1} \bigcup_{\substack{\{c_1, d_1, c_2, d_2, \dots, c_k, d_k\} \\ c_1 < d_1 < c_2 < d_2 < \dots < c_k < d_k}} R(\Pi_s^*, h(c_1, d_1; c_2, d_2; \dots; c_k, d_k)), \end{aligned} \quad (7.38)$$

where the second union is taken over all collections of disjoint segments $[c_1, d_1], [c_2, d_2], \dots, [c_k, d_k]$ in s of total length at least $\frac{l(g)}{2}$, and the graph $h(c_1, d_1; c_2, d_2; \dots; c_k, d_k)$ has k bonds $\{c_1, d_1\}, \{c_2, d_2\}, \dots, \{c_k, d_k\}$. The crucial point for us here is that the number of such collections is bounded from above by only

$$2^{|s|-1} (\equiv \text{the number of subsets of the set of } |s| - 1 \text{ elements}) \quad (7.39)$$

(while the number of all graphs h with vertices in s is of the order of $2^{\text{const } |s|^2}$, which is the reason why we have to pass to segments s from graphs h).

By definition of the measure q_s we have that

$$q_s \left(R \left(\Pi_s^*, h(c_1, d_1; c_2, d_2; \dots; c_k, d_k) \right) \right) = \prod_{i=1}^k q_{c_i, d_i} \left(R(c_i, d_i, h(c_i, d_i)) \right), \quad (7.40)$$

where the measure $q_{c_i, d_i} = q_{c_i} q_{d_i}$ (see (7.36)), and the family $R(c_i, d_i, h(c_i, d_i))$ corresponds to the graph with two vertices and one bond joining them, i.e., in accordance with the definition (7.17), it is the collection of systems $P = (\hat{\pi}_1, \hat{\pi}_2), \hat{\pi}_1 \in \mathbb{L}_{c_i}, \hat{\pi}_2 \in \mathbb{L}_{d_i}$, which are not mutually compatible (i.e. $(\hat{\pi}_1, \hat{\pi}_2) \in R(\mathbb{L}_{c_i}, \mathbb{L}_{d_i})$). To estimate the probability $q_{c,d}(R(c, d, h(c, d)))$ consider the vertical line $l_{cd} = \{(x_{cd}, y)\} \subset \mathbb{R}^2$, with the abscissa x_{cd} positioned halfway between the sets D_c and D_d . Then

$$\sum_{P=\{\hat{\pi}_c, \hat{\pi}_d\} \in R(c, d, h(c, d))} q_c(\hat{\pi}_c) q_d(\hat{\pi}_d) \leq q_c\{\hat{\pi}_c \cap l_{cd} \neq \emptyset\} + q_d\{\hat{\pi}_d \cap l_{cd} \neq \emptyset\}. \quad (7.41)$$

The event $\{\hat{\pi}_c \cap l_{cd} \neq \emptyset\}$ can be written as a sum:

$$\begin{aligned} \{\hat{\pi}_c \cap l_{cd} \neq \emptyset\} &= \{\pi_c \cap l_{cd} \neq \emptyset\} \cup \\ &\cup \left[\bigcup_{b \in \mathbb{Z}^2 \setminus l_{cd}} \{b \in \pi_c, \text{ for a blob } M \text{ of } \hat{\pi}_c \text{ we have } b \in \overline{M}, \overline{M} \cap l_{cd} \neq \emptyset\} \right]. \end{aligned} \quad (7.42)$$

Now,

$$\begin{aligned} q_c\{b \in \pi_c, \text{ for } M \in \hat{\pi}_c, b \in \overline{M}, \overline{M} \cap l_{cd} \neq \emptyset\} \\ = q_c\{b \in \pi_c\} q_c\{\text{for } M \in \hat{\pi}_c, b \in \overline{M}, \overline{M} \cap l_{cd} \neq \emptyset \mid b \in \pi_c\}. \end{aligned} \quad (7.43)$$

Next, we need the following simple

Lemma 7.7. *Consider an elementary structure $\mathbb{L}(D)$, and let $p(\hat{\pi})$ be the natural probability distribution on it:*

$$p(\hat{\pi}) = \frac{E(\hat{\pi})}{S_\beta(\mathbb{L}(D))}.$$

Let M be a fixed blob. Then the probability of the event that the dressed system $\hat{\pi}$ contains this blob M in its dressing can be estimated as follows:

$$p[\hat{\pi} : \hat{\pi} = \{\pi, M, M_1, \dots\}] \leq \exp\{-2(\beta - \beta_0)d(M)\}.$$

Proof. The proof is almost immediate. Indeed,

$$\begin{aligned} p[\hat{\pi} : \hat{\pi} = \{\pi, M, M_1, M_2, \dots\}] &= \frac{\sum_{\hat{\pi}=\{\pi, M, M_1, M_2, \dots\}} E(\hat{\pi})}{S_\beta(\mathbb{L}(D))} \\ &= \frac{\sum_{\hat{\pi}=\{\pi, M_1, M_2, \dots\} : \Delta(\pi) \cap M \neq \emptyset, M_i \neq M, \Psi(M, \pi) E(\hat{\pi})}{S_\beta(\mathbb{L}(D))} \\ &\leq \exp\{-2(\beta - \beta_0)d(M)\} \frac{\sum_{\hat{\pi}=\{\pi, M_1, M_2, \dots\} : \Delta(\pi) \cap M \neq \emptyset, M_i \neq M, E(\hat{\pi})}{S_\beta(\mathbb{L}(D))} \\ &\leq \exp\{-2(\beta - \beta_0)d(M)\}. \quad \square \end{aligned}$$

According to this lemma the last factor in (7.43) can be estimated as follows:

$$q_c\{\text{for } M \in \hat{\pi}_c, b \in \bar{M}, \bar{M} \cap l_{cd} \neq \emptyset | b \in \pi_c\} \leq \exp\{-4(\beta - \beta_0) \text{dist}(b, l_{cd})\}. \quad (7.44)$$

Hence

$$q_c\{\hat{\pi}_c \cap l_{cd} \neq \emptyset\} \leq \sum_{r \in \mathbb{Z}^1 : r \geq 0} \exp\{-4(\beta - \beta_0)r\} q_c\{\pi_c \cap (l_{cd} - r) \neq \emptyset\}.$$

Here $(l_{cd} - r)$ is the line l_{cd} shifted to the left by r units, and of course the probabilities $q_c\{\pi_c \cap (l_{cd} - r) \neq \emptyset\}$ equal to 1 once $r \geq \frac{1}{2} \text{dist}(D_c, D_d)$. For the remaining values of r these probabilities are estimated in (3.6), according to which we have

$$q_c\{\pi_c \cap (l_{cd} - r) \neq \emptyset\} \leq \exp\{-c\beta(\frac{1}{2} \text{dist}(D_c, D_d) - r)\}.$$

Together with (7.44) it shows that

$$\sum_{P=\{\hat{\pi}_c, \hat{\pi}_d\} \in R(c, d, h(c, d))} q_c(\hat{\pi}_c) q_d(\hat{\pi}_d) \leq 2 \exp\{-\frac{c}{4}\beta \text{dist}(D_c, D_d)\}.$$

Combining the last estimate, the estimates (7.41) and (7.39), the relations (7.40) and (7.38) and the estimate (7.37) we get:

$$|Q_\beta^1(s)| \leq 2^{|2s|} \exp\{-\frac{c}{4}\beta l(s)\} \leq \exp\{-c'\beta l(s)\},$$

where $l(s) = \sum_{k \in S} \text{dist}(D_{i_k}, D_{i_{k+1}})$. Together with the formula (7.33) the last estimate shows that the term \mathcal{R}^1 has the form needed for the machinery of the cluster expansion to be applicable. \square

We finish this subsection by proving the estimate (7.34), together with another statement from graph theory.

7.4.2. Two statements about graphs. Let $g \in G(\mathcal{I})$ be a graph on the vertices \mathcal{I} (with no multiple bonds). We will call it a *spanning* graph, if for every $k \in \mathcal{I}, k > 1$ there is at least one bond $\{i, j\}$ of g , such that $i \leq k - 1 < k \leq j$. In other words, the graph g is spanning, if every segment $[k - 1, k]$ is covered by at least one bond of g . An equivalent definition is that the partition $\Pi(g)$ of \mathcal{I} consists of one element, which is the set \mathcal{I} itself. Note that a spanning graph is not necessarily connected.

Let $g \in G(\mathcal{I})$ be a spanning graph. Consider the quantity

$$N(g) = \sum_{h \subset g} (-1)^{|h|},$$

where the summation goes over all subgraphs of g which are themselves spanning, and where $|h|$ is the number of bonds in h . Then we have

Lemma 7.8.

$$N(g) = -1, 0 \text{ or } 1.$$

Proof. The proof goes by induction on the number of points in \mathcal{I} . For $|\mathcal{I}| = 2$ we have $N(g) = -1$ for the only spanning graph $g \in G(\mathcal{I})$. Each bigger set \mathcal{I} is also treated by induction – this time in the number of bonds in g . If $|g| = 1$ then again $N(g) = -1$. Consider the general spanning graph g . We call a bond $\{k, l\}$ a subordinate to a bond $\{i, j\}$, if $i \leq k < l \leq j$. Suppose first that the graph g has a bond $\{i, j\}$ with a subordinate $\{k, l\}$. We claim then that the subsum $\sum_{h \subset g, \{i, j\} \in h} (-1)^{|h|} = 0$. Indeed, the map $h \rightarrow h \Delta \{k, l\}$ is a one-to-one map on the set of all spanning subgraphs of g , containing the bond $\{i, j\}$. Hence to evaluate $N(g)$ we can delete the bond $\{i, j\}$ from g . If the resulting graph is not spanning, then $N(g) = 0$. Otherwise it has less bonds than g , which permits the induction step in our case. Suppose next that no bond has a subordinate. That implies in particular that the graph g has exactly one bond incident to the site n . Let that bond be $\{k, n\}$. Consider the factor graph $f = g / \{k, \dots, n\}$. Note that there is a natural one-to-one correspondence between the bonds of f and the bonds of g excluding one bond $\{k, n\}$, which disappears after the factorization. It is immediate to see that this correspondence gives rise to a one-to-one correspondence between the spanning subgraphs of f and of g , so $N(f) = -N(g)$. Hence the proof is complete, since f has less sites than g . \square

Let A be a finite set, and $A = \cup A_i, i \in \mathcal{I}$ be its partition into disjoint subsets, which are called *connected components* of A . For every subset $C \subset A$ we define connected components C_i of C as those intersections $C_i = C \cap A_i$, which are nonempty. These components are indexed by $i \in \mathcal{I}(C) \subseteq \mathcal{I}$. Denote by $G(C)$ the set of all connected graph structures with some of the sets C_i as their vertices. (For example, a single set C_i is an element of $G(C)$.) For $g \in G(C)$ we denote by $\mathcal{I}(g) \subseteq \mathcal{I}(C)$ the subset of indices corresponding to the vertices of g . Let $G = \cup_C G(C)$. (The element g of G can be thought of as a collection of nonempty subsets $C_i \subseteq A_i, i \in \mathcal{J} \subseteq \mathcal{I}$, together with a structure of a connected graph on the set \mathcal{J} of vertices.)

Suppose the function f is defined on G . Consider the sum

$$S(f) = \sum_{C \subset A} (-1)^{|A \setminus C|} \sum_{g \in G(C)} f(g). \quad (7.45)$$

Note that in general the term $f(g)$ would appear more than once in $S(f)$. Indeed, if for example the set C consists from two connected components, $C = C_{i_1} \cup C_{i_2}$, while the graph $g \in G(C)$ has one vertex C_{i_1} and no bonds, then for any $e \in C_{i_2}$ we also have $g \in G(C \setminus e)$. Hence, some cancellations in (7.45) should be expected. Let the (integer) coefficients $c(g), g \in G$ be the result of these cancellations; in other words, define them to be the coefficients of the formal expansion of the sum $S(f)$:

$$\sum_{g \in G} c(g) f(g) = \sum_{C \subset A} (-1)^{|A \setminus C|} \sum_{g \in G(C)} f(g). \quad (7.46)$$

Lemma 7.9. *If $\mathcal{I}(g) \neq \mathcal{I}$, then $c(g) = 0$. Otherwise $c(g) = \pm 1$.*

Proof. Let $g \in G$ be an arbitrary element, and suppose a subset $C_0 \subset A$ is such that $g \in G(C_0)$, and moreover $\mathcal{I}(g) = \mathcal{I}(C_0) \neq \mathcal{I}$. Then the complement $\mathcal{I}_c(C_0) = \mathcal{I} \setminus \mathcal{I}(C_0)$ is nonempty. Suppose now that $C \subset A$ is a subset, for which the inclusion $g \in G(C)$ also holds. This is equivalent to the statement that the set $C = C_0 \cup B$, where B is any subset of the union $A_c(C) = \cup_{i \in \mathcal{I}_c(C_0)} A_i$. So

$$c(g) = \pm \sum_{B \subseteq A_c(C)} (-1)^{|B|} = 0.$$

On the other hand, if $\mathcal{I}(g) = \mathcal{I}$, then clearly the graph g appears in the sum (7.46) exactly once. \square

7.4.3. Interlaced structures: The term \mathcal{R}^2 . We start with the following statement, which will be used in this and the next subsection. Let $D \subset \mathbb{Z}^1$ be a finite set, which is supposed to be $(1 - \varepsilon)$ -disconnected. We consider again the smallest segment $[l_D, r_D] \subset \mathbb{Z}^1$ such that $D \subset [l_D, r_D]$, and so $\text{diam}(D) = l_D - r_D$. Let $D_i \subset D, i \in \mathcal{I}$ be a partition of D into $(1 - \varepsilon)$ -connected components of D . Consider the ensemble $\tilde{K}_T(D)$ of all compatible dressed systems $\hat{\pi}$, isolating the set D . Let us assign to each system $\hat{\pi}$ the weight $E(\hat{\pi})$, introduced in (7.29). We are interested in a certain event $\mathcal{A}(D, l)$ in this ensemble, which we describe next. Roughly speaking, $\mathcal{A}(D, l)$ happens when the system $\hat{\pi}$ “covers” the segment $[l_D, r_D + l]$. More precisely it means the following. We say that $\hat{\pi} \in \mathcal{A}(D, l)$ iff

- the partition $\Pi(g(\hat{\pi}))$ of \mathcal{I} contains precisely one element;
- the geometric projection of the set of bonds $b(\hat{\pi}) = \pi \cup (\cup_{M_i \in \hat{\pi}} \overline{M}_i) \subset \mathbb{R}^2$ on the x axis contains the segment $[l_D, r_D + l]$.

Lemma 7.10. *There exists $\beta' = \beta'(\beta, \varepsilon)$, $\beta' \rightarrow \infty$ as $\beta \rightarrow \infty$, such that*

$$\frac{\sum_{\hat{\pi} \in \mathcal{A}(D, l)} E(\hat{\pi})}{\prod_{j=1}^n S_\beta(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)\emptyset))} \leq \exp\{-\beta'(\text{diam}(D) + l)\}. \quad (7.47)$$

Proof. Let the dressed system $\hat{\pi} \in \mathcal{A}(D, l)$, $\hat{\pi} = \{\pi, M_1, \dots, M_k\}$, and $\pi = \{\gamma_1, \dots, \gamma_l\}$. The proof will be an adaptation of the Peierls argument to our setting. To implement it we will define first the $(-)$ -sets N_α of π . For that let us consider the configuration σ_π , which is defined by the property that its contours are precisely the contours $\gamma_1, \dots, \gamma_l$. The sets N_α are defined as the maximal connected components of the set $\{t \in \mathbb{Z}^2 : \sigma_\pi(t) = -1\} \setminus D$. So they are either in the upper or in the lower halfplane. We think about them as composed from unit closed plaquettes centered at the sites of the lattice \mathbb{Z}^2 . In the extreme case the collection N_α is empty, and that corresponds to the shortest possible collection $\{\gamma_1, \dots, \gamma_{l(D)}\}$ of contours, isolating D . The set of bonds belonging to the contours in this minimal collection will be denoted by Γ_D .

Consider the connected components \mathcal{T}_i of the union $(\cup_\alpha N_\alpha) \cup (\cup_1^k M_i)$. The elementary (Peierls) surgery of the system $\hat{\pi}$ corresponds to the removal of one such component $\mathcal{T} = (\cup_{\alpha \in A(\mathcal{T})} N_\alpha) \cup (\cup_{\beta \in B(\mathcal{T})} M_\beta)$ and results in a system $\hat{\pi}' = \{\pi', M'_1, \dots, M'_{k'}\}$, which is obtained from $\hat{\pi}$ in the following way. The collection $\{M'_1, \dots, M'_{k'}\}$ is just the initial collection $\{M_1, \dots, M_k\}$, from which the subcollection $\{M_\beta, \beta \in B(\mathcal{T})\}$ is removed. The family $\pi' = \{\gamma'_1, \dots, \gamma'_{l'}\}$ is defined to be the collection of all the contours of the configuration $\sigma_{\pi'}$, which is given by the relation

$$\sigma_{\pi'}(t) = \begin{cases} +1 & \text{for } t \in \cup_{\alpha \in A(\mathcal{T})} N_\alpha, \\ \sigma_\pi(t) & \text{otherwise.} \end{cases}$$

(So indeed it is just flipping some spins inside some contours.) The natural thing to do is to perform such a surgery only on these \mathcal{T} 's, which contain extra long contours. Such \mathcal{T} 's will be called *bridges*, and we define them as follows: the component \mathcal{T} is a bridge, iff it is adjacent to at least two different $(1 - \varepsilon)$ -connected components D_i of D . Let us denote the corresponding set of these indices i by $\mathcal{I}(\mathcal{T}) \subset \mathcal{I}$. The second (and the last) case when a component (with $|\mathcal{I}(\mathcal{T})| \geq 1$) is called a bridge is when it is responsible for the overhang over the segment $[r_D, r_D + l]$. (The component then picks up its surplus length from the overhang.) These bridges will be called *Avignon bridges*, or *A-bridges*, though in such cases the set $\mathcal{I}(\mathcal{T})$ might be one-element. (*Note.* We will *not* perform surgeries on A-bridges \mathcal{T} having single element sets $\mathcal{I}(\mathcal{T})$.) We denote by $\mathcal{B}(\hat{\pi})$ the collection of all bridges of $\hat{\pi}$. For every bridge \mathcal{T} we define the segment $S(\mathcal{I}(\mathcal{T})) \subseteq \mathcal{I}$ as the smallest one, containing the set $\mathcal{I}(\mathcal{T})$. For every \mathcal{T} we introduce the subset $D_{\mathcal{T}} = \cup_{i \in \mathcal{I}(\mathcal{T})} D_i \subset D$, the *base* of the bridge. Let $[l_{D_{\mathcal{T}}}, r_{D_{\mathcal{T}}}]$ be the smallest segment, containing the base $D_{\mathcal{T}}$. It follows from the condition $\hat{\pi} \in \mathcal{A}(D, l)$ that the collection of segments $\{S(\mathcal{I}(\mathcal{T})), \mathcal{T} \in \mathcal{B}(\hat{\pi})\}$ is a covering of \mathcal{I} .

We begin the proof by considering the case of dressed systems which have no A-bridges \mathcal{T} with single element sets $\mathcal{I}(\mathcal{T})$. So we introduce the subset $\mathcal{A}'(D, l) \subset \mathcal{A}(D, l)$ as the set of all dressed systems $\hat{\pi} \in \mathcal{A}(D, l)$ with no A-bridges \mathcal{T} with $|\mathcal{I}(\mathcal{T})| = 1$. By $\mathcal{A}''(D, l) \subset \mathcal{A}(D, l)$ we denote its complement.

Let $\mathcal{T} \in \mathcal{B}(\hat{\pi})$ be some bridge of $\hat{\pi} \in \mathcal{A}'(D, l)$, and $\hat{\pi}'$ be the result of the surgery. We are going to compare the two weights, $E(\hat{\pi})$ and $E(\hat{\pi}')$. To do it, we will introduce the boundary $\partial\mathcal{T}$ of \mathcal{T} to be the set

$$\partial\mathcal{T} = \partial(\cup_{\alpha \in A(\mathcal{T})} N_\alpha) \cup (\cup_{\beta \in B(\mathcal{T})} \delta M_\beta).$$

We split it into two subsets:

$$\begin{aligned} \partial^+\mathcal{T} &= \Gamma_D \cap \partial(\cup_{\alpha \in A(\mathcal{T})} N_\alpha), \\ \partial^-\mathcal{T} &= (\partial(\cup_{\alpha \in A(\mathcal{T})} N_\alpha) \setminus \partial^+\mathcal{T}) \cup (\cup_{\beta \in B(\mathcal{T})} \delta M_\beta). \end{aligned}$$

We are using these notation since the result of a surgery is the removal of $\partial^-\mathcal{T}$ and replacing it by $\partial^+\mathcal{T}$. Therefore if we introduce the quantity

$$l(\mathcal{T}) = |\partial^-\mathcal{T}| - |\partial^+\mathcal{T}|,$$

then we have:

$$E(\hat{\pi}) \leq E(\hat{\pi}') \exp\{-2\beta l(\mathcal{T})\}.$$

For a family \mathcal{B} of bridges we also define

$$l(\mathcal{B}) = \sum_{\mathcal{T} \in \mathcal{B}} l(\mathcal{T}).$$

Likewise we define

$$L(\mathcal{T}) = |\partial\mathcal{T}|, \quad L(\mathcal{B}) = \sum_{\mathcal{T} \in \mathcal{B}} L(\mathcal{T}).$$

Note that

$$L(\mathcal{T}) \leq l(\mathcal{T}) + 2|D_{\mathcal{T}}|.$$

Let us apply repeatedly the above estimates to the sequence $\mathcal{T}_i \in \mathcal{B}(\hat{\pi})$ of all bridges of $\hat{\pi}$, and denote by $\hat{\pi} = \hat{\pi}(\hat{\pi})$ the final system. Then we likewise have

$$E(\hat{\pi}) \leq E(\hat{\pi}) \exp\{-2\beta l(\mathcal{B}(\hat{\pi}))\}. \quad (7.48)$$

We need to have a lower estimate on $l(\mathcal{T})$ and $l(\mathcal{B})$. Since $|\mathcal{I}(\mathcal{T})| > 1$, the base $D_{\mathcal{T}}$ of the bridge \mathcal{T} is $(1 - \varepsilon)$ -disconnected, and its subsets $D_i, i \in \mathcal{I}(\mathcal{T})$, are its $(1 - \varepsilon)$ -connected components. We claim that

$$|\partial^-\mathcal{T}| - |\partial^+\mathcal{T}| \geq 2 \left(\text{diam}(D_{\mathcal{T}}) - \sum_{i \in \mathcal{I}(\mathcal{T})} |D_i| \right),$$

and

$$|\partial^-\mathcal{T}| - |\partial^+\mathcal{T}| \geq 2 \left(\text{diam}(D_{\mathcal{T}}) + l - \sum_{i \in \mathcal{I}(\mathcal{T})} |D_i| \right),$$

if the bridge \mathcal{T} is an A-bridge. It is most easily seen in the case when the set $\partial^+\mathcal{T}$ is the maximal possible; i.e. $\partial^+\mathcal{T} = \Gamma_D \cap \{(x, y) \in \mathbb{R}^2 : y = 1/2\}$. Then the set $\partial\mathcal{T}$ contains a double connection between points $l_{D_{\mathcal{T}}}$ and $r_{D_{\mathcal{T}}}$ (and an additional overhang of the length $\geq 2l$ in the case of an A-bridge), while always $|\partial^+\mathcal{T}| \leq \sum_{i \in \mathcal{I}(\mathcal{T})} |D_i|$. In the general case the set $\partial^+\mathcal{T}$ is less than $\Gamma_D \cap \{(x, y) \in \mathbb{R}^2 : y = 1/2\}$ by a subset Δ , say, but then this Δ can be added both to $\partial^+\mathcal{T}$ and $\partial^-\mathcal{T}$, which reduces this case to the previous special one. Hence,

$$|\partial^-\mathcal{T}| - |\partial^+\mathcal{T}| \geq 2\varepsilon \text{diam}(D_{\mathcal{T}}),$$

and

$$|\partial^-\mathcal{T}| - |\partial^+\mathcal{T}| \geq 2\varepsilon \text{diam}(D_{\mathcal{T}}) + 2l$$

for an A-bridge, since the set $D_{\mathcal{T}}$ is $(1 - \varepsilon)$ -disconnected. So, because the collection $S(\mathcal{I}(\mathcal{T}_i))$ of segments is a covering of \mathcal{I} , we have

$$l(\mathcal{B}) \geq 2\varepsilon \text{diam}(D) + 2l. \quad (7.49)$$

Let us sum the estimates (7.48) over all configurations $\hat{\pi}$ with $\mathcal{B}(\hat{\pi}) = \mathcal{B}$, where \mathcal{B} is a fixed collection of bridges. We have:

$$\sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \mathcal{B}} E(\hat{\pi}) \leq \exp\{-2\beta l(\mathcal{B})\} \sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \mathcal{B}} E(\hat{\pi}(\hat{\pi})). \quad (7.50)$$

But the systems $\hat{\pi}$ have no bridges. So every such system is an element of the product $\prod_{j=1}^n \mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset})$. Also, if $\hat{\pi}_1 \neq \hat{\pi}_2, \mathcal{B}(\hat{\pi}_1) = \mathcal{B}(\hat{\pi}_2) = \mathcal{B}$, then $\hat{\pi}(\hat{\pi}_1) \neq \hat{\pi}(\hat{\pi}_2)$. Hence

$$\frac{\sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \mathcal{B}} E(\hat{\pi}(\hat{\pi}))}{\prod_{j=1}^n S_{\beta}(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset}))} \leq 1. \quad (7.51)$$

So we arrive at an estimate:

$$\frac{\sum_{\hat{\pi} \in \mathcal{A}'(D, l)} E(\hat{\pi})}{\prod_{j=1}^n S_{\beta}(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset}))} \leq \sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \mathcal{B}} \exp\{-2\beta l(\mathcal{B})\}.$$

The estimate of the last sum is a standard combinatorics. Note first that a bridge \mathcal{T} is completely defined by the two sets of bonds: $\partial(\cup_{\alpha \in A(\mathcal{T})} N_\alpha)$ and $\cup_{\beta \in B(\mathcal{T})} \delta M_\beta$; together they form the connected set $\partial\mathcal{T}$. Let us call the bonds of the former as γ -bonds, and the bonds of the latter as M -bonds. Consider the set of all \mathcal{T} 's, such that they contain a given bond b , while $L(\mathcal{T}) = L$. Then the number of such bridges is bounded from above by 8^L ; the extra factor 2 comes from the option for a bond to be a γ -bond or an M -bond. Each bridge is attached to the set D along several bonds; let us choose the leftmost one for each. The number of different sets of bonds we can obtain in that way, is clearly less than $2^{|D|}$. If that collection is fixed, then the number of different collections \mathcal{B} of bridges with $L(\mathcal{B}) = L$ is bounded from above by the same quantity 8^L . So by using (7.49) we arrive at an estimate

$$\frac{\sum_{\hat{\pi} \in \mathcal{A}'(D, l)} E(\hat{\pi})}{\prod_{j=1}^n S_\beta(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset))} \leq \sum_{k \geq 2\varepsilon(r_D - l_D) + 2l} 2^{r_D - l_D} 8^{2(r_D - l_D) + k} \exp\{-2\beta k\}.$$

This estimate proves our lemma for the case of configurations which have no A-bridges \mathcal{T} with $|\mathcal{I}(\mathcal{T})| = 1$.

The argument for the remaining $\hat{\pi}$'s, which form the set $\mathcal{A}''(D, l)$, is essentially the same, apart from one modification. Let $\mathcal{T}_{i_1}, \dots, \mathcal{T}_{i_k}$ be all the A-bridges \mathcal{T} of the system $\hat{\pi}$ with $|\mathcal{I}(\mathcal{T})| = 1$. The base of each of them is just a single $(1 - \varepsilon)$ -connected component of D . Let $\bar{\mathcal{T}}$ be the one for which its base $D_{\bar{\mathcal{T}}}$ is the leftmost such component. The rest of the A-bridges can be ignored, so for the sake of simplicity of the exposition we will suppose that $\bar{\mathcal{T}}$ is the only A-bridge of $\hat{\pi}$. If we denote by \bar{i} the only element the set $\mathcal{I}(\bar{\mathcal{T}})$ has, then, according to our notation, $D_{\bar{\mathcal{T}}} = D_{\bar{i}}$. We introduce also the segment $[l_{D_{\bar{i}}}, r_{D_{\bar{i}}}]$ as the smallest one containing the set $D_{\bar{i}}$.

Let now $\bar{\mathcal{B}}(\hat{\pi})$ be the collection of the *remaining* bridges of $\hat{\pi}$. We again do the surgeries over all of them, denote by $\hat{\pi} = \hat{\pi}(\hat{\pi})$ the final system, and have

$$E(\hat{\pi}) \leq E(\hat{\pi}) \exp\{-2\beta l(\bar{\mathcal{B}}(\hat{\pi}))\}.$$

Note that the resulting dressed system $\hat{\pi}$ still contains the A-bridge $\bar{\mathcal{T}}$ (as its only bridge). Let $\hat{\pi}_{\bar{i}} = \hat{\pi}_{\bar{i}}(\hat{\pi}) \subset \hat{\pi}$ be the dressed subsystem, attached to $D_{\bar{\mathcal{T}}}$ (which subsystem develops the bridge $\bar{\mathcal{T}}$). Clearly, $\hat{\pi}_{\bar{i}} \in \mathbb{L}(D_{\bar{i}}, \{\bar{i}\}, \{D_{\bar{i}}\}, (g_{\bar{i}})_\emptyset)$. Let us write $\hat{\pi}$ as the disjoint union, $\hat{\pi} = \hat{\pi} \cup \hat{\pi}_{\bar{i}}$. Note that $E(\hat{\pi} \cup \hat{\pi}_{\bar{i}}) = E(\hat{\pi}) E(\hat{\pi}_{\bar{i}})$, and that the system $\hat{\pi}$ has no bridges. So the analog of (7.50) looks as follows:

$$\sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \bar{\mathcal{B}} \cup \bar{\mathcal{T}}} E(\hat{\pi}) \leq \exp\{-2\beta l(\bar{\mathcal{B}})\} \sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \bar{\mathcal{B}} \cup \bar{\mathcal{T}}} E(\hat{\pi}) E(\hat{\pi}_{\bar{i}}).$$

Instead of (7.51) we write:

$$\frac{\sum_{\hat{\pi}: \mathcal{B}(\hat{\pi}) = \bar{\mathcal{B}} \cup \bar{\mathcal{T}}} E(\hat{\pi}) E(\hat{\pi}_{\bar{i}})}{\prod_{j=1}^n S_\beta(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_\emptyset))} \leq \frac{\sum_{\hat{\pi}_{\bar{i}} \in \mathbb{L}(D_{\bar{i}}, \{\bar{i}\}, \{D_{\bar{i}}\}, (g_{\bar{i}})_\emptyset): \mathcal{B}(\hat{\pi}_{\bar{i}}) = \bar{\mathcal{T}}} E(\hat{\pi}_{\bar{i}})}{S_\beta(\mathbb{L}(D_{\bar{i}}, \{\bar{i}\}, \{D_{\bar{i}}\}, (g_{\bar{i}})_\emptyset))}.$$

The rhs of the last estimate (in contrast with the lhs of (7.47) !) can be interpreted as a probability of a certain event in the ensemble $\mathbb{L}(D_{\bar{i}}, \{\bar{i}\}, \{D_{\bar{i}}\}, (g_{\bar{i}})_\emptyset)$: namely, it is the event of observing the dressed system living on $D_{\bar{i}}$, having the $\bar{\mathcal{T}}$ -shaped overhang covering the segment $[r_{D_{\bar{i}}}, r_D + l]$. As in the relation (7.42), we conclude that

$$\frac{\sum_{\hat{\pi}_{\bar{i}} \in \mathbb{L}(D_{\bar{i}}, \{\bar{i}\}, \{D_{\bar{i}}\}, (g_{\bar{i}})_{\emptyset}) : E(\hat{\pi}_{\bar{i}})} E(\hat{\pi}_{\bar{i}})}{\hat{\pi}_{\bar{i}} \text{ hangs over } [r_{D_{\bar{i}}}, r_D + l]} \leq \exp \{-c' \beta (r_D + l - r_{D_{\bar{i}}})\}.$$

On the other hand,

$$l(\bar{\mathcal{B}}) \geq 2\varepsilon \text{diam}(D_1 \cup \dots \cup D_{\bar{i}}),$$

provided $\bar{i} > 1$, in which case $\varepsilon \text{diam}(D_1 \cup \dots \cup D_{\bar{i}}) + (r_D + l - r_{D_{\bar{i}}}) \geq \varepsilon \text{diam}(D) + l$. When $\bar{i} = 1$ we likewise have $r_D + l - r_{D_1} \geq \varepsilon \text{diam}(D) + l$, because D is not $(1 - \varepsilon)$ -connected. The rest of the argument for the \mathcal{A}'' case is the same. \square

We now can formulate and prove the statement about the structure of the term \mathcal{R}^2 , similar to the one about \mathcal{R}^1 .

Proposition 7.11. *In the notation of Proposition 7.5 there exists a function w^2 defined for all finite $(1 - \varepsilon)$ -disconnected subsets of \mathbb{Z}^1 , such that for*

$$\begin{aligned} \mathcal{R}^2 &= \mathcal{R}^2(D) \\ &= \left(\sum_{k=1}^{n-1} \sum_{\Pi = \Lambda_1, \dots, \Lambda_k} \sum_{\substack{g_1 \in G(\Lambda_1), \dots, g_k \in G(\Lambda_k) \\ \Pi(g_j) = \Lambda_j}} \prod_{j=1}^k S_{\beta}(\mathbb{L}(D(\Lambda_j), \Lambda_j, \{D_i, i \in \Lambda_j\}, g_j)) \right) \\ &\quad \times \left(\prod_{j=1}^n S_{\beta}(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset})) \right)^{-1} \end{aligned} \tag{7.52}$$

the following representation holds:

$$\mathcal{R}^2(D) = \sum_{\Pi} \prod_{\Lambda \in \Pi: |\Lambda| > 1} w^2(D_{\Lambda}).$$

The function w^2 satisfies the estimate:

$$w^2(D_{\Lambda}) \leq \exp \{-\beta' \text{diam}(D_{\Lambda})\}.$$

Proof. We remind the reader that each term in the expression for \mathcal{R}^2 corresponds to a collection of graphs $g_1 \in G(\Lambda_1), \dots, g_k \in G(\Lambda_k)$ with $\Pi(g_j) = \Lambda_j$, such that not all of the sets Λ_j are one-point sets. Note that all one-point sets Λ_j cancel out, and the remainder factors into the product of terms of the type

$$Q^2(\bar{\Lambda}) = \frac{\sum_{g \in G(\bar{\Lambda}): \Pi(g) = \bar{\Lambda}} \prod_{j \in \bar{\Lambda}} S_{\beta}(\mathbb{L}(D(\bar{\Lambda}), \bar{\Lambda}, \{D_i, i \in \bar{\Lambda}\}, g))}{\prod_{j \in \bar{\Lambda}} S_{\beta}(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset}))},$$

where $\bar{\Lambda} \subset \mathcal{I}$ is a nontrivial segment. But this is exactly the situation we were considering in the preceding lemma (with $l = 0$), and the result follows. \square

7.4.4. *The general term \mathcal{R}^3 .* Now we are in the position to consider the generic term in (7.22). The idea is to combine the methods of the two previous subsections. This term corresponds to a partition $\Pi \equiv \Pi_{\mathcal{I}} = (\Lambda_1, \Lambda_2, \dots, \Lambda_j, \dots, \Lambda_{|\mathcal{I}|}), j \in \mathcal{I}$ of the index set \mathcal{I} into consecutive segments, and is given by

$$\begin{aligned} \mathcal{R}^3 = \mathcal{R}_{\Pi}^3 = & - \sum_{\substack{g_1 \in G(\Lambda_1), \dots, g_k \in G(\Lambda_k): \\ \Pi(g_j) = \Lambda_j}} \sum_{P \in \overline{\mathbb{R}}(\Pi, \{g_j\})} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j) \\ & \times \left(\prod_{j=1}^n S_{\beta}(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset})) \right)^{-1}. \end{aligned} \quad (7.53)$$

Now we treat the inner sum of (7.53) in the same way we treated the sum (7.26). Namely,

$$\begin{aligned} & - \sum_{P \in \overline{\mathbb{R}}(\Pi, \{g_j\})} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j=1}^n S_{\beta}(\mathbb{L}(D_j, \{j\}, \{D_j\}, (g_j)_{\emptyset}))} = \\ & = \sum_{h \in G(\mathcal{J}), |h| \geq 0} (-1)^{|h|} \sum_{P \in R(\Pi, \{g_j\}, h)} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j=1}^n S_{\beta}(\mathbb{L}_j)}. \end{aligned}$$

Here we are dealing with the case $\Pi_{\mathcal{I}} \neq \Pi_{\mathcal{I}}^*$ (the latter being the partition into single points), and that is why the case $|h| = 0$ is included, in contrast with (7.26). Now we introduce the set $\overline{\mathcal{C}}(h)$ of all maximal connected subgraphs of h , *excluding isolated vertices, corresponding to trivial graphs g_j* . Then

$$\begin{aligned} & (-1)^{|h|} \sum_{P = \{\hat{\pi}_j: j \in \overline{h}\} \in R(\Pi, \{g_j\}, h)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j) \\ & = \prod_{k \in \overline{\mathcal{C}}(h)} (-1)^{|k|} \sum_{P = \{\hat{\pi}_j: j \in \overline{k}\} \in R(\Pi, \{g_j\}, k)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j). \end{aligned} \quad (7.54)$$

Here \overline{k} is a subset of \mathcal{J} . So we define the weight $Q_{\beta}^3(k)$ of a connected graph k with vertices among the points of the index set \mathcal{J} by

$$Q_{\beta}^3(k) = (-1)^{|k|} \frac{\sum_{P = \{\hat{\pi}_j: j \in \overline{k}\} \in R(\Pi, \{g_j\}, k)} \prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{j \in \overline{k}} S_{\beta}(\mathbb{L}_j)}. \quad (7.55)$$

We then pass to the weight

$$Q_{\beta}^3(s) = \sum_{h \in H(s)} \prod_{k \in \overline{\mathcal{C}}(h)} Q_{\beta}^3(k), \quad (7.56)$$

where $H(s) \subset G(\mathcal{J})$ is the subset of all graphs which have the segment $s \subset \mathcal{J}$ as the only nontrivial element of the partition $\Pi(h)$ of \mathcal{J} . Using (7.34), we have the estimate

$$|Q_{\beta}^3(s)| \leq \sum_{\substack{P = \{\hat{\pi}_j: j \in s\} \in \overline{\mathbb{R}}(\Pi, \{g_j\}): \\ g(P) \in H(s)}} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{i \in \overline{s}} S_{\beta}(\mathbb{L}_i)}, \quad (7.57)$$

where $\mathfrak{s} \subset \mathcal{I}$ is the full preimage of $s \subset \mathcal{J}$ under the natural projection of the index set \mathcal{I} onto \mathcal{J} ; of course, $|\mathfrak{s}| \geq |s|$.

Here comes the difference between the general case and the case of the \mathcal{R}^1 term: the ratios in (7.57) can not be interpreted as probabilities, unless $|\mathfrak{s}| = |s|$, in which case all the graphs g are trivial. So we use the lemma of the previous section instead. As in the \mathcal{R}^1 case, instead of estimating the individual terms in (7.57), we will estimate the sums

$$\sum_{\substack{P=\{\hat{\pi}_j:j \in s\} \in \overline{\mathbb{R}}(\Pi, \{g_j\}): \\ sp(g(P))=\gamma \in H(s)}} \frac{\prod_{\hat{\pi}_j \in P} E(\hat{\pi}_j)}{\prod_{i \in s^{-1}} S_\beta(\mathbb{L}_i)},$$

corresponding to different spanning graphs γ the families P might have. The rest of the proof follows literally the one for the \mathcal{R}^1 case and will be omitted. The result is the following

Proposition 7.12. *Consider the set of pairs (D', Π') , where D' is a finite subset of \mathbb{Z}^1 , split into its sufficiently separated components D'_i , $i \in \mathcal{I}'$, while Π' is a partition of \mathcal{I}' into consecutive segments. In the notation of Proposition 7.5 there exists a function w^3 defined for all pairs (D', Π') , such that the following representation holds:*

$$\mathcal{R}_\Pi^3(D) = \sum_{\tilde{\Pi}} \prod_{\tilde{\Lambda} \in \tilde{\Pi}} w^3(D_{\tilde{\Lambda}}, \tilde{\Lambda} \cap \Pi).$$

Here the summation is taken over all partitions $\tilde{\Pi}$ of the set \mathcal{I} into segments, such that Π is a strict refinement of $\tilde{\Pi}$, $D_{\tilde{\Lambda}} = \cup_{i \in \tilde{\Lambda}} D_i$, while the partition $\tilde{\Lambda} \cap \Pi$ is the partition of $\tilde{\Lambda}$ into subsets, which are elements of Π . The function w^3 satisfies the estimate:

$$w^3(D', \Pi') \leq \exp\{-\beta' \tilde{d}(D', \Pi')\}, \tag{7.58}$$

with $\beta' = \beta'(\beta, \varepsilon) \rightarrow \infty$ as $\beta \rightarrow \infty$ and with the function $\tilde{d}(D', \Pi')$ defined as follows. Let

$$\mathcal{I}' = \Lambda'_r \cup \Lambda'_{r+1} \cup \dots \cup \Lambda'_{r+s}, \quad s \geq 0 \tag{7.59}$$

be our partition Π' of the set \mathcal{I}' into consecutive segments. Then

$$\tilde{d}(D', \Pi') = \sum_{i=0}^{s-1} \text{dist}(D'_{\Lambda'_{r+i}}, D'_{\Lambda'_{r+i+1}}) + \sum_{i=0}^{s-*} \text{diam}(D'_{\Lambda'_{r+i}}), \tag{7.60}$$

where the last sum \sum^* is taken only over these Λ'_i , which are not one-element subsets of \mathcal{I}' . In the special case when Π' is a partition into singletons, the second term in (7.60) disappears, and we have $w^3(D', \Pi') = w^1(D')$. In the special case when Π' has just one element (i.e. $s = 0$ in (7.59)), while D' consists of more than one component, we have the first term in (7.60) disappearing, and $w^3(D', \Pi') = w^2(D')$. In the special case when Π' has just one element, but D' consists of one component, we have $w^3(D', \Pi') = 1$ (which is in line with (7.58), since in that case $\tilde{d}(D', \Pi') = 0$).

7.4.5. *Proof of the polymer representation.* Now we have all the ingredients needed to finish the proof of the representation for the partition function $Z_\beta(D)$. We just have to put together the results of Propositions 7.6, 7.11 and 7.12. We need to define for all segments Λ with $|\Lambda| > 1$ the weights $w(D_\Lambda)$, where $D_\Lambda \subset D$ is equipped with the partition $D_\Lambda = \cup_{i \in \Lambda} D_i$ into the sufficiently separated components of D . The definition is the following:

$$w(D_\Lambda) = \sum_{\Pi_\Lambda} w^3(D_\Lambda, \Pi_\Lambda),$$

where the summation goes over all partitions Π_Λ of the index set Λ . As was mentioned in Proposition 7.12, for extreme choices of the partition Π_Λ we get either the functional $w^1(D_\Lambda)$ or $w^2(D_\Lambda)$. Note that the number of summands in the last expression is at most $2^{|\Lambda|-1}$. On the other hand, the distances between the consecutive subsets D_i are not smaller than 2, hence the entropy term $2^{|\Lambda|-1}$ is beaten for large β . So the estimate (7.24) follows from (7.58), (7.60), the definition (7.4) and the fact that the starting set \tilde{D} and hence all D_Λ 's are $(1 - \varepsilon)$ -disconnected. \square

7.5. *Final estimate.* Now we can obtain the desired estimate on the interaction U . Recall that

$$U_{T,A}^{\beta,+} = \sum_{D \subseteq A} (-1)^{|A \setminus D|} Q_{T,D}^{\beta,+}. \quad (7.61)$$

Suppose now that the set A is $(1 - \varepsilon)$ -disconnected, and $A_i \subset A$, $i \in \mathcal{I}$, are its $(1 - \varepsilon)$ -connected components. Then we can write the formula (7.5) for $Q_{T,D}^{\beta,+}$ for every subset $D \subseteq A$, with $D_i = D \cap A_i$. Let us substitute all these formulas into (7.61). To do the cancellations we are going to use Lemma 7.9, which we apply for the following choice of the function $f(g)$, $g \in G(D)$:

- in case the graph g has for its vertices the components D_{i_1}, \dots, D_{i_k} of D , which are *consecutive* components of D , while the bonds of g are the following pairs of components of D : $\{D_{i_1}, D_{i_2}\}, \{D_{i_2}, D_{i_3}\}, \dots, \{D_{i_{k-1}}, D_{i_k}\}$ – we put

$$f(g) = G_\beta(D_{[i_1, i_k]});$$

in the special case when $k = 1$ and g has just one vertex D_i , we put

$$f(g) = Q_{T, D_i}^{\beta,+}.$$

- for all other graphs g we put $f(g) = 0$.

Lemma 7.9 then tells us that the only surviving terms are those which have points in every component of A :

$$U_{T,A}^{\beta,+} = \sum_{\substack{D \subseteq A: \\ D \cap A_i \neq \emptyset \text{ for all } i=1, \dots, n}} (-1)^{|A \setminus D|} G_\beta(D; \{D_k, k \in [1, n]\}).$$

The estimate (7.6) implies the following bound:

$$\left| U_{T,A}^{\beta,+} \right| \leq 2^{|A|} \exp \{-\beta' l(A)\}.$$

So, if the density $\rho(A) < 1 - 3\varepsilon$, then by Lemma 7.3 we have

$$\left| U_{T,A}^{\beta,+} \right| \leq \exp \{-\beta'' \text{diam}(A)\}. \quad \square$$

8. Proof of the Almost Gibbsianity

What is left now is the check of the DLR equation (1.5). More precisely, we have to define the sets $\overline{X}_V^{\mathcal{U}}$ of boundary conditions, for which the series (1.2) converges absolutely, and then to check the relation (1.5) for corresponding boundary conditions. Our choice of $\overline{X}_V^{\mathcal{U}}$ is the following: let $\delta > 0$ be any real, and define

$$\Omega(\delta) = \{ \sigma : \exists n = n(\sigma) \forall k > n \\ m_{[0,k]}(\sigma) > -1 + \delta, m_{[-k,0]}(\sigma) > -1 + \delta \},$$

$$\overline{X}_V^{\mathcal{U}} = \overline{X}_V^{\mathcal{U}}(\delta) = \{ \sigma : \exists n = n(\sigma) \forall k > n \\ m_{[0,k] \setminus V}(\sigma) > -1 + \delta, m_{[-k,0] \setminus V}(\sigma) > -1 + \delta \},$$

where for every finite set Y , every σ we denote by $m_Y(\sigma)$ the average magnetization of σ on Y :

$$m_Y(\sigma) = \frac{\sum_{t \in Y} \sigma_t}{|Y|}.$$

Let us introduce also the subsets $\Omega(\delta, n) \subset \Omega(\delta)$, $\overline{X}_V^{\mathcal{U}}(\delta, n) \subset \overline{X}_V^{\mathcal{U}}(\delta)$ by

$$\Omega(\delta, n) = \{ \sigma : \forall k > n \ m_{[0,k]}(\sigma) > -1 + \delta, m_{[-k,0]}(\sigma) > -1 + \delta \}, \\ \overline{X}_V^{\mathcal{U}}(\delta, n) = \{ \sigma : \forall k > n \ m_{[0,k] \setminus V}(\sigma) > -1 + \delta, m_{[-k,0] \setminus V}(\sigma) > -1 + \delta \}.$$

Clearly, $\Omega(\delta) = \cup_n \Omega(\delta, n)$, $\overline{X}_V^{\mathcal{U}}(\delta) = \cup_n \overline{X}_V^{\mathcal{U}}(\delta, n)$. In what follows, the arguments for the sets $\Omega(\delta)$ and $\overline{X}_V^{\mathcal{U}}(\delta)$ would be identical, and so we will present them only for the former case.

Proposition 8.1. *For every $\delta < 2$ there exists a value $\beta(\delta)$, such that for all $\beta > \beta(\delta)$, all V finite we have:*

$$\mathbb{P}_T^{\beta,+}(\Omega(\delta, n)) \uparrow \mathbb{P}_T^{\beta,+}(\Omega(\delta)) = 1, \\ \mathbb{P}_T^{\beta,+}(\overline{X}_V^{\mathcal{U}}(\delta, n)) \uparrow \mathbb{P}_T^{\beta,+}(\overline{X}_V^{\mathcal{U}}(\delta)) = 1.$$

The same statements hold for the finite volume Gibbs measures $p_{W_N}^{\beta,+}(\sigma_{W_N})d\sigma_{W_N}$ with (+)-boundary conditions, uniformly in N .

Proof. We will show that the measures of the complements $\mathbb{P}_T^{\beta,+}[(\Omega(\delta, n))^c]$ go to zero as $n \rightarrow \infty$, provided $\beta > \beta(\delta)$ is large enough. Let $\sigma \in (\Omega(\delta, n))^c$. By definition it means that the subset $(\sigma_{[-n,n]})^- \subset [-n, n]$ has at least $n(2 - \delta)$ elements. Note that the number of all possible subsets of $[-n, n]$ is bounded by 2^{2n} . We are left with the estimate of the $\mathbb{P}^{\beta,+}$ -probability in the usual 2D Ising model of the event

$$N_A = \{ \sigma = (\sigma_t, t \in \mathbb{Z}^2) : \sigma_A = -1, \sigma_{[-n,n] \setminus A} = +1 \},$$

where A is a subset of $[-n, n] \subset \mathbb{Z}^1$. If the event N_A takes place, then the set A is isolated from infinity by contours, surrounding it. Let us consider the family γ of all external contours among these. (Note that these exist with probability one with respect to the measure $\mathbb{P}^{\beta,+}$.) They are also separating A from infinity. There are two cases to consider:

1. each of the contours in γ intersects the segment $[-n, n]$;
2. there exist a contour in γ which does not intersect $[-n, n]$.

In the second case the family γ consists in fact from just one contour Γ , which surrounds the whole segment $[-n, n]$. If L is the rightmost point where Γ intersects the x axis, then the $\mathbb{P}^{\beta,+}$ -probability to observe such a contour is bounded from above by

$$\sum_{L \geq n} \sum_{l > 2(L+n)} 3^l \exp \{-2\beta l\} \leq \exp \{-\beta' n\}$$

for some β' diverging with β . In the first case let us fix an intersection point inside $[-n, n]$ for every contour in γ . Note that the number of different collections of such intersection points again does not exceed 2^{2n} . Once such a collection is fixed, the number of different contours of the total length l , passing through each point of this collection, is bounded from above by 3^l . Since γ surrounds A , the total length l has to be bigger than $2|A|$. Putting all these estimates together we get the following bound:

$$\mathbb{P}_T^{\beta,+} [(\Omega(\delta, n))^c] \leq 2^{4n} \sum_{l > 2n(2-\delta)} 3^l \exp \{-2\beta l\} \leq \exp \{-\beta' n(2-\delta)\}. \quad \square$$

We proceed by checking that the function $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})$ can indeed be defined by the formula (1.3), provided $\bar{\sigma}_{T \setminus V} \in \bar{X}_V^{\mathcal{U}}(\delta, n)$. For this we need to check the absolute convergence of the series (1.2) for any n , any $\bar{\sigma}_{T \setminus V} \in \bar{X}_V^{\mathcal{U}}(\delta, n)$. But it is easy to see that for every V and n one can find an $n'(V, n)$, such that for every $\bar{\sigma}_{T \setminus V} \in \bar{X}_V^{\mathcal{U}}(\delta, n)$ and every $\sigma_V \in X^V$ we have for any subset $A \in (\sigma_V \cup \bar{\sigma}_{T \setminus V})^-$ with the properties $A \cap V \neq \emptyset$, $A \cap ((-\infty, -n'] \cup [+n', +\infty)) \neq \emptyset$ that $\rho(A) < 1 - \frac{\delta}{2}$ (see (2.16)). So by Proposition 2.3 we obtain that the terms of the series (1.2) are exponentially small in the diam (A) , with the exponent β' , diverging as $\beta \rightarrow \infty$. Since the number of different A 's intersecting V and having diameter $\text{diam}(A) = l$ is bounded by $(|V| + l) 2^l$, the absolute convergence follows.

Next let us consider the question of measurability. Note that the function $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})$ is not yet defined everywhere thus far; let us put $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) = 0$ for all $\bar{\sigma}_{T \setminus V} \notin \bar{X}_V^{\mathcal{U}}$. Since due to Proposition 8.1, $\mathbb{P}_T^{\beta,+}(\bar{X}_V^{\mathcal{U}}(\delta)) = 1$, we can put instead of 0 any other constant. The argument of the last paragraph shows that the function $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})$ is continuous on every subspace $\bar{X}_V^{\mathcal{U}}(\delta, n)$ in the topology induced by the product topology on Ω (though not uniformly in n). Hence it is a measurable function on $\bar{X}_V^{\mathcal{U}}(\delta, n)$ with respect to the σ -algebra of subsets of $\bar{X}_V^{\mathcal{U}}(\delta, n)$ obtained from the σ -algebra \mathfrak{B}_T by taking the intersections of the subsets from \mathfrak{B}_T with $\bar{X}_V^{\mathcal{U}}(\delta, n)$. Since the subsets $\bar{X}_V^{\mathcal{U}}(\delta, n)$ are themselves elements of \mathfrak{B}_T , the \mathfrak{B}_T -measurability of $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})$ follows. (Note that this function is *not* continuous on Ω .)

The last thing we need to show in order to complete our argument is the statement that the two probability distributions, $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+)$ and $p_V^{\mathcal{U}^N}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V})$, are close enough, provided that the configuration $\bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+$ belongs to the family $\bar{X}_V^{\mathcal{U}}(\delta, n)$, while the number N , which may depend on n , is large enough.

Proposition 8.2. *Let the integers n, N be fixed, with $N \geq 8n, N \gg \text{diam}(V)$. Consider the set $\mathcal{S}(n, N)$ of all configurations $\bar{\sigma}_{[-N, N] \setminus V}$ such that $\bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+ \in \overline{X}_V^{\mathcal{M}}(\delta, n)$. Then the ratio of the two probability distributions satisfy the following estimate:*

$$\sup_{\bar{\sigma}_{[-N, N] \setminus V} \in \mathcal{S}(n, N)} \frac{p_V^{\mathcal{M}}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+)}{p_V^{\mathcal{M}^N}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V})} \leq \exp\{-\beta' N\}, \quad (8.1)$$

with β' diverging as $\beta \rightarrow \infty$.

Proof. Denote by Δ the set $(\bar{\sigma}_{[-N, N] \setminus V})^-$. Then the ratio

$$\frac{p_V^{\mathcal{M}}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+)}{p_V^{\mathcal{M}^N}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V})}$$

is a function of the quantities $U_{T,A}^{\beta,+}, U_{T,A}^{\beta,+,N}$, where $A \subseteq \Delta \cup V, A \cap V \neq \emptyset$. We would be done once we show that for every $B \subseteq V, B \neq \emptyset$,

$$\left| \sum_{\substack{A: A \subseteq \Delta \cup V, \\ A \cap V = B}} U_{T,A}^{\beta,+} - \sum_{\substack{A: A \subseteq \Delta \cup V, \\ A \cap V = B}} U_{T,A}^{\beta,+,N} \right| \leq \exp\{-\beta' N\}.$$

Note, however, that for every set A , entering the last sums, and such that $A \cap [-\frac{N}{8}, \frac{N}{8}]^c \neq \emptyset$, we have $\rho(A) \leq 1 - \frac{\delta}{2}$, and therefore $|U_{T,A}^{\beta,+}| \leq \exp\{-\beta' \text{diam}(A)\}, |U_{T,A}^{\beta,+,N}| \leq \exp\{-\beta' \text{diam}(A)\}$, according to Proposition 2.3. Hence the contribution of the corresponding sums is less than $\exp\{-\beta' \frac{N}{10}\}$. What is left now is the treatment of the sets $A \subset [-\frac{N}{8}, \frac{N}{8}]$, which would be possible due to the fact that in such a case the distance $\text{dist}(A, W_N^c) \geq \frac{7N}{8}$.

Recall that

$$U_{T,A}^{\beta,+,N} = \sum_{D \subseteq A} (-1)^{|A \setminus D|} Q_{T,D}^{\beta,+,N}, \quad (8.2)$$

$$Q_{T,D}^{\beta,+,N} = -\ln \left(\sum_{\pi \in K_T(D, W_N)} E_N(\pi) \right), \quad (8.3)$$

$$E_N(\pi) = \exp \left\{ -2\beta|\pi| - \sum_{M: M \subset W_N, M \cap \Delta(\pi) \neq \emptyset, M \cap T = \emptyset} \Phi(M) \right\}, \quad (8.4)$$

while

$$U_{T,A}^{\beta,+} = \sum_{D \subseteq A} (-1)^{|A \setminus D|} Q_{T,D}^{\beta,+}, \quad (8.5)$$

$$Q_{T,D}^{\beta,+} = -\ln \left(\sum_{\pi \in K_T(D)} E(\pi) \right), \quad (8.6)$$

$$E(\pi) = \exp \left\{ -2\beta|\pi| - \sum_{M: M \cap \Delta(\pi) \neq \emptyset, M \cap T = \emptyset} \Phi(M) \right\}. \quad (8.7)$$

So the difference between (8.2) and (8.5) comes from the following facts:

- the sum (8.6) contains extra terms over the sum (8.3);
- the families π , entering both sums, have different weights (8.4) and (8.7).

Note, however, that the difference

$$\sum_{\pi \in K_T(D, W_N)} E_N(\pi) - \sum_{\pi \in K_T(D, W_{N/2})} E_N(\pi) = O(\exp\{-2\beta N\}),$$

while each of the above sums is of the order not smaller than $\exp\{-4\beta|D|\} \geq \exp\{-\beta N\}$. Hence

$$\sum_{\pi \in K_T(D, W_N)} E_N(\pi) = (1 + O(\exp\{-\beta N\})) \sum_{\pi \in K_T(D, W_{N/2})} E_N(\pi).$$

Likewise,

$$\sum_{\pi \in K_T(D)} E(\pi) = (1 + O(\exp\{-\beta N\})) \sum_{\pi \in K_T(D, W_{N/2})} E(\pi).$$

Finally, for a given family $\pi \in K_T(D, W_{N/2})$ we have the representation

$$\frac{E(\pi)}{E_N(\pi)} = \exp \left\{ - \sum_{M: M \cap \Delta(\pi) \neq \emptyset, M \cap W_N \neq \emptyset, M \cap T = \emptyset} \Phi(M) \right\},$$

and since the last sum is $O(N^2 \exp\{-2\beta N\})$ uniformly in π , we have

$$\sum_{\pi \in K_T(D, W_{N/2})} E(\pi) = \sum_{\pi \in K_T(D, W_{N/2})} E_N(\pi) (1 + O(N^2 \exp\{-2\beta N\})).$$

Hence

$$Q_{T,D}^{\beta,+N} - Q_{T,D}^{\beta,+} = O(N^2 \exp\{-2\beta N\}) + O(\exp\{-\beta N\}),$$

and

$$\begin{aligned} & \left| \sum_{\substack{A: A \subseteq \Delta \cup V, \\ A \cap \bar{V} = B}} U_{T,A}^{\beta,+} - \sum_{\substack{A: A \subseteq \Delta \cup V, \\ A \cap \bar{V} = B}} U_{T,A}^{\beta,+N} \right| \\ & \leq 2^{\lfloor \frac{N}{4} \rfloor} O(\exp\{-\beta' N\}) + \exp\left\{-\beta' \frac{N}{10}\right\}. \quad \square \end{aligned}$$

Now we are in a position to check (1.5) for the measure $\mathbb{P}_T^{\beta,+}$. We first rewrite the integral $\int_B \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) \mathbb{P}_{T \setminus V}^{\beta,+}(d\bar{\sigma}_{T \setminus V})$ as a sum:

$$\begin{aligned} & \int_B \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) \mathbb{P}_{T \setminus V}^{\beta,+}(d\bar{\sigma}_{T \setminus V}) = \\ &= \int_{B \cap \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) \mathbb{P}_{T \setminus V}^{\beta,+}(d\bar{\sigma}_{T \setminus V}) \\ &+ \int_{B \setminus \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) \mathbb{P}_{T \setminus V}^{\beta,+}(d\bar{\sigma}_{T \setminus V}), \end{aligned}$$

and observe that the second integral goes to zero as $n \rightarrow \infty$, due to Proposition 8.1 and because the integrand is bounded. Because the function $p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V})$ is continuous (=quasilocal) on $\bar{X}_V^{\mathcal{U}}(\delta, n)$, we can find a value $N = N(\varepsilon)$ big enough, such that

$$\begin{aligned} & \left| \int_{B \cap \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) \mathbb{P}_{T \setminus V}^{\beta,+}(d\bar{\sigma}_{T \setminus V}) - \right. \\ & \left. \int_{B \cap \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+) \right) \times \right. \\ & \left. \times p_{W_N, [-N, N] \setminus V}^{\beta,+}(\bar{\sigma}_{[-N, N] \setminus V}) d\bar{\sigma}_{[-N, N] \setminus V} \right| < \varepsilon. \end{aligned}$$

(Here $p_{W_N}^{\beta,+}$ is just the finite volume Ising distribution with (+)-boundary conditions, while $p_{W_N, [-N, N] \setminus V}^{\beta,+}$ is its projection on the subset $[-N, N] \setminus V \subset T$.) Due to Proposition 8.2 we know that if N is large enough, then we can approximate arbitrarily close the probability distribution $p_V^{\mathcal{U}}(\cdot/\cdot)$ by $p_V^{\mathcal{U}^N}(\cdot/\cdot)$, provided that the conditioning belongs to the class $\bar{X}_V^{\mathcal{U}}(\delta, n)$:

$$\begin{aligned} & \left| \int_{B \cap \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V} \cup \sigma_{T \setminus [-N, N]}^+) \right) \times \right. \\ & \left. \times p_{W_N, [-N, N] \setminus V}^{\beta,+}(\bar{\sigma}_{[-N, N] \setminus V}) d\bar{\sigma}_{[-N, N] \setminus V} - \right. \\ & \left. \int_{B \cap \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}^N}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V}) \right) \times \right. \\ & \left. \times p_{W_N, [-N, N] \setminus V}^{\beta,+}(\bar{\sigma}_{[-N, N] \setminus V}) d\bar{\sigma}_{[-N, N] \setminus V} \right| < \varepsilon. \end{aligned}$$

Again by Proposition 8.1 we know that

$$\left| \int_{B \cap \bar{X}_V^{\mathcal{U}}(\delta, n)} \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}^N}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V}) \right) \times \right. \\ \left. \times p_{W_N, [-N, N] \setminus V}^{\beta, +}(\bar{\sigma}_{[-N, N] \setminus V}) d\bar{\sigma}_{[-N, N] \setminus V} - \int_B \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}^N}(\sigma_V / \bar{\sigma}_{[-N, N] \setminus V}) \right) p_{W_N, [-N, N] \setminus V}^{\beta, +}(\bar{\sigma}_{[-N, N] \setminus V}) d\bar{\sigma}_{[-N, N] \setminus V} \right| \\ < \varepsilon,$$

if n is large. But for the last integral we have the identity

$$\int_B \left(\sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathcal{U}^N}(\sigma_V / \bar{\sigma}_{T \setminus V}) \right) p_{W_N, [-N, N] \setminus V}^{\beta, +}(\bar{\sigma}_{[-N, N] \setminus V}) d\bar{\sigma}_{[-N, N] \setminus V} \\ = \int_B \phi(\sigma_{W_N} | V) p_{W_N}^{\beta, +}(\sigma_{W_N}) d\sigma_{W_N},$$

which is just the partial case of the DLR equation for the Ising model Gibbs measure (see (2.7), (2.10)). Since

$$\int_B \phi(\sigma_{W_N} | V) p_{W_N}^{\beta, +}(\sigma_{W_N}) d\sigma_{W_N} \rightarrow \int_B \phi(\sigma_V) \mathbb{P}_T^{\beta, +}(d\sigma)$$

as $N \rightarrow \infty$, that proves our statement.

9. Concluding Remarks

The prediction of Remark 2 turned out to be true. Recently in the paper by Bricmont, Kupiainen and Lefevre the corresponding result was proven for the case of the projection of the d -dimensional Ising model to the sublattice $b\mathbb{Z}^d$.

In the paper [MV] of Ch. Maes and Van de Velde a result close to the results of the present paper is obtained. The difference between [MV] and the present paper is that in [MV] the authors use the one-dimensionality of the problem considered in a more essential way. Namely, they use the observation that any one-dimensional potential $\mathcal{U} = (U_A(\sigma_A), A \subset \mathbb{Z}^1, 0 < |A| < \infty)$ can be reduced to the potential $\mathcal{U}^* = (U_{[a, b]}^*(\sigma_{[a, b]}), [a, b] \subset \mathbb{Z}^1, -\infty < a \leq b < +\infty)$. The reduction is given by the formula

$$U_{[a, b]}^*(\sigma_{[a, b]}) = \sum_{\substack{A \subset [a, b]; \\ a \in A, b \in A}} U_A(\sigma_{[a, b]} | A).$$

The potential \mathcal{U}^* , constructed in [MV], is related to the potential of the present paper by the above summation. Unfortunately, the paper [MV] contains several erroneous statements; it seems, though, that they can be corrected.

The subject of the almost Gibbsian fields is also treated in [ES, MS].

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