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Sojourn time for rank one perturbations

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We consider a self-adjoint, purely absolutely continuous operator M. Let P be a rank one operator $Pu=\langle\varphi,u\rangle\varphi$ such that for β_0 $H_{\beta_0}:=M+\beta_0 P$ has a simple eigenvalue E_0 embedded in its absolutely continuous spectrum, with corresponding eigenvector ψ. Let H_ω be a rank one perturbation of the operator H_{β_0}, namely, $H_\omega=M+(\beta_0+\omega)P$. Under suitable conditions, the operator H_ω has no point spectrum in a neighborhood of E_0, for ω small. Here, we study the evolution of the state ψ under the Hamiltonian H_ω, in particular, we obtain explicit estimates for its sojourn time $\tau_\omega(\psi)=\int_{-\infty}^{\infty}|\langle\psi,e^{-iH_\omega t}\psi\rangle|^2\,dt$. By perturbation theory, we prove that $\tau_\omega(\psi)$ is finite for $\omega\neq 0$, and that for ω small it is of order ω^{-2}. Finally, by using an analytic deformation technique, we estimate the sojourn time for the Friedrichs model in $\mathbb{R}^n$. *© 2006 American Institute of Physics.* [DOI: 10.1063/1.2174236]

I. INTRODUCTION

A rank one perturbation of an operator H_0 may drastically change the nature of its spectrum. See for instance Refs. 2 and 16. Here we study a class of perturbations for a self-adjoint operator H_0 having a simple eigenvalue E_0 embedded in its absolutely continuous spectrum. We impose general conditions on the rank one perturbation P, $Pu=\langle\varphi,u\rangle\varphi$, which guarantee that the operator $H_\omega=H_0+\omega P$ is purely absolutely continuous. Moreover if H_0 has a normalized eigenvector ψ_0 with corresponding eigenvalue E_0 then we explicitly estimate the sojourn time $\tau_\omega(\psi_0)$, precisely,

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$$\tau_\omega(\psi_0) \sim \frac{1}{\pi}\frac{1}{\omega^2}\frac{1}{|\langle P\psi_0,\psi_0\rangle|^2\mu'_{\psi_0}(E_0)}$$

under conditions which guarantee in particular suitable regularity of the spectral measure μ_{ψ_0} of M.

In the case where M is the multiplication operator by x and φ is analytic (in the sense of Sec. V) we can use the analytic translation technique to prove that

$$\tau_\omega(\psi_0) = \frac{1}{2|\Gamma|} + O\left(\frac{1}{|\omega|}\right),$$

where $1/2\Gamma$ is the Fermi golden rule term,

$$\frac{1}{2}\Gamma = \omega^2 \operatorname{Im}\langle P\psi_0, S(E_0+i0)P\psi_0\rangle.$$

Here, $S(E_0+i0)$ is the reduced resolvent of H_{β_0} at E_0 (Ref. 8, Chap. III, Sec. 6.5, for the definition of the reduced resolvent).

There are numerous works which describe resonances by analyzing the behavior of the survival amplitude, i.e., the function $\mathbb{R} \ni t \mapsto \langle e^{-itH}\psi_0, \psi_0\rangle$, which, in many cases, include explicit exponential decay laws for this quantity. We mention, Refs. 1, 3, 4, 6, 10, 11, 17, 19, 18, and 20, for example. On the other hand, as it was suggested in Ref. 10, the study of the sojourn time seems to be an approach to resonances more general than analytic continuation techniques. (See Refs. 7, 13, and 15.)

The present work is also an attempt to give a dynamical characterization of quantum resonances, by estimating directly the sojourn time, that is, the L^2 norm of this survival amplitude, which measures the total amount of time that well chosen states remain on itself. We expect that this direct approach of the sojourn time will allow less regular Hamiltonian than the ones considered with the survival amplitude method. For example, in the concrete Friedrichs model, see Sec. IV A, we need only that our perturbation P is twice differentiable with respect to the x variable, which seems to be a rather weak condition in view of the above quoted papers. This hope must be tested with a genuine potential perturbation. Also, contrary to what is usually done, we have not localized ψ_0 within an ad hoc spectral subspace of H_ω, i.e., $g(H_\omega)\psi_0$, where g is a function localized around the embedded eigenvalue E_0; this is mainly because our assumptions on M do not allow neither thresholds nor other eigenvalues than E_0.

This paper has the following structure: in the first two sections we establish some technical facts. The following section contains our main result, the asymptotics of the sojourn time. In the last section we use the analytic translation technique to establish the connection with resonance theory.

II. RANK ONE PERTURBATIONS OF SELF-ADJOINT OPERATORS

Although the content of this section is classical, we include it for the reader's convenience. Let M be a self-adjoint operator on a Hilbert space $\mathcal{H}$. We consider rank one perturbations of M,

$$H_\beta = M + \beta P, \tag{2.1}$$

where $\beta \in \mathbb{R}$ and $P = |\varphi\rangle\langle\varphi|$ denotes the orthogonal projection $Pu = \langle\varphi, u\rangle\varphi$ and $\|\varphi\| = 1$.

Let E_x^M be the resolution of the identity associated to M, that is $M = \int_{-\infty}^{\infty} x \, dE_x^M$ and let $\mu_\varphi(x) := \langle\varphi, E_x^M\varphi\rangle$. For $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ we consider the Borel transform of the measure μ_φ, that is

$$F_\varphi(z) = \langle\varphi, (M-z)^{-1}\varphi\rangle. \tag{2.2}$$

Lemma 2.1: Assume E_0 is not an eigenvalue of M. Given $\beta \neq 0$, the real number E_0 is an eigenvalue of H_β if and only if $\lim_{\epsilon\to 0}(M-E_0-i\epsilon)^{-1}\varphi$ exists in $\mathcal{H}$ and $F_\varphi(E_0+i0) = -1/\beta$.

Proof: If $H_\beta\psi_0 = E_0\psi_0$, ψ_0 normalized, then,

$$M\psi_0 + \beta\langle\varphi,\psi_0\rangle\varphi = E_0\psi_0. \tag{2.3}$$

Since E_0 is not an eigenvalue of M, we have that $\langle\psi_0,\varphi\rangle \neq 0$ and φ belongs to the range of $M - E_0$.

On the other hand,

$$(M - E_0)(M - E_0 - i\epsilon)^{-1}\psi_0 = -\beta\langle\varphi,\psi_0\rangle(M - E_0 - i\epsilon)^{-1}\varphi.$$

But $(M-E_0)(M-E_0-i\epsilon)^{-1}$ is a bounded operator which is strongly convergent to the identity as ϵ approaches 0. By computing such limit we have that $\psi := \lim_{\epsilon\to 0}(M-E_0-i\epsilon)^{-1}\varphi$ exists and it satisfies $\psi_0 + \beta\langle\varphi,\psi_0\rangle\psi = 0$. It follows that,

$$F_\varphi(E_0 + i0) = \langle\varphi,\psi\rangle = -\frac{1}{\beta}.$$

Conversely, if both conditions hold, then $\psi := \lim_{\epsilon\to 0}(M-E_0-i\epsilon)^{-1}\varphi$ satisfies $\langle\varphi,\psi\rangle = -1/\beta$. Consider $\psi(\epsilon) = (M-E_0-i\epsilon)^{-1}\varphi$. Then,

$$(M - E_0)\psi(\epsilon) = (M - E_0)(M - E_0 - i\epsilon)^{-1}\varphi,$$

converges to φ, when ϵ approaches 0. Since M is a closed operator, we obtain that ψ belongs to the domain of M and $(M-E_0)\psi = \varphi$.

Hence, $M\psi - \varphi = E_0\psi$. The eigenvalue equation (2.3) follows from the identity $-1 = \beta\langle\varphi,\psi\rangle$. □

Corollary 2.1: Let E_0 be an eigenvalue of H_β. Then E_0 is simple with eigenvector $\psi = \lim_{\epsilon\to 0}(M-E_0-i\epsilon)^{-1}\varphi$. Also,

$$\|\psi\|^2 = \int_{-\infty}^{\infty}\frac{d\mu_\varphi}{(x - E_0)^2} \quad \textit{and}\ \langle\psi,\varphi\rangle = \int_{-\infty}^{\infty}\frac{d\mu_\varphi}{x - E_0}. \tag{2.4}$$

Proof: The first part of the corollary follows from the proof of the Lemma 2.1. To prove formula (2.4), we note that since $\|\psi\|^2 = \lim_{\epsilon\to 0}\|(M-E_0-i\epsilon)^{-1}\varphi\|^2$ exists, by the monotone convergence theorem,

$$\int_{\mathbb{R}}\frac{d\mu_\varphi}{(x - E_0)^2} = \lim_{\epsilon\to 0}\int_{\mathbb{R}}\frac{d\mu_\varphi}{(x - E_0)^2 + \epsilon^2} = \|\psi\|^2.$$

So the integrals $\int_{\mathbb{R}}[d\mu_\varphi/(x-E_0)^2]$ and $\int_{\mathbb{R}}(d\mu_\varphi/|x-E_0|)$ are finite. The Lebesgue theorem then implies

$$\langle\psi,\varphi\rangle = \lim_{\epsilon\to 0}\int_{\mathbb{R}}\frac{d\mu_\varphi}{(x - E_0) + i\epsilon} + \int_{\mathbb{R}}\frac{d\mu_\varphi}{x - E_0}.$$

□

We now consider a family of rank one perturbations of the operator $H_{\beta_0} = M + \beta_0 P$ on $\mathcal{H}$, explicitly,

$$H_\beta = M + \beta|\varphi\rangle\langle\varphi|, \tag{2.5}$$

where $\varphi \in \mathcal{H}$ is a fixed unit vector. We are mainly interested on studying the time behavior of possible bound states of H_{β_0}, under the Hamiltonian H_β, for β near β_0.

Let us assume that $H_{\beta_0} = M + \beta_0|\varphi\rangle\langle\varphi|$ has an eigenvalue E_0 with corresponding eigenvector ψ. Because of Lemma 2.1 and Corollary 2.1, this means that $1 + \beta_0 \lim_{\epsilon\to 0} F_\varphi(E_0 + i\epsilon) = 0$ and

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\varphi}{(x-E_0)^2} < \infty.$$

Our goal is to study the time evolution of ψ under the perturbed Hamiltonian H_β for the parameter ω close to zero.

For this purpose, we study the function

$$\mathbb{R} \ni t \mapsto |\langle \psi, e^{-iH_\beta t}\psi\rangle|^2$$

which represents the probability of finding at time t the system in its initial state ψ, and

$$\tau_\omega(\psi) = \int_{-\infty}^{\infty} |\langle \psi, e^{-iH_\beta t}\psi\rangle|^2 \,\mathrm{d}t, \quad \omega = \beta - \beta_0 \tag{2.6}$$

which measures the total amount of time the state remains in its initial subspace $\{s\psi : s \in \mathbb{C}\}$. We observe that $|\langle \psi, e^{-iH_{\beta_0}t}\psi\rangle|^2 = 1$ for all t and so $\tau_0(\psi)$ is infinite.

We shall prove that for $\omega \neq 0$ and small, the sojourn time $\tau_\omega(\psi)$ is finite and of order ω^{-2}, when the operator H_β has no bound states [see assumption (H3)].

III. FINITENESS OF THE SOJOURN TIME OF ψ

Let H be a self-adjoint operator on a complex Hilbert space $\mathcal{H}$. For any vector ϕ in the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}(H)$ associated to H, we know that

$$\langle \phi, e^{-iHt}\phi\rangle = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-i\lambda t}\, \mathrm{Im}\langle \phi, (H-\lambda-i0)^{-1}\phi\rangle \mathrm{d}\lambda.$$

This allows us to express the sojourn time in terms of resolvent operators, explicitly,

$$\tau(\psi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathrm{Im}\langle \psi, (H-\lambda-i0)^{-1}\psi\rangle|^2 \,\mathrm{d}\lambda. \tag{3.1}$$

It is known that (2.6) and (3.1) are both valid expressions for the sojourn time, as soon as the survival amplitude $\mathbb{R} \ni t \mapsto (e^{-itH}\varphi, \varphi)$ is in $L^2(\mathbb{R})$, see Ref. 12.

In all what follows, we assume the following hypothesis.

(H0): *M is a purely absolutely continuous operator acting on the Hilbert space $\mathcal{H}$, $\varphi \neq 0$ a normalized element of $\mathcal{H}$, $P := |\varphi\rangle\langle\varphi|$* and $H_\beta := M + \beta P$.

(H1): *There exists $\beta_0 \neq 0$, such that H_{β_0} has a unique eigenvalue E_0, $H_{\beta_0}\psi = E_0\psi$.* Notice that, by Lemma 2.1, this implies in particular that $\psi := (M - E_0 \pm i0)^{-1}\varphi$ exists in $\mathcal{H}$ and $\langle \varphi, \varphi\rangle = -\beta_0^{-1}$.

(H2): *The function $\lambda \in \mathbb{R} \mapsto \langle \varphi, (M-\lambda-i0)^{-1}\varphi\rangle \in \mathbb{C}$ is continuous. We assume moreover that*

$$\lim_{|\lambda|\to\infty} \langle \varphi, (M-\lambda-i0)^{-1}\varphi\rangle = 0.$$

Notice that, by the first resolvent equation, this implies in particular that

$$\mathbb{R}\setminus\{E_0\} \ni \lambda \mapsto \langle \psi, (M-\lambda-i0)^{-1}\psi\rangle \in \mathbb{C}$$

exists as a continuous function.

(H3): *There exists $\omega_0 > 0$ such that for all $0 < |\omega| \leq \omega_0$, the function*

$$D(\lambda,\omega) := 1 + (\beta_0 + \omega)\langle \varphi, (M-\lambda-i0)^{-1}\varphi\rangle$$

does not vanish, for any $\lambda \in \sigma(M)$. Clearly, for such ω, the operator $H_{\beta_{0+\omega}}$ has no eigenvalues.

(H4): *The function $\lambda \mapsto \langle \psi, (M-\lambda-i0)^{-1}\psi\rangle \in \mathbb{C}$ is of class C^1 in a neighborhood of E_0, that is,*

$$g(\lambda) := \mathrm{Re}\langle \psi, (M-\lambda-i0)^{-1}\psi\rangle,$$

$$\mu_\varphi'(\lambda) := \frac{1}{\pi}\,\mathrm{Im}\langle\psi,(M-\lambda-i0)^{-1}\psi\rangle$$

exist and they are C^1 *functions in a neighborhood of* E_0.
(H5): *The function* μ_ψ' *belongs to* $L^2(\mathbb{R})$.
(H6): $\mu_\psi'(E_0)>0$.
Concerning the resolvent operators, we use the following notation:

$$R_\omega(z)=(H_\beta - z)^{-1}, \quad R_0(z)=(H_{\beta_0}-z)^{-1}, \quad R(z)=(M-z)^{-1},$$

where $\omega=\beta-\beta_0$. The Aronszjan-Krein formula[9,16] expresses the resolvent $R_\omega(z)$ in terms of $R_0(z)$ for any $z=\lambda+i\epsilon\in\mathbb{C}$, with $\epsilon\neq 0$. This formula gives

$$R_\omega = R_0 - \frac{\omega}{1+\omega\langle\varphi,R_0\varphi\rangle}R_0PR_0. \tag{3.2}$$

Similarly,

$$R_0 = R - \frac{\beta_0}{1+\beta_0\langle\varphi,R\varphi\rangle}RPR. \tag{3.3}$$

Again, we use the Borel transform $F_\varphi(z)=\langle\varphi,(M-z)^{-1}\varphi\rangle$, which, because of (3.3) gives

$$\langle\varphi,R_0(z)\varphi\rangle = \frac{F_\varphi(z)}{1+\beta_0F_\varphi(z)}.$$

Thus, by replacing in (3.2),

$$R_\omega = R_0 - \omega\frac{1+\beta_0F_\varphi}{1+(\beta_0+\omega)F_\varphi}R_0|\varphi\rangle\langle\varphi|R_0. \tag{3.4}$$

Since, $\mu_\varphi(\lambda)=\langle\varphi,E_\lambda^M\varphi\rangle$, for any positive ϵ we have that

$$F_\varphi(z)=\int_{-\infty}^{\infty}\frac{x-\lambda}{(x-\lambda)^2+\varepsilon^2}\mathrm{d}\mu_\varphi(x) + i\int_{-\infty}^{\infty}\frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}\mathrm{d}\mu_\varphi(x). \tag{3.5}$$

By our hypothesis, we have that $\mathrm{d}\mu_\varphi(\lambda)=\mu_\varphi'(\lambda)\mathrm{d}\lambda=(1/\pi)\mathrm{Im}\,F_\varphi(\lambda+i0)\mathrm{d}\lambda$, where μ_φ' denotes the Radon-Nikodym derivative of the measure μ_φ relative to the Lebesgue measure. So the limit when $\varepsilon\to 0$ in (3.5) exists thanks to (H6) and gives

$$F_\varphi(\lambda+i0) = PV\int_{-\infty}^{\infty}\frac{\mu_\varphi'(x)}{x-\lambda}\mathrm{d}x + i\pi\mu_\varphi'(\lambda), \tag{3.6}$$

where the first term on the right-hand side of (3.6) is the Cauchy principal value.

Lemma 3.1: Suppose that (H0), (H1), (H2), (H3), and (H5) hold. Then, for all $0<|\omega|\leqslant\omega_0$ *is finite and*

$$\tau_\omega(\psi) = \frac{2\pi\omega^4}{\beta_0^4}\int_{\mathbb{R}}\frac{|\mu_\psi'(\lambda)|^2}{|D(\omega,\lambda)|^4}\mathrm{d}\lambda, \tag{3.7}$$

where

$$D(\lambda,\omega) := 1+(\beta_0+\omega)F_\varphi(\lambda+i0). \tag{3.8}$$

Proof: From the formula (3.1), we only need to compute $\mathrm{Im}\langle\psi,R_\omega(\lambda+i0)\psi\rangle$ for $\lambda\neq E_0$.

Since $H_{\beta_0}\psi=E_0\psi$ and $\psi=(M-E_0-i0)^{-1}\varphi$, we have that $\beta_0\langle\psi,\varphi\rangle=-1$. Hence, the identity (3.4) gives

$$\langle \psi, R_\omega(z)\psi\rangle = \frac{1}{E_0 - z}\|\psi\|^2 - \omega \frac{1+\beta_0 F_\varphi(z)}{1+(\beta_0+\omega)F_\varphi(z)} \frac{1}{\beta_0^2 (E_0 - z)^2}.$$

Consider $z=\lambda+i\epsilon$ with $\epsilon>0$. Hypothesis (H2) allows us to compute the limit as $\varepsilon\to 0$ to obtain that

$$\langle \psi, R_\omega(\lambda+i0)\psi\rangle = \frac{1}{(E_0-\lambda)}\|\psi\|^2 - \frac{\omega}{\beta_0^2(E_0-\lambda)^2} \lim_{\varepsilon\to 0} \frac{1+\beta_0 F_\varphi(\lambda+i\epsilon)}{1+(\beta_0+\omega)F_\varphi(\lambda+i\epsilon)}.$$

By taking the imaginary part, it follows immediately that

$$\operatorname{Im}\langle \psi, R_\omega(\lambda+i0)\psi\rangle = -\frac{\omega}{\beta_0^2(E_0-\lambda)^2} \lim_{\varepsilon\to 0} \operatorname{Im} \frac{1+\beta_0 F_\varphi(z)}{1+(\beta_0+\omega)F_\varphi(z)}.$$

Now,

$$\operatorname{Im} \frac{1+\beta_0 F_\varphi(z)}{1+(\beta_0+\omega)F_\varphi(z)} = \frac{\omega \operatorname{Im} \overline{F_\varphi}(z)}{|1+(\beta_0+\omega)F_\varphi(z)|^2}.$$

Hence, by using that $[1/(E_0-\lambda)^2]\mu'_\varphi(\lambda)=\mu'_\psi(\lambda)$ we conclude that

$$\operatorname{Im}\langle \psi, R_\omega(\lambda+i0)\psi\rangle = -\frac{\omega^2}{\beta_0^2(E_0-\lambda)^2} \lim_{\varepsilon\to 0} \frac{\operatorname{Im} \overline{F_\varphi}(\lambda+i\epsilon)}{|1+(\beta_0+\omega)F_\varphi(\lambda+i\epsilon)|^2} = \frac{\omega^2}{\beta_0^2} \frac{\pi \mu'_\psi(\lambda)}{|D(\lambda,\omega)|^2}$$

for any $\lambda \neq E_0$, thus ending the proof. The integral of (3.7) makes sense, i.e., is finite, thanks to (H2), (H3), and (H5). □

IV. EXACT ASYMPTOTICS FOR τ_ω

In this section we prove the explicit asymptotics for the sojourn time in a general context and we will apply it to some concrete examples.

Theorem 4.1: *Assume that* (H0)–(H6) *holds. Then the sojourn time* $\tau_\omega(\psi)$ *has the following behavior:*

$$\lim_{\omega\to 0} \omega^2 \tau_\omega(\psi) = \frac{1}{\pi} \frac{\beta_0^2 \|\psi\|^{10}}{\mu'_\varphi(E_0)}.$$

Proof: By using the first resolvent equation and (H1), we can rewrite $D(\lambda,\omega)$ as

$$D(\lambda,\omega) = 1 + \beta\langle \varphi, R(\lambda+i0)\varphi\rangle = 1 - \frac{\beta}{\beta_0} + \beta(\lambda - E_0)\langle \psi, R(\lambda+i0)\varphi\rangle.$$

By (H4) and applying the resolvent equation twice, we obtain that

$$D(\lambda,\omega) = -\frac{\omega}{\beta_0} + \beta(\lambda-E_0)q(\lambda) + i\pi\beta(\lambda-E_0)^2\mu'_\psi(\lambda)$$

with

$$q(\lambda) := \|\psi\|^2 + (\lambda - E_0)g(\lambda).$$

By Lemma 3.1 we known that $\tau_\omega(\psi)$ is finite and it is represented by (3.7).

Consider the interval $I_\delta := \{\lambda \in \mathbb{R}, |\lambda - E_0| \leq \delta\}$ for some $\delta>0$. Let us split the integral in two parts,

$$A_\delta := \frac{2\pi\omega^6}{\beta_0^4}\int_{I_\delta}\frac{|\mu_\psi'(\lambda)|^2}{|D(\lambda,\omega)|^4}\mathrm{d}\lambda, \quad B_\delta := \frac{2\pi\omega^6}{\beta_0^4}\int_{\mathbb{R}\setminus I_\delta}\frac{|\mu_\psi'(\lambda)|^2}{|D(\lambda,\omega)|^4}\mathrm{d}\lambda.$$

We will prove first that $B_\delta \to 0$ as $\omega \to 0$ and second that A_δ tends to the desired limit using the Lebesgue dominated convergence theorem.

We get at once that $B_\delta = \mathcal{O}(\omega^6)$ since $|D(\lambda,\omega)| \geqslant c_\delta > 0$ on $\mathbb{R}\setminus I_\delta$, and μ_ψ' belongs to $L^2(\mathbb{R})$ by assumption (H5).

For $\omega \neq 0$ one defines

$$s_\omega(\lambda) := \omega^{-2}(-\omega\beta_0^{-1} + \beta(\lambda - E_0)q(\lambda)).$$

Thanks to (H4) we know that $g(\lambda)$ is a C^1 function in a neighborhood I_{δ_g} of E_0. Then s is a C^1 function with derivative

$$s_\omega'(\lambda) = \frac{\beta(q(\lambda) + (\lambda - E_0)q'(\lambda))}{\omega^2}$$

in I_{δ_g} and $s_\omega'(E_0) = \omega^{-2}\beta\|\psi\|^2$.

We choose $\omega_0 \leqslant |\beta_0|/2$ so that β does not vanish and therefore E_0 is not a critical point of s_ω. By the inverse mapping theorem we know that s_ω is locally in E_0 a C^1 diffeomorphism, i.e., we can choose $\delta_s > 0$ small enough so that the restriction of s_ω to I_{δ_s} is such a C^1-diffeomorphism. In view of the explicit dependence of s_ω with respect to ω one can choose this δ_s independently of ω. We continue to denote δ_s by δ. We note that

$$s \in \tilde{I}_\delta \Leftrightarrow \left| s + \frac{1}{\omega\beta_0} \right| < \delta\frac{|\beta q(\lambda(s))|}{\omega^2}.$$

Let $\tilde{I}_\delta \ni s \mapsto \lambda_\omega(s) \in I_\delta$ be the inverse of the previous change of variable. The quantities D and A_δ can now be written as

$$D(\lambda_\omega(s),\omega) = \omega^2 s + \frac{i\pi}{\beta}\left(\frac{\omega^2 s + \omega\beta_0^{-1}}{q(\lambda_\omega(s))}\right)^2 \mu_\psi'(\lambda_\omega(s)) =: \omega^2\tilde{D}(s,\omega)$$

with

$$\tilde{D}(s,\omega) = s + \frac{i\pi}{\beta}\left(\frac{\omega s + \beta_0^{-1}}{q(\lambda_\omega(s))}\right)^2 \mu_\psi'(\lambda_\omega(s))$$

and

$$\begin{aligned} A_\delta &= \frac{2\pi\omega^6}{\beta_0^4}\int_{\tilde{I}_\delta}\frac{|\mu_\psi'(\lambda_\omega(s))|^2}{\omega^8|\tilde{D}(s,\omega)|^4}\frac{\omega^2}{\beta(q(\lambda_\omega(s)) + (\lambda_\omega(s) - E_0)q'(\lambda_\omega(s)))}\mathrm{d}s \\ &= \frac{2\pi}{\beta\beta_0^4}\int_{\mathbb{R}} 1_{\tilde{I}_\delta}\frac{|\mu_\psi'(\lambda_\omega(s))|^2}{|\tilde{D}(s,\omega)|^4}\frac{1}{q(\lambda_\omega(s)) + (\lambda_\omega(s) - E_0)q'(\lambda_\omega(s))}\mathrm{d}s, \end{aligned}$$

where $1_{\tilde{I}_\delta}$ denotes the characteristic function of I_δ.

We know that $q(\lambda) = \|\psi\|^2 + (\lambda - E_0)g(\lambda)$ and g is assumed to be C^1 on I_{δ_g} so g is bounded there. Therefore we can choose $0 < \delta_q < \min\{\delta_g, \delta_s\}$ small enough such that $q(\lambda) \geqslant \frac{1}{2}\|\psi\|^2$ on I_{δ_q}. In view of

$$\forall s \in I_{\delta_s}, \quad \omega^2 s + \omega\beta_0^{-1} = \beta(\lambda(s) - E_0)q(\lambda(s))$$

we see at once that $\lim_{\omega\to 0}\lambda_\omega(s) = E_0$. Then, by assumption (H4),

$$\lim_{\omega\to 0}\mu'_\psi(s)=\mu'_\psi(E_0),\quad \lim_{\omega\to 0}\tilde{D}(s,\omega)=s+\frac{i\pi}{\beta_0^3\|\psi\|^4}\mu'_\psi(E_0)$$

and

$$\lim_{\omega\to 0}\frac{1}{(q(\lambda_\omega(s))+(\lambda_\omega(s)-E_0)q'(\lambda_\omega(s)))}=\frac{1}{\|\psi\|^2}.$$

Therefore,

$$\lim_{\omega\to 0}(\text{integrand})=\frac{2\pi}{\beta_0^5}\frac{|\mu'_\psi(E_0)|^2}{\left|s^2+\dfrac{\pi^2|\mu'_\psi(E_0)|^2}{\beta_0^6\|\psi\|^8}\right|^2}=\frac{1}{\|\psi\|^2}.$$

By formal integration one arrives at

$$\frac{2\pi}{\beta_0^5}\frac{|\mu'_\psi(E_0)|^2}{\|\psi\|^2}\int_{\mathbb{R}}\frac{1}{\left|s^2+\dfrac{|\pi\mu'_\psi(E_0)|^2}{\beta_0^6\|\psi\|^8}\right|^2}\mathrm{d}s=\frac{2\pi}{\beta_0^5}\frac{|\mu'_\psi(E_0)|^2}{\|\psi\|^2}\frac{\pi}{2\left(\dfrac{\pi|\mu'_\psi(E_0)|}{\beta_0^3\|\psi\|^4}\right)^3}=\frac{\beta_0^4}{\pi}\frac{\|\psi\|^{10}}{|\mu'_\psi(E_0)|}.$$

Then, it remains to justify the interchange of the limit with the integral.

First we have

$$|\tilde{D}(s,\omega)|^2=s^2+\frac{\pi^2}{\beta^2}\left(\frac{\omega s+\beta_0^{-1}}{q(\lambda_\omega(s))}\right)^4(\mu'_\psi(\lambda_\omega(s)))^2.$$

Due to assumptions (H4), (H6) we know that there exists a neighborhood of E_0 on which $\mu'_\psi>0$. Let K be a compact interval which contains 0 in its interior, the image of K in the λ variable is a compact interval which contains E_0 and shrinks as $\omega\to 0$. So for $|\omega|$ small enough we are sure that $\mu'_\psi(\lambda_\omega(s))^2\geqslant c>0$ with c independent of ω. Since we can also easily make that the quantity $(1/\beta^2)(\omega s+\beta_0^{-1}/q(\lambda_\omega(s)))^4$ is also bounded below uniformly with respect to ω we get that

$$\exists\,\omega_0>0,\quad \forall\, s\in K,\quad \forall\,|\omega|\leqslant\omega_0,\quad |\tilde{D}(s,\omega)|^2\geqslant c>0.$$

It follows that

$$\forall\, s\in\tilde{I}_\delta,\quad |\tilde{D}(s,\omega)|^2\geqslant\max\{s^2,c\}.$$

Then since g is C^1 on I_{δ_g} one has that $\lambda\mapsto q(\lambda)-(\lambda-E_0)q'(\lambda)$ is C^0 on I_{δ_g} and since $q(E_0)=\|\psi\|^2>0$ one can choose $\delta:=\delta_g$ small enough so that

$$\forall s\in\tilde{I}_\delta,\quad q(\lambda_\omega(s))+(\lambda_\omega(s)-E_0)q'(\lambda_\omega(s))\geqslant c>0.$$

Finally thanks to (H4) we can choose δ small enough so that $|\mu'_\psi|\leqslant c_1$ on I_δ. In conclusion we have obtained

$$\exists\,\delta>0,\quad \exists\,\omega_0>0,\quad \forall\, s\in\tilde{I}_\delta,\quad \forall\,|\omega|\leqslant\omega_0,\quad 0\leqslant\text{integrand}\leqslant\frac{4\pi}{\beta_0^5}\frac{c_1^2}{\max\{s^4,c^2\}c}$$

and since outside $\tilde{I}_\delta$ the integrand vanishes this bound is valid on $\mathbb{R}$. □

A. The Friedrichs model

The Friedrichs model corresponds to a very simple choice of the operator M, namely, the multiplication operator $M\varphi(x)=x\varphi(x)$ acting in $\mathcal{H}:=L^2(\mathbb{R})$. The operator M with domain $D(M)=\{\varphi:x\varphi\in L^2(\mathbb{R})\}$ is purely absolutely continuous with spectrum $\sigma(M)=\mathbb{R}$. Moreover, its spectral measure is given by $\mu'_\varphi(x)=\langle\varphi,E_x^M\varphi\rangle=|\varphi(x)|^2$.

We consider the rank one perturbation $H_\beta=M+\beta|\varphi\rangle\langle\varphi|$. In order to verify our hypothesis, we impose on the vector φ the following conditions.

(F0): φ is normalized in $\mathcal{H}$.

(F1): φ belongs to the Sobolev space $\mathcal{H}^2(\mathbb{R})$.

(F2): φ has a unique zero at $x=E_0$.

(F3): $\beta_0\neq 0$ and φ satisfy the relation

$$1+\beta_0\int_{-\infty}^{\infty}\frac{|\varphi(x)|^2}{x-E_0}\mathrm{d}x=0.$$

(F4): $\varphi'(E_0)\neq 0$, and $\varphi\in C^2$ in a neighborhood of E_0.

Clearly (H0) holds. On the other hand, conditions (F1)–(F3) imply hypothesis (H1)–H(5). Also, (F4) guarantees condition (H6). By Theorem 4.1 we deduce the following result.

Theorem 4.2 (Friedrichs model): *Assume that* (F1)–(F4) *holds. Then the sojourn time* $\tau_\omega(\psi)$ *has the following behavior:*

$$\lim_{\omega\to 0}\omega^2\tau_\omega(\psi)=\frac{1}{\pi}\frac{\beta_0^4\|\psi\|^{10}}{|\psi(E_0)|^2},$$

or else if $\psi_0:=\psi\|\psi\|^{-1}$ *denotes the normalized eigenvector then*

$$\tau_\omega(\psi_0)\overset{\omega\to 0}{\sim}\frac{1}{\pi}\frac{1}{\omega^2}\frac{1}{|\langle P\psi_0,\psi_0\rangle\psi_0(E_0)|^2}.$$

Actually conditions (F1)–(F4) can be relaxed by asking local properties, around E_0, for the vector φ.

B. Model $M=X^2$

Another choice of the operator M is the multiplication by x^2 in $\mathcal{H}:=L^2(\mathbb{R})$, which it is purely absolutely continuous with spectrum $\sigma(M)=[0,\infty)$. Its spectral measure is just

$$\mu'_\phi(\lambda)=\begin{cases}0 & \text{if } \lambda<0,\\ \dfrac{|\phi(\sqrt{\lambda})|^2+\ |\phi(-\sqrt{\lambda})|^2)}{2\sqrt{\lambda}} & \text{if } \lambda>0.\end{cases}$$

Again, we consider rank one perturbations $H_\beta=M+\beta|\varphi\rangle\langle\varphi|$. As we mentioned in the Introduction, in order to apply our results we must choose φ as $g(M)$ times an adequate function, where g is a suitable cutoff function. It is easy to verify that if we just take $g(x)=x^2$ and φ_0 satisfies

(1) φ_0 is in the Schwartz' class and $\varphi_0(x)>0$, for all $x\in\mathbb{R}$,
(2) $\int_{-\infty}^{\infty}(x^2-E_0)x^4|\varphi_0|^2=-1/\beta_0$,
(3) $\int_{-\infty}^{\infty}(x^2-E_0)x^4|\varphi_0|^2\neq\int_{-\infty}^{\infty}(x^2-E_0)^2x^2|\varphi_0|^2$, and
(4) $\int_{-\infty}^{\infty}(x^2-E_0)^2x^4|\varphi_0|^2=1$.

then, with E_0 positive and $\varphi(x)=g(M)(x^2-E_0)\varphi_0(x)=x^2(x^2-E_0)\varphi_0(x)$ all the hypothesis (H0)–(H6) are satisfied.

V. SOJOURN TIME AND RESONANCES DEFINED BY ANALYTIC TRANSLATION

This section is concerned with the connection between the sojourn time and the resonance width which can be shown for the Friedrichs model in $L^2(\mathbb{R}^n)$, $n \geq 1$. We first recall its definition.

Let $\mathbf{e}=(e_1,e_2,\ldots,e_n)$ be a unit vector in $\mathbb{R}^n$ and consider the multiplication operator by the function $\mathbf{e}\cdot x=\sum_{i=1}^n e_i x_i$, i.e., the operator

$$(M\phi)(x) = \mathbf{e}\cdot x\phi(x). \tag{5.1}$$

We denote by $\mathcal{D}$ its natural domain. Let now $\varphi \in L^2(\mathbb{R}^n)$, $\|\varphi\|=1$ and denote by P the projector on the vector φ. Then the Friedrichs operators,

$$H(\beta) = M + \beta P, \quad \beta \in \mathbb{R} \tag{5.2}$$

with domain $\mathcal{D}$ are well defined in $L^2(\mathbb{R}^n)$ as self-adjoint operators. We are interested in the study of the sojourn time associated to the dynamics defined from the family of Hamiltonians $\{H_\beta, \beta \in \mathbb{R}\}$ for a dense subset of vectors of $L^2(\mathbb{R}^n)$ (Ref. 12, p. 88). We shall estimate the time evolution under $H(\beta)$ and the sojourn time, following a method developed by Herbst in the context of Stark effect see, e.g., Refs. 3 and 4. From now we use the notation $H := H(\beta)$.

For $\theta \in \mathbb{R}$, define the following family of unitary transformations:

$$\forall \phi \in L^2(\mathbb{R}^n), \quad (U_\theta\phi)(x) := \phi(x-\theta\mathbf{e}). \tag{5.3}$$

Then, U_θ; $\theta \in \mathbb{R}$ is a strongly continuous one-parameter unitary group and

$$H_\theta := U_\theta H U_\theta^{-1} = M_\theta + \beta P_\theta, \quad M_\theta = \mathbf{e}\cdot x - \theta, \tag{5.4}$$

where P_θ is the projector on the span generated by the vector $\varphi_\theta := U_\theta\varphi$.

Our general assumption is the following.

(HA): There exists some $a>0$ such that the vector valued function, $\mathbb{R} \ni \theta \to \varphi_\theta \in L^2(\mathbb{R}^n)$ has an analytic extension in the strip $\mathcal{S}_a := \{z \in \mathbb{C}, |\mathrm{Im}\, z| < a\}$.

We denote by $\mathcal{D}_a$ the set of vectors satisfying the assumption (HA). This set is a dense subset of $L^2(\mathbb{R}^n)$.[5]

We extend P_θ, for $\theta \in S_a$ by

$$\forall \phi \in L^2(\mathbb{R}^n), \quad P_\theta\phi := \langle \varphi_{\bar\theta}, \phi\rangle \varphi_\theta$$

which is an analytic family of rank one operators. Then $\{H_\theta; \theta \in \mathcal{S}_a\}$ is a self-adjoint analytic family of type A operators.[8] Moreover, due to the Weyl theorem,[14] we have that $\sigma_{\mathrm{ess}}(H_\theta) = \sigma_{\mathrm{ess}}(H_{0,\theta}) = \mathbb{R} - i\,\mathrm{Im}\,\theta$. Our first technical result is the following.

Lemma 5.1: Suppose that (HA) *is satisfied. Then for* $a > \mathrm{Im}\,\theta > 0$ *and* $0 < \epsilon \leq 1$, *there exists a positive energy e such that*

$$\sup\{\|(H_\theta - z)^{-1}\|;\ z \in \mathbb{C},\ |\mathrm{Re}\, z| \geq e,\ \mathrm{Im}\, z \geq -\mathrm{Im}\,\theta(1-\epsilon)\} < \infty. \tag{5.5}$$

Lemma 5.1, together with the discreteness of the spectrum in $\{z \in \mathbb{C}, \mathrm{Im}\, z > -\mathrm{Im}\,\theta\}$ imply that for $a > \mathrm{Im}\,\theta \geq 0$, the operators H_θ have only a finite number of eigenvalues localized in the compact set $\{z \in \mathbb{C}, 0 \geq \mathrm{Im}\, z \geq -\mathrm{Im}\,\theta(1-\epsilon), |\mathrm{Re}\, z| \leq e\}$, $0 < \epsilon \leq 1$. Some of them can be real, in that case they correspond to embedded eigenvalues for the self-adjoint operator H while the nonreal eigenvalues correspond to resonances for the pair (M,H) (see, e.g., Refs. 14 and 4). Notice that the upper bound on the number of eigenvalues given by the proof below diverges when a tends to 0.

Proof: Because of the unitary property for real θ, it is sufficient to choose $\theta = i\eta$, $a > \eta > 0$. Then for $z \in \mathbb{C}$, $\mathrm{Im}\, z > -\eta$ we have

$$\|(M_{i\eta} - z)^{-1}\| \leq (\eta + \mathrm{Im}\, z)^{-1}. \tag{5.6}$$

Recall that by the Aronszjan-Krein formula we get

$$(H_{i\eta}-z)^{-1}=(M_{i\eta}-z)^{-1}-\beta\frac{(M_{i\eta}-z)^{-1}P_{i\eta}(M_{i\eta}-z)^{-1}}{1+\beta\langle\varphi_{-i\eta},(M_{i\eta}-z)^{-1}\varphi_{i\eta}\rangle}. \tag{5.7}$$

Hence the lemma is proven if we show that the denominator on the right-handside (rhs) of (5.7) is uniformly bounded below by a strictly positive constant. Then consider the following integral:

$$I=\int_{\mathbb{R}^n}\mathrm{d}x\frac{|\varphi_{i\eta}(x)|^2}{(|\mathbf{e}\cdot x-\mathrm{Re}\,z|^2+|\eta+\mathrm{Im}\,z|^2)^{1/2}}. \tag{5.8}$$

For $\delta>0$ let $\Omega_\delta:=\{x\in\mathbb{R}^n,|\mathbf{e}\cdot x-\mathrm{Re}\,z|\geqslant\delta\}$ and Ω_δ^c its complement, accordingly let

$$I_\delta=\int_{\Omega_\delta}\mathrm{d}x\frac{|\varphi_{i\eta}(x)|^2}{(|\mathbf{e}\cdot x-\mathrm{Re}\,z|^2+|\eta+\mathrm{Im}\,z|^2)^{1/2}}$$

and $I_\delta^c=I-I_\delta$. Since obviously

$$I_\delta\leqslant\frac{\|\varphi_{i\eta}\|^2}{\delta}$$

we can choose $\delta>0$ such that $I_\delta\leqslant 1/4$. On the other hand,

$$I_\delta^c\leqslant\frac{1}{|\eta+\mathrm{Im}\,z|}\int_{\Omega_\delta^c}|\varphi_{i\eta}(x)|^2\,\mathrm{d}x$$

but the rhs of this last inequality goes to zero as $|\mathrm{Re}\,z|$ goes to infinity, uniformly in $\mathrm{Im}\,z\geqslant-\eta(1-\epsilon)$. Hence there exists a positive energy e such that for $z\in\mathbb{C}$, $|\mathrm{Re}\,z|\geqslant e$, $\mathrm{Im}\,z\geqslant-\eta(1-\epsilon)$, $I_\delta^c\leqslant 1/4$ and then $1+|\langle\varphi_{-i\eta},(M_{i\eta}-z)^{-1}\varphi_{i\eta}\rangle|\geqslant 1/2$ which finishes to prove the lemma. □

We turn now on the dynamics defined by the operators (5.2). Let $\{E_j\}_{j=1,\ldots,N}$ be the real eigenvalues of H and $\{\Pi_j\}_{j=1,\ldots,N}$ the associated orthogonal eigenprojectors. We know from Refs. 14 and 4 that for $j=1,\ldots,N,\{\Pi_j(\theta),\theta\in\mathcal{S}_a\}$ are analytic families of projectors and that for $\mathrm{Im}\,\theta\neq 0$ they coincide with

$$\Pi_j(\theta)=-\frac{1}{2\pi i}\int_{|z-E_j|=\rho}(H_\theta-z)^{-1}\,\mathrm{d}z,\quad \rho>0\text{ and small enough}. \tag{5.9}$$

Fix $\mathrm{Re}\,\theta=0$, $\mathrm{Im}\,\theta=\eta$, $0<\eta<a$ and denote by $\{Z_j\}_{j=1,\ldots,M}$ the set of complex eigenvalues of $H_{i\eta}$ and by $\{\tilde{\Pi}_j(i\eta)\}_{j=1,\ldots,M}$, the associated eigenprojectors defined through the Cauchy integral formula as in (5.9). We have the following.

Theorem 5.1: *Assume* (HA). *Let* $0<\eta<a$ *as above and* $\phi\in\mathcal{D}_a$, *then there exist* $0<\epsilon<1$ *such that for all* $t\geqslant 0$,

$$\langle\phi,e^{-itH}\phi\rangle\overset{t\to\infty}{=}\sum_{j=1..N}e^{-itE_j}\langle\phi,\Pi_j\phi\rangle+\sum_{j=1..M}e^{-itZ_j}\langle\tilde{\phi}_{-i\eta},\tilde{\Pi}_j(i\eta)\tilde{\phi}_{i\eta}\rangle+O_{\phi,\eta}(e^{-(\eta(1-\epsilon))t}). \tag{5.10}$$

Here $\tilde{\phi}_{\pm i\eta}:=(1-\Pi(i\eta))\phi_{\pm i\eta}$ *and* $\Pi:=\Sigma_{j=1\ldots N}\Pi_j$. *If* H *has no eigenvalue take* $\Pi=0$ *on the rhs of (5.10) and similarly for the complex eigenvalues* Z_j *of* $H_{i\eta}$.

It is worth to notice, in particular to well understand (5.18), that due to the analyticity and unitary properties, the coefficients $\{\langle\tilde{\phi}_{-i\eta},\tilde{\Pi}_j(i\eta)\tilde{\phi}_{i\eta}\rangle\}_{j=1,\ldots,M}$ are η independent.

Proof: By using the spectral theorem and since H has no singular continuous spectrum (H is a rank one perturbation of the purely ac operator M, see Ref. 8), for every $\phi\in\mathcal{D}_a$, we have

$$\langle \phi, e^{-itH}\phi\rangle = \sum_{j=1..N} e^{-itE_j}\langle \phi, \Pi_j \phi\rangle + \int_{\mathbb{R}} \mathrm{d}\lambda\, Q_\phi(\lambda) e^{-it\lambda}. \tag{5.11}$$

Here

$$Q_\phi(\lambda) = \frac{1}{2i\pi}(G_{\tilde{\phi}}(\lambda + i0) - G_{\tilde{\phi}}(\lambda - i0)), \tag{5.12}$$

where

$$G_\phi(z) := \langle \phi, (H-z)^{-1}\phi\rangle. \tag{5.13}$$

From our previous discussion, it is clear that $\{G_{\tilde{\phi}}(z), \mathrm{Im}\, z > 0\}$ has a meromorphic extension in $\{\mathrm{Im}\, z > -\eta\}$ given by

$$G_{\tilde{\phi}}(z) = \langle \tilde{\phi}_{-i\eta}, (H_{i\eta} - z)^{-1}\tilde{\phi}_{i\eta}\rangle.$$

Similarly $\{G_{\tilde{\phi}}(z), \mathrm{Im}\, z < 0\}$ has a meromorphic extension in $\mathrm{Im}\, z < \eta$ with the expression

$$G_{\tilde{\phi}}(z) = \langle \tilde{\phi}_{i\eta}, (H_{-i\eta} - z)^{-1}\tilde{\phi}_{-i\eta}\rangle.$$

Moreover by using the formula

$$(H-z)^{-1} = -\frac{1}{z} - \frac{1}{z^2}H + \frac{1}{z^2}H(H-z)^{-1}H$$

and the analyticity properties evoked above, we have for $0 > \mathrm{Im}\, z > -\eta$, $|\mathrm{Re}\, z| \geq e$,

$$Q_\phi(z) = \frac{1}{2i\pi z^2}(\langle H_{-i\eta}\tilde{\phi}_{-i\eta}, (H_{i\eta} - z)^{-1}H_{i\eta}\tilde{\phi}_{i\eta}\rangle - \langle H\tilde{\phi}, (H-z)^{-1}H\tilde{\phi}\rangle) \tag{5.14}$$

which together with Lemma 5.1 immediately implies

$$|Q_\phi(z)| = Q_{\phi,\eta}\left(\frac{1}{|z|^2}\right) \tag{5.15}$$

if $0 > \mathrm{Im}\, z > -\eta(1-\epsilon)$, $0 < \epsilon \leq 1$, and $|\mathrm{Re}\, z|$ large enough. Then the integral on the rhs of (5.11) can be computed by using the Cauchy theorem,

$$\int_{\mathbb{R}} \mathrm{d}\lambda\, Q_\phi(\lambda) e^{-it\lambda} = \sum_{j=1..M} e^{-itZ_j}\ \mathrm{Res}\, G_{\tilde{\phi}}(z)|_{z=Z_j} + e^{-(\eta(1-\epsilon))t}\int_{\mathbb{R}} \mathrm{d}\lambda\, Q_\phi(\lambda - i(\eta(1-\epsilon))\ e^{-it\lambda}. \tag{5.16}$$

The parameter ϵ, $0 < \epsilon < 1$ is chosen such that the operator $H_{i\eta}$ has no complex eigenvalues on the line $\{\lambda - i(\eta - \epsilon); \lambda \in \mathbb{R}\}$. Standard arguments show that the first term of the rhs of (5.16) gives the second term of the rhs of (5.10). By using the estimate (5.15) we get that the second term of the rhs of (5.16) is $O_{\phi,\eta}(e^{-\eta(1-\epsilon)t})$ so that (5.16) implies the theorem. □

Theorem 5.1 provides a general framework to study sojourn times associated to the family of operators $\{H(\beta), \beta > 0\}$ given in (5.2), for the dense set of analytic vectors associated to the one parameter unitary group $\{U_\theta, \theta \in \mathbb{R}\}$.

According to the general context of this paper we consider the following situation. Suppose that $H(\beta_0)$ has only one real eigenvalue E_0 for some β_0. Lemma 2.1 gives a necessary and sufficient condition for this property to take place and Corollary 2.1 asserts that E_0 is simple. Let $\Pi := |\psi_0\rangle\langle\psi_0|$ be the orthogonal projector onto the corresponding eigenvector. We know that E_0 remains a simple eigenvalue of $H_{i\eta}(\beta_0)$ for $a > \eta > 0$. Denote by $\{Z_j\}_{j=1,\ldots,M}$ the (eventual) complex eigenvalues of $H_{i\eta}(\beta_0)$.

We also assume that, for β near β_0 and $\beta \neq \beta_0$, the operator $H(\beta)$ has only continuous spectrum. Then, $H_{i\eta}(\beta)$ has no real eigenvalues and therefore, by the usual perturbation theory, the operator $H_{i\eta}(\beta)$ has an eigenvalue $E(\beta)$ near E_0 with $\operatorname{Im} E(\beta) \neq 0$ and eigenvalues $\{Z_j(\beta)\}_{j=1,\ldots,M}$ near $\{Z_j\}_{j=1,\ldots,M}$.

Using the classical formulas of regular perturbation theory, see Ref. 8, Chap. 2, one gets with $\omega := \beta - \beta_0$,

$$\begin{aligned} E(\beta) &\overset{\omega\to 0}{=} E_0 + \omega\langle \psi_{0,-i\eta}, P_{i\eta}\psi_{0,i\eta}\rangle - \omega^2\langle \psi_{0,-i\eta}, P_{i\eta}S_{i\eta}(E_0)P_{i\eta}\psi_{0,i\eta}\rangle + \mathcal{O}_\eta(\omega^3) \\ &\overset{\omega\to 0}{=} E_0 + \omega\langle \psi_0, P\psi_0\rangle - \omega^2\langle P\psi_0, S(E_0+i0)P\psi_0\rangle + \mathcal{O}_\eta(\omega^3), \end{aligned}$$

where $S_{i\eta}(E_0)$ denotes the reduced resolvent of $H_{i\eta}(\beta_0)$ at E_0, see Ref. 8, Chap. III, Sec. 6.5 for the definition of the reduced resolvent. Let

$$\Gamma := 2\omega^2 \operatorname{Im}\langle P\psi_0, S(E_0+i0)P\psi_0\rangle, \tag{5.17}$$

be the resonance width of $E(\beta)$; one has $\operatorname{Im} E(\beta) = -\frac{1}{2}\Gamma + \mathcal{O}(\omega^3)$. One also obtains that $|Z_j(\beta) - Z_j| = \mathcal{O}(\omega)$ and therefore $|\operatorname{Im} Z_j(\beta)|$ remains uniformly away from zero for ω small enough. Finally the perturbation theory gives the expansion of the eigenprojector associated to $E(\beta)$,

$$\Pi(i\eta,\beta) \overset{\omega\to 0}{=} \Pi(i\eta,\beta_0) + \mathcal{O}_\eta(\omega) \quad \text{with } \Pi(i\eta,\beta_0) = |\psi_{0,i\eta}\rangle\langle\psi_{0,-i\eta}|.$$

Define now $\tau_\omega^\pm$ (the sojourn time in the future, respectively, in the past)

$$\tau_\omega^\pm(\phi) := \int_{\mathbb{R}_\pm} |\langle \phi, e^{-itH(\beta_0+\omega)}\phi\rangle|^2 \, \mathrm{d}t.$$

By using Theorem 5.1 one gets integrating (5.10) over $\mathbb{R}_+$ (notice that here $\tilde{\phi} = \phi$),

$$\tau_\omega^+(\phi) \overset{\omega\to 0}{=} \frac{|\langle \phi_{-\eta}, \Pi(i\eta,\beta)\phi_{i\eta}\rangle|^2}{-2\operatorname{Im} E(\beta)} + \mathcal{O}_{\phi,\eta}(1) \overset{\omega\to 0}{=} \frac{|\langle \phi, \psi_0\rangle|^4}{\Gamma} + \mathcal{O}_{\phi,\eta}(\omega^{-1}).$$

Since clearly $\tau_\omega^- = \tau_\omega^+$ we have proven the following.

Theorem 5.2: *In the conditions stated above let $\phi \in \mathcal{D}_a$ and assume in addition that Γ defined in (5.17) does not vanish. Then for $\beta = \beta_0 + \omega$,*

$$\tau_\omega(\phi) \overset{\omega\to 0}{=} \frac{|\langle \phi_{-i\eta}, \Pi(i\eta,\beta)\phi_{i\eta}\rangle|^2}{-\operatorname{Im} E(\beta)} + \mathcal{O}_{\phi,\eta}(1) \tag{5.18}$$

$$= 2\frac{|\langle \phi, \psi_0\rangle|^4}{\Gamma} + \mathcal{O}_\phi(\omega^{-1}), \tag{5.19}$$

where ψ_0 denotes the normalized eigenvector associated to the eigenvalue E_0 of $H(\beta_0)$.

Remark 5.1: (a) Let us illustrate this result in the one dimensional case. Again with the Aronszjan-Krein formula

$$S(E_0+i0) = R_0(E_0+i0) + (R_0(E_0+i0)\psi_0, \psi_0)\Pi - (R_0(E_0+i0)\Pi + \Pi R_0(E_0+i0))$$

so that

$$\begin{aligned} \frac{1}{2}\Gamma &= \omega^2 \operatorname{Im}\langle P\psi_0, S(E_0+i0)P\psi_0\rangle = \omega^2|(\varphi,\psi_0)|^4 \operatorname{Im}(R(E_0+i0)\psi_0,\psi_0) = \pi\omega^2|\langle \psi_0, P\psi_0\rangle|^2|\psi_0(E_0)|^2 \\ &= \frac{\pi\omega^2|\psi(E_0)|^2}{\beta_0^4\|\psi\|^6} = \frac{\pi\omega^2|\varphi'(E_0)|^2}{\beta_0^4\|\psi\|^6}. \end{aligned}$$

First we see that the assumption $\varphi'(E_0)\neq 0$ guarantees that $\Gamma\neq 0$. Then $\omega^2\tau_\omega(\psi_0) = \langle\psi_0, P\psi_0\rangle|^{-2}|\psi_0(E_0)|^{-2}+\mathcal{O}(\omega)$ which indeed is what we found in Theorem 4.1 with a more precise estimate here on the rate of decay of the remainder.

(b) Instead of requiring that $H(\beta)$ has no real eigenvalue for β near β_0 we could equivalently demand that the resonance width Γ is not zero since as we have seen above $E(\beta_0+\omega)=E_0 + \omega\langle\psi_0, P\psi_0\rangle - \frac{1}{2}\Gamma+\mathcal{O}(\omega^3)$.

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