Asymptotic properties of the differential equation $h^3(h'' + h') = 1$

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Abstract. We derive the complete asymptotic series, as $t \to +\infty$, for a general solution $h(t)$ of the nonlinear differential equation $h^3(h'' + h') = 1$. The equation originates from a physical model related to the Hall effect.

1. Introduction

The purpose of this article is to describe the asymptotic behaviour of solutions of the second-order ordinary nonlinear differential equation

$$h(t)^3\big(h''(t) + h'(t)\big) = 1 \tag{1}$$

with initial conditions

$$h(t_0) = h_0 > 0, \qquad h'(t_0) = h_1. \tag{2}$$

Before formulating the result let us describe our motivation and the origin of the problem which has its roots in the physical Hall effect.

In a classical mechanics description the issue is to study the dynamics of a point mass moving in a periodic planar potential and driven by an exterior electromagnetic field where the magnetic field is constant and the electric field circular and created by a linearly time dependent flux tube through the origin, see [8] for the origin of the model. The equations of motions are Hamiltonian. The time dependent Hamiltonian is

$$\frac{1}{2}\big(p - A(q,t)\big)^2 + V(q,t) \quad \text{on } \mathbb{R} \times \big(\mathbb{R}^2 \setminus \{0\}\big) \times \mathbb{R}^2$$

with

$$A(q,t) = \left(\frac{b}{2} + \frac{et}{|q|^2}\right)(q_2, -q_1).$$

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Here b and e are real parameters and V a smooth periodic function. In Newtonian form the equations of motion are

$$\ddot{q} = E(q) + b\mathbb{D}\dot{q} - \nabla V(q) \quad \text{in } \mathbb{R}^2 \setminus \{0\},$$

where $\mathbb{D}$ is rotation by $\pi/2$ and $E(q) = e|q|^{-2}\,\mathbb{D}q = -\partial_t A(q)$.

We shall prove elsewhere that if b and e are nonzero the solutions are diffusive with or without direction depending on the direction of the fields. In this article we discuss the particular case when $e = 1,\ b = 0$ and $V = 0$. In polar coordinates the Hamiltonian reads

$$\frac{1}{2}\left(p_r^2 + \frac{1}{r^2}(p_\phi + t)^2\right)$$

and the equations of motion become

$$p'_\phi = 0, \quad \phi' = \frac{p_\phi + t}{r^2}, \quad p'_r = \frac{(p_\phi + t)^2}{r^3}, \quad r' = p_r.$$

Consequently, p_ϕ is a constant and $r'' = r^{-3}(p_\phi + t)^2$. After a shift in time we arrive at the equation

$$r'' = \frac{t^2}{r^3}.$$

The substitution

$$r(t) = th(\ln t)$$

leads to Eq. (1).

In order to formulate our result we have to introduce some auxiliary notation. The polynomials introduced below will be needed for an inversion method such as the Lagrange inversion formula (see Chapter 2, pp. 21–33, in [6]). Let $s_{m,k} \in \mathbb{R}[a_1, a_2, \ldots, a_k]$ be polynomials defined as follows:

$$s_{m,k}(a_1, a_2, \ldots, a_k) = \sum_{i_1+i_2+\cdots+i_m=k} a_{i_1} a_{i_2} \cdots a_{i_m}, \tag{3}$$

$m = 1, 2, \ldots, k = 0, 1, 2, \ldots$. Clearly, $s_{m,k}(a_1, a_2, \ldots, a_k) = 0$ if $k < m$, and we set by definition $s_{0,k} = \delta_{0,k}$. The polynomials obey the recursive rule

$$s_{m+1,k}(a_1, a_2, \ldots, a_k) = \sum_{j=m}^{k-1} s_{m,j}(a_1, a_2, \ldots, a_j) a_{k-j} \quad \text{for } m+1 \leqslant k.$$

In the space of formal power series, $\mathbb{R}[[x]]$, it holds

$$\left(\sum_{k=1}^{\infty} a_k x^k\right)^m = \sum_{k=m}^{\infty} s_{m,k}(a_1, a_2, \ldots, a_k) x^k, \quad m = 0, 1, 2, \ldots.$$

This implies that if

$$a=\sum_{k=1}^{\infty}a_kx^k,\qquad f=\sum_{k=0}^{\infty}f_kx^k\in\mathbb{R}[[x]],$$

then

$$\sum_{m=0}^{\infty}f_ma^m=\sum_{k=0}^{\infty}g_kx^k\quad\text{where } g_k=\sum_{m=0}^{k}f_ms_{m,k}(a_1,a_2,\dots,a_k).\tag{4}$$

Furthermore, set

$$\sigma_k^0(a_1,a_2,\dots,a_k)=\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}s_{j,k}(a_1,a_2,\dots,a_k)\tag{5}$$

and

$$\sigma_k^m(a_1,a_2,\dots,a_k)=\sum_{j=0}^{k}\binom{-m}{j}s_{j,k}(a_1,a_2,\dots,a_k)\quad\text{for } m\geqslant 1.\tag{6}$$

Then it holds, in $\mathbb{R}[[x]]$,

$$\ln\left(1+\sum_{k=1}^{\infty}a_kx^k\right)=\sum_{k=1}^{\infty}\sigma_k^0(a_1,a_2,\dots,a_k)x^k,$$
$$\left(1+\sum_{k=1}^{\infty}a_kx^k\right)^{-m}=1+\sum_{k=1}^{\infty}\sigma_k^m(a_1,a_2,\dots,a_k)x^k\quad\text{for } m\geqslant 1.$$

Further, let $f(z)=\sum_{n=0}^{\infty}\beta_nz^n$ be the formal solution of the differential equation

$$f'(z)=-\frac{1}{z^2}f(z)^2\big(1-f(z)\big)+\frac{3}{4}\frac{f(z)}{z}.$$

This first order differential equation is in turn derived by setting $f(z)=1/g(z)$ in the differential equation

$$g'(z)=\frac{1}{z^2}\left(1-\frac{1}{g(z)}\right)-\frac{3}{4}\frac{g(z)}{z}.$$

The original second order differential equation (1) can be reduced to this last differential equation as explained in detail in Section 4.1. The sequence of real numbers $\{\beta_n\}_{n=0}^{\infty}$ is defined recursively,

$$\begin{aligned}&\beta_0=1,\\&\beta_{n+1}=\left(n-\frac{3}{4}\right)\beta_n+\sum_{\substack{j,k=0\\j+k=n+1}}^{n}\beta_j\beta_k-\sum_{\substack{j,k,\ell=0\\j+k+\ell=n+1}}^{n}\beta_j\beta_k\beta_\ell.\end{aligned}\tag{7}$$

Here are several first values:

$$\beta_1=-\frac{3}{4},\quad \beta_2=-\frac{21}{16},\quad \beta_3=-\frac{165}{32},\quad \beta_4=-\frac{7245}{256},\ldots.$$

For a fixed constant $c\in\mathbb{R}$ we introduce a sequence of polynomials, $p_n(c;z)\in\mathbb{R}[z]$, $n\in\mathbb{Z}_+$, by the recursive rule

$$p_0(c;z)=3z-c \tag{8}$$

and

$$p_n=3\sigma_n^0(p_0,p_1,\ldots,p_{n-1})+\sum_{k=1}^{n-1}\frac{4^{k+1}\beta_{k+1}}{k}\sigma_{n-k}^k(p_0,p_1,\ldots,p_{n-k-1})+\frac{4^{n+1}\beta_{n+1}}{n}. \tag{9}$$

This can be rewritten with the aid of the polynomials $s_{m,k}$,

$$\begin{aligned}p_n={}&3\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j}s_{j,n}(p_0,p_1,\ldots,p_{n-1})\\&+\sum_{k=0}^{n-1}\frac{4^{n-k+1}\beta_{n-k+1}}{n-k}\sum_{j=0}^{k}\binom{-n+k}{j}s_{j,k}(p_0,p_1,\ldots,p_{k-1}).\end{aligned}$$

For $n\geqslant 1$, the degree of $p_n(c;z)$ is less or equal to n (this can be easily shown by induction when using the fact that for any monomial $a_{i_1}^{s_1}a_{i_2}^{s_2}\cdots a_{i_\ell}^{s_\ell}$ occurring in $\sigma_k^m(a_1,a_2,\ldots,a_k)$ it holds $\sum i_j s_j=k$). Here are several first polynomials $p_n(z)$,

$$\begin{aligned}p_1(c;z)&=9z-21-3c,\\p_2(c;z)&=-\frac{27}{2}z^2+(90+9c)z-228-30c-\frac{3}{2}c^2,\\p_3(c;z)&=27z^3-\left(\frac{621}{2}+27c\right)z^2+\left(1638+207c+9c^2\right)z-3540-546c-\frac{69}{2}c^2-c^3.\end{aligned}$$

Now we are able to formulate the result.

Theorem 1. *For any initial data* $(t_0,h_0,h_1)\in\mathbb{R}\times\,]0,\infty[\,\times\mathbb{R}$ *there exists a unique solution* $h(t)$ *to the problem* (1), (2) *on the real line. Moreover, there exists a constant* $c=c(t_0,h_0,h_1)\in\mathbb{R}$ *such that*

$$h(t)=(4t)^{1/4}\left(1+\sum_{k=1}^{n}\frac{q_k(c;\ln(4t))}{t^k}+\mathrm{O}\left(\left(\frac{\ln(t)}{t}\right)^{n+1}\right)\right) \tag{10}$$

as $t\to+\infty$ *for all* $n\in\mathbb{Z}_+$, *where*

$$q_k=\sum_{m=1}^{k}\frac{1}{4^k}\binom{\frac{1}{4}}{m}s_{m,k}(p_0,p_1,\ldots,p_{k-1}).$$

The degree of $q_k(c;z)$ *is less than or equal to* k.

Remarks. (i) Several first polynomials $q_k(c; z)$ are

$$q_1(c; z) = \frac{3}{16}z - \frac{1}{16}c,$$
$$q_2(c; z) = -\frac{27}{512}z^2 + \left(\frac{9}{64} + \frac{9}{256}c\right)z - \frac{21}{64} - \frac{3}{64}c - \frac{3}{512}c^2,$$
$$q_3(c; z) = \frac{189}{8192}z^3 - \left(\frac{135}{1024} + \frac{189}{8192}c\right)z^2 + \left(\frac{549}{1024} + \frac{45}{512}c + \frac{63}{8192}c^2\right)z$$
$$- \frac{57}{64} - \frac{183}{1024}c - \frac{15}{1024}c^2 - \frac{7}{8192}c^3.$$

(ii) In the final step of the proof, in Subsection 4.5, we shall show the following invariance property of the asymptotic expansion. Set

$$A_n(c; t) = (4t)^{1/4}\left(1 + \sum_{k=1}^{n} \frac{q_k(c; \ln(4t))}{t^k}\right),$$

with $n \in \mathbb{Z}_+$ and $c, t \in \mathbb{R}$. Then for all $s \in \mathbb{R}$ it holds true that

$$A_n(c; t + s) = A_n(c - 4s; t) + t^{1/4}\,\mathrm{O}\left(\left(\frac{\ln(t)}{t}\right)^{n+1}\right) \quad \text{as } t \to +\infty.$$

(iii) For the definition of the constant c see equality (27).

The remainder of the paper contains all necessary steps to prove Theorem 1. We shall proceed as follows. In Section 2 we show the completeness and derive the first term of the asymptotic series. In Section 3 we make use of the fact that the second order differential equation can be reduced to a first order differential equation and we investigate the asymptotic properties of the latter equation with the aid of the centre manifold theorem. These results are used in Section 4 to derive the asymptotic properties of the original second order differential equation and this way we complete the proof of Theorem 1. Section 5 contains an additional remark on the asymptotics of the Lambert function.

2. Basic properties of the differential equation

The differential equation (1) is equivalent to the dynamical system

$$(x', y') = \left(y, \frac{1}{x^3} - y\right) \quad \text{on } M =]0, \infty[\times\mathbb{R}. \tag{11}$$

Proposition 2. *The flow of* (11) *is complete and so for all initial data* $(t_0, h_0, h_1) \in \mathbb{R}\times\,]0, \infty[\, \times\mathbb{R}$ *there exists a unique globally defined positive solution* h *of* (1) *with initial conditions*

$$h(t_0) = h_0, \qquad h'(t_0) = h_1.$$

Proof. We use the following criterion (cf. [1, Chapter 2.1.20]): *The flow of a C^1 vector field ξ on a manifold M is complete if there is a proper map $f \in C^1(M,\mathbb{R})$ which meets the estimate*

$$\exists A>0, B>0,\ \forall p \in M, \quad \left|\xi \cdot f(p)\right| \leqslant A\left|f(p)\right| + B.$$

In our case $M =]0,\infty[\times\mathbb{R}$, $\xi = y\partial_x + (x^{-3}-y)\partial_y$ and we choose $f(x,y) = x^2+y^2+x^{-2}$. With this choice we have

$$\left|\xi \cdot f(x,y)\right| = \left|2xy - 2y^2\right| \leqslant x^2 + 3y^2 \leqslant 3f(x,y).$$

Moreover, for any bounded set $S \subset \mathbb{R}$ the inverse image $f^{-1}(S)$ is bounded and separated from the border of the half-plane M: there exists $\varepsilon > 0$ such that $f^{-1}(S) \subset [\,\varepsilon,\infty\,] \times \mathbb{R}$. This implies that f is in fact a proper map and the proposition is proven. □

Proposition 3. *Let $(x(t), y(t))$, with $t \in [t_0,\infty[$, be a solution of the dynamical system* (11). *Then there exists $T \in [t_0,\infty[$ such that*

$$\forall t, s,\ t \geqslant s \geqslant T, \quad \sqrt{2}\big(t+c(s)\big)^{1/4} \leqslant x(t) \leqslant y(s) + \sqrt{2}\big(t+c(s)\big)^{1/4}$$

where $c(s) = \frac{1}{4}x(s)^4 - s$.

Proof. We shall first show that $y'(t)$ is negative for all sufficiently large t. Note that

$$y'' = -\left(1+3\frac{y}{x}\right)y' - 3\frac{y^2}{x} = -\left(1+3\frac{y}{x}+3y^2x^2\right)y' - 3y^3x^2.$$

Hence for all t, $y'(t)=0$ implies $y''(t)<0$. Consequently, it is sufficient to show that there is at least one t such that $y'(t) \leqslant 0$. Suppose on the contrary that $y'(t) > 0$ for all t. Notice that (11) implies

$$y(t) = y(t_0) + \frac{t-t_0}{x(t_0)^3} - \int_{t_0}^{t}\left(3\frac{t-s}{x(s)^4}+1\right)y(s)\,\mathrm{d}s.$$

Therefore there exists $s \geqslant t_0$ such that $y(s) > 0$. It follows that for all $t \geqslant s$, both $x(t)$ and $y(t)$ are increasing positive functions and $y''(t) \leqslant -3y(s)^3x(s)^2 < 0$, a contradiction.

Let now $T \geqslant t_0$ be such that $y'(t) < 0$ for all $t > T$. Let us fix $s \geqslant T$. Due to (11), for any $t > T$ we have $(x^4/4)' = x^3y > 1$. Consequently, if $t \geqslant s$ then

$$x(t) \geqslant \sqrt{2}\left(t+\frac{1}{4}x(s)^4 - s\right)^{1/4} = \sqrt{2}\big(t+c(s)\big)^{1/4}.$$

To show the other inequality set, for $t \geqslant s$, $z = x-\varphi$ where $\varphi(t) = \sqrt{2}(t+c)^{1/4}$ and $c = c(s)$. Note that $\varphi' = \varphi^{-3}$ and $\varphi'' < 0$. Furthermore, $\varphi(s) = x(s)$, i.e., $z(s) = 0$, and $z'(s) = y(s) - x(s)^{-3} = -y'(s) > 0$. We find that

$$\left(\mathrm{e}^t z'\right)' = \mathrm{e}^t\left(\frac{1}{x^3} - \frac{1}{\varphi^3} - \varphi''\right) \leqslant -\mathrm{e}^t\varphi''.$$

Integrating twice this inequality one easily deduces that

$$z'(t) \leqslant \mathrm{e}^{-t+s} z'(s) - \int_s^t \mathrm{e}^{-t+u} \varphi''(u)\,\mathrm{d}u$$

and

$$z(t) \leqslant z(s) + \left(1-\mathrm{e}^{-t+s}\right) z'(s) - \int_s^t \left(1-\mathrm{e}^{-t+u}\right) \varphi''(u)\,\mathrm{d}u \leqslant z(s) + z'(s) + \varphi'(s).$$

Hence $x(t) - \varphi(t) \leqslant x'(s) = y(s)$. This completes the proof. □

Corollary 4. *If $h(t)$ is a solution of* (1) *on $[t_0, \infty[$ with the initial conditions $h(t_0) = h_0 > 0$, $h'(t_0) = h_1$, then there exists $T \geqslant t_0$ such that $h'(t) > 0$ for all $t > T$.*

Corollary 5. *If $h(t)$ is a solution of* (1) *on $[t_0, \infty[$ with the initial conditions $h(t_0) = h_0 > 0$, $h'(t_0) = h_1$, then*

$$h(t) = \sqrt{2}t^{1/4} + \mathrm{O}(1) \quad \textit{as } t \to +\infty.$$

Remark. This means that if we restrict ourselves in what follows to the initial condition (2) with $h_1 > 0$ we do not loose the generality as far as the asymptotics is concerned. Furthermore, due to the invariance of the differential equation in time we can set $t_0 = 0$. This fact will be used in the course of the proof. First we verify Theorem 1 for the particular case when $t_0 = 0$ and $h_1 > 0$ and then, in Subsection 4.5, we shall extend the result to the general initial condition.

3. A reduced differential equation of first order

In accordance with the remark at the end of Section 2 we assume that $t_0 = 0$ and $h_1 > 0$. The second-order differential equation (1) is invariant in t and this is why it can be reduced to a first-order differential equation. Actually, using the substitution $h(t) = (G^{-1}(4t))^{1/4}$, $z_0 = 4/h_0^4$ and $g_0 = h_0^3 h_1$ where

$$G(x) = \int_{h_0^4}^x \frac{\mathrm{d}s}{g(4/s)}$$

we arrive at a first-order nonlinear differential equation, namely

$$\left(1 - \frac{3}{4} z g(z) - z^2 g'(z)\right) g(z) = 1, \quad g(z_0) = g_0. \tag{12}$$

Here we confine ourselves to noting that this substitution means that

$$z = \frac{4}{h(t)^4}, \qquad g = h(t)^3 h'(t). \tag{13}$$

However we shall carry out the computations relating (1), (2) and (12) in detail later, in Subsection 4.1. In the current section we shall concentrate on the study of (12) on the interval $[0, z_0]$, assuming that $z_0 > 0$ and $g_0 > 0$.

3.1. Domain of the left maximal solution $g(z)$

We shall need two equivalent forms of the differential equation (12), namely

$$g'(z) = \frac{1}{z^2}\left(1 - \frac{1}{g(z)}\right) - \frac{3}{4}\frac{g(z)}{z} \tag{14}$$

and

$$\left(z^{3/4}g(z)\right)' = z^{-5/4}\left(1 - \frac{1}{g(z)}\right). \tag{15}$$

Remark. In what follows we use repeatedly the following elementary argument: *if ψ and φ are two differentiable functions on $]a, b[$ and the equality $\psi(z) = \varphi(z)$ implies $\psi'(z) < \varphi'(z)$, for all $z \in]a, b[$, then the two functions coincide in at most one point $z \in]a, b[$.*

Proposition 6. *Let $g(z)$ be the left maximal solution of* (12). *Then $g(z)$ is defined and positive on* $]0, z_0]$, *and*

$$\lim_{z\downarrow 0} g(z) = 1. \tag{16}$$

Proof. Let γ be the minimal nonnegative number such that $g(z)$ is defined on $]\gamma, z_0]$. Eq. (12) clearly excludes the possibility that $g(z) = 0$ for some $z \in]\gamma, z_0]$. So $g(z)$ is positive on $]\gamma, z_0]$. Our goal is to show that $\gamma = 0$ and (16) holds true. We split the proof into six claims.

(i) $\exists \xi \in]\gamma, z_0]$ s.t. $g(\xi) \geqslant 1$.

Suppose that $g(z) < 1, \forall z \in]\gamma, z_0]$. Then, by (14), $g'(z) < 0$ on $]\gamma, z_0]$ and so there exists

$$\lim_{z\downarrow\gamma} g(z) = g_1$$

with $g_0 < g_1 \leqslant 1$. Hence, by minimality of γ, it should hold $\gamma = 0$. According to (15),

$$\left(z^{3/4}g(z)\right)' < 0 \Rightarrow g(z) > g_0\left(\frac{z_0}{z}\right)^{3/4}, \quad \forall z \in]0, z_0],$$

a contradiction.

In the remainder of the proof we choose $\xi \in]\gamma, z_0]$ to be the largest number such $g(\xi) \geqslant 1$.

(ii) $g(z) > 1, \forall z \in]\gamma, \xi[$.

Actually, $g(z) = 1$ implies $g'(z) < 0$.

(iii) $\gamma = 0$.

If $\gamma > 0$ then, by (15), $|(z^{3/4}g(z))'| \leqslant \gamma^{-5/4} < \infty$ for all $z \in]\gamma, \xi[$. This means that $z^{3/4}g(z)$ is absolutely continuous on $]\gamma, \xi[$ and so $\lim_{z\downarrow\gamma} g(z)$ exists and is finite, a contradiction with the minimality of γ.

(iv) $\exists \eta \in]0, \xi[$ s.t. $g'(z) \neq 0, \forall z \in]0, \eta[$.

Equalling the right-hand side of (14) to zero we get a quadratic equation with respect to $g(z)$. Its solution is a couple of functions,

$$\varphi_1(z) = \frac{2}{1+\sqrt{1-3z}}, \qquad \varphi_2(z) = \frac{2}{1-\sqrt{1-3z}},$$

defined on the interval $]0,\xi[\,\cap\,]0,\frac{1}{3}[$. Clearly, $\varphi_1'(z) > 0$ and $\varphi_2'(z) < 0$ everywhere on that interval. This implies that the right-hand side of (14) vanishes in a point z from that interval if and only if $g(z) = \varphi_1(z)$ or $g(z) = \varphi_2(z)$ and in such a case either $0 = g'(z) < \varphi_1'(z)$ or $0 = g'(z) > \varphi_2'(z)$. Thus $\varphi_1(z)$ coincides with $g(z)$ in at most one point z, and the same is true for $\varphi_2(z)$. Consequently, there exists $\eta \in\,]0,\xi[\,\cap\,]0,\frac{1}{3}[$ such that $g'(z)$ does not vanish on $]0,\eta[$.

We choose η having the property stated in claim (iv).

(v) $g'(z) > 0$, $\forall z \in\,]0,\eta[$.

If $g'(z) < 0$, $\forall z \in\,]0,\eta[$, then $g_1 = \lim_{z\downarrow 0} g(z)$ exists (finite or infinite) and $g_1 > g(\eta) > 1$. On the other hand, in virtue of (14),

$$z^2 g'(z) = 1 - \frac{1}{g(z)} - \frac{3}{4} z g(z) < 0. \tag{17}$$

Equality (15) implies that $(z^{3/4} g(z))' > 0$ on $]0,\eta[$ and so

$$g(z) < g(\eta)\left(\frac{\eta}{z}\right)^{3/4}, \quad \forall z \in\,]0,\eta[.$$

Consequently, $\lim_{z\downarrow 0} z g(z) = 0$. Sending z to 0 in (17) gives

$$1 - \frac{1}{g_1} \leqslant 0,$$

a contradiction.

(vi) $\lim_{z\downarrow 0} g(z) = 1$.

It follows from claim (v) that $g_1 = \lim_{z\downarrow 0} g(z)$ exists and $1 \leqslant g_1 < g(\eta)$. Suppose that $g_1 > 1$. Then one concludes from (14) that there exists $\delta \in\,]0,\eta[$ s.t.

$$g'(z) > \frac{d}{z^2}, \quad \forall z \in\,]0,\delta[, \quad \text{where } d = \frac{1}{2}\left(1 - \frac{1}{g_1}\right) > 0.$$

This implies

$$g(z) < g(\delta) + \frac{d}{\delta} - \frac{d}{z}, \quad \forall z \in\,]0,\delta[,$$

a contradiction.

Corollary 7. *The maximal solution $g(z)$ satisfies the integral identity*

$$g(z)^2 = 2z^{-3/2} \int_0^z s^{-1/2}\big(g(s) - 1\big)\, \mathrm{d}s, \quad \forall z \in\,]0, z_0]. \tag{18}$$

Proof. Rewrite (14) as

$$\left(z^{3/2}g^2\right)' = 2z^{-1/2}(g-1),$$

and integrate from 0 to z when taking into account (16). □

3.2. Asymptotics of the left maximal solution $g(z)$ at $z=0$

Let us consider Eq. (12) (without the initial condition) in the space of formal power series $\mathbb{C}[z]$. Its solution

$$\tilde{g}(z) = \sum_{k=0}^{\infty} \alpha_k z^k \in \mathbb{C}[z] \tag{19}$$

is unique, with the coefficients being determined by the recursive relation

$$\alpha_0 = 1, \quad \alpha_{k+1} = \left(\frac{1}{2}k + \frac{3}{4}\right) \sum_{j=0}^{k} \alpha_j \alpha_{k-j} \quad \text{for } k \geqslant 0. \tag{20}$$

Several first coefficients are

$$\alpha_0 = 1,\ \alpha_1 = \frac{3}{4},\ \alpha_2 = \frac{15}{8},\ \alpha_3 = \frac{483}{64}, \ldots .$$

Proposition 8. *The left maximal solution $g(z)$ has an asymptotic series, as $z \downarrow 0$, that is equal to*

$$\sum_{k=0}^{\infty} \alpha_k z^k.$$

Proof. Consider the system

$$\begin{aligned} u' &= \Phi(u,w) = -u^2, \\ w' &= \Psi(u,w) = -1 + \frac{1}{w+3u/4} + \frac{3}{4}uw + \frac{21}{16}u^2, \end{aligned} \tag{21}$$

on the interval $t \geqslant 0$. A particular solution is $u(t) \equiv 0$, $w(t) \equiv 1$. It can be shown to be stable provided we restrict ourselves to the initial conditions with $u(0) \geqslant 0$. An analysis of the system is possible with the aid of the centre manifold theorem [2,3]. The derivative of the mapping $(u,w) \mapsto (\Phi(u,w), \Psi(u,w))$ at the point $(0,1)$ equals $\mathrm{diag}(0,-1)$. According to the theorem, a centre manifold $w = f(u)$ through the point $(u,w) = (0,1)$ exists. Recall that, by definition, this is an invariant manifold for the system of differential equations such that $f'(0) = 0$.

The function f can be taken from the class C^k for arbitrary $k \in \mathbb{N}$ provided we restrict ourselves to a sufficiently small neighbourhood of the origin. The Taylor series of f at the origin, $\tilde{f}(u) \in \mathbb{R}[u]$, is well defined as a formal power series,

$$\tilde{f}(u) = 1 + \sum_{k=2}^{\infty} f_k u^k,$$

and satisfies the differential equation

$$\tilde{f}'(u)\Phi\big(u,\tilde{f}(u)\big)-\Psi\big(u,\tilde{f}(u)\big)=0. \tag{22}$$

This equation determines $\tilde{f}(u)$ unambiguously.

According to the theory, the stability properties of the system (21) are the same as those of the reduced equation

$$v'=\Phi\big(v,f(v)\big)=-v^2. \tag{23}$$

Since the zero solution of (23) is stable provided we restrict ourselves to the initial conditions with $v(0)\geqslant 0$ an analogous claim is actually true for the distinguished constant solution of (21). In that case the theory yields an asymptotic result: *if* (u,w) *is any solution of* (21) *on the positive half-line with* $(u(0),w(0))$ *sufficiently close to* $(0,1)$ *then there exists a solution* v *of* (23) *and* $\gamma>0$ *such that*

$$u(t)=v(t)+\mathrm{O}\big(\mathrm{e}^{-\gamma t}\big),\qquad w(t)=h\big(v(t)\big)+\mathrm{O}\big(\mathrm{e}^{-\gamma t}\big).$$

Since both $u(t)$ and $v(t)$ satisfy the same Eq. (23) it readily follows that these functions coincide. Thus the asymptotic series of $w(t)$, as $t\to\infty$, is the same as that of $f(u(t))$.

Let $g(z)$ be a left maximal solution of (14) and $a>0$ be a point from the domain of g. To conclude the proof it suffices to note that

$$u(t)=\frac{1}{t+b},\quad w(t)=g\big(u(t)\big)-\frac{3}{4}u(t),\quad \text{with } b=\frac{1}{a},$$

is a solution of the system (21), with the initial condition $u(0)=a$, $w(0)=g(a)-(3/4)a$. According to Lemma 6 we can take a as small as we please making the point $(u(0),w(0))$ arbitrarily close to $(0,1)$. Therefore the asymptotic behaviour of $g(z)$, as $z\downarrow 0$, is determined by the formal power series

$$\varphi(z)=\tilde{f}(z)+\frac{3}{4}z.$$

A simple manipulation shows that if $\tilde{f}(z)$ satisfies (22) then $\varphi(z)$ obeys (14). Hence $\varphi(z)=\tilde{g}(z)$, with $\tilde{g}(z)$ being the formal series (19), (20), as claimed. □

Corollary 9. *The left maximal solution* $g(z)$, *after having been defined at* $z=0$ *by* $g(0)=1$, *belongs to* $C^\infty([\,0,z_0\,])$.

Proof. Observe that consecutive differentiation of Eq. (14) jointly with Proposition 8 imply that, for any $m\in\mathbb{Z}_+$, $z^{m+1}g^{(m)}(z)$ has an asymptotic series at $z=0$ which we shall call $\sum_{k=0}^{\infty}\alpha_k^m z^k$. We have to show that $g\in C^m$, $\forall m$, and this in turn amounts to showing that $\alpha_k^m=0$ for $k<m+1$. Let us proceed by induction in m. The case $m=0$ was the content of Proposition 6. Assume now that $g\in C^m$. Then $\alpha_k^m=0$ for $k<m+1$, and the mean value theorem implies that

$$\liminf_{z\downarrow 0} g^{(m+1)}(z)\leqslant\frac{\mathrm{d}g^{(m)}(0_+)}{\mathrm{d}z}=\alpha_{m+2}^m\leqslant\limsup_{z\downarrow 0} g^{(m+1)}(z). \tag{24}$$

On the other hand, since $z^{m+2}g^{(m+1)}(z)$ has an asymptotic series the limit $\lim_{z\downarrow 0} g^{(m+1)}(z)$ always exists and equals either $\pm\infty$ or α_{m+2}^{m+1} depending on whether there exists an index $k < m+2$ s.t. $\alpha_k^{m+1} \neq 0$ or not. However the property (24) clearly excludes the first possibility. $\square$

4. Asymptotics of a solution $h(t)$ of the second order differential equation

Except of the last subsection, we still consider the particular case when $t_0 = 0$ and $h_1 > 0$ (see the remark at the end of Section 2). We shall proceed to the case of general initial condition only at the very end of the proof, in Subsection 4.5.

4.1. Reduction of the second order differential equation

Let us now complete some computations concerning the reduction of the second order differential equation (1), (2) to a first order differential equation. Let $g(z)$ be the left maximal solution of the first order differential equation

$$\left(1 - \frac{3}{4}zg(z) - z^2g'(z)\right)g(z) = 1, \quad g(z_0) = g_0,$$

where

$$z_0 = \frac{4}{h_0^4}, \qquad g_0 = h_0^3h_1.$$

From Section 3 we know that $g(z)$ is a positive function from the class $C^\infty([0, z_0])$ (Proposition 6 and Corollary 9). Consider the function

$$G(x) = \int_{h_0^4}^{x} \frac{\mathrm{d}s}{g(4/s)}, \quad h_0^4 \leqslant x < \infty. \tag{25}$$

Then $G \in C^\infty([h_0^4, \infty[)$, G is strictly increasing, $G(h_0^4) = 0$, and, due to (16), $\lim_{x\to\infty} G(x) = \infty$. So the inverse function satisfies $G^{-1} \in C^\infty([0, \infty[)$ with $G^{-1}(0) = h_0^4$. Set

$$h(t) = \left(G^{-1}(4t)\right)^{1/4}.$$

Then $h(t)$ solves the problem (1), (2).

Actually, $G(h(t)^4) = 4t$, $G'(h^4) = g(4/h^4)^{-1}$, and so

$$h^3h' = \frac{1}{4G'(h^4)}\frac{\mathrm{d}G(h^4)}{\mathrm{d}t} = g\left(\frac{4}{h^4}\right). \tag{26}$$

Differentiating (26) once more gives

$$3h^2(h')^2 + h^3h'' = -16h^{-5}h'g'\left(\frac{4}{h^4}\right).$$

Denote for brevity $z=4/h^4$. Hence

$$h^3(h'+h'')=g(z)-3h^{-4}(h^3h')^2-16h^{-8}(h^3h')g'(z)=g(z)-\frac{3}{4}zg(z)^2-z^2g(z)g'(z)=1.$$

Furthermore, $h(0)=(G^{-1}(0))^{1/4}=h_0$ and

$$h'(0)=G^{-1}(0)^{-3/4}\frac{\mathrm{d}}{\mathrm{d}s}G^{-1}(0)=h_0^{-3}\left(\frac{\mathrm{d}}{\mathrm{d}x}G(h_0^4)\right)^{-1}=h_0^{-3}g\left(\frac{4}{h_0^4}\right)=h_1.$$

4.2. Asymptotics of $G(x)$

First let us find, in $\mathbb{C}[z]$, the reciprocal element to the formal power series $\tilde{g}(z)=\sum_{k=0}^{\infty}\alpha_k z^k$ defined in (19), (20). Set

$$\tilde{g}(z)^{-1}=\sum_{n=0}^{\infty}\beta_n z^n.$$

The formal power series $\tilde{g}(z)$ solves the differential equation (14) and so an easy calculation shows that $\tilde{f}(z)=\tilde{g}(z)^{-1}$ solves the differential equation

$$f'(z)=-\frac{1}{z^2}f(z)^2\big(1-f(z)\big)+\frac{3}{4}\frac{f(z)}{z}.$$

On the other hand, this differential equation implies a recursive rule on the coefficients β_n, namely rule (7) preceding the formulation of Theorem 1.

Lemma 10. *The asymptotic series at infinity of the function $G(x)$ defined in* (25) *is given by*

$$G(x)\sim x-3\ln(x)+c-4\sum_{k=1}^{\infty}\frac{\beta_{k+1}}{k}\left(\frac{4}{x}\right)^k,$$

where

$$c=c(0,h_0,h_1)=\int_{h_0^4}^{\infty}\left(\frac{1}{g(4/s)}-1+\frac{3}{s}\right)\mathrm{d}s-h_0^4+3\ln\big(h_0^4\big). \tag{27}$$

Proof. It holds

$$G(x)=\int_{h_0^4}^{\infty}\left(\frac{1}{g(4/s)}-1+\frac{3}{s}\right)\mathrm{d}s+\int_{h_0^4}^{x}\left(1-\frac{3}{s}\right)\mathrm{d}s-\int_{x}^{\infty}\left(\frac{1}{g(4/s)}-1+\frac{3}{s}\right)\mathrm{d}s.$$

According to Proposition 8 we have the asymptotics at infinity,

$$\frac{1}{g(4/s)}-1+\frac{3}{s}\sim\sum_{k=2}^{\infty}\beta_k\left(\frac{4}{s}\right)^k.$$

The claim then follows straightforwardly. □

4.3. Asymptotics of $G^{-1}(x)$

Let us now focus on the inverse function G^{-1}.

Lemma 11. *There exists $x_\star$ such that for all $x > x_\star$ it holds true that*

$$0 \leqslant G^{-1}(x) - x \leqslant \frac{x}{x-4}\bigl(x - G(x)\bigr). \tag{28}$$

Proof. Choose $y_\star \geqslant G(h_0^4) > 0$ so that, for all $y \geqslant y_\star$,

$$0 \leqslant 1 - \frac{1}{g(4/y)} \leqslant \frac{4}{y} \quad \text{and} \quad y - G(y) \geqslant 0. \tag{29}$$

This is possible due to Proposition 8 and Lemma 10. Set $x_\star = \max\{4, y_\star\}$. For $x > x_\star$ fixed define a sequence $\{y_n\}_{n=0}^\infty$ by the recursive rule

$$y_0 = x, \qquad y_{n+1} = x + y_n - G(y_n).$$

Due to our choice, $y_n \geqslant x \geqslant x_\star \geqslant y_\star$ for all n. Using (25) one finds that

$$y_{n+2} - y_{n+1} = y_{n+1} - y_n - \int_{y_n}^{y_{n+1}} \frac{\mathrm{d}s}{g(4/s)} = \int_{y_n}^{y_{n+1}} \left(1 - \frac{1}{g(4/s)}\right) \mathrm{d}s.$$

In virtue of (29), the sequence satisfies

$$0 \leqslant y_{n+2} - y_{n+1} \leqslant 4\bigl(\ln(y_{n+1}) - \ln(y_n)\bigr) \leqslant \frac{4}{x}(y_{n+1} - y_n).$$

Since $y_1 - y_0 = x - G(x)$ we get

$$0 \leqslant y_{n+1} - y_n \leqslant \left(\frac{4}{x}\right)^n \bigl(x - G(x)\bigr), \quad \forall n.$$

By the choice of $x_\star$ we have $4 < x$ and, consequently, the sequence $\{y_n\}$ is convergent. The limit $y = \lim y_n$ solves $0 = x - G(y)$ and so $y = G^{-1}(x)$. Moreover,

$$0 \leqslant y - y_n = \sum_{k=n}^{\infty}(y_{k+1} - y_k) \leqslant \sum_{k=n}^{\infty} \left(\frac{4}{x}\right)^k \bigl(x - G(x)\bigr) = \frac{x}{x-4}\bigl(x - G(x)\bigr)\left(\frac{4}{x}\right)^n.$$

The particular case $n = 0$ in this relation is nothing but our claim. □

Combining Lemma 11 with Lemma 10 one immediately gets

Corollary 12. *It holds true that, as $x \to +\infty$,*

$$G^{-1}(x) = x + \mathrm{O}\bigl(\ln(x)\bigr).$$

Recall that in (5), (6) we have introduced polynomials $\sigma_k^m(a_1, a_2, \ldots, a_k)$ labelled by indices $m \geqslant 0$ and $k \geqslant 1$.

Proposition 13. *For all $n \in \mathbb{Z}_+$ it holds true that, as $x \to \infty$,*

$$G^{-1}(x) = x + p_0\big(c; \ln(x)\big) + \sum_{k=1}^{n} \frac{p_k(c; \ln(x))}{x^k} + \mathrm{O}\bigg(\Big(\frac{\ln(x)}{x}\Big)^{n+1}\bigg), \tag{30}$$

where the polynomials $p_n(c; z)$ have been defined in (8), (9) *and the constant c is given by equality* (27).

Proof. Corollary 12 implies

$$\ln\big(G^{-1}(x)\big) = \ln(x) + \mathrm{O}\Big(\frac{\ln(x)}{x}\Big), \qquad \frac{1}{G^{-1}(x)} = \frac{1}{x} + \mathrm{O}\Big(\frac{\ln(x)}{x^2}\Big). \tag{31}$$

Combining (31) with Lemma 10 one derives the relation

$$x = G^{-1}(x) - 3\ln\big(G^{-1}(x)\big) + c - 4\sum_{k=1}^{n} \frac{\beta_{k+1}}{k}\Big(\frac{4}{G^{-1}(x)}\Big)^k + \mathrm{O}\Big(\frac{1}{x^{n+1}}\Big), \tag{32}$$

valid for all $n \geqslant 0$. Setting $n = 0$ in (32) one arrives at the case $n = 0$ in (30). To finish the proof one can proceed, in the obvious way, by induction in n when repeatedly using relation (32). □

4.4. Asymptotics of $h(t)$ for particular initial data

We already know that $h(t) = (G^{-1}(4t))^{1/4}$ solves the problem $h(t)^3(h''(t) + h'(t)) = 1$, $h(0) = h_0$, $h'(0) = h_1$. Using the known asymptotics of $G^{-1}(x)$ we get

$$h(t) = (4t)^{1/4}\bigg(1 + \sum_{k=1}^{n} \frac{p_{k-1}(c; \ln(4t))}{4^k t^k} + \mathrm{O}\Big(\frac{\ln(t)^n}{t^{n+1}}\Big)\bigg)^{1/4}$$

and consequently

$$h(t) = (4t)^{1/4}\bigg(1 + \sum_{k=1}^{n} \frac{q_{k-1}(c; \ln(4t))}{t^k} + \mathrm{O}\bigg(\Big(\frac{\ln(t)}{t}\Big)^{n+1}\bigg)\bigg),$$

where

$$q_k = \sum_{m=1}^{k} \frac{1}{4^k}\binom{\frac{1}{4}}{m} s_{m,k}(p_0, p_1, \ldots, p_{k-1}).$$

So q_k are exactly the polynomials introduced in Theorem 1. It is also easy to see that the degree of $q_k(c; z)$ is less than or equal to k since the same is true for the polynomials p_n with $n \geqslant 1$ and $\deg p_0 = 1$. This observation in fact completes the proof of Theorem 1 in the case when $t_0 = 0$ and $h_1 > 0$.

4.5. General initial conditions

Consider first a solution $h(t)$ of (1) with the initial conditions $h(0)=h_0$, $h'(0)=h_1$, assuming that h_1 is positive. Then, as we already know, the asymptotic behaviour of $h(t)$ is described by Theorem 1, i.e., equality (10) holds true with $c=c(0,h_0,h_1)$. Choose $s\in\mathbb{R}$ and set $\tilde{h}(t)=h(t+s)$. Then $\tilde{h}(t)$ solves equation (1) and satisfies the initial conditions $\tilde{h}(0)=\tilde{h}_0=h(s)$, $\tilde{h}'(0)=\tilde{h}_1=h'(s)$. But $h'(s)$ is positive for s sufficiently small and so equality (1) applies to $\tilde{h}(t)$ as well, with c being replaced by $\tilde{c}=c(0,\tilde{h}_0,\tilde{h}_1)$. Equating the asymptotics of $h(t+s)$ to the asymptotics of $\tilde{h}(t)$ one arrives at the equality

$$\begin{aligned}&\big(4(t+s)\big)^{1/4}\left(1+\sum_{k=1}^{n}\frac{q_k(c;\ln(4(t+s)))}{(t+s)^k}\right)\\&\quad=(4t)^{1/4}\left(1+\sum_{k=1}^{n}\frac{q_k(\tilde{c};\ln(4t))}{t^k}+\mathrm{O}\left(\left(\frac{\ln(t)}{t}\right)^{n+1}\right)\right)\end{aligned}\tag{33}$$

valid for $t\to+\infty$ and every $n\in\mathbb{Z}_+$. From (33) it is not difficult to derive the relation between c and $\tilde{c}$, it reads

$$\tilde{c}=c-4s.\tag{34}$$

Thus the invariance of the differential equation (1) is reflected in an invariance of the asymptotic expansion of its solutions, as expressed by relations (33), (34). It is also clear that these relations must hold true not only for s small but even for all $s\in\mathbb{R}$.

Choose now arbitrary initial data $(t_0,h_0,h_1)\in\mathbb{R}\times]0,\infty[\times\mathbb{R}$ and let $h(t)$ be the corresponding solution. Then, as we know from Corollary 4, $h'(t)>0$ for all sufficiently large t. Fix $s>t_0$ such that $h'(s)>0$ and set $\tilde{h}(t)=h(t+s)$. We use once more the already proven fact that $\tilde{h}(t)$ satisfies equality (10), with c being replaced by $\tilde{c}=c(0,h(s),h'(s))$. This implies that the asymptotic behaviour of $h(t)=\tilde{h}(t-s)$ is given by

$$h(t)=\big(4(t-s)\big)^{1/4}\left(1+\sum_{k=1}^{n}\frac{q_k(\tilde{c};\ln(4(t-s)))}{(t-s)^k}+\mathrm{O}\left(\left(\frac{\ln(t)}{t}\right)^{n+1}\right)\right),$$

$n\in\mathbb{Z}_+$. But in that case one deduces from (33), (34) that $h(t)$ satisfies equality (10) as well, with $c=\tilde{c}+4s$. Theorem 1 is proven.

5. Additional remark: comparison with the asymptotics of $-W_{-1}(-\mathrm{e}^{-x})$

This is a digression whose aim is to emphasise a rather close similarity of the asymptotic behaviour of the function $G^{-1}(x)$ with that of Lambert function. The Lambert function $W(z)$ – which was introduced by Euler in 1779, see [7] – gives the principal solution for w in $z=w\,\mathrm{e}^w$ and $W_k(z)$ gives the k-th solution. Surprisingly, it is not documented in some standard text books and reference books on special functions though we may have missed some sources. On the other hand, the Lambert function seems to have attracted even in a rather recent period some attention, particularly from the computational and combinatorial point of view (see [5] for a summary). Let us just briefly recall that $W(z)$ is analytic in a neighbourhood of $z = 0$ with the convergence radius equal to e^{-1},

$$W(z) = \sum_{k=1}^{\infty} \frac{(-1)^k k^{k-1}}{k!} z^k.$$

The coefficients have a combinatorial interpretation when counting distinct oriented trees.

Consider now the equation

$$y - \ln(y) = x,$$

or, equivalently,

$$y\,\mathrm{e}^{-y} = \mathrm{e}^{-x}.$$

It is elementary to see that for $x \in]1, +\infty[$ there are exactly two real solutions, $y_1(x)$ and $y_2(x)$, with $y_1(x) \in]0, 1[$ and $y_2(x) \in]1, +\infty[$. The both solutions can be expressed with the aid of the Lambert function, namely

$$y_1(x) = -W\big(-\mathrm{e}^{-x}\big), \qquad y_2(x) = -W_{-1}\big(-\mathrm{e}^{-x}\big).$$

The aim of this remark is to point out that the asymptotics of the second solution, i.e., $-W_{-1}(-\mathrm{e}^{-x})$, as $x \to +\infty$, can be derived in a way quite similar to what we have done in Subsection 4.3 when treating the function $G^{-1}(x)$. To this end let us recursively define polynomials $\tilde{p}_k(z)$,

$$\tilde{p}_0(z) = z, \quad \tilde{p}_{k+1}(z) = \sigma_{k+1}^0\big(\tilde{p}_0(z), \tilde{p}_1(z), \ldots, \tilde{p}_k(z)\big).$$

For $k \geqslant 1$, the degree of the polynomial $\tilde{p}_k(z)$ is k. Here are several first polynomials:

$$\tilde{p}_1(z) = z,\ \tilde{p}_2(z) = z - \frac{1}{2}z^2,\ \tilde{p}_3(z) = z - \frac{3}{2}z^2 + \frac{1}{3}z^3, \ldots.$$

Proposition 14. *It holds, as $x \to +\infty$,*

$$-W_{-1}\big(-\mathrm{e}^{-x}\big) = x + \mathrm{O}(\ln x)$$

and, for $n \geqslant 0$,

$$-W_{-1}\big(-\mathrm{e}^{-x}\big) = x + \sum_{k=0}^{n} \frac{\tilde{p}_k(\ln x)}{x^k} + \mathrm{O}\bigg(\Big(\frac{\ln x}{x}\Big)^{n+1}\bigg).$$

The proposition can be proven using a similar approach as the one used in the proof of Lemma 11. In fact, this asymptotic expansion is well known and is in agreement with what has been published in [6,4] and [5] though the derivation and presentation here is somewhat different.

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References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Addison-Wesley, 1978.
[2] J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.
[3] C. Chicone, *Ordinary Differential Equations with Applications*, Springer-Verlag, New York, 1999.
[4] L. Comtet, Inversion de $y^\alpha \mathrm{e}^\gamma$ et $y \log^\alpha y$ au moyen de nombres de Stirling, *C. R. Acad. Sci. Paris* **270** (1970), 1085–1088.
[5] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, On the Lambert W Function, *Adv. Comput. Math.* **5** (1996), 329–359.
[6] N.G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland, 1961.
[7] L. Euler, *De Serie Lambertina Plurismique Eius Insignibus Proprietatibus*, reprinted in: L. Euler, *Opera Omnia*, Ser. 1, "Opera Mathematica", Vol. 6, Teubner-Verlag, 1921.
[8] B. Halperin, Quantized Hall conductance, current-carrying edge states and the existence of extended states in a two-dimensional disordered potential, *Phys. Rev. B* **25** (1982), 2185ff.