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RESONANT CYCLOTRON ACCELERATION OF PARTICLES BY A TIME PERIODIC SINGULAR FLUX TUBE*

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Abstract. We study the dynamics of a classical nonrelativistic charged particle moving on a punctured plane under the influence of a homogeneous magnetic field and driven by a periodically time-dependent singular flux tube through the hole. We observe an effect of resonance of the flux and cyclotron frequencies. The particle is accelerated to arbitrarily high energies even by a flux of small field strength which is not necessarily encircled by the cyclotron orbit; the cyclotron orbits blow up and the particle oscillates between the hole and infinity. We support this observation by an analytic study of an approximation for small amplitudes of the flux which is obtained with the aid of averaging methods. This way we derive asymptotic formulas that are afterwards shown to represent a good description of the accelerated motion even for fluxes which are not necessarily small. More precisely, we argue that the leading asymptotic terms may be regarded as approximate solutions of the original system in the asymptotic domain as the time tends to infinity.

Key words. electron-cyclotron resonance, singular flux tube, averaging method, leading asymptotic term

AMS subject classifications. 70K28, 70K65, 34E10, 34C11, 34D05

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1. Introduction.

Consider a classical point particle of mass m and charge e moving on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ in the presence of a homogeneous magnetic field of magnitude b. Suppose further that a singular magnetic flux line whose strength $\Phi(t)$ is oscillating with frequency Ω intersects the plane at the origin. The equations of motion in phase space $\mathbb{P} = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ are generated by the time-dependent Hamiltonian

$$H(q,p,t) = \frac{1}{2m}\left(p - eA(q,t)\right)^2, \text{ with } A(q,t) = \left(-\frac{b}{2} + \frac{\Phi(t)}{2\pi|q|^2}\right)q^{\perp}, \tag{1.1}$$

where $(q,p) \in \mathbb{P}$, $t \in \mathbb{R}$. Here and throughout we denote $x^{\perp} = (-x_2, x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Our aim is to understand the dynamics of this system for large times. Of particular interest is the growth of energy as well as the drift of the guiding center.

Our main result in the present paper is to exhibit and to prove a resonance effect whose origin can qualitatively be understood as follows. The Lorentz force equals

$$eb\, q'(t)^{\perp} + eE(t), \quad \text{where} \quad E(t) = -\frac{\Phi'(t)}{2\pi|q|^2}\, q^{\perp}.$$

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In the induction free case when $\Phi'(t) = 0$, the particle moves along a circle of fixed center (cyclotron orbit) with the cyclotron frequency $\omega_c = |eb|/m$ and the cyclotron radius depending bijectively on the energy. If $\Phi'(t)$ is nonzero but small, then the energy, the frequency, and the guiding center of the orbit become time-dependent. For the time derivative of the energy computed in polar coordinates (r, θ) one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t} H = \frac{\partial}{\partial t} H = e\, q'(t) \cdot E(t) = -\frac{e}{2\pi}\, \theta'(t)\Phi'(t), \tag{1.2}$$

and so the acceleration rate is given by

$$\gamma = \lim_{\tau \to \infty} \frac{H(\tau)}{\tau} = -\frac{e}{2\pi} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \theta'(t)\Phi'(t) dt. \tag{1.3}$$

One may expect that $\theta'(t)$ is close to a periodic function of frequency ω_c (which is the case when $\Phi'(t) = 0$). If the frequencies ω_c and Ω are resonant, i.e., if there exist indices n and m in the support of the Fourier transforms of $\theta'(t)$ and $\Phi'(t)$, respectively, such that $n\omega_c + m\Omega = 0$, then one may speculate that γ is positive.

Our original motivation to study this problem was to understand the dynamics of so-called quantum charge pumps [7, 9] already on a classical level and to gain detailed intuition on the dynamical mechanisms in the simplest case. We suggest, however, that particle acceleration mechanisms might be of actual interest in various other models, for example, in interstellar physics [3, 5]. We speculate that the equations of motion in this domain might exhibit ingredients similar to those studied here.

We remark, too, that our results could be of interest in accelerator physics. While the betatron principle uses a linearly time-dependent flux tube to accelerate particles on cyclotron orbits around the flux [6], the resonance effect we observe in the present paper has the feature that acceleration can be achieved with arbitrarily small field strength. A second aspect is that, in contrast to the case of a linearly increasing flux, cyclotron orbits which do not encircle the flux tube are accelerated as well. In fact, for the linear case it was shown in [2] that outside the flux tube one has the usual drift of the guiding center, without acceleration, along the field lines of the averaged potential.

In the case of resonant frequencies one readily observes an accelerated motion numerically; see Figure 1.1 for an illustration. A typical resonant trajectory is a helix-like curve whose center goes out at the same rate as the radius grows. At the moment, a complete treatment of the equations of motion is out of reach. Therefore we first apply a resonant averaging method to derive a Hamiltonian which is formally a first order approximation in a small coupling constant of the flux tube. We then analyze its flow and show a certain type of asymptotic behavior at large times with the aid of differential topological methods. Though this approach gives only an existence result, we use the analysis to derive the leading asymptotic terms.

Then, in the second stage of our analysis, these asymptotic terms are used to formulate a simplified version of the evolution equations. We decouple the equations by substituting the anticipated leading asymptotic terms into the right-hand sides (RHSs). The decoupled system admits a rigorous asymptotic analysis whose conclusions turn out to be fully consistent with the formulas derived by the averaging methods. This holds under the assumption that the singular flux tube has a correct sign of the time derivative when the particle passes next to the origin. It should be emphasized that the singular magnetic flux is not assumed to be very small in the analysis of the decoupled system.

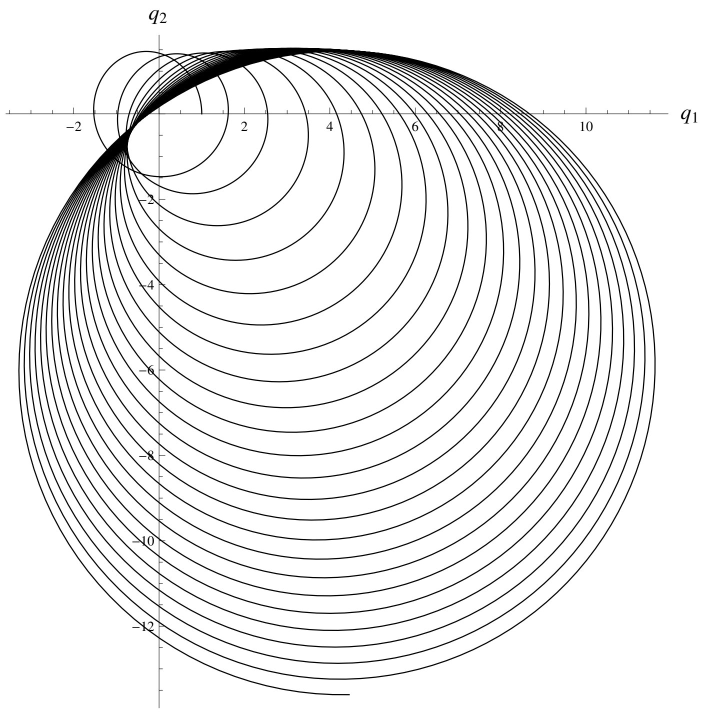

FIG. 1.1. *The solution $q(t)$ of the equations of motion in the plane for $t \in [0, 150]$, with $\Phi(t) = 2\pi\epsilon f(\Omega t)$, $f(t) = \sin(t) - (1/3)\cos(2t)$, for the values of parameters $\epsilon = 0.35$, $b = 1$, $\Omega = 1$, and with the initial conditions $q(0) = (1, 0)$, $q'(0) = (0, 1.617)$.*

The paper is organized as follows. In section 2 we summarize our main results. Namely, in subsection 2.1 we introduce the Hamiltonian equations of motion in action-angle coordinates, in subsection 2.2 we study the averaged dynamics resulting from the Poincaré–von Zeipel elimination method, and in subsection 2.3 we rederive the leading asymptotic terms by introducing a decoupled system. Finally, in subsection 2.4 we interpret the formulas derived so far in guiding center coordinates. Proofs, derivations, and additional details are postponed to sections 3, 4, 5, and 6.

2. Main results.

2.1. The Hamiltonian. In view of the rotational symmetry of the problem we prefer to work with polar coordinates, $q = r(\cos\theta, \sin\theta)$. Denoting by p_r and p_θ the momenta conjugate to r and θ, respectively, one has

$$p_r = p \cdot q/|q|, \; p_\theta = p \cdot q^\perp \equiv q \wedge p.$$

Using that the vector potential $A(q, t)$ is proportional to $q^\perp$, one finds for the Hamiltonian (1.1) in polar coordinates

$$H(r, \theta, p_r, p_\theta, t) = \frac{1}{2m}\left(p_r^2 + \left(\frac{1}{r}\left(p_\theta - \frac{e\Phi(t)}{2\pi}\right) + \frac{eb}{2}\,r\right)^2\right). \tag{2.1}$$

The angular momentum p_θ is an integral of motion, and thus the analysis of the system effectively reduces to a one-dimensional radial motion with time-dependent coefficients. From now on we set $e = m = 1$, and so the cyclotron frequency equals b. Put

$$a(t) = p_\theta - \frac{1}{2\pi}\,\Phi(t). \tag{2.2}$$

In the radial Hamiltonian one may omit the term $ba(t)/2$ not contributing to the equations of motion and thus arrive at the expression

$$H_{\mathrm{rad}}(r, p_r, t) = \frac{p_r^2}{2} + \frac{a(t)^2}{2r^2} + \frac{b^2r^2}{8}. \tag{2.3}$$

For definiteness we assume that $b > 0$.

The time-independent Hamiltonian system, with $a = \text{const}$, is explicitly solvable. In particular, one finds the corresponding action-angle coordinates (I, φ) depending on a as a parameter. Substituting the given function $a(t)$ for a, one gets a time-dependent transformation of coordinates. This is an essential step in the analysis of the time-dependent Hamiltonian (2.3) since action-angle coordinates are appropriate for employing averaging methods. Postponing the derivation to section 3, here we give the transformation equations:

$$r = \frac{2}{\sqrt{b}}\left(I + \frac{|a|}{2} + \sqrt{I(I+|a|)}\,\sin(\varphi)\right)^{1/2}, \tag{2.4}$$

$$p_r = \frac{\sqrt{bI(I+|a|)}\,\cos(\varphi)}{\left(I + \frac{|a|}{2} + \sqrt{I(I+|a|)}\,\sin(\varphi)\right)^{1/2}}, \tag{2.5}$$

and, conversely,

$$I = \frac{1}{2b}\left(p_r^2 + \left(\frac{|a(t)|}{r} - \frac{br}{2}\right)^2\right), \tag{2.6}$$

$$\tan(\varphi) = -\frac{1}{brp_r}\left(p_r^2 + \frac{a(t)^2}{r^2} - \frac{b^2r^2}{4}\right). \tag{2.7}$$

Furthermore, expressing the Hamiltonian in action-angle coordinates, one obtains

$$H_c(\varphi, I, t) = bI - |a(t)|' \arctan\left(\frac{\sqrt{I}\cos(\varphi)}{\sqrt{I+|a(t)|} + \sqrt{I}\sin(\varphi)}\right), \tag{2.8}$$

and the corresponding Hamiltonian equations of motion take the form

$$\varphi' = b - \frac{\cos(\varphi)aa'}{2\sqrt{I(I+|a|)}\left(2I + |a| + 2\sqrt{I(I+|a|)}\,\sin(\varphi)\right)}, \tag{2.9}$$

$$I' = -\frac{|a|'}{2}\left(1 - \frac{|a|}{2I + |a| + 2\sqrt{I(I+|a|)}\,\sin(\varphi)}\right). \tag{2.10}$$

Using action-angle coordinates, one can give a rough qualitative description of trajectories in the resonant case. From (2.4) we see that

$$r^2 = \frac{1}{2}\,(r_+^2 + r_-^2) + \frac{1}{2}\,(r_+^2 - r_-^2)\sin(\varphi),$$

where

$$r_{\pm}=\sqrt{\frac{2}{b}}\left(\sqrt{I+|a|}\pm\sqrt{I}\right)$$

are extremal points of the trajectory (see section 3). As formulated more precisely in what follows, resonant trajectories are characterized by a linear increase of $I(t)$ for large times while $\varphi(t)\approx bt+\text{const}$. Thus, as the angle φ increases, the radius r oscillates between r_- and r_+ (though r_-, r_+ themselves are also time-dependent). Moreover, since $a(t)$ is bounded, $I\to\infty$ implies $r_+\to\infty$ and $r_-(t)=2|a(t)|/(b\,r_+(t))\to 0$ as $t\to+\infty$. Therefore the trajectory in the q-plane periodically returns to the origin and then again escapes far away from it while its extremal distances to the origin converge, respectively, to zero and infinity. We refer again to Figure 1.1 for a typical trajectory in the q-plane in the case of resonant frequencies.

Concluding this subsection, let us make more precise some assumptions. For the sake of definiteness we shall focus on the case when p_θ is positive and greater than the amplitude of $\Phi(t)/(2\pi)$, and so $a(t)$ is an everywhere strictly positive function. Let us stress, however, that this restriction is not essential for the resonance effect as the radial Hamiltonian (2.3) depends only on $a(t)^2$, and thus the sign of $a(t)$ is irrelevant for the motion in the radial direction. On the other hand, as discussed in subsection 2.4, the sign of $a(t)$ determines whether the orbit encircles the singular magnetic flux or not.

Furthermore, the frequency $\Omega>0$ of the singular flux tube is treated as a parameter of the model, and so we write

$$\Phi(t)=2\pi\epsilon f(\Omega t), \tag{2.11}$$

where f is a 2π-periodic real function possibly obeying additional assumptions. Hence

$$a(t)=p_\theta-\epsilon f(\Omega t),$$

where the coupling constant ϵ is supposed to be positive and, if desired, playing the role of a small parameter.

2.2. The dynamics generated by the first order averaged Hamiltonian. In order to study occurrences of resonant behavior, we apply the Poincaré–von Zeipel elimination method, which takes into account possible resonances, as explained in detail, for instance, in [1]. In Proposition 2.2 we provide a detailed information on the resonance effect for the dynamics generated by the first order averaged Hamiltonian.

We start from introducing some basic notation. Let $\mathbb{T}^d=(\mathbb{R}/2\pi\mathbb{Z})^d$ be the d-dimensional torus. For $f(\varphi)\in C(\mathbb{T}^d)$ and $k\in\mathbb{Z}^d$ we denote the kth Fourier coefficient of f by the symbol

$$\mathscr{F}[f(\varphi)]_k=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}f(\varphi)\,e^{-ik\cdot\varphi}\,\mathrm{d}\varphi.$$

We introduce $\operatorname{supp}\mathscr{F}[f]$ as the set of indices corresponding to nonzero Fourier coefficients of $f(\varphi)$. For $f\in C(\mathbb{T}^d)$ and $\mathbb{L}\subset\mathbb{Z}^d$ put

$$\langle f(\varphi)\rangle_{\mathbb{L}}=\sum_{k\in\mathbb{L}}\mathscr{F}[f]_k\,e^{ik\cdot\varphi}.$$

For example, in the formulation of Proposition 2.2 we use the averaged function $\langle f(\varphi)\rangle_{\mathbb{Z}\nu}$, with $\nu \in \mathbb{N}$. Assuming that $f \in C(\mathbb{T}^1)$ let us note that it can alternatively be expressed, without directly referring to the Fourier series, as

$$\langle f(\varphi)\rangle_{\mathbb{Z}\nu} = \frac{1}{\nu}\sum_{j=0}^{\nu-1} f\left(\varphi + \frac{2\pi}{\nu}\,j\right). \tag{2.12}$$

To verify the formula observe that the RHS of (2.12) again belongs to $C(\mathbb{T}^1)$. To check that its Fourier series coincides with the left-hand side (LHS) of (2.12) it suffices to consider exponential functions $f(\varphi) = e^{ik\varphi}$, with $k \in \mathbb{Z}$. In that case one finds that the RHS of (2.12) reproduces the function $f(\varphi)$ if ν is a divisor of k and vanishes otherwise.

In this subsection we assume that $\Phi(t)$ is given by (2.11), where $\epsilon > 0$ is regarded as a small parameter and the 2π-periodic real function $f(\varphi)$ fulfills

$$\sum_{k=1}^{\infty} k\,|\mathscr{F}[f(\varphi)]_k| < \infty. \tag{2.13}$$

This implies that $f \in C^1(\mathbb{T}^1)$.

Following a standard approach to time-dependent Hamiltonian systems, we pass to the extended phase space by introducing a new phase $\varphi_2 = \Omega t$ and its conjugate momentum I_2 thus obtaining an equivalent autonomous system on a larger space. To have a unified notation we rename the old variables φ, I as φ_1, I_1, respectively, and set $\omega_1 = b$, $\omega_2 = \Omega$. With this new notation, we write $I = (I_1, I_2)$, $\varphi = (\varphi_1, \varphi_2)$ (changing the meaning of the symbols I and φ in the current subsection).

The Hamiltonian on the extended phase space is defined as

$$K(\varphi_1, \varphi_2, I_1, I_2) = \omega_2 I_2 + H_c(\varphi_1, I_1, \varphi_2/\omega_2).$$

Recalling (2.8) and the conventions for p_θ and Φ, one gets

$$K(\varphi, I, \epsilon) = \omega_1 I_1 + \omega_2 I_2 + \epsilon F(\varphi, I, \epsilon), \tag{2.14}$$

where

$$F(\varphi, I, \epsilon) = \omega_2 f'(\varphi_2) \arctan\left(\frac{\sqrt{I_1}\cos(\varphi_1)}{\sqrt{I_1 + p_\theta - \epsilon f(\varphi_2)} + \sqrt{I_1}\sin(\varphi_1)}\right). \tag{2.15}$$

Our discussion focuses on the resonant case when

$$\lambda := \frac{\omega_2}{\omega_1} = \frac{\mu}{\nu}\,, \text{ with } \mu, \nu \in \mathbb{N} \text{ coprime,} \tag{2.16}$$

and, moreover, ν fulfills

$$\operatorname{supp} \mathscr{F}[f] \cap (\mathbb{Z}\nu \setminus \{0\}) \neq \emptyset. \tag{2.17}$$

Note that (2.17) happens if and only if $\langle f(\varphi)\rangle_{\mathbb{Z}\nu}$ is not a constant function. Discussion of the nonresonant case is, at least on the level of the averaged dynamics, simple and we avoid it. Let us just remark that, in that case, it is not difficult to see that trajectories as well as the energy for the averaged Hamiltonian are bounded.

The Hamiltonian (2.14) is appropriate for application of the Poincaré–von Zeipel elimination method, whose basic scheme is briefly recalled in subsection 4.1. The idea is to eliminate from $K(\varphi, I, \epsilon)$, with the aid of a canonical transformation, a subgroup of angle variables which are classified as nonresonant, and thus to arrive at a new reduced Hamiltonian $\mathcal{K}(\psi, J, \epsilon)$ depending only on the so-called resonant angle variables and the corresponding actions. Note, however, that a good deal of information about the system is contained in the canonical transformation itself.

To achieve this goal, one works with formal power series in the parameter ϵ, and the construction is in fact an infinite recurrence. In particular, $\mathcal{K}(\psi, J, \epsilon) = \mathcal{K}_0(\psi, J) + \epsilon\mathcal{K}_1(\psi, J) + \cdots$, where the leading term remains untouched,

$$\mathcal{K}_0(\psi, J) = K_0(\psi, J) = \omega_1 J_1 + \omega_2 J_2. \tag{2.18}$$

In practice one has to interrupt the recurrence at some order, which means replacing the true Hamiltonian system by an approximate averaged system. In our case we shall be content with a truncation at the first order.

In our model, one can apply the substitution $\chi_1 = \mu\psi_1 - \nu\psi_2$, $\chi_2 = \tilde{\nu}\psi_1 + \tilde{\mu}\psi_2$, where $\tilde{\mu}, \tilde{\nu} \in \mathbb{Z}$ are such that $\tilde{\mu}\mu + \tilde{\nu}\nu = 1$ ($\tilde{\mu}, \tilde{\nu}$ exist since μ, ν are coprime). The phase χ_2 is classified as nonresonant and can be eliminated while the phase χ_1 is resonant and survives the canonical transformation. The procedure is explained in more detail in subsections 4.2 and 4.3. Recalling (2.18) combined with (2.16), here we just give the resulting formula for the averaged Hamiltonian truncated at the first order,

$$\mathcal{K}_{(1)}(\psi, J, \epsilon) = \frac{\omega_1}{\nu}\,(\nu J_1 + \mu J_2) + \epsilon\,\mathcal{K}_1(\psi, J), \tag{2.19}$$

where

$$\mathcal{K}_1(\psi, J) = -\frac{\omega_1}{2} \sum_{n \in \mathbb{Z}\setminus\{0\}} \mathscr{F}[f]_{-n\nu}\, i^{n\mu} \beta(J_1)^{|n|\mu}\, e^{in(\mu\psi_1 - \nu\psi_2)} \tag{2.20}$$

and

$$\beta(J_1) = \sqrt{\frac{J_1}{J_1 + p_\theta}}\,. \tag{2.21}$$

One observes that the averaged four-dimensional Hamiltonian system admits a reduction to a two-dimensional subsystem. Indeed, as it should be, $\mathcal{K}_1(\psi, J)$ depends on the angles only through the combination $\mu\psi_1 - \nu\psi_2 = \chi_1$. It follows that the action $\nu J_1 + \mu J_2$ is an integral of motion for the Hamiltonian $\mathcal{K}_{(1)}(\psi, J, \epsilon)$. The reduced system then depends on the coordinates χ_1, J_1.

Moreover, by inspection of the series (2.20) one finds that the Hamiltonian for the reduced subsystem can be expressed in terms of a single complex variable $z = \beta(J_1)^\mu e^{i\chi_1}$ and takes the form $\mathcal{Z}(\chi_1, J_1) = \mathrm{Re}[h(z)]$, with h being a holomorphic function. We refer to subsection 4.3 for the proof of the following abstract theorem.

THEOREM 2.1. *Let $h \in C^1(\overline{B_1})$ and suppose $h(z)$ is a nonconstant holomorphic function on the open unit disk B_1. Let $\varrho : [0, +\infty[\, \to [0, 1[$ be a smooth function such that $\varrho'(x) > 0$ for $x > 0$, $\varrho(0) = 0$ and $\lim_{x\to+\infty} \varrho(x) = 1$. Let $\mathcal{Z}(\chi_1, J_1)$ be the Hamilton function on $\mathbb{R} \times\,]0, +\infty[$ defined by*

$$\mathcal{Z}(\chi_1, J_1) = \mathrm{Re}\big[h\big(\varrho(J_1)e^{i\chi_1}\big)\big]\,.$$

Then, for almost all initial conditions $(\chi_1(0), J_1(0))$*, the corresponding Hamiltonian trajectory fulfills*

$$\lim_{t\to+\infty} \chi_1(t) = \chi_1(\infty) \in \mathbb{R}, \quad \lim_{t\to+\infty} J_1(t) = +\infty, \tag{2.22}$$

and

$$\lim_{t\to+\infty} J_1'(t) = \operatorname{Im}\Big[e^{i\chi_1(\infty)}\, h'\Big(e^{i\chi_1(\infty)}\Big)\Big] > 0. \tag{2.23}$$

Theorem 2.1 yields the desired information about the asymptotic behavior of the averaged Hamiltonian system in action-angle coordinates (J, ψ). One has to go back to the original action-angle coordinates (I, φ), however, since in these coordinates the dynamics of the studied system can be directly interpreted. This means inverting the canonical transformation $(I, \varphi) \mapsto (J, \psi)$ which resulted from the Poincaré–von Zeipel elimination method. Let us remark that the generating function of this canonical transformation is truncated at the first order as well. Doing so, one derives the following result, whose proof can again be found in subsection 4.3.

Proposition 2.2. *Suppose that conditions* (2.16), (2.17) *characterizing the resonant case are satisfied. Let* $(\varphi_1(t), I_1(t))$ *be a trajectory for the first order averaged Hamiltonian* (2.19) *but expressed in the original angle-action coordinates, as introduced in subsection* 2.1*. Then, for almost all initial conditions* $(\varphi_1(0), I_1(0))$*,*

$$\lim_{t\to+\infty} (\varphi_1(t) - \omega_1 t) = \phi(\infty) \in \mathbb{R}, \quad \lim_{t\to+\infty} \frac{I_1(t)}{t} = C > 0, \tag{2.24}$$

and

$$C = -\frac{\varepsilon\omega_2}{2}\, f_\nu'\Big(-\Big(\phi(\infty) + \frac{\pi}{2}\Big)\lambda\Big), \ \textit{with } f_\nu(\varphi) = \langle f(\varphi)\rangle_{\mathbb{Z}\nu}.$$

2.3. A decoupled system and the leading asymptotic terms. In the preceding subsection we studied an approximate dynamics derived with the aid of averaging methods. It turned out that a typical trajectory reaches for large times an asymptotic domain characterized by formulas (2.24). Being guided by this experience as well as by numerical experiments that we have carried out, we suggest that formulas (2.24) in fact give the correct leading asymptotic terms for the complete system.

To support this suggestion we now derive the leading asymptotic terms and a bound on the order of error terms for the original Hamiltonian equations (2.9), (2.10) while assuming that the system already follows the anticipated asymptotic regime. At the same time, we relax the assumption on smallness of the parameter ϵ.

Applying the substitution

$$F(t) = 2I(t) + |a(t)|, \ \phi(t) = \varphi(t) - bt$$

to equations (2.9), (2.10) we obtain the system of differential equations

$$F'(t) = \frac{a(t)\, a'(t)}{F(t) + \sqrt{F(t)^2 - a(t)^2}\,\sin(bt + \phi(t))}\,, \tag{2.25}$$

$$\phi'(t) = -\frac{a(t)\, a'(t)\cos(bt + \phi(t))}{\sqrt{F(t)^2 - a(t)^2}\,\Big(F(t) + \sqrt{F(t)^2 - a(t)^2}\,\sin(bt + \phi(t))\Big)}\,, \tag{2.26}$$

where $a(t) = p_\theta - \epsilon f(\Omega t)$ (see (2.2) and (2.11)). The real function $f(t)$ is supposed to be continuously differentiable and 2π-periodic. The only assumptions imposed on $\epsilon > 0$ in this subsection are that ϵ is not too big so that the function $a(t)$ has no zeros and is everywhere of the same sign as p_θ. Recall that for definiteness p_θ is supposed to be positive. Clearly, the functions $a(t)$ and $a'(t)$ are bounded on $\mathbb{R}$.

Equations (2.25) and (2.26) are nonlinear and coupled together. To decouple them we replace $\phi(t)$ on the RHS of (2.25) and $F(t)$ on the RHS of (2.26) by the respective leading asymptotic terms (2.24), as learned from the averaging method. This is done under the assumption that the solution has already reached the domain $F(t) \geq F_0 \gg p_\theta$, where $F(t)$ is sufficiently large and starts to grow. Thus, to formulate a problem with decoupled equations, we replace $\phi(t)$ in (2.25) by the expected limit value $\phi \equiv \phi(\infty) \in \mathbb{R}$, i.e., the simplified equation reads

$$F'(t) = \frac{a(t)\, a'(t)}{F(t) + \sqrt{F(t)^2 - a(t)^2}\, \sin(bt + \phi)}\,. \tag{2.27}$$

As stated in Proposition 2.3 below, a solution $F(t)$ of (2.27) actually grows linearly for large times. Conversely, (2.26) is analyzed in Proposition 2.4 under the assumption that $F(t)$ grows linearly. In that case $\phi(t)$ is actually shown to approach a constant value as t tends to infinity. For derivation of Proposition 2.4 the periodicity of the functions $a(t)$ is not important. It suffices to assume that it takes values from a bounded interval separated from zero. For the proofs see section 5.

Proposition 2.3. *Suppose $f \in C^2(\mathbb{R})$ is 2π-periodic and*

$$\frac{\Omega}{b} \in \mathbb{N},\ f'\!\left(-\frac{\Omega}{b}\left(\phi + \frac{\pi}{2}\right)\right) < 0.$$

Then any solution of (2.27) *such that $F(0)$ is sufficiently large fulfills*

$$F(t) = \epsilon\,\Omega \left| f'\!\left(-\frac{\Omega}{b}\left(\phi + \frac{\pi}{2}\right)\right)\right| t + O\big(\log(t)^2\big) \quad \text{as } t \to +\infty.$$

Proposition 2.4. *Suppose $a(t) \in C^1(\mathbb{R})$ fulfills*

$$A_1 \leq a(t) \leq A_2,\ |a'(t)| \leq A_3$$

for some positive constants A_1, A_2, A_3. Furthermore, suppose $F(t) \in C(\mathbb{R})$ has the asymptotic behavior

$$F(t) = \alpha t + O\big(\log(t)^2\big) \quad \text{as } t \to +\infty, \tag{2.28}$$

with a positive constant α. If $\phi(t)$ obeys the differential equation (2.26) *on a neighborhood of $+\infty$, then there exists a finite limit $\lim_{t\to+\infty} \phi(t) = \phi(\infty)$ and*

$$\phi(t) = \phi(\infty) + O\!\left(\frac{\log(t)}{t}\right) \quad \text{as } t \to +\infty. \tag{2.29}$$

Remarks. (i) Let us point out an essential difference between (2.25) and (2.26). Note that for all $F, a, s \in \mathbb{R}$ such that $F \geq |a| > 0$ one has

$$\left|\frac{a\cos(s)}{F + \sqrt{F^2 - a^2}\,\sin(s)}\right| \leq 1, \tag{2.30}$$

and, consequently, it follows from (2.26) that

$$|\phi'(t)| \leq \frac{|a'(t)|}{\sqrt{F(t)^2 - a(t)^2}}\,. \tag{2.31}$$

Thus the RHS of (2.26) is inversely proportional to $F(t)$. The point is that if the denominator in (2.30) becomes very small for those s for which $\sin(s) = -1$, then one gets compensation by the vanishing numerator. Nothing similar can be claimed, however, for (2.25).

(ii) The replacement of the phase $\phi(t)$ by a constant in (2.25) was quite crucial for derivation of the result stated in Proposition 2.3. In fact, suppose that $F(t)$ is sufficiently large. In the case of the original equation (2.25), too, essential contributions to the increase of $F(t)$ are achieved at the moments of time for which $\sin(bt+\phi(t)) = -1$. If $\phi(t)$ equals a constant, then these moments of time are well defined and the growth of $F(t)$ can be estimated. On the contrary, without a sufficiently precise information about $\phi(t)$ one loses any control on the growth of $F(t)$.

Propositions 2.3 and 2.4 can also be interpreted in the following way. Let us pass from the differential equations (2.25), (2.26) to the integral equations

$$F(t) - F(0) - \int_0^t \frac{a(s)\,a'(s)}{F(s) + \sqrt{F(s)^2 - a(s)^2}\,\sin(bs + \phi(s))}\,\mathrm{d}s = 0,$$

$$\phi(t) - \phi(\infty) - \int_t^\infty \frac{a(s)\,a'(s)\cos(bs + \phi(s))}{\sqrt{F(s)^2 - a(s)^2}\left(F(s) + \sqrt{F(s)^2 - a(s)^2}\sin(bs + \phi(s))\right)}\,\mathrm{d}s = 0.$$

Suppose $\phi(\infty)$ satisfies $\alpha = -\epsilon\,\Omega f'(-(\Omega/b)(\phi(\infty) + \pi/2)) > 0$. If $F(0)$ is sufficiently large, then the functions

$$F(t) = \alpha t + F(0),\ \ \phi(t) = \phi(\infty),\ \ t > 0,$$

can be regarded as an approximate solution of this system of integral equations with errors of order $O(\log(t)^2)$ for the first equation and of order $O(\log(t)/t)$ for the second one.

One has to admit, however, that this argument still does not represent a complete mathematical proof of the asymptotic behavior of the action-angle variables $I(t) = (F(t) - |a(t)|)/2$, $\varphi(t) = bt + \phi(t)$. So far we have analyzed either the dynamics generated by an approximate averaged Hamiltonian in subsection 2.2 or a simplified decoupled system in the current subsection. Moreover, in the latter case it should be emphasized that the simplified equations were derived under the essential assumption that the dynamical system had already reached the regime characterized by an acceleration with an unlimited energy growth (this is reflected by the assumption that $F(0)$ is sufficiently large). Nevertheless, on the basis of this analysis as well as on the basis of numerical experiments we formulate the following conjecture.

Conjecture 2.5. *If $\Omega/b \in \mathbb{N}$, then the regime of acceleration for the original (true) dynamical system is described by the asymptotic behavior*

$$I(t) = Ct + O\big(\log(t)^2\big),\ \ \varphi(t) = bt + \phi(\infty) + O\bigg(\frac{\log(t)}{t}\bigg) \quad \text{as } t \to +\infty, \tag{2.32}$$

where $\phi(\infty)$ is a real constant and

$$C = -\frac{1}{2}\,\epsilon\Omega f'(-\xi) > 0,\ \text{with}\ \ \xi = \frac{\Omega}{b}\left(\phi(\infty) + \frac{\pi}{2}\right). \tag{2.33}$$

2.4. Guiding center coordinates. Being given the asymptotic relations (2.32), (2.33) it is desirable to describe the accelerated motion in terms of the original Cartesian coordinates q. The description becomes more transparent if the motion is decomposed into a motion of the guiding center X and a relative motion of the particle with respect to this center which is characterized by a gyroradius vector R and a gyrophase ϑ [8]. Since the presented results have a direct physical interpretation, in this subsection we make an exception and give the formulas using now all physical constants (including m and e). Additional details and derivations are postponed to section 6.

Let $v(q,p,t) = (p - eA(q,t))/m$ be the velocity. We write $q = X + R$, where, by definition,

$$X(q,p,t) = q + \frac{1}{\omega_c}\,v^{\perp}(q,p,t),\ \ R(q,p,t) = q - X(q,p,t) = -\frac{1}{\omega_c}\,v^{\perp}(q,p,t)$$

are the guiding center field and the gyroradius field, respectively. In what follows, we use the polar decompositions

$$q = r\,(\cos\theta, \sin\theta),\ \ X = |X|(\cos\chi, \sin\chi),\ \ R = |R|(\cos\vartheta, \sin\vartheta).$$

Concerning the geometrical arrangement, one has the relation

$$\vartheta = \varphi + \chi - \frac{\pi}{2} \pmod{2\pi} \tag{2.34}$$

(with φ being introduced in subsection 2.1; for the derivation see section 6).

The quantities X, R were introduced (under different names) and studied in [2], where one can also find several formulas given below, notably those given in (2.36). Observe that $|p|^2 = p_r^2 + p_\theta^2/r^2$ and, according to (2.3) and (2.6),

$$H = H_{\mathrm{rad}} + \frac{\omega_c a}{2} = \omega_c\left(I + \frac{1}{2}\,(|a| + a)\right), \quad \text{with } a(t) = p_\theta - \frac{e}{2\pi}\,\Phi(t). \tag{2.35}$$

Using these relations one derives that

$$\frac{m\omega_c^2}{2}\,|X|^2 = H - \omega_c\,p_\theta + \frac{e\omega_c}{2\pi}\,\Phi, \quad \frac{m\omega_c^2}{2}\,|R|^2 = H. \tag{2.36}$$

Observe from (2.36) that if $a(t)$ is an everywhere positive function, then $|R(t)| > |X(t)|$ and so the center of coordinates always stays in the domain encircled by the spiral-like trajectory. On the contrary, if $a(t)$ is an everywhere negative function, then the center of coordinates is never encircled by the trajectory.

For a Hamiltonian trajectory $(q(t), p(t))$ put $H(t) = H(q(t), p(t), t)$. Suppose again that $p_\theta > 0$ and hence $a(t) > 0$. Applying Conjecture 2.5 and recalling (2.11), one has

$$I(t) = -\frac{e}{4\pi}\,\Phi'\!\left(-\frac{1}{\omega_c}\left(\phi(\infty) + \frac{\pi}{2}\right)\right) t + O\big(\log(t)^2\big) \quad \text{as } t \to +\infty. \tag{2.37}$$

Using (2.35) and recalling definition (1.3) of the acceleration rate γ, one finds the positive value

$$\gamma = \lim_{\tau\to\infty} \frac{H(\tau)}{\tau} = -\frac{e\omega_c}{4\pi}\,\Phi'\!\left(-\frac{1}{\omega_c}\left(\phi(\infty) + \frac{\pi}{2}\right)\right) > 0. \tag{2.38}$$

From (2.36), (2.35), and (2.37) one also deduces the asymptotic behavior of the guiding center and the gyroradius,

(2.39)
$$|X(t)| = \sqrt{\frac{2\gamma t}{m\omega_c^2}} + O\bigg(\frac{\log(t)^2}{\sqrt{t}}\bigg),\ |R(t)| = \sqrt{\frac{2\gamma t}{m\omega_c^2}} + O\bigg(\frac{\log(t)^2}{\sqrt{t}}\bigg) \quad \text{as } t \to +\infty.$$

Now it is obvious why the resonance also has consequences for the drift. The growing energy H comes along with a growing distance of the guiding center from the origin. Thus in q-space the particle oscillates around a drifting center which goes out at the same rate as the radius grows.

From Conjecture 2.5, with a bit of additional analysis (see section 6), one can also derive consequences for the asymptotic behavior of the angle variables. One has, as $t \to +\infty$,

$$\chi(t) = D\log(\omega_c t) + \chi(\infty) + o(1), \quad \vartheta(t) = \omega_c t + D\log(\omega_c t) + \vartheta(\infty) + o(1), \tag{2.40}$$

where D, $\chi(\infty)$, and $\vartheta(\infty)$ are real constants. Relations (2.39) and (2.40) give complete information about the asymptotic behavior of the trajectory $q(t) = X(t) + R(t)$.

Finally, in connection with formula (2.38) let us point out that the studied system behaves for large times almost as a "kicked" system. Note that, for a fixed $a > 0$,

$$\frac{a}{x - \sqrt{x^2 - a^2}\,\cos(t)} = \sum_{n=-\infty}^{+\infty} \left(\frac{x-a}{x+a}\right)^{|n|/2} e^{int} \;\to\; 2\pi\,\delta_{2\pi}(t) \quad \text{as } x \to +\infty,$$

where $\delta_A(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nA)$ stands for the A-periodic prolongation of the Dirac δ function. Since $H'(t) = -e\theta'(t)\Phi'(t)/(2\pi)$ (see (1.2)), $\varphi(t) \approx \omega_c t + \phi(\infty)$ and, in view of (2.4),

$$\theta' = \frac{a}{mr^2} + \frac{\omega_c}{2} = \frac{\omega_c a}{2\Big(2I + a + 2\sqrt{I(I+a)}\,\sin\varphi\Big)} + \frac{\omega_c}{2},$$

one has

$$H'(t) \approx -\frac{e}{2}\,\Phi'\bigg(-\frac{1}{\omega_c}\Big(\phi(\infty) + \frac{\pi}{2}\Big)\bigg)\,\delta_{2\pi/\omega_c}\bigg(t + \frac{1}{\omega_c}\Big(\phi(\infty) + \frac{\pi}{2}\Big)\bigg).$$

Thus the system gains the main contributions to the energy growth in narrow intervals around the instances of time for which $\omega_c t = -\phi(\infty) - \pi/2 \pmod{2\pi}$. According to (2.34), these are exactly those instances of time when $\vartheta(t) - \chi(t) \approx \pi \pmod{2\pi}$, i.e., when the particle passes next to the singular flux line.

3. Derivation of the Hamiltonian in action-angle coordinates. In this section we derive the transformation equations from coordinates r, p_r to action-angle coordinates I, φ.

Let us first note that the equations of motion for the radial Hamiltonian (2.3) have a well-defined unique solution for any initial condition and for all times. They are equivalent to the nonlinear second order differential equation

$$r'' + \frac{b^2}{4}\,r = \frac{a(t)^2}{r^3}\,. \tag{3.1}$$

PROPOSITION 3.1. *Suppose $a(t)$ is a real continuously differentiable function defined on $\mathbb{R}$ having no zeros. Then for any initial condition $r(t_0) = r_0$, $r'(t_0) = r_1$,* *with* $(t_0, r_0, r_1) \in \mathbb{R} \times]0, +\infty[\times \mathbb{R}$*, there exists a unique solution of the differential equation* (3.1) *defined on the whole real line* $\mathbb{R}$ *and satisfying this initial condition.*

Proof. Suppose $r(t)$ is a solution of the Hamiltonian equations (3.1). Put $p_r = r'$ and $H(t) = H_{\mathrm{rad}}(r(t), p_r(t), t)$. Then

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}H(t)\right| = \frac{|a(t)a'(t)|}{r(t)^2} \leq \left|\frac{2a'(t)}{a(t)}\right| H(t).$$

From here one readily concludes that if $r(t)$ is defined on a bounded interval $M \subset \mathbb{R}$, then there exist constants R_1, R_2, $0 < R_1 \leq R_2 < +\infty$, such that $R_1 \leq r(t) \leq R_2$ for all $t \in M$. From the general theory of ordinary differential equations it follows that any solution $r(t)$ of (3.1) can be continued to the whole real line. ∎

As a first step of the derivation of the transformation we introduce the action-angle coordinates for a frozen time. Assume for a moment that $a(t) = a$ is a constant and denote

$$V(r) = \frac{a^2}{2r^2} + \frac{b^2 r^2}{8}.$$

Suppose a fixed energy level E is greater than the minimal value $V_{\min} = b|a|/2$. Then the motion is constrained to a bounded interval $[r_-, r_+]$, and one has

$$E - V(r) = \frac{b^2}{8r^2}\,(r_+^{\,2} - r^2)(r^2 - r_-^{\,2}).$$

The action equals

$$I(E) = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2(E - V(\rho))}\,\mathrm{d}\rho = \frac{b}{8}\,(r_+ - r_-)^2.$$

Hence

$$r_\pm = \frac{2}{\sqrt{b}}\left(I + \frac{|a|}{2} \pm \sqrt{I(I + |a|)}\right)^{1/2} = \sqrt{\frac{2}{b}}\left(\sqrt{I + |a|} \pm \sqrt{I}\right).$$

Using the generating function,

$$S(r, I) = \int_{r_-}^{r} \sqrt{2(E - V(\rho))}\,\mathrm{d}\rho = \frac{b}{2}\int_{r_-}^{r} \frac{1}{\rho}\sqrt{(r_+^{\,2} - \rho^2)(\rho^2 - r_-^{\,2})}\,\mathrm{d}\rho,$$

one derives the canonical transformation of variables between (r, p_r) and the angle-action variables (φ, I). One has

$$\begin{aligned}\frac{\partial S}{\partial I} &= \frac{b}{4}\int_{r_-}^{r} \frac{1}{\rho}\left(\sqrt{\frac{\rho^2 - r_-^{\,2}}{r_+^{\,2} - \rho^2}}\,\frac{\mathrm{d}r_+(I)^2}{\mathrm{d}I} - \sqrt{\frac{r_+^{\,2} - \rho^2}{\rho^2 - r_-^{\,2}}}\,\frac{\mathrm{d}r_-(I)^2}{\mathrm{d}I}\right)\mathrm{d}\rho \\ &= \frac{\pi}{2} - \arctan\left(\frac{r_+^{\,2} + r_-^{\,2} - 2r^2}{2\sqrt{(r_+^{\,2} - r^2)(r^2 - r_-^{\,2})}}\right).\end{aligned}$$

For the angle variable $\varphi = \partial S/\partial I - \pi/2$ one obtains

$$\sin(\varphi) = \frac{1}{\sqrt{I(I + |a|)}}\left(\frac{br^2}{4} - I - \frac{|a|}{2}\right).$$

Furthermore,

$$p_r = \frac{\partial S}{\partial r} = \frac{b}{2r}\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)} = \frac{2}{r}\sqrt{I(I + |a|)}\,\cos(\varphi).$$

Relations (2.4), (2.5), (2.6), (2.7) readily follow.

Let us now switch to the time-dependent case. The Hamiltonian transforms according to the rule

$$H_c(\varphi, I, t) = H_{\mathrm{rad}}\big(r(\varphi, I, t), p_r(\varphi, I, t), t\big) + \left.\frac{\partial S(u, I, t)}{\partial t}\right|_{u=r(\varphi,I,t)}.$$

One computes

$$\left.\frac{\partial S(u, I, t)}{\partial t}\right|_{u=r(\varphi,I,t)} = \frac{b\,|a|'}{4}\int_{r_-}^{r(\varphi,I,t)} \frac{1}{\rho}\left(\sqrt{\frac{\rho^2 - r_-^2}{r_+^2 - \rho^2}}\,\frac{\partial r_+^2}{\partial |a|} - \sqrt{\frac{r_+^2 - \rho^2}{\rho^2 - r_-^2}}\,\frac{\partial r_-^2}{\partial |a|}\right)\mathrm{d}\rho.$$

Simplifying the expression and dropping those terms which are independent of φ and I, one finally arrives at the expression (2.8).

4. Details on the von Zeipel elimination method and the proofs.

4.1. A summary of basic formulas. This is a short summary of basic steps in the Poincaré–von Zeipel elimination procedure. Recall the notation from the beginning of subsection 2.2.

Consider a completely integrable Hamiltonian in action-angle coordinates, $K_0(I) = \omega \cdot I$, where I runs over a domain in $\mathbb{R}^d$, $\varphi \in \mathbb{T}^d$, and $\omega \in \mathbb{R}_+^d$ is a constant vector of frequencies. One is interested in a perturbed system with a small Hamiltonian perturbation so that the total Hamiltonian reads

$$K(\varphi, I, \epsilon) = K_0(I) + \epsilon\, K_*(\varphi, I, \epsilon) = K_0(I) + \epsilon\, K_1(\varphi, I) + \epsilon^2\, K_2(\varphi, I) + \cdots,$$

where ϵ is a small parameter. The function $K_*(\varphi, I, \epsilon)$ is assumed to be analytic in all variables

Let $\mathbb{K}$ be the lattice of indices in $\mathbb{Z}^d$ corresponding to resonant frequencies and let $\mathbb{K}^c$ be its complement, i.e.,

$$\mathbb{K} = \{n \in \mathbb{Z}^d;\, n \cdot \omega = 0\},\ \mathbb{K}^c = \mathbb{Z}^d \setminus \mathbb{K}.$$

One applies a formal canonical transformation of variables, $(I, \varphi) \mapsto (J, \psi)$, so that the Fourier series in the angle variables ψ of the resulting Hamiltonian $\mathcal{K}(\psi, J, \epsilon)$ has nonzero coefficients only for indices from the lattice $\mathbb{K}$. The canonical transformation is generated by a function $S(\varphi, J, \epsilon)$ regarded as a formal power series with coefficient functions $S_j(\varphi, J)$ and the absolute term $S_0(\varphi, J) = \varphi \cdot J$. Similarly, the new Hamiltonian $\mathcal{K}(\psi, J, \varepsilon)$ is sought in the form of a formal power series with coefficient functions $\mathcal{K}_j(\psi, J)$. One arrives at the system of equations $\mathcal{K}_0(J, \varphi) = K_0(J) = \omega \cdot J$ and

$$\mathcal{K}_j(\varphi, J) = \omega \cdot \frac{\partial S_j(\varphi, J)}{\partial \varphi} + P_j(\varphi, J),\ j \geq 1,$$

where $P_1(\varphi, J) = K_1(\varphi, J)$ and the terms P_j for $j \geq 2$ are determined recursively. The formal von Zeipel Hamiltonian is defined by the equalities $\mathcal{K}_j(\psi, J) = \langle P_j(\psi, J)\rangle_{\mathbb{K}}$ for $j \geq 1$. Coefficients $S_j(\varphi, J)$ are then solutions of the first order differential equations

$$\omega \cdot \frac{\partial S_j(\varphi, J)}{\partial \varphi} = -\,\langle P_j(\varphi, J)\rangle_{\mathbb{K}^c},\ j \geq 1.$$

In practice one truncates $\mathcal{K}(\psi, J, \epsilon)$ at some order $m \geq 1$ of the parameter ϵ. Let us define the mth order averaged Hamiltonian

$$\mathcal{K}_{(m)}(\psi, J, \epsilon) = \mathcal{K}_0(J) + \epsilon\, \mathcal{K}_1(\psi, J) + \cdots + \epsilon^m \mathcal{K}_m(\psi, J).$$

Similarly, let $S_{(m)}(\varphi, J, \epsilon)$ be the truncated generating function. If $(\psi(t), J(t))$ is a solution of the Hamiltonian equations for $\mathcal{K}_{(m)}(\psi, J, \epsilon)$, and if $(\varphi(t), I(t))$ is the same solution after the inverted canonical transformation generated by $S_{(m)}(\varphi, J, \epsilon)$, then $(\varphi(t), I(t))$ is expected to approximate well the solution of the original system (governed by the Hamiltonian $K(\varphi, I, \epsilon)$) for times of order $1/\epsilon^m$ (see [1] for a detailed discussion).

4.2. Derivation of the first order averaged Hamiltonian. If the ratio ω_2/ω_1 is irrational, then the lattice $\mathbb{K}$ is trivial, $\mathbb{K} = \{0\}$, and the von Zeipel method amounts to the ordinary averaging method in the angle variables φ. Here we focus on the complementary case (2.16) when $\omega_2/\omega_1 = \mu/\nu$, with μ and ν being coprime positive integers.

Recalling (2.14), (2.15), we have $K(\varphi, I, \epsilon) = \omega_1 I_1 + \omega_2 I_2 + \epsilon K_1(\varphi, I) + \epsilon^2 \widetilde{K}(\varphi, I, \epsilon)$, where $\widetilde{K}(\varphi, I, \epsilon)$ is an analytic function in ϵ,

$$K_1(\varphi, I) = \omega_2 f'(\varphi_2) F_1(\varphi_1, I_1),\ \ F_1(\varphi_1, I_1) = \arctan\left(\frac{\sqrt{I_1}\cos(\varphi_1)}{\sqrt{I_1 + p_\theta} + \sqrt{I_1}\sin(\varphi_1)}\right).$$

One finds that

$$\mathscr{F}[F_1(\varphi_1, I_1)]_k = \frac{i^{k-1}}{2k}\left(\frac{I_1}{I_1 + p_\theta}\right)^{|k|/2} \quad \text{for } k \neq 0,\ \ \mathscr{F}[F_1(\varphi_1, I_1)]_0 = 0.$$

Obviously, the Fourier image of $K_1(\varphi, I)$ takes nonzero values only for indices (k, ℓ) such that $k \in \mathbb{Z} \setminus \{0\}$, $\ell \in \operatorname{supp} \mathscr{F}[f] \setminus \{0\}$, and

$$\mathscr{F}[K_1(\varphi, I)]_{(k,\ell)} = i\ell\omega_2\, \mathscr{F}[f(\varphi_2)]_\ell\, \mathscr{F}[F_1(\varphi_1, I_1)]_k.$$

Now we can describe the von Zeipel canonical transformation of the first order. The resonant lattice is $\mathbb{K} = \mathbb{Z}\,(\mu, -\nu)$, and, according to the general scheme, one has $\mathcal{K}_1(\psi, J) = \langle K_1(\psi, J)\rangle_{\mathbb{K}}$. With the above computations, this immediately leads to (2.20), (2.21), and, consequently, (2.19).

An important piece of information is also the Poincaré–von Zeipel canonical transformation generated by $S_{(1)}(\varphi, J, \epsilon)$. $S_1(\varphi, J)$ is a solution to the differential equation $\omega \cdot \partial S_1/\partial\varphi = -K_1 + \mathcal{K}_1$. Seeking $S_1(\varphi, J)$ in the form

$$S_1(\varphi, J) = 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty} \mathscr{F}[f']_k G_k(\varphi_1, J_1)\, e^{ik\varphi_2}\right), \tag{4.1}$$

one finally arrives at the countable system of equations

$$\left(\frac{\partial}{\partial\varphi_1} + ik\lambda\right) G_k(\varphi_1, J_1) = \lambda \sum_{n\in\mathbb{Z}\setminus\{0\},\ n\neq -k\lambda} \frac{i^{n+1}}{2n}\, \beta(J_1)^{|n|}\, e^{in\varphi_1},\ \ k \geq 1.$$

For the solution we choose

$$G_k(\varphi_1, J_1) = \lambda \sum_{n\in\mathbb{Z}\setminus\{0\},\ n\neq -k\lambda} \frac{i^n}{2n(n + k\lambda)}\, \beta(J_1)^{|n|}\, e^{in\varphi_1}. \tag{4.2}$$

Of course, if $k\lambda \notin \mathbb{Z}$, then the restriction $n \neq -k\lambda$ is void. On the other hand, if $k\lambda \in \mathbb{Z}$, and this happens if and only if $k \in \mathbb{Z}\nu$, then the solution $G_k(\varphi_1, J_1)$ is not unique.

4.3. Proofs of Theorem 2.1 and Proposition 2.2. The purpose of this subsection is to give some details concerning the dynamics generated by the Hamiltonian $\mathcal{K}_{(1)}(\psi, J, \epsilon)$, as defined in (2.19), and the proofs of Theorem 2.1 and Proposition 2.2.

We start from the reduction to a two-dimensional Hamiltonian subsystem. Since μ and ν are coprime there exist $\tilde{\mu}, \tilde{\nu} \in \mathbb{Z}$ such that $\tilde{\mu}\mu + \tilde{\nu}\nu = 1$. Put

$$\mathbf{R} = \begin{pmatrix} \mu & -\nu \\ \tilde{\nu} & \tilde{\mu} \end{pmatrix}$$

and consider the canonical transformation $\chi = \mathbf{R}\psi$, $J = \mathbf{R}^{\mathrm{T}} L$ generated by the function $L \cdot \mathbf{R}\psi$. In particular,

$$\chi_1 = \mu\psi_1 - \nu\psi_2,\ L_2 = \nu J_1 + \mu J_2,\ J_1 = \mu L_1 + \tilde{\nu} L_2.$$

Since $\mathcal{K}_1(\psi, J)$ in (2.20) depends only on the angle χ_1, and not on χ_2, the action L_2 is an integral of motion. Let us define

$$\mathcal{Z}(\chi_1, J_1) = \varepsilon\mu\,\mathcal{K}_1(\mathbf{R}^{-1}\chi, J).$$

Then

$$\begin{aligned} \chi_1'(t) &= \varepsilon\,\frac{\partial\mathcal{K}_1(\psi, J)}{\partial J_1}\,\frac{\partial J_1}{\partial L_1} = \frac{\partial\mathcal{Z}(\chi_1, J_1)}{\partial J_1}, \\ J_1'(t) &= -\varepsilon\,\frac{\partial\mathcal{K}_1(\psi, J)}{\partial\psi_1} = -\frac{1}{\mu}\,\frac{\partial\mathcal{Z}(\chi_1, J_1)}{\partial\chi_1}\,\frac{\partial\chi_1}{\partial\psi_1} = -\frac{\partial\mathcal{Z}(\chi_1, J_1)}{\partial\chi_1}. \end{aligned}$$

Thus the time evolution in coordinates χ_1, J_1 is governed by the Hamiltonian $\mathcal{Z}(\chi_1, J_1)$.

From the form of the series (2.20) one can see that the Hamiltonian $\mathcal{Z}(\chi_1, J_1)$ can be expressed, in a compact way, in terms of a holomorphic function. Set

$$h(z) = -\varepsilon\mu\omega_1 \sum_{n=1}^{\infty} \mathscr{F}[f]_{-n\nu}\, i^{n\mu}\, z^n \tag{4.3}$$

and

$$\varrho(x) = \beta(x)^\mu = \left(\frac{x}{x + p_\theta}\right)^{\mu/2}, \quad x > 0.$$

Then, by assumption (2.13), $h(z)$ is holomorphic on the open unit disk $B_1 \subset \mathbb{C}$ and $h \in C^1(\overline{B_1})$. Moreover, one can write $\mathcal{Z}(\chi_1, J_1) = \mathrm{Re}[h(\varrho(J_1)e^{i\chi_1})]$ (note that $\mathscr{F}[f]_{n\nu} = \overline{\mathscr{F}[f]_{-n\nu}}$ since f is real). The Hamiltonian equations of motion read

$$\chi_1'(t) = \frac{\varrho'(J_1)}{\varrho(J_1)}\,\mathrm{Re}[z\,h'(z)],\ J_1'(t) = \mathrm{Im}[z\,h'(z)],\ \text{with } z = \varrho(J_1)e^{i\chi_1}. \tag{4.4}$$

Concerning the asymptotic behavior of Hamiltonian trajectories $(\chi_1(t), J_1(t))$, as $t \to +\infty$, one can formulate a proposition under somewhat more general circumstances, as is done in Theorem 2.1.

Proof of Theorem 2.1. Set $R(z) = \mathrm{Re}[h(z)]$, $z \in \overline{B_1}$. Then

$$\mathrm{d}R_z \equiv (\mathrm{Re}[h'(z)], -\mathrm{Im}[h'(z)]).$$

Hence $\mathrm{d}R_z = 0$ if and only if $h'(z) = 0$, and the set of critical points of R in B_1 has no accumulation points in B_1 and is at most countable. By Sard's theorem, almost all $y \in \mathbb{R}$ are regular values of $R|\partial B_1$. If y is a regular value of both R and $R|\partial B_1$ then the level set $R^{-1}(y)$ is a compact one-dimensional C^1 submanifold with boundary in $\overline{B_1}$, $\partial R^{-1}(y) = R^{-1}(y) \cap \partial B_1$, and $R^{-1}(y)$ is not tangent to ∂B_1 at any point. Moreover, $R^{-1}(y) \cap B_1$ is a smooth submanifold of B_1 [4]. By the classification of compact connected one-dimensional manifolds [4], every component of $R^{-1}(y)$ is diffeomorphic either to a circle or to a closed interval. But the first possibility is excluded because $R(z)$ is a harmonic function. In fact, if $\overline{U} \subset B_1$, U is an open set, $\partial U \simeq S^1$ is a smooth submanifold of B_1, and $R(z)$ is constant on ∂U, then $R(z)$ is constant on U and so is $h(z)$. Consequently, $h(z)$ is constant on B_1, a contradiction with our assumptions. Thus every component Γ of $R^{-1}(y)$ is diffeomorphic to a closed interval, $\partial\Gamma = \{a, b\} = \Gamma \cap \partial B_1$, and Γ is tangent to ∂B_1 neither at a nor at b. Let $z \in B_1$ be such that $\mathrm{d}R_z \neq 0$. By the local submersion theorem [4], R is locally equivalent at z to the canonical submersion $\mathbb{R}^2 \to \mathbb{R}$. Hence z possesses an open neighborhood U such that $R(U)$ is an open interval. We know that almost every $y \in R(U)$ is a regular value both of R and $R|\partial B_1$. By the Fubini theorem, for almost every $w \in U$, $R(w)$ is a regular value both of R and $R|\partial B_1$. The same claim is true for almost all $w \in B_1$ because the set of critical points of R in B_1 is at most countable. It follows that for almost all $(\chi_1, J_1) \in \mathbb{R} \times]0, +\infty[$, $R(\varrho(J_1)e^{i\chi_1}) \neq R(0)$ is a regular value of both R and $R|\partial B_1$.

Suppose now that an initial condition $(\chi_1(0), J_1(0))$ has been chosen so that $y = R(\varrho(J_1(0))e^{i\chi_1(0)}) \neq R(0)$ is a regular value of both R and $R|\partial B_1$. Let Γ be the component of $R^{-1}(y)$ containing the point $\varrho(J_1(0))e^{i\chi_1(0)}$. Since the Hamiltonian $\mathcal{Z}(\chi_1, J_1)$ is an integral of motion the Hamiltonian trajectory $z(t) = \varrho(J_1(t))e^{i\chi_1(t)}$ is constrained to the submanifold $\Gamma \subset \overline{B_1}$. We have to show that $z(t)$ reaches the boundary ∂B_1 as $t \to +\infty$. The tangent vector to the trajectory at the point $z(t)$ equals

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = i\varrho(J_1(t))\varrho'(J_1(t))\,\overline{h'(z(t))}\,.$$

Since $0 \notin \Gamma$, $\varrho'(J_1) > 0$ for all $J_1 > 0$ and $h'(z)$ has no zeros on Γ (because y is a regular value) it follows that $z(t)$ leaves any compact subset of B_1 in a finite time. It remains to show that $z(t)$ does not reach ∂B_1 in a finite time. But by equations of motion (4.4), $|J_1'(t)| \leq \max_{z \in \partial B_1} |h'(z)|$ and so $J_1(t)$ cannot grow faster than linearly.

This reasoning shows (2.22). From (4.4) and (2.22) it follows (2.23); one has only to justify the sign of the limit. Obviously, the limit must be nonnegative. Denote $\partial R = R|\partial B_1$. Then ∂R can be regarded as a function of the angle variable, $\partial R(x) = \mathrm{Re}[h(e^{ix})]$, and one has

$$(\partial R)'\big(\chi_1(\infty)\big) = -\,\mathrm{Im}\Big[e^{i\chi_1(\infty)}\,h'\Big(e^{i\chi_1(\infty)}\Big)\Big] \neq 0$$

because $y = \partial R(\chi_1(\infty))$ is a regular value of ∂R. ∎

Finally, to derive Proposition 2.2 one has to apply the inverted canonical transformation, from (ψ, J) to (φ, I), generated by $S_{(1)}(\varphi, J, \epsilon) = \varphi \cdot J + \epsilon S_1(\varphi, J)$. Hence

$$\psi = \varphi + \epsilon\,\frac{\partial S_1(\varphi, J)}{\partial J}\,,\ I = J + \epsilon\,\frac{\partial S_1(\varphi, J)}{\partial \varphi}\,. \tag{4.5}$$

For the proof we need a couple of auxiliary estimates.

Lemma 4.1. *For all β, $0 \leq \beta < 1$, and all $a \in \mathbb{R} \setminus \mathbb{Z}$,*

$$\sup_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \frac{\beta^{|n+j|}}{|n-a|} \leq \frac{1}{\mathrm{dist}(a, \mathbb{Z})} + 2 + 6|\log(1-\beta)|, \tag{4.6}$$

and for all $a \in \mathbb{Z}$,

$$\sup_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z},\ n\neq a} \frac{\beta^{|n+j|}}{|n-a|} \leq 1 + 3|\log(1-\beta)|. \tag{4.7}$$

Proof. Notice that inequality (4.6) is invariant if a is replaced either by $-a$ or by $k+a$, $k \in \mathbb{Z}$. Thus we can restrict ourselves to the interval $0 < a \leq 1/2$. Then $|a| = \operatorname{dist}(a, \mathbb{Z})$ and $|n-a| \geq |n|/2$. This observation reduces (4.6) to (4.7) with $a = 0$. Similarly, inequality (4.7) is invariant if a is replaced by $k+a$, $k \in \mathbb{Z}$. It follows that, in both cases, it suffices to show (4.7) for $a = 0$ and with the range of j restricted to nonnegative integers.

Suppose that $j \geq 0$. Splitting the range of summation in n into the subranges $n \leq -j-1$, $-j \leq n \leq -1$, and $1 \leq n$, one gets

$$\sum_{n\in\mathbb{Z},\ n\neq 0} \frac{\beta^{|n+j|}}{|n|} \leq 2|\log(1-\beta)| + \sum_{m=1}^{j} \frac{\beta^{j-m}}{m}\,.$$

Furthermore,

$$\sum_{1\leq m\leq j/2} \frac{\beta^{j-m}}{m} \leq \sum_{1\leq m\leq j/2} \frac{\beta^{m}}{m} \leq |\log(1-\beta)|$$

and

$$\sum_{j/2<m\leq j} \frac{\beta^{j-m}}{m} \leq \sum_{j/2<m\leq j} \frac{1}{m} \leq 1.$$

This shows the lemma. ∎

Proof of Proposition 2.2. If (2.17) is true, then $h(z)$ defined in (4.3) obeys the assumptions of Theorem 2.1, and so for almost all initial conditions $(\chi_1(0), J_1(0))$, equalities (2.22) and (2.23) hold. In particular, Theorem 2.1 implies that

$$1 - \beta(J_1(t)) = O(t^{-1}) \quad \text{as } t \to +\infty. \tag{4.8}$$

Putting $f_\nu(\varphi) = \langle f(\varphi)\rangle_{\mathbb{Z}\nu}$ one also has

$$\lim_{t\to+\infty} \frac{J_1(t)}{t} = \operatorname{Im}\Big[e^{i\chi_1(\infty)}\, h'\Big(e^{i\chi_1(\infty)}\Big)\Big] = -\frac{\varepsilon\omega_2}{2}\, f_\nu'\bigg(-\frac{\chi_1(\infty)}{\nu} - \frac{\pi\lambda}{2}\bigg) > 0. \tag{4.9}$$

Further, using (4.2) one can estimate (recalling that $\lambda = \mu/\nu$)

$$\left|\frac{\partial G_k(\varphi_1, J_1)}{\partial J_1}\right| \leq \frac{\lambda}{2}\,\beta'(J_1) \sum_{n\in\mathbb{Z}\setminus\{0\},\ n\neq -k\lambda} \frac{1}{|n+k\lambda|}\,\beta(J_1)^{|n|-1} \leq \frac{\mu\beta'(J_1)}{1-\beta(J_1)}\,.$$

Hence, using (2.21),

$$\left|\frac{\partial G_k(\varphi_1, J_1)}{\partial J_1}\right| \leq \frac{\mu}{2p_\theta}\, \frac{1-\beta(J_1)}{\beta(J_1)}\,. \tag{4.10}$$

As a next step let us estimate $|\partial G_k/\partial\varphi_1|$. Clearly,

$$\left|\frac{\partial G_k(\varphi_1, J_1)}{\partial \varphi_1}\right| \leq \frac{\lambda}{2} \sum_{n\in\mathbb{Z}\setminus\{0\},\ n\neq -k\lambda} \frac{1}{|n+k\lambda|}\,\beta(J_1)^{|n|}\,.$$

Writing $k\lambda = -j - a$, with $j \in \mathbb{Z}$ and $a = s/\nu$, $s = 0, 1, \ldots, \nu - 1$, one can apply Lemma 4.1 to show that

$$\left|\frac{\partial G_k(\varphi_1, J_1)}{\partial \varphi_1}\right| \le c' + c''|\log(1 - \beta(J_1))|, \tag{4.11}$$

where the constants c', c'' do not depend on k and φ_1, J_1.

From definitions (4.1), (4.2) and from assumption (2.13) one can readily see that $S(\varphi, J)$ is C^1 in φ_2 and C^∞ in φ_1, J_1 (and does not depend on J_2). Moreover, from (4.10) and (4.8) it follows that

$$\frac{\partial S_1(\varphi(t), J(t))}{\partial J_1} = O(t^{-1}) \quad \text{as } t \to +\infty. \tag{4.12}$$

Similarly, estimate (4.11) implies

$$\frac{\partial S_1(\varphi(t), J(t))}{\partial \varphi_1} = O(\log(t)) \quad \text{as } t \to +\infty. \tag{4.13}$$

Now one can deduce the asymptotic behavior of $\varphi_1(t)$ and $I_1(t)$. Observe that $\psi_2' = \omega_2$ and $S_1(\varphi, J)$ does not depend on J_2, and hence $\psi_2(t) = \varphi_2(t) = \omega_2 t$. Furthermore, $\psi_1 = (\chi_1 + \nu\psi_2)/\mu$ and so

$$\lim_{t \to +\infty} (\psi_1(t) - \omega_1 t) = \frac{1}{\mu}\, \chi_1(\infty). \tag{4.14}$$

Put $\phi(\infty) = \chi_1(\infty)/\mu$. To conclude the proof it suffices to recall the transformation rules (4.5) and to take into account (4.9) jointly with (4.13) and (4.14) jointly with (4.12). ∎

5. Proofs of Propositions 2.3 and 2.4. Analyzing (2.27) we prefer to work with a rescaled time (or one can choose the units so that $b = 1$) and, simplifying the notation, we consider a differential equation of the form

$$g'(t) = \frac{\varrho(t)}{g(t) + \sqrt{g(t)^2 - a(t)^2}\sin(t + \phi)}, \tag{5.1}$$

where $\varrho(t)$, $a(t)$ are continuously differentiable real functions, $a(t)$ is strictly positive, and ϕ is a real constant. In the resonant case the functions $\varrho(t)$, $a(t)$ are supposed to be 2π-periodic, which means for the original data that $\Omega \in \mathbb{N}$ (and $b = 1$).

In the first step we estimate the growth of a solution on an interval of length $\pi/2$. Let $\|f\| = \max|f(t)|$ denote the norm in $C([0, \pi/2])$, and put

$$A = \min_{0 \le t \le \pi/2} a(t) > 0.$$

Consider for a moment the differential equation

$$h'(t) = \frac{\varrho(t)}{h(t) - \sqrt{h(t)^2 - a(t)^2}\,\cos(t)} \tag{5.2}$$

on the interval $[0, \pi/2]$ with an initial condition $h(0) = h_0 > \|a\|$. The goal is to show that for large values of h_0, an essential contribution to the growth of a solution $h(t)$ on this interval comes from a narrow neighborhood of the point $t = 0$.

Remark 5.1. If $\varrho(t)$ is nonnegative on the interval $[0,\pi/2]$, then a solution $h(t)$ to (5.2) surely exists and is unique. In the general case, the existence and uniqueness is guaranteed, provided the initial condition h_0 is sufficiently large. From (5.2) one derives that $|h'(t)| \leq 2\|\varrho\| h(t)/A^2$ and so

$$\exp\bigl(-2\|\varrho\|t/A^2\bigr)\,h_0 \leq h(t) \leq \exp\bigl(2\|\varrho\|t/A^2\bigr)\,h_0 \tag{5.3}$$

as long as $h(t)$ makes sense. Consequently, a sufficient condition for the existence of a solution is $h_0 > \exp(\pi\|\varrho\|/A^2)\|a\|$.

LEMMA 5.2. *Let $\varrho, a \in C^1([0,\pi/2])$ be real functions, $\varrho(0) \neq 0$ and $a(t) > 0$ on $[0,\pi/2]$. Consider the set of solutions $h(t)$ to the differential equation* (5.2) *on the interval $[0,\pi/2]$ with a variable initial condition $h(0) = h_0$ for h_0 sufficiently large. Then*

$$h\Bigl(\frac{\pi}{2}\Bigr) = h_0 + \frac{\pi\varrho(0)}{a(0)} + O\bigl(h_0^{-1}\log(h_0)\bigr) \quad \textit{as } h_0 \to +\infty.$$

Proof. Let us fix η, $0 < \eta \leq \pi/2$, so that $|\varrho(t)| > 0$ on the interval $[0,\eta\,[$; i.e., $\varrho(t)$ does not change its sign on that interval. Thus any solution $h(t)$ to (5.2) is strictly monotone on $[0,\eta]$. For $\eta \leq t \leq \pi/2$ one can estimate $|h'(t)| \leq C/h(t)$, where $C = \|\varrho\|/(1-\cos(\eta))$. In view of (5.3) it follows that

$$h(\pi/2) - h(\eta) = O\left(h_0^{-1}\right) \text{ as } h_0 \to +\infty. \tag{5.4}$$

Set

$$h_1 = \min\{h(0), h(\eta)\},\ h_2 = \max\{h(0), h(\eta)\},\ \Delta = h(\eta) - h_0.$$

Then $|\Delta| = h_2 - h_1$. Set, for $x \geq a > 0$,

$$\Psi(x,a,t) = \frac{1}{x - \sqrt{x^2-a^2}\,\cos(t)}\,.$$

One has $h'(t) = \varrho(t)\,\Psi(h(t),a(t),t)$. If $x \geq 2u/\sqrt{3} > 0$, then $\sqrt{x^2-u^2} \geq x/2$ and

$$\left|\frac{\partial}{\partial x}\Psi(x,u,t)\right| = \frac{\bigl|\sqrt{x^2-u^2} - x\cos(t)\bigr|}{\sqrt{x^2-u^2}\,\bigl(x - \sqrt{x^2-u^2}\cos(t)\bigr)^2} \;\leq\; \frac{2}{x}\,\Psi(x,u,t), \tag{5.5}$$

$$\left|\frac{\partial}{\partial u}\Psi(x,u,t)\right| = \frac{u\cos(t)}{\sqrt{x^2-u^2}\,\bigl(x - \sqrt{x^2-u^2}\cos(t)\bigr)^2} \;\leq\; \frac{3}{u}\,\Psi(x,u,t). \tag{5.6}$$

Observe also that, for $x \geq u > 0$,

$$\int_0^{\pi/2} \Psi(x,u,t)\,\mathrm{d}t = \frac{2}{u}\,\arctan\!\left(\frac{x+\sqrt{x^2-u^2}}{u}\right) \leq \frac{\pi}{u}\,. \tag{5.7}$$

Assuming that h_0 is sufficiently large, and using (5.5), (5.7) one can estimate

$$\begin{aligned}\left|\Delta - \int_0^\eta \varrho(t)\Psi(h_0,a(t),t)\,\mathrm{d}t\right| &\leq \int_0^\eta |\varrho(t)|\,|\Psi(h(t),a(t),t) - \Psi(h_0,a(t),t)|\,\mathrm{d}t\\ &\leq \frac{2\|\varrho\|}{h_1}\int_0^{\pi/2}\left(\int_{h_1}^{h_2}\Psi(x,A,t)\,\mathrm{d}x\right)\mathrm{d}t\\ &\leq \frac{2\pi\|\varrho\|}{Ah_1}\,|\Delta|.\end{aligned}$$

In view of (5.3) it follows that

$$\Delta = \big(1 + O(h_0^{-1})\big) \int_0^{\eta} \varrho(t)\Psi(h_0, a(t), t)\,\mathrm{d}t.$$

Furthermore, with the aid of (5.6) one finds that

$$|\varrho(t)\Psi(h_0, a(t), t) - \varrho(0)\Psi(h_0, a(0), t)| \le C'\Psi(h_0, A, t)t,$$

where

$$C' = \left(1 + \frac{9\|\varrho\|^2}{A^2}\right)^{1/2} \sqrt{\|\varrho'\|^2 + \|a'\|^2}\,.$$

Note that

$$\int_0^{\eta} \Psi(h_0, A, t)\,t\,\mathrm{d}t \le \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin(t)}{h_0 - \sqrt{h_0^2 - A^2}\cos(t)}\,\mathrm{d}t = O\big(h_0^{-1}\log(h_0)\big)$$

and

$$\begin{aligned}\int_0^{\eta} \varrho(0)\Psi(h_0, a(0), t)\,\mathrm{d}t &= \varrho(0) \int_0^{\pi/2} \Psi(h_0, a(0), t)\,\mathrm{d}t + O\big(h_0^{-1}\big) \\ &= \frac{2\varrho(0)}{a(0)} \arctan\!\left(\frac{h_0 + \sqrt{h_0^2 - a(0)^2}}{a(0)}\right) + O\big(h_0^{-1}\big) \\ &= \frac{\pi\varrho(0)}{a(0)} + O\big(h_0^{-1}\big)\,.\end{aligned}$$

Altogether this means that

$$h(\eta) - h_0 = \Delta = \frac{\pi\varrho(0)}{a(0)} + O\big(h_0^{-1}\log(h_0)\big)\,.$$

Recalling (5.4), the lemma follows. ∎

Consider the mapping $\mathcal{H} : h(0) \mapsto h(\pi/2)$, where $h(t)$ runs over solutions to the differential equation (5.2). From the general theory of ordinary differential equations it is known that $\mathcal{H}$ is a C^1 mapping well defined on a neighborhood of $+\infty$. Lemma 5.2 claims that $\mathcal{H}(x) = x + \pi\varrho(0)/a(0) + O(x^{-1}\log(x))$. On the basis of similar arguments, the inverse mapping $\mathcal{H}^{-1} : h(\pi/2) \mapsto h(0)$ is also well defined and C^1 on a neighborhood of $+\infty$. From the asymptotic behavior of $\mathcal{H}(x)$ one readily derives that $\mathcal{H}^{-1}(y) = y - \pi\varrho(0)/a(0) + O(y^{-1}\log(y))$. These considerations make it possible to reverse the roles of the boundary points 0 and $\pi/2$. Moreover, splitting the interval $[0, 2\pi]$ into four subintervals of length $\pi/2$, one arrives at the following lemma.

Lemma 5.3. *Let $\varrho, a \in C^1([0, 2\pi])$ be real functions, $\varrho(\pi) \ne 0$ and $a(t) > 0$ on $[0, 2\pi]$. Consider the set of solutions $h(t)$ to the differential equation*

$$h'(t) = \frac{\varrho(t)}{h(t) + \sqrt{h(t)^2 - a(t)^2}\,\cos(t)}$$

on the interval $[0, 2\pi]$ with a variable initial condition $h(0) = h_0$ for h_0 sufficiently large. Then

$$h(2\pi) = h_0 + \frac{2\pi\varrho(\pi)}{a(\pi)} + O\big(h_0^{-1}\log(h_0)\big) \quad \textit{as } h_0 \to +\infty.$$

In the next step, applying repeatedly Lemma 5.3 one can show that solutions of the differential equation (5.1) in the resonant case $\Omega \in \mathbb{N}$ (with $b = 1$) grow linearly with time, provided the initial condition is sufficiently large and the phase ϕ belongs to a certain interval.

PROPOSITION 5.4. *Suppose $\varrho(t)$, $a(t)$ are continuously differentiable 2π-periodic real functions, $\phi \in \mathbb{R}$, $a(t)$ is everywhere positive, and*

$$\varrho\Big(-\phi - \frac{\pi}{2}\Big) > 0.$$

Let $g(t)$ be a solution of the differential equation (5.1) *on the interval $t \geq 0$ with the initial condition $g(0) = g_0 \geq 1$. If g_0 is sufficiently large, then*

$$g(t) = \frac{\varrho\big(-\phi - \frac{\pi}{2}\big)}{a\big(-\phi - \frac{\pi}{2}\big)}\, t + O\big(\log(t)^2\big) \quad \text{as } t \to +\infty.$$

Proof of Proposition 2.3. This is just an immediate corollary of Proposition 5.4. It suffices to go back to the original notation and (2.27) by applying the substitution $F(t) = g(bt)$, $a(t) = \tilde{a}(bt)$, $\varrho(t) = \tilde{a}(t)\tilde{a}'(t)$. Equation (2.27) transforms into (5.1) (with $a(t)$ being replaced by $\tilde{a}(t)$), and Proposition 5.4 is directly applicable and gives the result. □

Proof of Proposition 2.4. Put $\zeta(t) = bt - \phi(t)$. Recall that $\phi'(t)$ obeys the estimate (2.31). Choose $t_* \in \mathbb{R}$ such that $|\phi'(t)| \leq b/2$ for all $t \geq t_*$. Hence the function $\zeta(t)$ is strictly increasing and $b/2 \leq \zeta'(t) \leq 3b/2$. Moreover, we choose t_* sufficiently large so that

$$F(t+s) \leq \sqrt{2}\, F(t) \text{ for } 0 \leq s \leq 3\pi b, \text{ and } F(t) \geq \sqrt{2}\, A_2 \quad \forall t \geq t_*. \tag{5.8}$$

Fix $\ell \in \mathbb{N}$ such that $\zeta(2\pi(\ell+1)) \geq t_*$. Put $\tau_k = \zeta(2\pi(\ell+k))$, $k \in \mathbb{N}$. Note that $\pi b \leq \tau_{k+1} - \tau_k \leq 3\pi b$. For a given $k \in \mathbb{N}$ put

$$F_1 = \min_{t \in [\tau_k, \tau_{k+1}]} F(t), \ F_2 = \max_{t \in [\tau_k, \tau_{k+1}]} F(t).$$

One has

$$\begin{aligned}\int_{\tau_k}^{\tau_{k+1}} |\phi'(t)|\, \mathrm{d}t &\leq \frac{2}{b} \int_{\zeta(2\pi(\ell+k))}^{\zeta(2\pi(\ell+k+1))} |\phi'(t)|\zeta'(t)\, \mathrm{d}t \\ &\leq \frac{2A_2A_3}{b\sqrt{F_1^2 - A_2^2}} \int_{2\pi(\ell+k)}^{2\pi(\ell+k+1)} \frac{|\cos s|}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s}\, \mathrm{d}s,\end{aligned}$$

where $\tilde{F} = F \circ \zeta^{-1}$, $\tilde{a} = a \circ \zeta^{-1}$. Put $M_+ = 2\pi(\ell+k) + [\,0, \pi\,]$, $M_- = 2\pi(\ell+k) + [\,\pi, 2\pi\,]$. One has

$$\int_{M_+} \frac{|\cos s|}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s}\, \mathrm{d}s \leq 2 \int_0^{\pi/2} \frac{\cos s}{F_1 + \sqrt{F_1^2 - A_2^2}\, \sin s}\, \mathrm{d}s \leq \frac{2\log 2}{\sqrt{F_1^2 - A_2^2}}.$$

For $s \in M_-$ one can estimate

$$\frac{1}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\, \sin s} \leq \frac{F_2 + \sqrt{F_2^2 - A_1^2}\, |\sin s|}{F_1^2 \cos^2 s + A_1^2 \sin^2 s} < \frac{2}{F_1 + \sqrt{F_1^2 - A_1^2}\, \sin s},$$

where we have used that

$$\frac{F_2}{\sqrt{F_2^2 - A_1^2}} \leq \frac{F_1}{\sqrt{F_1^2 - A_1^2}}\,,\ F_2^2 - A_1^2 \leq 2F_1^2 - A_1^2 \leq 4F_1^2 - 5A_1^2 < 4(F_1^2 - A_1^2),$$

as it follows from (5.8). Thus one arrives at the estimates

$$\int_{M_-} \frac{|\cos s|}{\tilde{F}(s) + \sqrt{\tilde{F}(s)^2 - \tilde{a}(s)^2}\,\sin s}\,\mathrm{d}s \leq 4\int_0^{\pi/2} \frac{\cos s}{F_1 - \sqrt{F_1^2 - A_1^2}\,\sin s}\,\mathrm{d}s$$
$$\leq \frac{4}{\sqrt{F_1^2 - A_1^2}} \log\left(\frac{2F_1^2}{A_1^2}\right)$$

and

$$\int_{\tau_k}^{\tau_{k+1}} |\phi'(t)|\,\mathrm{d}t \leq \frac{32A_2A_3}{bF_1^2}\,\log\left(\frac{2F_1}{A_1}\right).$$

Referring to the asymptotic behavior (2.28), one concludes that there exists a constant $C_* > 0$ such that

$$\int_{\tau_1}^{\infty} |\phi'(t)|\,\mathrm{d}t = \sum_{k=1}^{\infty} \int_{\tau_k}^{\tau_{k+1}} |\phi'(t)|\,\mathrm{d}t \leq C_* \sum_{j=\ell+1}^{\infty} \frac{\log(j)}{j^2} < \infty.$$

Hence the limit $\lim_{t\to+\infty} \phi(t) = \phi(\infty) \in \mathbb{R}$ does exist and (2.29) follows. □

6. Some derivations related to the guiding center coordinates. We keep the notation introduced in subsection 2.4, but for the sake of simplicity we again put $m = e = 1$. The physical constants can readily be reestablished if necessary.

In particular, $X = q + (1/b)v^{\perp}$, $R = -(1/b)v^{\perp}$, and a direct computation yields

$$|R|^2 - |X|^2 = \frac{2a}{b}\,,\ |v|^2 = p_r^2 + \frac{b^2r^2}{4} + \frac{a^2}{r^2} + ba.$$

Using (2.4), (2.5) one derives the equalities

$$|X|^2 = \frac{1}{b}\,(2I + |a| - a),\ |R|^2 = \frac{1}{b}\,(2I + |a| + a) \tag{6.1}$$

and

$$r = \left(|X|^2 + |R|^2 + 2|X||R|\sin(\varphi)\right)^{1/2},\ p_r = \frac{b}{r}\,|X||R|\cos(\varphi). \tag{6.2}$$

On the other hand, one has

$$r^2 = |X|^2 + |R|^2 + 2X\cdot R = \frac{2}{b}\left(2I + |a| + 2\sqrt{I(I+|a|)}\,\cos(\vartheta - \chi)\right). \tag{6.3}$$

By comparison of (6.3) with (6.2) one shows equality (2.34).

Further we sketch derivation of the asymptotic relations (2.40). To this end, let us compute the derivative $\chi'(t)$. This can be done by differentiating the equality

$$r\,(\cos\theta, \sin\theta) = |X|(\cos\chi, \sin\chi) + |R|(\cos\vartheta, \sin\vartheta)$$

and then taking the scalar product with the vector $(-\sin\theta, \cos\theta)$. One has

$$\theta' = \frac{\partial H}{\partial p_\theta} = \frac{a}{r^2} + \frac{b}{2},$$

where H is the Hamiltonian (2.1) expressed in polar coordinates. Using (2.34) and some straightforward manipulations, one finally arrives at the differential equation

$$\chi' = \frac{|R|a'\cos\varphi}{|X|br^2}\,. \tag{6.4}$$

Equation (6.4) admits an asymptotic analysis with the aid of methods similar to those used in section 5. In order to spare some space we omit the details. We still assume that $\Omega/b \in \mathbb{N}$. Recalling (2.32), (2.33) one observes that the main contribution to the growth of $\chi(t)$ over a period $T = 2\pi/b$ equals

$$\chi((n+1)T) - \chi(nT) \sim \frac{1}{4CnT}\lim_{\alpha\to 0}\int_0^T \frac{a'(t)\cos(bt+\phi(\infty))}{1+\sqrt{1-\alpha^2}\,\sin(bt+\phi(\infty))}\,\mathrm{d}t.$$

Proceeding this way, one finally concludes that the first equality in (2.40) holds, with $\chi(\infty)$ being a real constant and

$$D = \frac{1}{4\pi f'(-\xi)}\int_0^\pi \left(f'\!\left(\frac{\Omega}{b}\,t-\xi\right) - f'\!\left(-\frac{\Omega}{b}\,t-\xi\right)\right)\frac{\sin(t)}{1-\cos(t)}\,\mathrm{d}t.$$

Being given the Fourier series $f'(t) = \sum_{k=1}^\infty (a_k\cos(kt) + b_k\sin(kt))$, one can also express

$$D = \frac{1}{2}\sum_{k=1}^\infty (a_k\sin(k\xi) + b_k\cos(k\xi)) \Bigg/ \sum_{k=1}^\infty (a_k\cos(k\xi) - b_k\sin(k\xi))\,,$$

as it follows from the equality, valid for any $n \in \mathbb{N}$, that

$$\int_0^\pi \frac{\sin(nt)\sin(t)}{1-\cos(t)}\,\mathrm{d}t = \pi.$$

Moreover, (2.32), (2.34), and the first equality in (2.40) imply the second equality in (2.40), where one has to put $\vartheta(\infty) = \phi(\infty) + \chi(\infty) - \pi/2$.

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