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A constant of quantum motion in two dimensions in crossed magnetic and electric fields*

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Abstract
We consider the quantum dynamics of a single particle in the plane under the influence of a constant perpendicular magnetic and a crossed electric potential field. For a class of smooth and small potentials we construct a non-trivial invariant of motion. To do so we prove that the Hamiltonian is unitarily equivalent to an effective Hamiltonian which commutes with the observable of kinetic energy.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Consider a particle of mass m and charge e in the plane under the influence of a constant magnetic field of strength B and an electric potential. We choose the units of magnetic length $\sqrt{\frac{\hbar}{|eB|}}$, the gyration time $\frac{m}{|eB|}$ and the energy gap $\frac{\hbar|eB|}{m}$. The dynamics are generated by

$$H = H_{La} + V \quad \text{in} \quad L^2(\mathbb{R}^2),$$

with

$$H_{La} = \frac{1}{2}\left(-\mathrm{i}\nabla - \frac{q^\perp}{2}\right)^2$$

with the operator core Schwartz space $\mathcal{S}(\mathbb{R}^2)$. V is the multiplication operator by a function $V(q)$ and $(q_1, q_2)^\perp := (-q_2, q_1)$. For the gaussian

$$g(q) := \mathrm{e}^{-\frac{q^2}{2}} \qquad (q \in \mathbb{R}^2)$$

* We dedicate this work to the memory of Pierre Duclos.

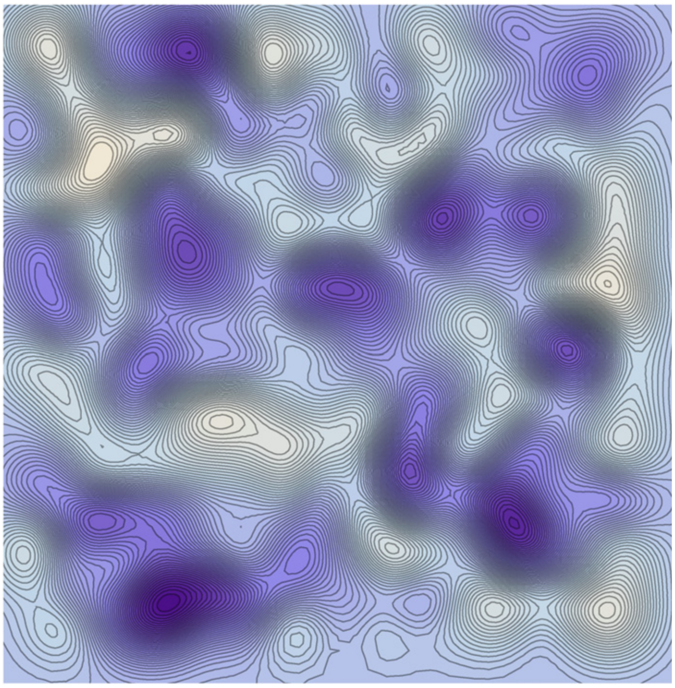

Figure 1. $V(q) = \sum_{i\in[-10,10]^2} u(i)g(q-i)$, u(i) i.i.d. random variables.

we consider the class of functions defined by convolution with a real-valued finite measure μ, $g * \mu(q) := \int_{\mathbb{R}^2} g(q-q')\mathrm{d}\mu(q')$:

$$\mathcal{G} := \left\{ V : \mathbb{R}^2 \to \mathbb{R};\, V = g * \mu,\, \int_{\mathbb{R}^2} d|\mu| < \infty \right\}.$$

Our main result is that there exists a non-trivial integral of motion; thus, in this weak sense, the two-dimensional system is integrable.

Theorem 1.1. *For $V \in \mathcal{G}$ small enough there exists a unitary operator U such that*

$$[U^{-1}(H_{La} + V)U, H_{La}] = 0.$$

In particular, $UH_{La}U^{-1}$ is an invariant of the flow $\mathrm{e}^{-\mathrm{i}Ht}$ for all $t \in \mathbb{R}$.

The meaning of ‘small enough’ will be made precise in the following. A potential in $\mathcal{G}$ is depicted in figure 1.

It is folklore in plasma physics that in slowly varying fields the classical particle gyrates on a cycloid whose centre drifts along the contour lines of the averaged potential and whose kinetic energy is an approximate invariant up to a certain time [CB, Nei]. On the other hand classically chaotic motion may occur in relevant regimes of parameters [GWNO].

In the realm of quantum physics the corresponding invariance is, in the large magnetic field limit, an essential ingredient for the current understanding of the integer quantum Hall effect [L, BESB, ASS, HS, G, GKS, CC, ABJ], while the interesting physics happen in the lowest Landau level. Coupling between Landau bands may, however, lead to non-negligible effects [PG].

Several methods to construct an approximately invariant subspace, making precise the notion of lowest Landau level, and an effective Hamiltonian are known, see [N, T, BDP]. They may lead to estimates valid to exponential order in a small parameter and valid for exponentially long times. See [BG] for an application of these ideas to a propagation problem. Our aim here is to study all Landau levels simultaneously as well as the possibility to go to the limit of all times.

Our strategy is to employ a superconvergent iterative partial diagonalization procedure which was originally introduced in quantum problems to discuss stability of non-resonant time periodic problems [B, C, DŠ]; see [ADE] for an application to a condensed matter problem. It coincides in first order with the above-mentioned 'space adiabatic' algorithms. In particular the effective Hamiltonian restricted to the lowest Landau level is in first order $\langle V\rangle(x, D)$, a pseudodifferential operator whose symbol is the potential averaged over the Landau orbits. At higher orders the algorithm differs; on one hand it exhibits quadratic convergence, and on the other hand it is an unsolved, and to our opinion important problem, whether our higher order effective operators are pseudodifferential.

The partial diagonalization procedure is roughly described as follows: the diagonal part of an operator H with respect to a fixed, orthogonal, mutually disjoint family of projections $\{P_n\}$ is defined by $\mathbf{D}H := \sum_n P_n H P_n$. In fact, as the dimension of P_n is not supposed to be one, or even to be finite, $\mathbf{D}H$ is block diagonal. Suppose that the off-diagonal part $\mathbf{O}H := H - \mathbf{D}H$ is small with respect to $\mathbf{D}H$. The equation

$$e^W H e^{-W} - \mathbf{D}H = \mathcal{O}(\|\mathbf{O}H\|^2)$$

is satisfied by an anti-self-adjoint operator W of order $\mathcal{O}(\|\mathbf{O}H\|)$ solving

$$\mathbf{O}H + [W, \mathbf{D}H] = 0.$$

If one takes P_n the projections on the Landau levels, then because of the gap a solution of this equation can be found if the coupling between the bands decays sufficiently fast. This is the case for the potentials of our class $\mathcal{G}$. The procedure can then be iterated by replacing H by $e^W H e^{-W}$. The convergence of the transformed H to a block-diagonal operator which, because of the degeneracy, commutes with H_{La} is quadratic.

We remark that in order to be really applicable to the quantum Hall effect our result should be extended to a more general class of potentials than stated above. This should in principle be possible. However, a delicate control of $[H_{La}, V]$ is needed. The method does not work for the extensively studied purely periodic problem $V(q_1, q_2) = \cos q_1 + \cos q_2$.

The plan of the paper is to set up the iterative algorithm in section 2. The class of potentials $\mathcal{G}$ and control of the necessary norms will be discussed in section 3. Section 4 contains the proof of theorem 1.1 and a discussion related to special cases.

2. An iterative partial diagonalization algorithm

As discussed in the introduction, the task is to partially diagonalize the operator $H_{La} + V$.

Recall that $H_{La} = \sum_{n\in\mathbb{N}_0}(n + 1/2)P_n^{La}$ with infinite dimensional projections P_n^{La}. We consider H which is of the same type as H_{La}; in order to cover situations where H is already an effective Hamiltonian at finite order, we assume that it has either a finite number of bands σ_n or an infinite number such that

$$dist(\sigma_n, \sigma_m) \geqslant \mathfrak{g}|n - m|.$$

Definition 2.1. *A self-adjoint operator H is of class $\mathcal{C}_\mathfrak{g}$ for a $\mathfrak{g} > 0$ if for a complete family of orthogonal, mutually disjoint projections $\{P_n\}_{n\in I\subset\mathbb{N}_0}$ which commute with H; it holds for*

$$\sigma_n := spect\big(P_n H P_n|_{Ran P_n}\big):$$
$$\min\sigma_{n+1} - \max\sigma_n \geqslant \mathfrak{g}.$$

Figure 2. Typical spectrum of H.

For the same family of projections P_n and a bounded operator V we use the notation

$$\mathbf{D}V := \sum_{n\in I} P_n V P_n, \qquad \mathbf{O}V := V - \mathbf{D}V.$$

To organize the estimates we shall make frequent use of the notations

$$\langle a\rangle := \max(1, |a|), \qquad \|\ \| : \text{the operator norm, and}$$
$$\|A\|_l := \sup_{n,m\in I}\langle n-m\rangle^l \|P_n A P_m\|.$$

We prove (extending [DLŠV])

Theorem 2.2. *Let $H \in \mathcal{C}_\mathfrak{g}$ and V be a bounded self-adjoint operator such that*

$$\|V\|_1 \leqslant \frac{\mathfrak{g}}{8}.$$

Then there exists a unitary $\mathcal{U}$ such that $\mathcal{U}^{-1}D(H)\subset D(H)$ with the property that for

$$H_\infty := \mathcal{U}(H+V)\mathcal{U}^{-1}, \qquad D(H_\infty) = D(H)$$

it holds

$$[H_\infty, P_n] = 0.$$

Proof. Define $H_0 := H + V = \mathbf{D}H_0 + \mathbf{O}H_0$. Assume $\|V\|_1 \leqslant \frac{\mathfrak{g}}{8}$. Then $\mathbf{D}H_0 \in \mathcal{C}_{1/4}$, $\|\mathbf{O}H_0\|_1 = \|OV\|_1 < \infty$. By lemma 2.3 there exists a bounded solution W_0 of

$$[\mathbf{D}H_0, W_0] = \mathbf{O}H_0, \qquad \mathbf{D}W_0 = 0.$$

Define $\mathcal{U}_0 := e^{W_0}$.

Remark. $\mathcal{U}_0$ is unitary, $D(H_0)\subset D(H)$, $W_0D(H_0)\subset D(H_0)$, $\mathcal{U}_0D(H_0)\subset D(H_0)$.

Suppose that for $s \in \mathbb{N}$ diagonalization has been done up to $H_s, \mathcal{U}_{s-1}$ such that $\|\mathbf{D}H_s - H\| \leqslant \mathfrak{g}/4$, $\|\mathbf{O}H_s\|_1 < \infty$. To go to step $s+1$, use lemma 2.3 to solve

$$[\mathbf{D}H_s, W_s] = \mathbf{O}H_s, \qquad \mathbf{D}W_s = 0$$

for a bounded W_s and define $\mathcal{U}_s := e^{W_s}\mathcal{U}_{s-1}$,

$$H_{s+1} := e^{W_s}H_se^{-W_s} = e^{L_{W_s}}(H_s) = \sum_{k=0}^\infty \frac{L^k_{W_s}(H_s)}{k!},$$

with the notation $L_W(A) := [W, A]$. Now

$$L_{W_s}(\mathbf{D}H_s) = -\mathbf{O}H_s;$$

thus, for $k \geqslant 1$

$$L^k_{W_s}(H_s) = -L^{k-1}_{W_s}(\mathbf{O}H_s) + L^k_{W_s}(\mathbf{O}H_s)$$

so that

$$H_{s+1} = \mathbf{D}H_s + \phi\big(L_{W_s}\big)(\mathbf{O}H_s), \tag{1}$$

with

$$\phi(x) := e^x - \frac{1}{x}(e^x - 1) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} x^k \qquad (x \in \mathbb{R}).$$

It holds by corollary 2.5 that

$$\big\| L_{W_s}(\mathbf{O}H_s)\big\|_1 \leqslant \|W_s\mathbf{O}H_s\|_1 + \|\mathbf{O}H_s W_s\|_1 \leqslant 2K\|W_s\|_2\|\mathbf{O}H_s\|_1.$$

Thus, by induction and lemma 2.3

$$\Big\| L^k_{W_s}(\mathbf{O}H_s)\Big\|_1 \leqslant \left(\frac{\pi 2K}{\mathfrak{g}}\right)^k \|\mathbf{O}H_s\|_1^{k+1},$$

which implies

$$\begin{aligned}\big\|\phi\big(L_{W_s}\big)(\mathbf{O}H_s)\big\|_1 &\leqslant \sum_{k=1}^{\infty} \frac{k}{(k+1)!}\left(\frac{2\pi K}{\mathfrak{g}}\|\mathbf{O}H_s\|_1\right)^k \|\mathbf{O}H_s\|_1 \\ &= \frac{\mathfrak{g}}{2\pi K}\psi\left(\frac{2\pi K}{\mathfrak{g}}\|\mathbf{O}H_s\|_1\right),\end{aligned}$$

with $\psi(x) := x\phi(x) = (x-1)e^x + 1 \quad (x \geqslant 0)$. Remark that ψ is non-negative, $\psi(x) \leqslant x \quad (x < 1)$ and 0 is a superattractive fixed point. By (1)

$$\frac{2\pi K}{\mathfrak{g}}\|\mathbf{O}H_{s+1}\|_1 \leqslant \psi\left(\frac{2\pi K}{\mathfrak{g}}\|\mathbf{O}H_s\|_1\right)$$

thus

$$\|\mathbf{O}H_s\|_1 \leqslant \lambda^{2^s}, \tag{2}$$

with λ proportional to $\|\mathbf{O}V\|_1$ small enough. For the diagonal part it holds

$$\|\mathbf{D}H_{s+1} - \mathbf{D}H_s\| = \|\mathbf{D}H_{s+1} - \mathbf{D}H_s\|_1 \leqslant \big\|\phi\big(L_{W_s}\big)(\mathbf{O}H_s)\big\|_1 \leqslant \frac{\mathfrak{g}}{2\pi K}\psi\left(\frac{2\pi K}{\mathfrak{g}}\|\mathbf{O}H_s\|_1\right)$$

and thus for an x proportional to $\|\mathbf{O}V\|_1$

$$\|\mathbf{D}H_{s+1} - H\| \leqslant \frac{\mathfrak{g}}{8} + \frac{\mathfrak{g}}{2\pi K}\int_0^x \psi \leqslant \frac{\mathfrak{g}}{4},$$

so the iteration is well defined.

In particular $s \mapsto \mathbf{D}H_s - H$ is a norm convergent sequence. Equation (2) and lemma 2.3 imply that $\|W_s\|$ converges superexponentially to 0. By (1) this implies in turn as $\mathbf{O}V$ is bounded and $\|\mathbf{O}V\|_1$ is small enough:

$$\|\mathbf{O}H_{s+1}\| \leqslant \big\|\phi\big(L_{W_s}\big)(\mathbf{O}H_s)\big\| \leqslant \underbrace{\sum \frac{k}{(k+1)!}2^k\|W_s\|^k}_{\leqslant 1}\|\mathbf{O}H_s\|;$$

thus, $\mathbf{O}H_s$ converges in operator norm, and furthermore as

$\|\mathcal{U}_{s+1} - \mathcal{U}_s\| = \big\|e^{W_{s+1}} - \mathbb{I}\big\|$ one concludes that $\mathcal{U}_s \to_{\|.\|} \mathcal{U}$. By construction $H_\infty = \mathbf{D}H_\infty$, which commutes with the projections. □

We now prove some claims which where used in the preceding proof: the existence of a bounded solution of the commutator equation is assured by

Lemma 2.3. *Let $H \in \mathcal{C}_{\mathfrak{g}}$ and V be a bounded self-adjoint operator such that $P_n V P_n = 0\ \forall n$ and such that $\|V\|_1 < \infty$. Then there exists a bounded anti-self-adjoint W such that*

$$[H, W] = V, \qquad \mathbf{D}W = 0,$$

so that

$$\|W\| \leqslant \frac{\pi\zeta(2)}{\mathfrak{g}}\|V\|_1,$$
$$\|W\|_2 \leqslant \frac{\pi}{2\mathfrak{g}}\|V\|_1.$$

Proof. In [BR] for bounded operators A, B, C there exists a solution X of $AX - XB = C$ such that

$$\|X\| \leqslant \frac{\pi}{2}\frac{\|C\|}{\mathrm{dist}(\mathrm{spect}(A), \mathrm{spect}(B))}.$$

It follows that there exists W_{nm} such that

$$P_n H P_n W_{nm} - W_{nm} P_m H P_m = P_n V P_m,$$

with $W_{nm} = P_n W_{nm} P_m$ and

$$\|W_{nm}\| \leqslant \frac{\pi}{2}\|P_n V P_m\|\frac{1}{\mathfrak{g}\langle n-m\rangle} \leqslant \frac{\pi}{2\mathfrak{g}}\frac{\|V\|_1}{\langle n-m\rangle^2}.$$

Now define $W := \sum_{n\neq m} W_{nm}$ in norm convergence; then

$$\|W\| \leqslant \sup_n \sum_m \|P_n W P_m\| = \sup_n \sum_m \frac{1}{\langle n-m\rangle^2}\frac{\pi\|V\|_1}{2\mathfrak{g}}$$

from which the claim follows. □

Lemma 2.4. *Let $n, m \in \mathbb{N}_0$ and $K := 3 + 2\zeta(2)$. It holds*

$$\sum_{j\geqslant 0, j\neq n, j\neq m} \frac{1}{\langle j-n\rangle^2\langle j-m\rangle} \leqslant \frac{K}{\langle m-n\rangle}. \tag{3}$$

Proof. The case where $n = m$ is evident. In what follows, we will simply write j as the index for the sum except of $j \geqslant 0$, $j \neq n$, $j \neq m$:

$$\sum_j \frac{1}{\langle j-n\rangle^2\langle j-m\rangle} = \frac{1}{\langle m-n\rangle}\left(\sum_j \frac{\langle j-n+m-j\rangle}{\langle j-n\rangle^2\langle j-m\rangle}\right)$$
$$\leqslant \frac{1}{\langle m-n\rangle}\left(\sum_j \frac{1}{\langle j-n\rangle^2} + \sum_j \frac{1}{\langle j-n\rangle\langle j-m\rangle}\right).$$

The left term of the r.h.s. is bounded by $2\zeta(2)$. It remains to prove that the second term of the r.h.s. is bounded. Recall the series expansion of the digamma function ψ_0:

$$\psi_0(x+1) + \gamma = \sum_{j=1}^{\infty} \frac{x}{j(j+x)} \qquad (x \in \mathbb{N}). \tag{4}$$

Now, since it is symmetric in m and n, we can assume that $m > n$ and define $a := m - n \geqslant 1$. Then the second term of the r.h.s. becomes

$$\sum_j \frac{1}{\langle j-n\rangle\langle j-m\rangle} = \sum_{j\geqslant -n} \frac{1}{\langle j\rangle\langle j-a\rangle}$$
$$= \sum_{j=-n}^{a-1} \frac{1}{\langle j\rangle\langle j-a\rangle} + \sum_{j\geqslant a+1} \frac{1}{\langle j\rangle\langle j-a\rangle}$$
$$= \sum_{j=-n}^{a-1} \frac{1}{\langle j\rangle\langle j-a\rangle} + \frac{\gamma + \psi_0(1+a)}{a},$$

where we used (4). But

$$\sum_{j=-n}^{a-1} \frac{1}{\langle j\rangle\langle j-a\rangle} = \sum_{j=1}^{n} \frac{1}{\langle j\rangle\langle j+a\rangle} + \sum_{j=1}^{a-1} \frac{1}{\langle j\rangle\langle j-a\rangle}$$
$$\leqslant \frac{\gamma + \psi_0(1+a)}{a} + |a-1| \sup_{j\in[1,a-1]\cap\mathbb{N}} \frac{1}{|j||j-a|}.$$

Since $a \geqslant 1$, both of these terms are bounded by 1, so the claim of the lemma follows. □

We have the following corollary.

Corollary 2.5. *For operators A and B such that $\|A\|_2 < \infty$ and $\|B\|_1 < \infty$ it holds*

$$\|AB\|_1 \leqslant K\|A\|_2\|B\|_1,$$

where K was defined in 2.4.

Proof. Since $\{P_n\}_{n\in I}$ is a complete family of orthogonal and mutually disjoint projectors, we can write

$$P_n A B P_m = \sum_{l\in I}(P_n A P_l)(P_l B P_m).$$

Thus,

$$\|AB\|_1 \leqslant \sup_{n,m}\langle m-n\rangle \sum_{l\geqslant 0} \frac{\|A\|_2}{\langle n-l\rangle^2}\frac{\|B\|_1}{\langle l-m\rangle}.$$

The result follows from lemma 2.4. □

3. The class $\mathcal{G}$

We show that the basic decay estimate is satisfied for potentials in the class $\mathcal{G}$ defined in the introduction and give some examples.

Proposition 3.1. *For $V \in \mathcal{G}$ and P_n the eigenprojections of H_{La} on its nth level it holds in operator norm on $L^2(\mathbb{R}^2)$ for a $d > 0$ and all $n, m \in \mathbb{N}_0$:*

$$\|P_n V P_m\| \leqslant \frac{d}{\langle n-m\rangle}.$$

Proof.

$$\|P_n V P_m\| = \|P_n \int g(.-y)\mathrm{d}\mu(y) P_m\|$$
$$\leqslant \int |\mathrm{d}\mu(y)| \|P_n g(.-y) P_m\|.$$

Consider the unitary magnetic translations on $L^2(\mathbb{R}^2)$ defined for $a \in \mathbb{R}^2$ by

$$T(a)\psi(q) = e^{\frac{i}{2}q\wedge a}\psi(q-a).$$

It holds that $[T(a), H_{La}] = 0$ and $T(a)\,g\,T^*(a) = g(.-a)$. Thus,

$$\|P_n g(.-y)P_m\| = \|P_n g P_m\| \quad \forall y \in \mathbb{R}^2,$$

and the result follows from proposition 3.5 to be proven below. □

Remark that $V = g * \mu$ extends necessarily to an entire analytic function; its Fourier transform $\widehat{V}$ has the property that $\widehat{V}(p)\exp\frac{p^2}{2}$ is the Fourier transform of a finite measure. We elaborate on this in order to point out that $\mathcal{G}$ contains sufficiently many potentials to be of interest for applications to the quantum hall effect.

Definition 3.2. *Denote by $\mathcal{A}$ the class of functions $V : \mathbb{R}^2 \to \mathbb{R}$ such that*

(i) V has an extension to an entire function on $\mathbb{C}^2$;

(ii) $\mathbb{R}^2 \ni y \mapsto \mathrm{e}^{-\frac{y^2}{2}}V(iy) \in L^1(\mathbb{R}^2)$;

(iii) for $\widetilde{V}(p) := \mathrm{e}^{\frac{p^2}{2}}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}py}\mathrm{e}^{-\frac{y^2}{2}}V(iy)\frac{\mathrm{d}y}{(2\pi)^2}$, it holds $\widetilde{V} \in L^1(\mathbb{R}^2)$.

Proposition 3.3. *For $V \in \mathcal{A}$ it holds*

$$V = g * (\widetilde{V}\mathrm{d}q).$$

Proof. By Fourier's theorem, it holds for $q \in \mathbb{R}^2$

$$\mathrm{e}^{-\frac{q^2}{2}}V(\mathrm{i}q) = \int_{\mathbb{R}^2\times\mathbb{R}^2}\mathrm{e}^{\mathrm{i}p(q-y)}\mathrm{e}^{-\frac{y^2}{2}}V(\mathrm{i}y)\frac{\mathrm{d}y}{(2\pi)^2}\mathrm{d}p.$$

Thus,

$$V(iq) = \int_{\mathbb{R}^2}\mathrm{e}^{\frac{(q+\mathrm{i}p)^2}{2}}\widetilde{V}(p)\mathrm{d}p$$

and the claims follows as both sides are analytic in q. □

We list some examples of potentials in $\mathcal{G}$ which contain in particular Anderson-type models on a finite portion of the probe.

Corollary 3.4. *$\mathcal{G}$ contains*

(i) for p a real polynomial, $\alpha \in (0, 1)$, $k_1, k_2 \in \mathbb{R}$:

$$p\big(q_1, q_2, \mathrm{e}^{\mathrm{i}k_1q_1}, \mathrm{e}^{\mathrm{i}k_2q_2}\big)\mathrm{e}^{-\alpha\frac{q^2}{2}},$$

(ii) $\sum_{i\in\mathbb{Z}^2}\mu_i\, g(q-i)$ with $\mu \in l^1(\mathbb{Z}^2, \mathbb{R})$.

The function $q \mapsto \mathrm{e}^{\mathrm{i}kq}$ for $k \in \mathbb{R}^2$ does not belong to $\mathcal{G}$.

Proof. For 1. it is sufficient to verify that

$$\mathbb{R} \ni y \mapsto \mathrm{e}^{-\frac{y^2}{2}}f(\mathrm{i}y) \in L^1(\mathbb{R})$$

and

$$\mathbb{R} \ni x \mapsto \mathrm{e}^{\frac{x^2}{2}}\int\mathrm{e}^{-\mathrm{i}xy}\mathrm{e}^{-\frac{y^2}{2}}f(\mathrm{i}y)\mathrm{d}y \in L^1(\mathbb{R})$$

for $f(y) = y^n\mathrm{e}^{-\alpha\frac{y^2}{2}}$ and $f(y) = \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-\alpha\frac{y^2}{2}}$, $n \in \mathbb{N}$ and $k \in \mathbb{R}$.

This is by standard properties of the Fourier transform. In the first case,

$$\left|f(\mathrm{i}y)\mathrm{e}^{-\alpha\frac{y^2}{2}}\right| = |y|^n\mathrm{e}^{-(1-\alpha)\frac{y^2}{2}} \in L^1(\mathbb{R})$$

and

$$\left|\mathrm{e}^{\frac{x^2}{2}}\int \mathrm{e}^{-\mathrm{i}xy}y^n\mathrm{e}^{-(1-\alpha)\frac{y^2}{2}}\,\mathrm{d}y\right| = \frac{1}{\sqrt{1-\alpha}}\left|\mathrm{poly}(x)\mathrm{e}^{\frac{x^2}{2}}\mathrm{e}^{-\frac{1}{1-\alpha}\frac{x^2}{2}}\right| \in L^1(\mathbb{R}).$$

In the second case

$$\left|f(\mathrm{i}y)\mathrm{e}^{-\alpha\frac{y^2}{2}}\right| \leqslant \mathrm{e}^{|k||y|}\mathrm{e}^{-(1-\alpha)\frac{y^2}{2}} \in L^1(\mathbb{R})$$

$$\left|\mathrm{e}^{\frac{x^2}{2}}\int \mathrm{e}^{-\mathrm{i}xy}\mathrm{e}^{-ky}\mathrm{e}^{-(1-\alpha)\frac{y^2}{2}}\,\mathrm{d}y\right| = \frac{1}{\sqrt{1-\alpha}}\mathrm{e}^{\frac{k^2}{2(1-\alpha)}}\mathrm{e}^{\frac{x^2}{2}}\mathrm{e}^{-\frac{1}{1-\alpha}\frac{x^2}{2}} \in L^1(\mathbb{R}).$$

In 2., one deals with the pure point measure

$$\sum_{i\in\mathbb{Z}^2}\mu_i\delta(x-i).$$

Finally, one has

$$\mathrm{e}^{\mathrm{i}kx} = g*\mu,$$

with $\mu = \mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{\frac{k^2}{2}}\mathrm{d}x$ which is not a finite measure. □

It remains to treat the case of the gaussian potential g which turns out to be non-trivial. We follow the strategy designed in [W].

Proposition 3.5. *For* $g(q) = \exp\left(-\frac{q^2}{2}\right)$, $(q\in\mathbb{R}^2)$ *and* P_n *the eigenprojections of* H_{La} *on its nth level it holds in operator norm on* $L^2(\mathbb{R}^2)$ *for a* $c>0$ *and all* $n, m\in\mathbb{N}_0$*:*

$$\|P_n g P_m\| \leqslant \frac{c}{\langle n-m\rangle}.$$

Proof. We use the representation of P_n by angular momentum eigenfunctions:

$$P_n = \sum_{l\geqslant -n}|\psi_{n,l}\rangle\langle\psi_{n,l}|$$

$$\psi_{n,l}(r,\Theta) := (-1)^n\sqrt{\frac{n!}{2^l(n+l)!}}r^l\mathrm{e}^{\mathrm{i}\Theta l}L_n^l\left(\frac{r^2}{2}\right)\frac{\mathrm{e}^{-\frac{r^2}{4}}}{\sqrt{2\pi}}, \qquad (5)$$

where the Laguerre polynomials are defined by

$$L_n^l(x) := \sum_{j=0}^n\frac{(-x)^j}{j!}\binom{n+l}{n-j} \quad (l\geqslant 0)$$

$$L_n^l(x) := \frac{(n+l)!}{n!}(-x)^{|l|}L_{n+l}^{|l|}(x) \quad (0\geqslant l\geqslant -n).$$

Then

$$P_n g P_m = \sum_{l\geqslant -n\wedge m}|\psi_{n,l}\rangle\langle\psi_{n,l}, g\psi_{m,l}\rangle\langle\psi_{m,l}|;$$

thus,

$$|\langle\psi, P_n g P_m\varphi\rangle| \leqslant \sup_{l\geqslant -n\wedge m}|\langle\psi_{n,l}, g\psi_{m,l}\rangle|\|\psi\|\|\varphi\|$$

and the claim follows from equation (6) and estimate (9) to be proven in the following two lemmas. □

Lemma 3.6. *For* $g(q) = \exp\left(-\frac{q^2}{2}\right)$, $(q \in \mathbb{R}^2)$ *and* $\psi_{n,l}$ *defined in (5) it holds for* $n, m \in \mathbb{N}_0, l \geqslant -(n \wedge m)$:

$$|\langle \psi_{n,l}, g\psi_{m,l}\rangle| = \frac{1}{2^{l+m+n+1}} \frac{(l+m+n)!}{\sqrt{(l+m)!(l+n)!n!m!}}. \tag{6}$$

Proof. By definition

$$|\langle \psi_{n,l}, g\psi_{m,l}\rangle| = \frac{1}{2^l}\sqrt{\frac{n!m!}{(l+n)!(l+m)!}} \int_0^\infty \mathrm{e}^{-r^2} r^{2l} L_n^l L_m^l \left(\frac{r^2}{2}\right) r\,dr.$$

Consider first $l \geqslant 0$. To study the dependence of the integral on l, m, n we use that the family $n \mapsto L_n^l(x)$ is orthogonal in $L^2(\mathbb{R}_+, d\nu_l)$, $d\nu_l := x^l \mathrm{e}^{-x} dx$ and the identity:

$$L_n^l L_m^l \left(\frac{x}{2}\right) = \sum_{s\geqslant 0} B_s^{n,m,l} L_s^l(x).$$

As $L_0^l \equiv 1\, \forall l$, one has

$$\int_0^\infty L_n^l L_m^l \left(\frac{x}{2}\right) d\nu_l(x) = \sum_{s\geqslant 0} B_s^{n,m,l} \int_0^\infty L_s^l L_0^l(x) d\nu_l(x)$$
$$= B_0^{n,m,l} \int_0^\infty d\nu_l(x) = B_0^{n,m,l}\Gamma(l+1).$$

It was proven in [Ca] that

$$g_s(x, y, l) = \frac{\left(\frac{x}{2(1-x)} + \frac{y}{2(1-y)}\right)^s}{(1-x)^{l+1}(1-y)^{l+1}\left(1 + \frac{x}{2(1-x)} + \frac{y}{2(1-y)}\right)^{l+s+1}}$$
$$= \sum_{n,m} B_s^{n,m,l} x^m y^n.$$

Thus,

$$B_0^{n,m,l} = \frac{1}{n!m!} \partial_x^m \partial_y^n g_0(x, y, l)\Big|_{x=y=0} = \frac{1}{2^{m+n}} \frac{(l+m+n)!}{l!m!n!}$$

from which the claim follows for $l \geqslant 0$.

Now for $l < 0$, one has

$$\psi_{n,l} = \overline{\psi}_{n+l,-l};$$

thus,

$$|\langle \psi_{n,l}, V\psi_{m,l}\rangle| = |\langle \psi_{n+l,-l}, V\psi_{m+l,-l}\rangle|$$

and the result follows for $l < 0$. □

The following lemma might be known to probabilists, we know of no reference though.

Lemma 3.7. *For a* $c > 0$, *it holds*

$$\frac{(m+n)!}{2^{m+n}n!m!} \leqslant c \frac{m+n}{\sqrt{(m+n)^2 - (m-n)^2}} \mathrm{e}^{-\frac{m-n}{2(m+n)}} \quad (\mathbb{N} \ni m, n \geqslant 1), \tag{7}$$

$$\frac{(m+n)!}{2^{m+n}n!m!} \leqslant \frac{c}{\langle m-n\rangle} \qquad (m,n \in \mathbb{N}_0), \tag{8}$$

$$\frac{(l+m+n)!}{2^{l+m+n+1}\sqrt{(l+m)!(l+n)!n!m!}} \leqslant \frac{c}{\langle m-n\rangle} \qquad (m,n \in \mathbb{N}_0, e \geqslant -m \wedge n). \tag{9}$$

Proof. Denote by C a positive constant whose value may change from line to line.

We use Stirling's and a concavity inequality:

$$\frac{1}{C} \leqslant \frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{n}} \leqslant C \qquad (n \geqslant 1) \tag{10}$$

$$(1+x)\log(1+x) + (1-x)\log(1-x) \geqslant x^2 \qquad (x \in [0,1)), \tag{11}$$

to estimate

$$a_{m,n} := \frac{(m+n)!}{2^{m+n}n!m!}.$$

For $m, n \geqslant 1$ it holds by (10):

$$a_{m,n} \leqslant C\frac{(m+n)^{(m+n)}}{m^m n^n 2^{m+n}}\sqrt{\frac{1}{n}+\frac{1}{m}} \leqslant C\frac{a^a}{(a+b)^{\frac{a+b}{2}}(a-b)^{\frac{a-b}{2}}}\sqrt{\frac{a}{a^2-b^2}},$$

with $a := m+n$, $b := m-n$ in

$$G := \{(a,b) \in \mathbb{Z}^2; a \geqslant 2, |b| \leqslant |a-2|\}.$$

Note that the case $n = m$ follows from the first inequality of the previous line. Using (11) with $x := \frac{b}{a}$ it follows

$$a_{m,n} \leqslant \sqrt{\frac{a}{a^2-b^2}}\mathrm{e}^{-\frac{b^2}{2a}},$$

which implies (7). Consider now

$$G_< := G \cap \left\{(a,b) \in \mathbb{Z}^2; \frac{b^2}{a^2} < \frac{1}{2}\right\};$$

then

$$\langle m-n\rangle^2 a_{m,n}^2 \leqslant C\frac{1}{1-\frac{b^2}{a^2}}\frac{b^2}{a}\mathrm{e}^{-\frac{b^2}{a}} < 2 \qquad ((a,b) \in G_<).$$

In $G \setminus G_<$, it holds

$$\frac{a}{2} \leqslant \frac{b^2}{a} \leqslant a \quad \text{as well as} \quad 1 - \frac{b^2}{a^2} \geqslant \frac{2}{a};$$

thus,

$$\langle m-n\rangle^2 a_{m,n}^2 \leqslant Ca^2\mathrm{e}^{-\frac{a}{2}} \leqslant C$$

which proves (8) as it is evident for $n = 0$.

Denote now

$$b_{l,m,n} := \left(\frac{(l+m+n)!}{2^{l+m+n}}\right)^2 \frac{1}{(l+m)!(l+n)!m!n!}.$$

From the identity $b_{l,m,n} = a_{l+m,n}\,a_{l+n,m}$ it follows for $l+m, l+n, m, n \geqslant 1$

$$b_{l,m,n} \leqslant C\sqrt{\frac{a}{a^2-b^2}}\sqrt{\frac{a}{a^2-c^2}}\mathrm{e}^{-\frac{b^2+c^2}{2a}},$$

with $a := l + m + n$, $b := l + m - n$, $c := l + n - m$, $(a, b) \in G$ and $(a, c) \in G$. It follows

$$\langle m - n\rangle^2 b_{l,m,n} \leqslant C\left|\frac{b-c}{\sqrt{a}}\right|^2 \frac{1}{\sqrt{1-\frac{b^2}{a^2}}\sqrt{1-\frac{c^2}{a^2}}} \mathrm{e}^{-\frac{b^2+c^2}{2a}}.$$

In G, one has $\frac{1}{1-\frac{b^2}{a^2}} \leqslant \frac{a}{2}$ so if $\frac{b^2}{a^2} \geqslant \frac{1}{2}$, then $\frac{a}{2} \leqslant \frac{b^2}{a} \leqslant a$ and, denoting any polynomial by poly:

$$\langle m - n\rangle^2 b_{l,m,n} \leqslant C\text{poly}\left(\frac{c}{\sqrt{a}}, \sqrt{a}\right) \mathrm{e}^{-\frac{a}{2}} \mathrm{e}^{-\frac{c^2}{2a}} \leqslant C.$$

Now if $\frac{b^2}{a^2} < \frac{1}{2}$, then either $\frac{c^2}{a^2} \geqslant \frac{1}{2}$ and

$$\langle m - n\rangle^2 b_{l,m,n} \leqslant C\text{poly}\left(\sqrt{a}, \frac{b}{\sqrt{a}}\right) \mathrm{e}^{-\frac{b^2+c^2}{2a}} \leqslant C$$

or $\frac{c^2}{a^2} < \frac{1}{2}$ and

$$\langle m - n\rangle^2 b_{l,m,n} \leqslant C\text{poly}\left(\frac{b}{\sqrt{a}}, \frac{c}{\sqrt{a}}\right) \mathrm{e}^{-\frac{b^2+c^2}{2a}} \leqslant C.$$

For the cases where one of $l + m, l + n, m, n$ is zero, remark first that

$$b_{l,n,n} = |a_{l+n,n}|^2 \leqslant \frac{C}{\langle l\rangle^2} \leqslant C$$

by (8) and secondly that for $l + m = 0$, $n \geqslant m$

$$\langle m - n\rangle^2 b_{l,m,n} = \frac{1}{2^n}\langle m - n\rangle^2 a_{l+n,m} \leqslant \frac{\langle m-n\rangle^2}{2^{n-m}} \frac{C}{2^m\langle n-2m\rangle} \leqslant C,$$

which covers all cases. □

4. Application of the algorithm

We prove theorem 1.1, and then give an illustration.

Proof. (of theorem 1.1)

Choose $H = H_{La}$ and $\{P_n^{La}\}_{n\in\mathbb{N}_0}$ its eigenprojections. Then $H_{La} \in \mathcal{C}_1$. By proposition 3.1 $\|V\|_1$ is finite for $V \in \mathcal{G}$. So for a $V \in \mathcal{G}$ with $\|V\|_1 \leqslant \frac{1}{8}$ by theorem 2.2 there exists $\mathcal{U}$ unitary such that $[\mathcal{U}(H_{La} + V)\mathcal{U}^{-1}, P_n^{La}] = 0$; thus, $[\mathcal{U}(H_{La} + V)\mathcal{U}^{-1}, H_{La}] = 0$. □

4.1. Quadratic Hamiltonians

We discuss the case where V is a polynomial of degree at most 2 for a sufficiently high magnetic field. Though this case is not covered directly by theorem 1.1, the iterative algorithm can be applied to the Hamiltonian matrix which defines the operator. This results in the construction of an integral of motion which is the quantization of a classical integral, independent of the Hamiltonian function. The following operations are to be understood first on vectors in $\mathcal{S}(\mathbb{R}^2)$ then on the appropriate extensions. Denote by $D := -\mathrm{i}\nabla$, the velocity and centre operators

$$v := D - \frac{q^\perp}{2}, \quad c := -D^\perp + \frac{q}{2},$$

and recall the commutation relations

$$[v_1, v_2] = i, \quad [c_2, c_1] = i, \quad [c_i, v_j] = 0.$$

The linear case is trivial, nevertheless it is instructive:

$$V(q) = -\langle E, q\rangle = -(E_1 q_1 + E_2 q_2),$$

define $W_0 := \mathrm{i}\langle E, v\rangle$, $\mathcal{U}_0 = e^{W_0}$. From the Weyl relations

$$\mathrm{e}^{\mathrm{i}\langle E,v\rangle} v \mathrm{e}^{-\mathrm{i}\langle E,v\rangle} = v - E^{\perp}$$

it follows.

Proposition 4.1. *For $E \in \mathbb{R}^2$, $V(q) = -\langle E, q\rangle$, $\mathcal{U}_0 = \mathrm{e}^{\mathrm{i}\langle E,v\rangle}$, it holds*

(i) $\mathcal{U}_0(H_{La} + V)\mathcal{U}_0^{-1} = H_{La} - \langle E, c\rangle - \frac{1}{2}E^2 = \mathbf{D}H - \frac{1}{2}E^2$
(ii) $\mathcal{U}_0^{-1} H_{La}\mathcal{U}_0 = \frac{1}{2}(v + E^{\perp})^2$.

Now consider the quadratic case, $V(q) = \frac{1}{2}\langle q, V''q\rangle$ for a real symmetric 2×2 matrix V''.

The Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\left(D - \frac{q^{\perp}}{2}\right)^2 + \frac{1}{2}\langle q, V''q\rangle \\ &= \frac{1}{2}\left\langle \begin{pmatrix} q \\ D \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \mathbb{H} \begin{pmatrix} q \\ D \end{pmatrix} \right\rangle \end{aligned}$$

with

$$\mathbb{H} = \begin{pmatrix} \sigma^t/2 & -\mathbb{I} \\ \mathbb{I}/4 + V'' & \sigma^t/2 \end{pmatrix},$$

where we denote $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbb{I}$ the 2×2 identity matrix.

$\mathbb{H}$ is a Hamiltonian matrix with respect to the symplectic structure defined by $\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$. Its eigenvalues are

$$\{\pm \underbrace{\mathrm{i}\sqrt{P + Q}}_{:=\lambda}, \pm \underbrace{\sqrt{Q - P}}_{:=\mu}\},$$

with $P := 1 + \mathrm{tr}V''$, $Q = P^2 - 4\det V''$. For V'' small enough, $P > 0$, $Q > 0$; thus, $\lambda \in \mathrm{i}\mathbb{R}$. In the case of an hyperbolic fixed point, $\det V'' < 0$ so $\mu \in \mathbb{R}$. In the elliptic case, $\mu \in \mathbb{R}$ if $\det V''$ is small enough and in the parabolic case, $\mu = 0$. In all three cases one knows from normal form theory (see [Wi]) that there exists a symplectic transformation decoupling the degrees of freedom. We state the explicit result for the cases of the quantum dot and antidot, i.e. $V'' = \pm\varepsilon^2\mathbb{I}$, which one verifies by the direct calculation.

Proposition 4.2. *Let $1/2 > \varepsilon > 0$, $V(q) = \pm\frac{\varepsilon^2}{2}(q_1^2 + q_2^2)$. Define $\Omega := \sqrt{1 \pm 4\varepsilon^2}$ and the unitary $\mathcal{U}\psi(q) = \frac{1}{\sqrt{\Omega}}\psi(q/\sqrt{\Omega})$, then it holds*

(i) $\mathcal{U}(H_{La} + V)\mathcal{U}^{-1} = \frac{1+\Omega}{2}H_{La} + \frac{\Omega-1}{2}\frac{c^2}{2} = \mathbf{D}H + \frac{\Omega-1}{2}\left(H_{La} + \frac{c^2}{2}\right) \pm \frac{\varepsilon^2}{2}\frac{c^2}{2}$,
(ii) and for the constant of motion:

$$\mathcal{U}^{-1}H_{La}\mathcal{U} = \frac{1}{2}\left(\frac{1}{\sqrt{\Omega}}D - \sqrt{\Omega}\frac{q^{\perp}}{2}\right)^2.$$

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