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Magnetic Bloch analysis and Bochner Laplacians

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Hamiltonians for a particle on a manifold in a magnetic field are constructed as Bochner Laplacians. We show for the case of a torus and a given magnetic field that they are in one to one correspondence with the constituents in the Bloch decomposition of the unique Hamiltonian on the universal covering.

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1. Introduction

We consider a Schrödinger Hamiltonian on a Riemannian manifold with magnetic field and its relation to the corresponding Hamiltonian on the universal covering manifold. This problem is motivated by and our results may be useful for some questions around models for the Quantum Hall Effect [TKNN, ASY, AKPS]. We review first the well known geometrical construction of the Hilbert space of quantum mechanical states and the Hamiltonian of the system for magnetic fields with integral flux. In this setting the Hilbert space consists of L^2-sections in a hermitian line bundle with connection over the manifold of configurations; the curvature of the connection is the magnetic field, and the dynamics is generated by the Bochner Laplacian. The construction is not unique if the manifold is not simply connected. The family of hermitian line bundles with connection (Hilbert spaces and Hamiltonians) is parametrized by Aharonov–Bohm-like fluxes through the "holes" of the manifold, mathematically: by certain cohomology groups.

This geometric description is known. It has been used by physicists since it was pointed out by Wu and Yang [WY], who worked out the case of the sphere (the Dirac monopole). Our aim here is to emphasize the aspect of non-uniqueness, which is much less discussed in the physics literature. We use explicitly methods of differential and algebraic geometry to relate the non-uniqueness to the Bloch

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decomposition of the corresponding setting on the covering space. This program is worked out in detail for the case of the two dimensional torus. In particular we show that the Bloch decomposition of the Hilbert space of a particle moving in the two-plane leads to a Hilbert bundle over the Brillouin zone (or in more general terms: the Jacobi torus), whose fibre operators are in one to one correspondence with the family of Bochner Laplacians arising in the geometrical construction. Geometrically speaking we show that the summation over all non-equivalent Hamiltonians on the torus gives the Hamiltonian on its universal cover.

Quantization of a particle on a Riemannian manifold in the presence of a magnetic field is technically prequantization in the terminology of "Geometric Quantization" [K, Wo]. In this context it is a known technique to start the quantization procedure on the covering space and then to "push it down" to the original manifold. This method has recently been used in ref. [AdPW] for quantization of Chern–Simons Gauge Theory. It might be useful to comment on some similarities and differences between our article and the one just mentioned. Here we start with a Riemannian manifold with integral two-form $\boldsymbol{b}$ and classify first all possible hermitian line bundles with connection and curvature $\boldsymbol{b}$ (theorem 2). After that all compatible connections for a given hermitian line bundle with curvature $\boldsymbol{b}$ are classified (theorem 3). In the article of Axelrod et al. the prequantum line bundle is given at the outset and quantization is discussed in terms of all possible complex structures. Here the manifold is configuration space. There it is phase space.

Integrality conditions for integrated curvature (magnetic fields) have a long history. In the physics literature it starts with Dirac's article [D]; in mathematics it appears in our context in ref. [W] and goes back to the Gauss–Bonnet theorem. Extensions of Dirac Quantization to Yang–Mills and Wess–Zumino models using modern concepts are presented in ref. [A]. Bochner Laplacians on manifolds are of course the object of many articles. Let us just mention the recent paper by Kuwabara [Ku], where the spectrum is related to the closed orbits of the corresponding classical dynamics.

The paper is organized as follows: in section 1 we recall the geometric description of quantum mechanics; section 2 contains the explicit calculation for the torus example; in section 3 the Bloch analysis for periodic magnetic fields is carried out.

1. Quantum mechanics on a manifold with magnetic field

We recall the well known geometric method for constructing the Hilbert space and the Hamiltonian of a system under the influence of a magnetic field. Let the configuration space be an oriented Riemannian manifold M and the magnetic field be given by a closed real two-form $\boldsymbol{b}$ on M.

If $\boldsymbol{b}=\mathrm{d}\boldsymbol{a}$ the Hamiltonian is formally given by $(\mathrm{d}-\mathrm{i}\boldsymbol{a})^*(\mathrm{d}-\mathrm{i}\boldsymbol{a})$. This always works locally. The use of local gauge transformations allows one to globalize the construction if $\boldsymbol{b}$ is not exact:

Take a cover $\{U_j\}$ of M such that there exist real one forms $\boldsymbol{a}_j$ on U_j and functions f_{jk} on $U_j \cap U_k$ with $\boldsymbol{b}=\mathrm{d}\boldsymbol{a}_j$ on U_j, $\boldsymbol{a}_j-\boldsymbol{a}_k=\mathrm{d}f_{jk}$ on $U_j \cap U_k$. Wavefunctions φ_j, φ_k are gauge transformed in $U_j \cap U_k$ by $c_{jk} := \exp(\mathrm{i}f_{jk})$, i.e., $\varphi_j = c_{jk}\varphi_k$. This is sensible if the cocycle conditions $c_{jk}c_{kl}=c_{jl}$ can be satisfied on $U_j \cap U_k \cap U_l$.

The geometrical object which formalizes this idea is a hermitian line bundle with connection whose curvature is $\boldsymbol{b}$ [WY]. The questions of existence and uniqueness were studied in refs. [S, W, K] and imply quantization conditions on the physical system. Let us fix notation.

In the following all mappings and all manifolds are supposed to be infinitely differentiable. Let M be a manifold and $\pi: \mathscr{B} \to M$ a vector bundle with fibre $\mathbb{C}$; $T_{\mathbb{C}}M$ ($T^*_{\mathbb{C}}M$) the complexified (dual) tangent bundle of M; $S(\mathscr{B})$ the $C^\infty(M, \mathbb{C})$ module of sections.

If there is a hermitian structure $\langle\cdot,\cdot\rangle$ on $\mathscr{B}$ and a compatible connection ∇, $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ is called a hermitian line bundle with connection (HLBC). Two HLBsC $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$, $(\mathscr{B}', \nabla', \langle\cdot,\cdot\rangle')$ with the same base M are called equivalent if there exists a diffeomorphism $h: \mathscr{B} \to \mathscr{B}'$ with $\pi' h = \pi$ such that for $m \in M$ the induced mappings $h_m: \pi^{-1}(m) \to \pi^{-1}(m)$ are linear isomorphisms, $\nabla'_X h \circ s = h \circ \nabla_X s$ ($X \in S(T_{\mathbb{C}}M)$, $s \in S(\mathscr{B})$) and for $b \in \mathscr{B}$: $\langle h(b), h(b)\rangle = \langle b, b\rangle$.

Two connections ∇, ∇' on a HLB $(\mathscr{B}, \langle\cdot,\cdot\rangle)$ are called equivalent if $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ is equivalent to $(\mathscr{B}, \nabla', \langle\cdot,\cdot\rangle)$. The curvature $\boldsymbol{b}$ of ∇ is the two-form $\boldsymbol{b}(X, Y)s := \mathrm{i}(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s$, $Y, X \in S(T_{\mathbb{C}}M)$, $s \in S(\mathscr{B})$.

$\boldsymbol{b}$ is called integral if $\boldsymbol{b}/2\pi$ is equivalent to an element of the cohomology with integer coefficients; for compact M this is the case iff the integral of $\boldsymbol{b}/2\pi$ over a singular two-cycle has an integer value. For such a field it is possible to construct a HLBC over M whose curvature is $\boldsymbol{b}$:

Theorem 1. *Consider a manifold M and a two-form $\boldsymbol{b}$ on M. A HLBC with curvature $\boldsymbol{b}$ exists iff $\boldsymbol{b}$ is real, closed, and integral.*

Proof. The cohomology theories of Čech and de Rham and their equivalence are used; the reader might consult refs. [W, G, BT] for this machinery, [W, K, Wo] for the proof. □

Remark. Moreover the following result is well known: The group of equivalence classes of complex line bundles over a manifold M is isomorphic to $H^2(M, \mathbb{Z})$; the isomorphism is given by the first Chern class. If in addition there is a hermitian structure and a connection on the line bundle (LB), its curvature is the natural image of this Chern class in $H^2(M, \mathbb{R})$. Note that the first Chern class (as an element of $H^2(M, \mathbb{Z})$) is determined by the curvature only up to torsion elements.

With the question of existence of a HLBC with curvature $\boldsymbol{b}$ settled, the first step towards the construction of the Hilbert space of states for a quantum mechanical particle on the manifold M in the magnetic field $\boldsymbol{b}$ has been accomplished. Now we address two questions of uniqueness:

Given a manifold M and a real closed integral two-form $\boldsymbol{b}$; firstly: how many non-equivalent possibilities do we have to construct a HLBC with curvature $\boldsymbol{b}$; secondly: given a hermitian line bundle (HLB) which admits a connection with curvature $\boldsymbol{b}$, what is the classification of the non-equivalent connections with curvature $\boldsymbol{b}$ on this HLB? We should like to stress that the two questions coincide if the Chern class is uniquely determined by the curvature $\boldsymbol{b}$.

The answer may be stated in the language of cohomology theory [Wo]. Denote by $H^k(M, G)$ the singular cohomology with coefficients in G [G]. Then the following holds:

Theorem 2. *Given a manifold M and a real closed integral two-form $\boldsymbol{b}$. The set of equivalence classes of hermitian line bundles with connection and curvature $\boldsymbol{b}$ is in bijection with $H^1(M, S^1)$.*

Theorem 3. *Given a HLB over M and a real closed integral two-form $\boldsymbol{b}$. The set of equivalence classes of connections with curvature $\boldsymbol{b}$ is in bijection with $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.*

Remarks.

- From the Aharonov–Bohm effect [AB] one knows for a particle in $\mathbb{R}^3\setminus$(cylinder): if one adds a vector potential to $\mathrm{d}-i\boldsymbol{a}$, the physics remains unchanged iff the derivative of the added potential is zero in the configuration space and its flux through the cylinder is an integral multiple of 2π. In this sense we may regard theorem 3 as a description of a generalized Aharonov–Bohm effect.
- $H^1(M, S^1)$ and $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$ are not isomorphic in general as one learns from the example $M=\mathbb{R}\mathrm{P}^3$: the exactness of

$$1 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\exp(i\cdot)} S^1 \to 1$$

entails the exactness of

$$0 \to H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{R}) \to H^1(M, S^1) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \to \dots .$$

As an example we consider $\mathbb{R}\mathrm{P}^3$ (which is not simply connected). It is known [BT] that $H^2(\mathbb{R}\mathrm{P}^3, \mathbb{Z})=\mathbb{Z}_2$ and $H^2(\mathbb{R}\mathrm{P}^3, \mathbb{R})=0$. It follows that $H^1(\mathbb{R}\mathrm{P}^3, \mathbb{R}) \to H^1(\mathbb{R}\mathrm{P}^3, S^1)$ is not surjective.
- If M is a closed surface in $\mathbb{R}^3$ the integrality condition implies quantization of

the magnetic flux through M. For the special case $M=S^2$, and $\boldsymbol{b}=$const. this is Dirac's famous result on the quantization of the magnetic monopole [D].

Given an integral magnetic field $\boldsymbol{b}$ on M the Hamiltonian H of the system is by definition the covariant Laplacian on a HLBC $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ with curvature $\boldsymbol{b}$. This is constructed as follows:

If M is oriented, the Riemannian metric $(\cdot,\cdot)$ and the volume form ω induce the Hilbert spaces $(L^2(\mathscr{B}), \langle\ ,\ \rangle_{\mathscr{B}})$ and $(L^2(T^*_{\mathbb{C}}M\otimes\mathscr{B}), \langle\ ,\ \rangle_{\otimes})$. ∇, its formal adjoint ∇^* and the Bochner Laplacian $\nabla^*\nabla$ are defined on the smooth sections with compact support. $\nabla^*\nabla$ is then symmetric and positive.

For an oriented Riemannian manifold M, a closed real integral two-form $\boldsymbol{b}$, and a HLBC $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ with curvature $\boldsymbol{b}$ we make the

Definition. The Hamiltonian of a particle on the configuration space M in an integral magnetic field $\boldsymbol{b}$ is the Friedrichs extension of the Bochner Laplacian $\nabla^*\nabla$.

Remarks.

- If M is complete, $\nabla^*\nabla$ is essentially self-adjoint by generalization of a result of ref. [Ch] for the flat Laplacian. Compact Riemannian manifolds are complete.
- An equivalence of HLBsC defines a unitary mapping $U: L^2(\mathscr{B})\to L^2(\mathscr{B}')$ with $UHU^{-1}=H'$. If $H^1(M, S^1)$ is non-trivial, a HLBC is not uniquely (up to equivalence) determined by the magnetic field $\boldsymbol{b}$. Therefore the Hamiltonian H (and the physics) is not unique (up to unitary equivalence). Note that this non-uniqueness has nothing to do with domain questions of H (as an operator on sections of $\mathscr{B}$).
- If $M=\mathbb{R}^n$ every closed $\boldsymbol{b}$ is integral and for $\boldsymbol{a}$ with $\mathrm{d}\boldsymbol{a}=\boldsymbol{b}$, H is represented by $(\mathrm{d}-\mathrm{i}\boldsymbol{a})^*(\mathrm{d}-\mathrm{i}\boldsymbol{a})$.

2. Schrödinger particle on the torus in a magnetic field

From now on we specify M as the two-torus (a manifold with non-trivial $H^1(M, S^1)$). Let $\Gamma\subset\mathbb{R}^2$ be the lattice given by integral combinations of linearly independent vectors e_1, e_2. We consider a particle on the torus $M:=\mathbb{R}^2/\Gamma$; M is a compact, therefore complete, oriented manifold with the natural induced Riemannian structure. Let furthermore a magnetic field be given by a closed integral two-form $\boldsymbol{b}$ on M.

Next we shall show that any Hamiltonian arising as a Bochner Laplacian is unitarily equivalent to a self-adjoint realization of $-(\partial_{x_1}-\mathrm{i}a_{x_1})^2-(\partial_{x_2}-\mathrm{i}a_{x_2})^2$ in L^2 (unit cell) with appropriate boundary conditions. This representation is widely used but we are not aware of a reference.

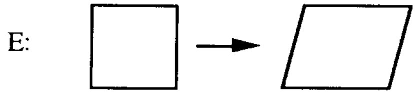

Fig. 1. The map E.

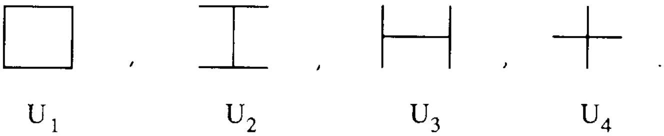

Fig. 2. A contractible cover.

M may be represented as a rectangle with opposite sides identified; this we represent for the sake of notation as the image of the unit square (with opposite sides identified) under a map E (fig. 1).

Let $\{V_j := EU_j\}_{j\in\{1,\dots,4\}}$ be the contractible cover of M defined in fig. 2. Here the drawn lines represent the points in $M\setminus V_j$.

Let $\{(V_j, \psi_j, \boldsymbol{a}_j)\}$ be a trivialization of $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ with transition functions c_{jk} and the ψ_j be chosen normalized, e.g., $\langle\psi_j, \psi_j\rangle \equiv 1$ in V_j. A section $\sigma\in S(\mathscr{B})$ is determined by a function $\varphi_1\in C^\infty(V_1)$ via $\sigma(x)=\psi_1(x, \varphi_1(x))$ $(x\in V_1)$; the following holds:

Proposition 4. *Given* $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ *over* M.

(i) The operation

$$\sigma\to\varphi_1 \quad (\sigma\in S(\mathscr{B}))$$

has a unique continuation to a unitary

$$U: L^2(\mathscr{B})\to L^2(V_1)\ .$$

(ii) Define the euclidean components of $\boldsymbol{a}_1$ *as* a_{x_1}, a_{x_2}, *and* h *the closure of the essentially self-adjoint operator*

$$-(\partial_{x_1} - \mathrm{i}a_{x_1})^2 - (\partial_{x_2} - \mathrm{i}a_{x_2})^2$$

defined on functions in $C^\infty(V_1)\cap C^2(\overline{V_1})$ *which satisfy the boundary conditions*

$$\begin{aligned}&(\partial_n - \mathrm{i}a(n))^\alpha\varphi(E(1,q))\\&\quad= c_{12}(E(1,q))c_{21}(E(0,q))(\partial_n - \mathrm{i}a(n))^\alpha\varphi(E(0,q))\ ,\\&(\partial_n - \mathrm{i}a(n))^\alpha\varphi(E(p,1))\\&\quad= c_{13}(E(p,1))c_{31}(E(p,0))(\partial_n - \mathrm{i}a(n))^\alpha\varphi(E(p,0))\\&\quad\text{for } \alpha\in\{0,1\},\ p, q\in[0,1]\ ,\end{aligned}$$

where n *denotes the normal vector field on the boundary of* V, ∂_n *the normal derivative.*

Then it holds for the Hamiltonian H (the Bochner Laplacian) that

$$UHU^{-1}=h\,.$$

Proof. The boundary conditions follow from

$$\sigma(x)=\psi_2(x,\varphi_2(x))=\psi_1(x,\varphi_1(x))\quad(x\in V_1\cap V_2)\,,$$
$$c_{21}\varphi_1(E(1,q))=\varphi_2(E(1,q))=\varphi_2(E(0,q))=c_{21}\varphi_1(E(0,q))\,,$$

and the periodicity of $\boldsymbol{a}_2$ in the p-variable. □

Remark. Changing the gauge of $(\mathscr{B}, \nabla, \langle\cdot,\cdot\rangle)$ (which means: changing ψ_j and $\boldsymbol{a}_j$) leads to an operator on $L^2(V_1)$ which is unitarily equivalent to h. The passage to an equivalent $(\mathscr{B}', \nabla', \langle\cdot,\cdot\rangle')$ has the same effect.

3. Bloch analysis; summation over all connections

Consider a smooth real-valued function B on $\mathbb{R}^2$ which is periodic on the lattice Γ, the two-form $\boldsymbol{b}:=B\,\mathrm{d}x_1\wedge\mathrm{d}x_2$ which is induced by the natural volume and a vector potential $\boldsymbol{a}$ with $\mathrm{d}\boldsymbol{a}=\boldsymbol{b}$. Then the Bochner Laplacian on the trivial bundle $\mathbb{R}^2\times\mathbb{C}$ with connection and curvature $\boldsymbol{b}$ is equivalent to the closure H of the essentially self-adjoint operator $\sum_j (D_{x_j}-a_{x_j})^2$ on $C_0^\infty(\mathbb{R}^2)$ $(D_{x_j}:=-\mathrm{i}\partial_{x_j})$.

We now present our main result. We shall show that H may be represented as a summation of Bochner Laplacians over the family of inequivalent HLBsC and curvature $\boldsymbol{b}$ over the torus $\mathbb{R}^2/\Gamma$; or, equivalently, over all connections with curvature $\boldsymbol{b}$ on a fixed HLB.

To do this, we employ ideas from Bloch analysis and the theory developed in sections 1, 2. By Bloch analysis we mean the reduction of H to the eigenspaces of the abelian group of magnetic translations which commutes with H. This group was introduced by Zak [Z] (for the case $B=\text{const.}$). It is defined as follows:

Denote the fundamental cell of Γ by

$$\begin{aligned}C&:=\{x\in\mathbb{R}^2;\,x=\langle x_1,x_2\rangle=pe_1+qe_2(p,q\in[0,1))\}\\&=E([0,1)^2)\,,\end{aligned}$$

divide the function B in its constant and oscillating parts $B=B_c+B_{osc}$, where

$$B_c:=\frac{1}{\cancel{\mathrm{vol}\,C}}\int_C \boldsymbol{b}\,,$$

split $\boldsymbol{b}=\boldsymbol{b}_c+\boldsymbol{b}_{osc}$ in the obvious way, and choose the gauge $\boldsymbol{a}:=\boldsymbol{a}_c+\boldsymbol{a}_{osc}$ with $\boldsymbol{a}_c(x_1,x_2):=(B_c/2)(x_1\,\mathrm{d}x_2-x_2\,\mathrm{d}x_1)$, $\boldsymbol{a}_{osc}(x+m)=\boldsymbol{a}_{osc}(x)$, $\mathrm{d}\boldsymbol{a}_{osc}=\boldsymbol{b}_{osc}$. Then $T(m)$ is defined for $m\in\Gamma$ as an operator on $L^2(\mathbb{R}^2)$ by

$$T(m)\psi(x) := \mathrm{e}^{-\mathrm{i}a_c(x)(m)}\psi(x-m)\ , \quad \psi\in C_0^\infty(\mathbb{R}^2)$$

[where m as usual is identified with the constant vector field $m(x)=m$].

These magnetic translations fulfill the Weyl relations:

$$T(m)T(l)=\mathrm{e}^{-\mathrm{i}b_c/2(m,l)}T(m+l) \quad (m,l\in\Gamma)\ ;$$

of course: $\boldsymbol{b}_c(m,l)=B_c(m_1l_2-m_2l_1)$.

For $m\in\Gamma$, $\psi\in C_0^\infty(\mathbb{R}^2)$ it holds that

$$\begin{aligned}
&[T(m), (\partial_{x_1}-\mathrm{i}a_{x_1})]\psi(x)\\
&=\mathrm{i}\mathrm{e}^{-\mathrm{i}a_c(x)(m)}\psi(x-m)[a_{x_1}(x)-a_{x_1}(x-m)+\partial_{x_1}a_c(x)(m)]\\
&=\mathrm{i}\mathrm{e}^{-\mathrm{i}a_c(x)(m)}\psi(x-m)[a_{cx_1}(x)-a_{cx_1}(x-m)+\partial_{x_1}a_c(x)(m)]\\
&=0\ ,
\end{aligned}$$

$$[T(m), (\partial_{x_2}-\mathrm{i}a_{x_2})]=0\ .$$

It follows that

$$[T(m),H]=0 \quad \text{on } C_0^\infty(\mathbb{R}^2)\ .$$

Remark. $T(m)=\mathrm{e}^{-\mathrm{i}(m,D+a_c)}$ is the magnetic translation generated by $D+a_c$. The oscillating part of $\boldsymbol{a}$ plays no role.

From now on we shall assume that the magnetic flux is quantized:

$$\Phi := B_c\times \mathrm{vol}\ C\in 2\pi\mathbb{Z}\ ;$$

the group is then abelian and it is natural to split $L^2(\mathbb{R}^2)$ into the eigenspaces of $\{T(m)\}$: For $\psi\in C_0^\infty(\mathbb{R}^2)$, $k\in\mathbb{R}^2$, $x\in\mathbb{R}^2$ we define (with $km=k_1m_1+k_2m_2$)

$$U\psi(k,x) := \sum_{m\in\Gamma}\mathrm{e}^{-\mathrm{i}km}\mathrm{e}^{-\mathrm{i}m_1m_2\Phi/2}T(m)\psi(x)\ ;$$

it holds for $m\in\Gamma$, $k\in\mathbb{R}^2$ that

$$T(m)U\psi(k,x)=\mathrm{e}^{+\mathrm{i}m_1m_2\Phi/2}\mathrm{e}^{\mathrm{i}km}U\psi(k,x)\ .$$

So $U\psi$ is determined by its values on $\mathbb{R}^2/\Gamma^*\times C$ (Γ^* denotes the 2π dual lattice of Γ). We will regard $U\psi$ as a function on this domain.

We then have

Theorem 5. *U extends to an isomorphism,*

$$U: L^2(\mathbb{R}^2)\to \int_{\oplus\mathbb{R}^2/\Gamma^*} L^2(C)\,\frac{\mathrm{d}k^2}{|\mathbb{R}^2/\Gamma^*|}\ .$$

Proof. For $\psi \in C_0^\infty(\mathbb{R}^2)$ the function

$$\tilde{\psi}(x, m) := e^{-ia_C(x)(m)} e^{-im_1 m_2 \Phi/2} \psi(x-m)$$

is in $\ell^2(\Gamma)$ for $x \in C$, and so by Plancherel's theorem:

$$\int_{\oplus \mathbb{R}^2/\Gamma^*} \frac{dk^2}{|\mathbb{R}^2/\Gamma^*|} \, |U\psi(k, x)|^2 = \|\tilde{\psi}(x, \cdot)\|^2_{\ell^2(\Gamma)} .$$

Now

$$\|U\psi\|^2_{\oplus} = \int_C dx^2 \|\tilde{\psi}(x, \cdot)\|^2_{\ell^2} = \sum_{m \in \Gamma} \int_C |\psi(x-m)|^2 = \|\psi\|^2 ;$$

this proves injectivity.

Now we give explicitly the adjoint U^*: For $\varphi \in \int_{\otimes} L^2(C)\, dk^2$ define

$$\tilde{\varphi}(x, m) := \int_{\mathbb{R}^2/\Gamma^*} e^{ikm} e^{-ia_C(x)(m)} e^{-im_1 m_2 \Phi/2} \varphi(k, x) \frac{dk^2}{|\mathbb{R}^2/\Gamma^*|}$$

$$(x \in C, m \in \Gamma)$$

and

$$U^*\varphi(x) := \tilde{\varphi}(x - [x], -[x]) \quad (x \in \mathbb{R}^2)$$

($[x]$ denotes the integral part of x with respect to the basis $\{e_1, e_2\}$). Then $U^*\varphi \in L^2(\mathbb{R}^2)$ and $(U^*\varphi, \psi)_{L^2} = (\varphi, U\psi)_{\oplus}$. □

In order to describe the corresponding decomposition of H we introduce for $k \in \mathbb{R}^2/\Gamma^*$ the operator

$$H(k) := \sum_j (D_{x_j} - a_{x_j})^2$$

defined on the core

$$\begin{aligned} B(H(k)) = \{\varphi \in C^\infty(C) \cap C^2(\bar{C}) ;\ & [(-i\partial_n - \boldsymbol{a}(n))]^\alpha \varphi(x-m) \\ & = e^{ikm} e^{-ia_C(x)(m)} e^{im_1 m_2 \Phi/2} [(-i\partial_n - \boldsymbol{a}(n))]^\alpha \varphi(x) \\ & \Leftrightarrow [(-i\partial_n - \boldsymbol{a}(n))]^\alpha T(m)\varphi \\ & = e^{ikm} e^{im_1 m_2 \Phi/2} [(-i\partial_n - \boldsymbol{a}(n))]^\alpha \varphi \\ & \text{for } m = \langle m_1, m_2 \rangle \in \Gamma, \alpha \in \{0, 1\}, \quad x \in \partial C \text{ s.t. } x + m \in \partial C\} . \end{aligned}$$

Then the following holds:

Theorem 6.

$$UHU^{-1}=\int_{\oplus\mathbb{R}^2/\Gamma^*} H(k)\,\frac{\mathrm{d}k^2}{|\mathbb{R}^2/\Gamma^*|}\,.$$

Proof. It is sufficient to show the equality

$$UH=\int_{\oplus} H(k)\,\frac{\mathrm{d}k^2}{|\mathbb{R}^2/\Gamma^*|}\,U$$

on the core $C_0^\infty(\mathbb{R}^2)$ of H. First we remark that for $\psi\in C_0^\infty(\mathbb{R}^2)$ it holds that

$$U\psi\in\int_{\oplus} B(H(k))\,\mathrm{d}k^2\,.$$

We have already verified that $U\psi(k,\cdot)$ is an eigenfunction of the group $\{T(m)\}$ so the required boundary conditions hold for $U\psi$; the fact that $\{T(m)\}$ commutes with $(-\mathrm{id}-\boldsymbol{a})$ implies their validity for the covariant derivatives. Secondly, as $H(k)$ commutes with $\{T(m)\}$ we have

$$\begin{aligned}
&\int_{\oplus} H(k)\,\mathrm{d}k^2 U\psi\\
&=\int_{\oplus}\Big(\sum_j (D_j-a_j)^2\sum_m \mathrm{e}^{ikm}\mathrm{e}^{\mathrm{i}\Phi m_1 m_2/2}T(m)\psi\Big)\,\mathrm{d}k^2\\
&=\int_{\oplus}\Big(\sum_m \mathrm{e}^{ikm}\mathrm{e}^{\mathrm{i}\Phi m_1 m_2/2}T(m)\sum_j (D_j-a_j)^2\psi\Big)\,\mathrm{d}k^2\\
&=UH\psi\,.
\end{aligned}$$

□

The Bloch analysis may be regarded from a geometrical point of view; this is based on the observation that for $k\in\mathbb{R}^2/\Gamma^*$ there exists a HLBC over the torus $\mathbb{R}^2/\Gamma$ such that $H(k)$ is unitarily equivalent to its Bochner Laplacian. We shall construct such a HLBC for $k=0$; after that we shall prove the assertion for all k.

Choose a HLBC which is defined by the following data in the p, q coordinates (we give only the parts which are relevant for our purpose):

$$\boldsymbol{a}_1:=\boldsymbol{a}_{\mathrm{c}}+\boldsymbol{a}_{\mathrm{osc}}\,,$$

$$E^*\boldsymbol{a}_2(p,q):=-\Phi q\,\mathrm{d}p+E^*\boldsymbol{a}_{\mathrm{osc}}\,,\qquad E^*\boldsymbol{a}_3(p,q):=\Phi p\,\mathrm{d}q+E^*\boldsymbol{a}_{\mathrm{osc}}\,,$$

with

$$E^{*}\boldsymbol{a}_1(p,q) := -(\Phi/2)(p\,\mathrm{d}q - q\,\mathrm{d}p):$$

$$c_{12}(E(p,q)) := \mathrm{e}^{\mathrm{i}(\Phi/2)pq}, \qquad c_{13}(E(p,q)) := \mathrm{e}^{-\mathrm{i}(\Phi/2)pq}.$$

By proposition 4 the Bochner Laplacian on this HLBC is unitarily equivalent to the closure of $\sum_j (D_{x_j} - a_{x_j})^2$ defined on the $C^\infty(V_1) \cap C^2(\overline{V_1})$ functions which satisfy the boundary conditions

$$\begin{aligned}
&(\partial_n - \mathrm{i}a(n))^\alpha \varphi(E((0,q)+(1,0)))\\
&\quad = c_{12}(E(1,q))c_{21}(E(0,q))(\partial_n - \mathrm{i}a(n))^\alpha \varphi(E(0,q))\\
&\quad = \exp(\mathrm{i}(\Phi/2)q)(\partial_n - \mathrm{i}a(n))^\alpha \varphi(E(0,q)),\\
&(\partial_n - \mathrm{i}a(n))^\alpha \varphi(E((p,0)+(0,1)))\\
&\quad = \exp(-\mathrm{i}(\Phi/2)p)(\partial_n - \mathrm{i}a(n))^\alpha \varphi(E(p,0))\\
&\qquad \text{for } \alpha \in \{0,1\},\ p,q \in [0,1].
\end{aligned}$$

But this operator is $H(0)$! So for the connection on this bundle it holds that

$$\nabla^* \nabla \cong H(0).$$

For general k we have

Theorem 7. *For $k \in \mathbb{R}^2/\Gamma^*$ there exists one and only one (equivalence class of) HLBsC such that*

$$\nabla^* \nabla \cong H(k).$$

Proof. By theorem 2 the set of all (equivalence classes of) HLBsC is isomorphic to $H^1(\mathbb{R}^2/\Gamma, S^1)$. We now construct an explicit bijection of this space to $\mathbb{R}^2/\Gamma^*$, which makes the role of the boundary conditions transparent.

Denote by $H^k(\{U_j\}, G)$ the kth Čech cohomology group relative to $\{U_j\}$ with coefficients in the locally constant functions with values in the abelian group G. We make use of the fact that $H^k(\{U_j\}, G)$ is isomorphic to $H^k(M, G)$, the singular cohomology with coefficients in G [G]. For the machinery the reader might refer to refs. [W, G, BT].

An element $f \in H^0(\mathbb{R}^2/\Gamma, S^1) \cong H^0(\{U_j\}, S^1)$ is characterized by $f = (f_1, f_2, f_3, f_4)$, where f_j are constant functions on U_j.

An element $f \in H^1(\mathbb{R}^2/\Gamma, S^1) \cong H^1(\{U_j\}, S^1)$ is characterized by $f = (f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34})$, where f_{jk} are locally constant functions on $U_j \cap U_k$ with $f_{jk} f_{kl} f_{lj} = 1$ on $U_j \cap U_k \cap U_l$. They are determined by their values on the connected components of $U_j \cap U_k$.

These regions can be visualized as shown in fig. 3 (cf. fig. 2). For example, f_{14}

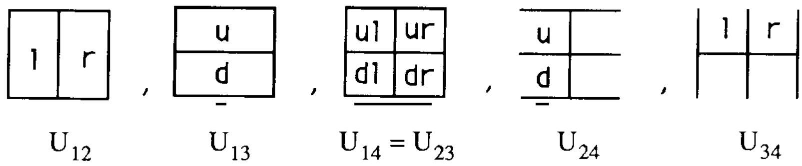

Fig. 3. Intersections of the cover.

is determined by the tuple $(ur, ul, dl, dr)_{14} \in (\mathrm{S}^1)^4$. The coboundary operator

$$\delta: H^0(\{U_j\}, \mathrm{S}^1) \to H^1(\{U_j\}, \mathrm{S}^1)$$

is defined by

$$\delta((f_1, f_2, f_3, f_4))_{jk} = f_j f_k^{-1} .$$

We claim that f may be represented in the following way:

$$\begin{aligned} f &= ((l, r)_{12}, (u, d)_{13}, (ur, ul, dl, dr)_{14}, \ldots) \\ &= ((\alpha, 1), (\beta, 1), (\beta, \beta\alpha, \alpha, 1), (\beta, \beta\alpha^{-1}, \alpha^{-1}, 1), (\beta, 1), \\ &\quad (\alpha, 1))\delta((1, \gamma, \eta, \zeta)) , \end{aligned} \tag{1}$$

with

$$\alpha := l_{12} r_{21} , \quad \beta := u_{13} d_{31} , \quad \gamma := r_{21} , \quad \eta := d_{31}, \quad \zeta := dr_{41}$$

(where, for example, dr_{41} is the inverse of the value of f_{14} in the lower right component of U_{14}).

In order to indicate how this is derived, we check this identity on f_{23},

$$\delta((1, \gamma, \eta, \zeta))_{23} = \gamma\eta^{-1}(1, 1, 1, 1) = dr_{23}(1, 1, 1, 1) .$$

Using the cocycle conditions one gets

$$\begin{aligned} &(\beta, \beta\alpha^{-1}, \alpha^{-1}, 1)\delta((1, \gamma, \eta, \zeta))_{23} \\ &\quad = (u_{13} d_{31} dr_{23}, u_{13} d_{31} l_{21} r_{12} dr_{23}, l_{21} r_{12} dr_{23}, dr_{23}) \\ &\quad = (u_{13} r_{21}, u_{13} d_{31} l_{21} d_{13}, l_{21} d_{13}, dr_{23}) \\ &\quad = (ur_{23}, ul_{23}, dl_{23}, dr_{23}) = f_{23} . \end{aligned}$$

Define for $\alpha, \beta \in \mathrm{S}^1$ the $H^1(\mathbb{R}^2/\Gamma, \mathrm{S}^1)$ element

$$f(\alpha, \beta) := ((\alpha, 1), (\beta, 1), (\beta, \beta\alpha, \alpha, 1), (\beta, \beta\alpha^{-1}, \alpha^{-1}, 1), (\beta, 1), (\alpha, 1)) .$$

Using (1) one obtains that the map

$$\mathbb{R}^2/\Gamma^* \to H^1(\mathbb{R}^2/\Gamma, \mathrm{S}^1) , \quad k \to f(\mathrm{e}^{-ike_1}, \mathrm{e}^{ike_2})$$

is bijective.

From the description of the non-equivalent HLBsC given by theorem 2 we can now conclude: For every $k \in \mathbb{R}^2/\Gamma^*$ there exists a unique (equivalence class of)

HLBsC determined by the data

$$(c_{ij}(k), \boldsymbol{a}_j) := (f_{ij}(\mathrm{e}^{-\mathrm{i}ke_1}, \mathrm{e}^{\mathrm{i}ke_2})c_{ij}, \boldsymbol{a}_j)\ ,$$

where $(c_{ij}, \boldsymbol{a}_j)$ are the data chosen above.

Fix now k in $\mathbb{R}^2/\Gamma^*$. From the structure of $f(\alpha, \beta)$ and an argumentation analogous to the one made above for the $k=0$ case, one sees that $\nabla^*\nabla$ on the HLBC determined by k is unitarily equivalent to $\sum_j (D_{x_j} - a_{x_j})^2$ with the jump conditions: $\mathrm{e}^{\mathrm{i}ke_1}\mathrm{e}^{\mathrm{i}(\Phi/2)q}$ in the e_1 direction and $\mathrm{e}^{\mathrm{i}ke_2}\mathrm{e}^{-\mathrm{i}(\Phi/2)p}$ in the e_2 direction. This operator is $H(k)$.

We now give an interpretation of Bloch analysis in geometric language:

Corollary 8. *Given a real closed two-form* $\boldsymbol{b}$ *on* $\mathbb{R}^2$ *which projects to an integral two-form* $\hat{\boldsymbol{b}}$ *on* $\mathbb{R}^2/\Gamma$. *Then the direct integral of the Bochner Laplacians over all non-equivalent HLBsC over the torus with curvature* $\hat{\boldsymbol{b}}$ *is unitarily equivalent to the unique Bochner Laplacian on the HLBC with curvature* $\boldsymbol{b}$ *on its universal cover.*

For the torus one may adopt a slightly different point of view:

Fix a HLB so that a connection with curvature $\boldsymbol{b}$ exists (i.e., the Chern class of the bundle equals the (cohomology class of) $\boldsymbol{b}$). By theorem 3 the set of all (equivalence classes of) connections is $H^1(\mathbb{R}^2/\Gamma, \mathbb{R})/H^1(\mathbb{R}^2/\Gamma, \mathbb{Z})$. By considerations analogous to those which led to theorem 7 one obtains

$$\mathbb{R}^2/\Gamma^* \cong H^1(\mathbb{R}^2/\Gamma, \mathbb{R})/H^1(\mathbb{R}^2/\Gamma, \mathbb{Z}) \cong H^1(\mathbb{R}^2/\Gamma, \mathrm{S}^1)$$

and for every k in $\mathbb{R}^2/\Gamma^*$ there is exactly one connection (up to equivalence) with $\nabla^*\nabla \cong H(k)$.

So we have:

Given a real closed two-form $\boldsymbol{b}$ *on* $\mathbb{R}^2$ *which projects to an integral two-form* $\hat{\boldsymbol{b}}$ *on* $\mathbb{R}^2/\Gamma$. *Then the direct integral of the Bochner Laplacians over all non-equivalent connections on a suitable HLB over the torus with curvature* $\hat{\boldsymbol{b}}$ *is unitarily equivalent to the unique Bochner Laplacian on the HLBC with curvature* $\boldsymbol{b}$ *on its universal cover.*

Again we remark that for manifolds with torsion like $\mathbb{R}\mathrm{P}^3$ these two points of view are not equivalent.

To summarize the content of section 3: We carried out a Bloch analysis for particles in a periodic magnetic field and gave a geometric reinterpretation in terms of a sum over all connections.

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