Dynamics of a classical Hall system driven by a time-dependent Aharonov-Bohm flux

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We study the dynamics of a classical particle moving in a punctured plane under the influence of a homogeneous magnetic field, an electric background, and driven by a time-dependent singular flux tube through the hole. We exhibit a striking (de)localization effect: when the electric background is absent we prove that a linearly time-dependent flux tube opposite to the homogeneous flux eventually leads to the usual classical Hall motion: the particle follows a cycloid whose center is drifting orthogonal to the electric field; if the fluxes are additive, the drifting center eventually gets pinned by the flux tube whereas the kinetic energy is growing with the additional flux. © *2007 American Institute of Physics.*
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I. INTRODUCTION

The motivation to study the dynamics of this classical system is to sharpen our intuition on its quantum counterpart which is, following Laughlin's[13] and Halperin's[11] proposals, widely used for an explanation of the integer quantum Hall effect. Of special interest is how the topology influences on the dynamics. In the mathematical physics literature Bellissard *et al.*[5] and Avron *et al.*[3,4] used an adiabatic limit of the model to introduce indices. The indices explain the quantization of charge transport observed in the experiments.[12] See Refs. 6, 9, 7, 8, and 10 for recent developments. We discussed aspects of the adiabatics of the quantum system in Ref. 2, its quantum and semiclassical dynamics will be treated elsewhere. The dynamics of the classical system without magnetic field were discussed in Ref. 1. We state the model and our main results.

Consider a classical point particle of mass $m>0$ and charge $e>0$ moving in the punctured plane $\mathbb{R}^2\backslash(0)$. Suppose that a magnetic flux line with time varying strength Φ pierces the origin and further the presence of a homogeneous magnetic field of strength B orthogonal to the plane and an interior electric field with smooth bounded potential V.

Let

$$\Phi:\mathbb{R}\to\mathbb{R} \text{ and } V:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R} \text{ be smooth functions.}$$

Denote $q^{\perp}:=(-q_2,q_1)$. The force on the particle at $q\in\mathbb{R}^2\backslash(0)$ with velocity $\dot{q}$ is

$$-e\left(B\dot{q}^{\perp}-\frac{\partial_t\Phi}{2\pi}\frac{q^{\perp}}{|q|^2}+\partial_q V\right)=e(\dot{q}\wedge\mathrm{rot}(A)-\partial_t A),$$

with

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$$A(\Phi(t),q) := \left(\frac{B}{2} - \frac{\Phi(t)}{2\pi|q|^2}\right)q^\perp + t\partial_q V(q).$$

Remark that the electromotive force induced by the flux line has circulation $e\partial_t\Phi$, constant torque $e\partial_t\Phi/2\pi$, vanishing rotation, and is long range with a $1/r$ singularity at the origin, we call it the circular part. V is smooth on the entire plane so the circulation of the corresponding field is zero. The total magnetic flux through a circle of radius R is $\pi BR^2 - \Phi$ if it encircles the flux line, else πBR^2. So the two fluxes are "opposite" if B and Φ have the same sign.

The equations of motions are Hamiltonian. For a point $(q,p)=((q_1,q_2),(p_1,p_2))$ in phase space

$$\mathbb{P} = \mathbb{R}^2 \setminus (0) \times \mathbb{R}^2,$$

the time-dependent Hamiltonian is

$$\mathcal{H}(t,B;q,p) = \frac{1}{2m}(p - eA(\Phi(t),q))^2.$$

Suppose

$$\Phi(-t) = -\Phi(t),$$

then there is the time reversal symmetry

$$\mathcal{H}(-t,-B;q,-p) = \mathcal{H}(t,B;q,p).$$

So in order to fix the ideas we convene that growing time means growing flux opposite to the homogeneous flux and suppose furthermore

$$B > 0, \quad \partial_t\Phi(t) \geqslant \Phi_0 > 0.$$

Recall that when only the constant magnetic field is present, the particle follows the Landau orbits: circles around a fixed center with the cyclotron frequency

$$\omega = \frac{eB}{m}$$

and radius R such that the magnetic flux through the Landau circle satisfies $e\omega/2\pi(\pi BR^2)=\mathcal{H}$. With the above convention this motion is clockwise.

If $V=0$ then intuitively the physics is the following: the additional field makes the center $q - (\dot{q}^\perp/\omega)$ drift orthogonal to the electric field. The constant torque exerced by the circular part accelerates orbits which encircle the origin, the orbit shrinks with growing flux and grows with decreasing flux; orbits not encircling the flux line should have constant radius.

We have the following results for the case $\Phi=\Phi_0 t$, $V=0$ (see Corollary 7.2).

a. The above intuition is asymptotically correct. Furthermore

$$\mathcal{H}(t,B) \sim_{\Phi(t)\to-\infty} \frac{e\omega}{2\pi}|\Phi(t)|$$

$$\mathcal{H}(t,B) \to_{\Phi(t)\to+\infty} \text{const}.$$

b. In the accelerating regime the center c is eventually trapped by the flux line, the particle is spiraling outward

$$c(t) \to_{\Phi(t)\to-\infty} \text{const},$$

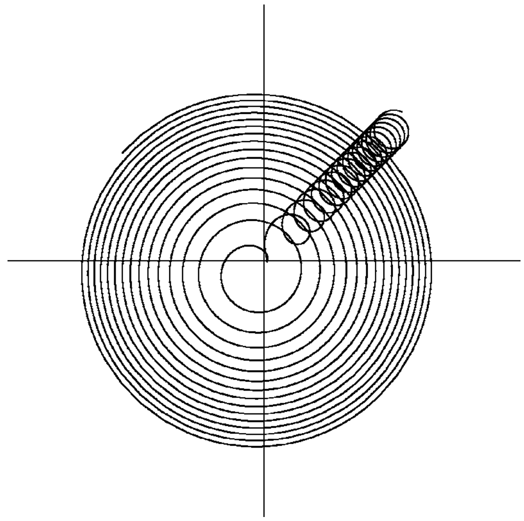

FIG. 1. Typical trajectory of the Hamiltonian $1/2(p-((1/2)q^{\perp}-s(q^{\perp}/q^2)))^2$.

$$\sqrt{\frac{\pi B}{|\Phi(t)|}}q(t)\sim_{\Phi(t)\to-\infty}(\cos(-\omega t),\sin(-\omega t)).$$

c. In the decelerating regime the orbit ends up drifting diffusively orthogonal to the field

$$\sqrt{\frac{\pi B}{\Phi(t)}}q(t)\to_{\Phi(t)\to\infty}\text{const}.$$

In addition we expand the solution up to an error $\mathcal{O}(1/t^{3/2})$ [see Theorem (5.1)]. We further compute the adiabatic limit (i.e., the solution of the equations of motions averaged over the Landau orbits) in the perspective to obtain information on the transition between the two dynamics. We find [see Theorem (6.2)] that *in the adiabatic limit* the transition between the two dynamics is sharp and that the center gets stuck after a finite time if there is no electric background; it is a challenging problem to study if the adiabatic limit provides an approximation of the true dynamics.

For a general increasing flux and a background field whose torque is controlled by the constant torque of the circular part we show [see Corrollary (7.1)]

$$\liminf_{\Phi(t)\to-\infty}\frac{2\pi}{e\omega}\frac{\mathcal{H}(t,B)}{|\Phi(t)|}>0,\quad \liminf_{\Phi(t)\to\infty}\frac{\pi Bc^2(t)}{\Phi(t)}>0.$$

We may state our observation as follows: States if submitted to an accelerating flux line will eventually become energy conducting; if no electric background is present they get trapped by the flux tube.

We give two numerical illustrations in Figs. 1 and 2.

II. DYNAMICS OF THE FROZEN SYSTEM

Upon scaling q, p, t to dimensionless coordinates, which we call q, p, s, we work with the Hamilitonian, $H(\varepsilon s)$ where

$$H(s;q,p)=\frac{1}{2}(p-a(s,q))^2,$$

with

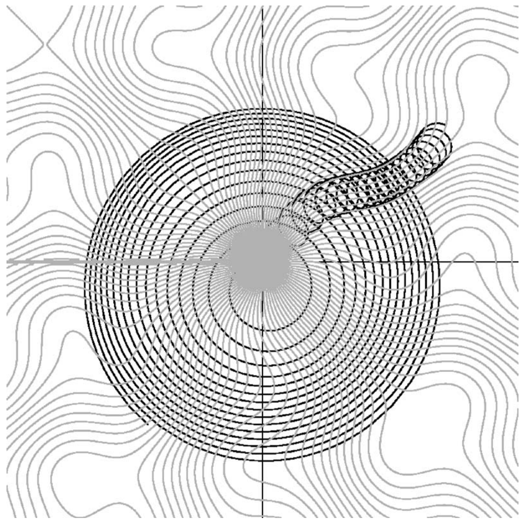

FIG. 2. Typical trajectory of the Hamiltonian $1/2(p-((1/2)q^{\perp}-s(q^{\perp}/q^2)+s\partial_q V))^2$ with the background potential V chosen to be $V(x,y)=1/10(\sin x+\sin y)$ on a region $[-10,10]^2$. The background shows the potential lines of $V(x,y)-\arg(x,y)$.

$$a(s,q) := \frac{1}{2}q^{\perp} + \underbrace{\left(-N(s)\frac{q^{\perp}}{q^2} + s\partial_q \tilde{V}(q)\right)}_{=:a_E(s,q)}$$

and N, $\tilde{V}$ smooth functions,

$$N(-s) = -N(s), \quad \partial_s N \geqslant 1,$$

and ε a dimensionless parameter. We discuss the scaling in Sec. VII.

The function a_E is smooth on $\mathbb{R}^2\setminus(0)$ with $\mathrm{rot}(a_E)=0$. Define $E(s):\mathbb{R}^2\setminus(0)\to\mathbb{R}^2$ by

$$E(s) := -\partial_s a_E(\varepsilon s) = \varepsilon\left((\partial_s N)(\varepsilon s)\frac{q^{\perp}}{q^2} - \partial_q \tilde{V}(q)\right). \tag{1}$$

We discuss first the solution of the equation of motions for a frozen time $\sigma\in\mathbb{R}$. As $\partial_s a_E(\sigma;q)=0$, the solution of the frozen equations generated by the Hamiltonian $H(\sigma)$ goes along the lines of the classical Landau problem [which means the case $\Phi=0$, $V(q)=0$].

For $\sigma\in\mathbb{R}$ define

1. the velocity field: $v(\sigma):\mathbb{P}\to\mathbb{R}^2$, $v(\sigma;q,p):=p-a(\sigma;q)$;
2. the center: $c(\sigma):\mathbb{P}\to\mathbb{R}^2$, $c(\sigma;q,p):=q-v^{\perp}(\sigma;q,p)$;
3. the angular momentum: $L:\mathbb{P}\to\mathbb{R}$, $L(q,p):=q\wedge p$.
   Denote the Poisson bracket: $\{f,g\}=\partial_q f\partial_p g-\partial_p f\partial_q g$.

We list some useful formulas.

Proposition 2.1: *The following identities hold as functions on phase space* $\mathbb{P}$ *for all* $\sigma\in\mathbb{R}$:

1. $\{v_1,v_2\}=1$, $\{c_1,c_2\}=-1$, $\{c,c^2/2\}=c^{\perp}$, $\{c_i,v_j\}=0$;
2. $H=\frac{1}{2}v^2$, $\{v,H\}=-v^{\perp}$, $\{c,H\}=0$;
3.

$$\frac{1}{2}c^2(\sigma) = H(\sigma) + L - q\wedge a_E(\sigma) = H + L + (N(\sigma) - \sigma q\wedge\partial_q\tilde{V}); \tag{2}$$

4. *the frozen flow* $(q(\sigma;s),p(\sigma;s))$ *defined by* $\partial_s q(\sigma;s)=\partial_p H(\sigma)$, $\partial_s p(\sigma;s)=-\partial_q H(\sigma)$,

$(q(\sigma;0),p(\sigma;0))=(q,p)$ *is*

$$q(\sigma;s)=c(\sigma)+\cos(s)v^{\perp}(\sigma)+\sin(s)v(\sigma)$$

$$p(\sigma;s)=\frac{1}{2}(c^{\perp}(\sigma)+\cos(s)v(\sigma)-\sin(s)v^{\perp}(\sigma))+a_E(\sigma;q(\sigma;s)).$$

Proof:

a. 1, 2, 3: $\{v_1,v_2\}=\{p_1-a_1(\sigma,q),p_2-a_2(\sigma,q)\}=\mathrm{rot}(a(\sigma))=1$, $\{q_i,v_j\}=\delta_{ij}$. $H=\frac{1}{2}v^2$ so $\{q,H\}=v,\{v,H\}=-v^{\perp}$. $c^2=q^2+v^2+2q\wedge v$; on the other hand, $L=q\wedge v+\frac{1}{2}q^2+q\wedge a_E(\sigma;q)$.

b. 4: The force is $-\dot{q}^{\perp}$ independently of σ; Newton's equation $\ddot{q}=-\dot{q}^{\perp}$ is readily verified. On the other hand, $p=v+a=a+c^{\perp}-q^{\perp}=c^{\perp}-\frac{1}{2}q^{\perp}+a_E(\sigma;q)$. So $p(s)$ follows from $q(s)$. □

Remark 2.1:

1. *Since the energy* $H(\sigma)=\frac{1}{2}v(\sigma)^2$ *is conserved under the frozen flow, the projections of the trajectories to* q *space are circles around* $c(\sigma)$ *with radius* $\sqrt{2H(\sigma)}$. *An orbit encircles the origin (has nontrivial homotopy) in* $\mathbb{R}^2\backslash(0)$ *if and only if*

$$c^2<2H\Leftrightarrow L-q\wedge a_E(\sigma;q)<0;$$

2. $\frac{1}{2}c^2=\frac{1}{2}(c^{\perp})^2$ *is the Hamiltonian for the reversed magnetic field.*

III. GENERAL FEATURES

Set $\varepsilon=1$ and denote by abuse of notation $O(s):=O(s;q(s),p(s))$ for an observable O. We have the following general qualitative behavior.

Proposition 3.1: *Suppose that there exists* $a\in[0,1)$ *such that for all* s,q:

$$|q\wedge\partial_q\widetilde{V}(q)|\leqslant\partial_sN(s)a,$$

then for any initial condition there exists a unique "hitting" time s_0 *such that*

$$\pm c^2(s)\geqslant\pm 2H(s),\quad \pm s>\pm s_0.$$

Furthermore,

$$\liminf_{N(s)\to-\infty}\frac{2H(s)}{|N(s)|}\geqslant(1-a)>0,$$

$$\liminf_{N(s)\to\infty}\frac{c^2(s)}{2N(s)}\geqslant(1-a)>0.$$

For radially symmetric potentials it holds

$$|q\wedge\partial_q\widetilde{V}(q)|=0\Rightarrow\frac{c^2(s)}{2}-H(s)=s-s_0. \tag{3}$$

Proof: By Eq. (2) and as $\{c^2,H\}=0$,

$$\frac{d}{ds}\left(\frac{c^2}{2}-H\right)=\partial_s\left(\frac{c^2}{2}-H\right)=(\partial_sN-q\wedge\partial_q\widetilde{V})\geqslant\partial_sN(1-a)\geqslant(1-a),$$

this gives the first claim. The second follows from positivity,

$$\frac{c^2(s)}{2} \geqslant H(s) + \left(\frac{c^2}{2} - H\right)(0) + (1-a)N(s) \geqslant \left(\frac{c^2}{2} - H\right)(0) + (1-a)N(s),$$

which implies

$$\liminf_{s\to\infty} \frac{c^2(s)}{2N(s)} = \liminf_{N(s)\to\infty} \frac{c^2(s)}{2N(s)} \geqslant (1-a).$$

Analogously,

$$\liminf_{s\to-\infty} \frac{H(s)}{-N(s)} = \liminf_{N(s)\to-\infty} \frac{H(s)}{|N(s)|} \geqslant (1-a).$$

IV. ACTION ANGLE COORDINATES

In order to discuss the dynamics we introduce action angle coordinates. The frozen dynamics as discussed in Proposition 2.1 suggests to take as coordinates the angles and absolute values of c and $v^\perp$, i.e., with the notation,

$$e(\theta) := (\cos\theta, \sin\theta),$$

$$q = c + v^\perp = |c|\frac{c}{|c|} + |v|\frac{v^\perp}{|v|} =: |c|e(\varphi_1) + |v|e(-\varphi_2),$$

$$p = \frac{1}{2}(c^\perp + v) + a_E(\sigma;q) = \frac{1}{2}(|c|e^\perp(\varphi_1) - |v|e^\perp(-\varphi_2)) + a_E(\sigma;q).$$

Motivated by this we define for $\sigma \in \mathbb{R}$,

$$q(\sigma;\varphi,I) := \sqrt{2I_1}e(\varphi_1) + \sqrt{2I_2}e(-\varphi_2) =: q(\varphi,I),$$

$$p(\sigma;\varphi,I) := \frac{1}{2}(\sqrt{2I_1}e^\perp(\varphi_1) - \sqrt{2I_2}e^\perp(-\varphi_2)) + a_E(\sigma;q(\sigma;\varphi,I)),$$

and denote by $\mathcal{C}$ the nullset $\{(\varphi,I);\varphi_1+\varphi_2=\pi, I_1=I_2\}$ where $q(\sigma;\varphi,I)=0$, by $\mathcal{D}$ the nullset $\{(q,p);v^2=0 \text{ or } c^2=0\}$. Thus for each frozen time $\sigma \in \mathbb{R}$ the transformation to action angle coordinates $T(\sigma)$ is defined by

$$T(\sigma): S^1 \times S^1 \times \{(I_1,I_2); I_1 \geqslant 0, I_2 \geqslant 0\} \setminus \mathcal{C} \to \mathbb{P} \setminus \mathcal{D},$$

$$T(\sigma;\varphi,I) = T(\sigma;\varphi_1,\varphi_2,I_1,I_2) := (q(\sigma;\varphi,I), p(\sigma;\varphi,I)).$$

We have the following.

Lemma 4.1:

1. *$T(\sigma)$ is a canonical diffeomorphism.*
2. *$T^{-1}(\sigma)$ is determined by*

$$I_1(\sigma) = \frac{c^2(\sigma)}{2} = \frac{1}{2}\left(p - \left(-\frac{1}{2}q^\perp + a_E(\sigma;q)\right)\right)^2,$$

$$I_2(\sigma) = H(\sigma) = \frac{1}{2}\left(p - \left(\frac{1}{2}q^\perp + a_E(\sigma;q)\right)\right)^2,$$

$$e(\varphi_1(\sigma)) = \frac{c}{|c|}(\sigma) = \frac{(1/2)q - p^\perp + a_E^\perp(\sigma;q)}{\sqrt{2(H(\sigma) + L - q \wedge a_E(\sigma;q))}},$$

$$e(-\varphi_2(\sigma)) = \frac{v^\perp}{|v|}(\sigma) = \frac{(1/2)q + p^\perp - a_E^\perp(\sigma;q)}{\sqrt{2H(\sigma)}}.$$

Proof: These identities follow immediately from Proposition 2.1:

$$\{I_1,I_2\} = 0, \quad \{e(\varphi_1),e(\varphi_2)\} = 0, \quad \{I_1,e(\varphi_2)\} = 0 = \{I_2,e(\varphi_1)\},$$

$$\{e(\varphi_1),I_1\} = \frac{1}{|c|}\left\{c,\frac{c^2}{2}\right\} = \frac{c^\perp}{|c|} = e^\perp(\varphi_1).$$

On the other hand, $\{e(\varphi_1),I_1\}=e^\perp(\varphi_1)\{\varphi_1,I_1\}$, so $\{\varphi_1,I_1\}=1$. Similarly, $\{\varphi_2,I_2\}=1$. □

We now write the equations of motion for time-dependent flux in these action angle coordinates. As $\mathrm{rot}(E)=0$ there exists a (possibly multivalued) function which we denote by $m=m(s;q)$ such that

$$\partial_q m(s) = E(s) = -\partial_s a_E(\varepsilon s).$$

Then $T(s)$ is generated by m:

$$\partial_s T(s;\varphi,I) = (0,\partial_s a_E(\varepsilon s,q(\varphi,I)) \; = (\partial_p m,-\partial_q m)\circ T(s;\varphi,I).$$

Denote by $U(s):\mathbb{P}\to\mathbb{P}$ the Hamiltonian flow of $H(\epsilon s)$ defined by $U(s) \coloneqq (q(s),p(s))$,

$$\dot{q}(s) = \partial_p H, \dot{p}(s) = -\partial_q H, \quad (q(0),p(0)) = (q,p),$$

then for the flow $\hat{U}(s)=(\varphi(s),I(s))$ in action angle coordinates defined by

$$T(s)\circ\hat{U}(s) = U(s)\circ T(s=0),$$

it holds

$$\dot{\varphi}(s) = \partial_I K\circ\hat{U}(s), \quad \dot{I}(s) = -\partial_\varphi K\circ\hat{U}(s), \quad (\varphi(0),I(0)) = (\varphi,I),$$

where the Hamiltonian in action angle coordinates, $K=H\circ T - m\circ T$, is

$$K(s;\varphi,I) = I_2 - m(s;q(\varphi,I))$$

and the equations of motion are (with the notation $\langle\cdot,\cdot\rangle$ for the scalar product)

$$\dot{\varphi}(s) = \partial_I K = \begin{pmatrix}0\\1\end{pmatrix} - \langle E(s,q(\varphi,I)),\partial_I q\rangle, \tag{4}$$

$$\dot{I}(s) = -\partial_\varphi K = \langle E(s;q(\varphi,I)),\partial_\varphi q\rangle. \tag{5}$$

Remark 4.1: *Another way to derive these equations is to start from Newton's equation*,

$$\ddot{q} = -\dot{q}^\perp + E(s;q).$$

From the very definition of c and v one gets

$$\dot{c} = -E^\perp(c+v^\perp), \quad \dot{v} = -v^\perp + E(c+v^\perp),$$

which in action angle coordinates gives Eqs. (4) and (5).

V. LARGE TIME ASYMPTOTICS, POTENTIAL FREE CASE

For the case $N(s)=s$, $V=0$ we can precise the large time asymptotics and develop the solution up to order $\mathcal{O}(1/s^{3/2})$ We have

$$E(s)=\frac{q^{\perp}}{q^2}.$$

So $m(q)=\arg(q)$. Observe that

$$K=K(\varphi,I)=I_2-\arg(\sqrt{2I_1}e(\varphi_1)+\sqrt{2I_2}e(-\varphi_2))$$

is an integral of motion.

Theorem 5.1: *Let $V=0$, $N(s)=s$. Denote by $I=(I_1,I_2)$, $\varphi=(\varphi_1,\varphi_2)$ the solution of the equations of motion (4) and (5),*

$$\dot{\varphi}(s)=\partial_I K(\varphi(s),I(s)),\quad I(0)=(I_1^0,I_2^0),$$

$$\dot{I}(s)=-\partial_{\varphi}K(\varphi(s),I(s)),\quad \varphi(0)=(\varphi_1^0,\varphi_2^0),$$

then the following asymptotic behaviors hold.

a. *In the future, $s\to\infty$.*
The following limits exist and define the constants $a_0>0$, b_0:

$$\lim_{s\to\infty} I_2(s)=:\frac{a_0^2}{4},\quad \lim_{s\to\infty}(\varphi_1(s)+\varphi_2(s)-s)=:b_0,\quad \lim_{s\to\infty}(I_2(s)-\varphi_1(s))=K.$$

The asymptotics are

$$I_2(s)=\frac{a_0^2}{4}-\frac{a_0}{2}\sin(s+b_0)\frac{1}{\sqrt{s}}+\frac{1}{4}\left(1+\frac{a_0^2}{2}\sin(2(s+b_0))\right)\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^{3/2}}\right),$$

$$I_1(s)=I_2(s)+(s-s_0),$$

$$\varphi_1(s)=\frac{a_0^2}{4}-K-\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^{3/2}}\right),$$

$$\varphi_2(s)=s+b_0-\frac{a_0^2}{4}+K-\frac{1}{a_0}\cos(s+b_0)\frac{1}{\sqrt{s}}+\frac{1}{8}\left(-1+2\cos(2(s+b_0))-\frac{4}{a_0^2}\sin(2(s+b_0))\right)\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^{3/2}}\right),$$

with s_0 defined as in Eq. (3).

b. *In the past, $s\to-\infty$.*
The following limits exist and define the constants $\tilde{a}_0>0$, $\tilde{b}_0$:

$$\lim_{s\to-\infty} I_1(s)=:\frac{\tilde{a}_0^2}{4},\quad \lim_{s\to-\infty}(\varphi_1(s)+\varphi_2(s)-s)=:\tilde{b}_0,\quad \lim_{s\to-\infty}(I_2(s)+\varphi_2(s))=K.$$

The asymptotics are

$$I_1(s)=\frac{\tilde{a}_0^2}{4}+\frac{\tilde{a}_0}{2}\sin(s+\tilde{b}_0)\frac{1}{\sqrt{|s|}}-\frac{1}{4}\left(1-\frac{\tilde{a}_0^2}{2}\sin(2(s+\tilde{b}_0))\right)\frac{1}{s}+\mathcal{O}\left(\frac{1}{|s|^{3/2}}\right),$$

$$I_2(s) = I_1(s) - (s - s_0),$$

$$\varphi_1(s) = s_0 + \tilde{b}_0 + \frac{\tilde{a}_0^2}{4} - K + \frac{1}{\tilde{a}_0}\cos(s + \tilde{b}_0)\frac{1}{\sqrt{|s|}},$$

$$-\frac{1}{8}\left(1 - 2\cos(2(s + \tilde{b}_0)) - \frac{4}{\tilde{a}_0^2}\sin(2(s + \tilde{b}_0))\right)\frac{1}{s} + \mathcal{O}\left(\frac{1}{|s|^{3/2}}\right),$$

$$\varphi_2(s) = s - s_0 - \frac{\tilde{a}_0^2}{4} + K - \frac{1}{4s} + \mathcal{O}\left(\frac{1}{|s|^{3/2}}\right).$$

Proof: We give an outline of the main steps of the proof for the case $t \to \infty$. Some particular computations in the proof turned out to be quite tedious and thus computer algebra systems were employed to facilitate them.

Suppose $t > 0$.

Step 1. From Eq. (2) we know $I_1(s) - I_2(s) = (s - s_0)$. So the equations of motion only involve $J := I_1 + I_2$ and $\psi := \varphi_1 + \varphi_2$ and transform to

$$\dot{\psi} = 1 + \frac{s \sin \psi}{\sqrt{J^2 - s^2}(J + \sqrt{J^2 - s^2}\cos \psi)}, \quad \dot{J} = \frac{s}{J + \sqrt{J^2 - s^2}\cos \psi},$$

Step 2. Do a second transformation,

$$x_1 := \sqrt{J^2 - s^2}\cos \psi, \quad x_2 := 1 + \sqrt{J^2 - s^2}\sin \psi,$$

the J, ψ equations transform to

$$\dot{x}_1 - \frac{x_1}{s} + x_2 = F(s, x_1, x_2), \quad \dot{x}_2 - x_1 = 0,$$

with

$$F(s, x_1, x_2) := 1 - \frac{x_1}{s} - \frac{s}{\sqrt{x_1^2 + (x_2 - 1)^2 + s^2} + x_1}.$$

The corresponding homogeneous system is equivalent to

$$\ddot{x}_1 - \frac{\dot{x}_1}{s} + \left(1 + \frac{1}{s^2}\right)x_1 = 0 \quad \text{or } s\ddot{y} + \dot{y} + \text{sy} = 0,$$

with y defined by $x_1 = sy$. The latter is Bessel's equation of order 0 so one has two independent solutions of the homogeneous system:

$$\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} sJ_0(s) \\ sJ_1(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} sY_0(s) \\ sY_1(s) \end{pmatrix}$$

with the Bessel functions J_m (Y_m) of the first (second) kind.

Step 3. Transform the x-differential equation to the integral equation,

$$x_1(s) = c_1 sJ_0(s) + c_2 sY_0(s) - \frac{\pi s}{2}\int_s^\infty (Y_0(s)J_1(\tau) - J_0(s)Y_1(\tau))F(\tau, x_1(\tau), x_2(\tau))d\tau,$$

$$x_2(s) = c_1 s J_1(s) + c_2 s Y_1(s) - \frac{\pi s}{2}\int_s^\infty (Y_1(s)J_1(\tau) - J_1(s)Y_1(\tau))F(\tau, x_1(\tau), x_2(\tau))d\tau,$$

where the numbers c_1, c_2 involve the initial conditions.

The equation is of the form $x=\mathcal{K}(x)$; the solution is constructed as the limit of the sequence $x_{n+1}=\mathcal{K}(x_n)$ starting from $x_0=0$. To verify the convergence one can apply yet another substitution $x(s)=y(s)/\sqrt{s}$, $G(s,y)=s^{-1/2}F(s,s^{-1/2}y)$. Consequently, the integral equation takes the form

$$y(s) = y_0(s) - \int_s^\infty \mathcal{F}(s,\tau)G(\tau, y_1(\tau), y_2(\tau))d\tau,$$

where

$$y_{0j}(s) = c_1\sqrt{s}J_{j-1}(s) + c_2\sqrt{s}Y_{j-1}(s), \quad j = 1,2,$$

$$\mathcal{F}_j(s,\tau) = \frac{\pi}{2}\sqrt{s\tau}(Y_{j-1}(s)J_1(\tau) - J_{j-1}(s)Y_1(\tau)), \quad j = 1,2.$$

Considering the new integral equation in the Banach space $L^\infty([s_*,\infty[)\otimes\mathbb{R}^2$, one can show that the iteration process is indeed contracting provided $s_*\geqslant 1$ is sufficiently large. It is then straightforward to derive from the integral equation the asymptotic expansion of the solution $x(s)$. One finds that

$$x(s) = a_0 e(s+b_0)\sqrt{s} + \left(\frac{a_0^3}{8}e(s+b_0) - \frac{5}{8}a_0 e^\perp(t+b_0)\right)\frac{1}{\sqrt{s}} + \mathcal{O}\left(\frac{1}{s}\right).$$

Step 4. Transforming back first to the J, ψ then to I_1, I_2, φ_1, φ_2 variables gives the claimed asymptotic expansion. □

VI. AVERAGED DYNAMICS

In the perspective to investigate the behavior at the transition point between the two dynamics in the case of small epsilon we analyze the equation averaged over the fast angle φ_2. This might provide a good approximation for the actions for small ε for times of order $1/\varepsilon$.[14] We consider again the case $N(s)=s$. We set up the averaged equations and show that *in the adiabatic limit* the energy grows if and only $I_1 < I_2$.

We then solve the equations explicitly for the case $V=0$ and show that in the adiabatic limit the transition between the pinned and the "Hall dynamics" happens for a unique value of the driving flux and that the center moves if and only if the Landau orbit does not encircle the origin.

We apply averaging with respect to the fast angle φ_2 to the system (4) and (5).

$$E(s) = \varepsilon\left(\frac{q^\perp}{q^2} - \partial_q \tilde{V}(q)\right).$$

Further, choose m and thus K,

$$m(q) = \varepsilon(\arg(q) - \tilde{V}),$$

$$K(\varphi, I) = I_2 - m(\sqrt{2I_1}e(\varphi_1) + \sqrt{2I_2}e(-\varphi_2)).$$

Denote the average of a function f on the phase space by

$$f_{\rm av}(\varphi_1,I) := \frac{1}{2\pi}\int_0^{2\pi} f(\varphi_1,\varphi_2,I)d\varphi_2.$$

In particular, for a function f defined on the plane thus depending only on the variable q we denote

$$f_{\rm av}(\varphi_1,I) = \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{2I_1}e(\varphi_1)+\sqrt{2I_2}e(-\varphi_2))d\varphi_2.$$

Making use of the identities

$$\left\langle \frac{q^\perp}{q^2},\partial_I q\right\rangle = \frac{\sin(\varphi_1+\varphi_2)}{q^2}\begin{pmatrix}\sqrt{I_2/I_1}\\ -\sqrt{I_1/I_2}\end{pmatrix},\quad \left\langle \frac{q^\perp}{q^2},\partial_\varphi q\right\rangle = \begin{pmatrix}[(I_1-I_2)/q^2]+1/2\\ [(I_1-I_2)/q^2]-1/2\end{pmatrix},$$

the system (4) and (5) reads

$$\dot\varphi(s) = \begin{pmatrix}0\\1\end{pmatrix} - \varepsilon\frac{\sin(\varphi_1+\varphi_2)}{2(I_1+I_2+2\sqrt{I_1I_2}\cos(\varphi_1+\varphi_2))}\begin{pmatrix}\sqrt{I_2/I_1}\\ -\sqrt{I_1/I_2}\end{pmatrix} + \varepsilon\partial_I\tilde V(q(I,\varphi)),$$

$$\dot I(s) = \varepsilon\frac{I_1-I_2}{2(I_1+I_2+2\sqrt{I_1I_2}\cos(\varphi_1+\varphi_2))}\begin{pmatrix}1\\1\end{pmatrix} + \frac{\varepsilon}{2}\begin{pmatrix}1\\-1\end{pmatrix} - \varepsilon\partial_\varphi\tilde V(q(I,\varphi)).$$

The averaged quantities are readily calculated: using

$$\left(\frac{1}{q^2}\right)_{\rm av} = \frac{1}{2|I_1-I_2|},\quad \left(\frac{\sin(\varphi_1+\varphi_2)}{q^2}\right)_{\rm av} = 0,$$

one finds for the averaged vector field

$$(\partial_I K)_{\rm av}(\varphi_1,I) = \begin{pmatrix}0\\1\end{pmatrix} + \varepsilon\partial_I\tilde V_{\rm av}(\varphi_1,I) - (\partial_\varphi K)_{\rm av}(\varphi_1,I) = \varepsilon\begin{pmatrix}\chi(I_1>I_2)\\ -\chi(I_1<I_2)\end{pmatrix} - \varepsilon\begin{pmatrix}\partial_{\varphi_1}\tilde V_{\rm av}(\varphi_1,I)\\ 0\end{pmatrix}, \tag{6}$$

where we used the binary function χ: $\chi(\mathrm{True}) := 1$, $\chi(\mathrm{False}) := 0$.

Remark 6.1: *Remark that the averaged vector field is the Hamiltonian vector field derived from the from the "averaged" Hamiltonian* $K_{\rm av}$. *Indeed, using the splitting of* $\arg(q)$, *which is a multivalued function defined on the covering space of* $\mathbb{R}^2\backslash(0)$, *into a linear and oscillating part*,

$$\arg(q(\varphi,I)) = \begin{cases}\varphi_1 + \arg\left((1,0)+\sqrt{\dfrac{I_2}{I_1}}e(-\varphi_1-\varphi_2)\right) & \textit{if } I_1>I_2\\[2ex] -\varphi_2 + \arg\left((1,0)+\sqrt{\dfrac{I_1}{I_2}}e(\varphi_1+\varphi_2)\right) & \textit{if } I_2>I_1,\end{cases}$$

and the equality

$$\int_0^{2\pi}\arg((1,0)+ae(s))\mathrm{d}s = 0 \quad \text{for } 0\le a<1,$$

one finds that for

$$K_{\rm av}(\varphi,I) := I_2 - \varepsilon((\varphi_1\chi(I_1>I_2) - \varphi_2\chi(I_1<I_2)) - \tilde V_{\rm av}(\varphi_1,I)),$$

it holds $\partial_\varphi K_{\rm av} = (\partial_\varphi K)_{\rm av}$, $\partial_I K_{\rm av} = (\partial_I K)_{\rm av}$.

The result on the averaged dynamics now is as follows.

Theorem 6.1: *Let $N(s)=s$. Denote by $J=(J_1,J_2)$, $\psi=(\psi_1,\psi_2)$ the solution of the averaged equations (6)*

$$\dot{\psi}(s)=\partial_I K_{\mathrm{av}}(\psi(s),J(s)),\quad J(0)=(J_1^0,J_2^0),$$

$$\dot{J}(s)=-\partial_\varphi K_{\mathrm{av}}(\psi(s),J(s)),\quad \psi(0)=(\psi_1^0,\psi_2^0),$$

then it holds the following.

1. *Let $V=0$, denote $\Delta J=J_2^0-J_1^0$ then*

$$J(s)=\min\{J_1^0,J_2^0\}+(\varepsilon s-\Delta J)\begin{pmatrix}\chi(\varepsilon s>\Delta J)\\ -\chi(\varepsilon s<\Delta J)\end{pmatrix},$$

$$\psi(s)=\begin{pmatrix}\psi_1^0\\ \psi_2^0+s\end{pmatrix}.$$

2. *For any V and any $s_1,s_2\in R$,*

$$|J_2(s_2)-J_2(s_1)|=\varepsilon\left|\int_{s_1}^{s_2}\chi(J_1(u)<J_2(u))\mathrm{d}u\right|.$$

Proof: Using that for $V=0$ it holds $J_1(s)-J_2(s)-\varepsilon s=\Delta J$ the first assertion follows by inspection. The second assertion follows from integration of Eq. (6). □

Remark 6.1:

1. *Loosely speaking the second assertion of the theorem means that, on the average, one has*

   $$|\textit{energy change}|=|\textit{flux change through the orbit during stay time}|,$$

   where the stay time means the time where the "orbit surrounds the origin." This should be like this as the change in energy equals the work of the electric field along the orbit:

   $$H(s;q(s))-H(s_0;q(s_0))=\int_{s_0}^{s}\langle a_E(s),ds\rangle.$$

2. *In situations where the averaging approximation is valid one gets estimates of the type $|I(s)-J(s)|=\mathcal{O}(\varepsilon)$ for $|s|\leqslant\mathcal{O}(1/\varepsilon)$. Because of the singular behavior of the averaged equation it is a challenging problem to investigate if such an estimate is true or not and how the error would depend on the initial conditions.*
3. *In the case when a smooth potential is present in view of the second assertion of the above theorem, it would be interesting to investigate if the kinetic energy only fluctuates by small amounts as soon as the particle is in the region $c^2/2>H$.*
4. *Remark that a finite time adiabatic analysis would apply also to the case where Φ is a switching function which is linear for some time.*

VII. SCALING

Let T, $\mathcal{L}$, B, $[\Phi]$ be units of time, length, magnetic field, and flux. Define dimensionless parameters $T\omega=:\varepsilon^{-1}$; $[\Phi]/(2\pi B\mathcal{L}^2)=:\eta$, denote $N(s):=\Phi(sT)/[\Phi]$ and choose $\mathcal{L}$ such that $\eta=1$ then

$$\mathcal{H}(t;q,p)=\frac{\omega e}{2\pi}[\Phi]H(t/T,q/\mathcal{L},p/(e\mathcal{L}B)),$$

where

$$H(s;q,p) = \frac{1}{2}(p - a(s,q))^2,$$

with

$$a(s,q) := \frac{1}{2}q^{\perp} + \underbrace{\left(-N(s)\frac{q^{\perp}}{q^2} + s(\partial_q \widetilde{V})(q)\right)}_{=:a_E(s,q)}$$

and

$$\widetilde{V}(q) := \frac{2\pi T}{e[\Phi]}V(\mathcal{L}q).$$

The scaled function $(q_{\mathrm{sc}}(s), p_{\mathrm{sc}}(s)) := (q/\mathcal{L}(s/\omega), p/(eB\mathcal{L})(s/\omega))$ then solves the Hamilton equations for the Hamiltonian $H(\varepsilon s)$.

Corollary 7.1: *(To Proposition 7.1). Suppose that the torque of the background field is smaller than the circular one, i.e., that there exists $a \in [0,1)$ such that for all t, q:*

$$e|q \wedge \partial_q V(q)| \leqslant e\frac{\partial_t \Phi(t)}{2\pi}a,$$

then for any initial condition there exists a unique hitting time t_0 such that

$$\pm \pi B c^2(t) \ \geqslant\ \pm \frac{2\pi}{e\omega}\mathcal{H}(t,B), \quad \pm t > \pm t_0.$$

Furthermore for any initial condition we have

$$\liminf_{\Phi(t)\to -\infty} \frac{2\pi}{e\omega}\frac{\mathcal{H}(t,B)}{|\Phi(t)|} \ \geqslant\ (1-a) > 0,$$

$$\liminf_{\Phi(t)\to \infty} \frac{\pi B c^2(t)}{\Phi(t)} \ \geqslant\ (1-a) > 0.$$

□

We have

$$\mathcal{H}(t,B) = \frac{e\omega}{2\pi}[\Phi]I_2(\varepsilon\omega t), \quad \mathcal{H}(t,-B) = \frac{e\omega}{2\pi}[\Phi]I_1(\varepsilon\omega t),$$

$$q(t) = Lq_{\mathrm{sc}}(\omega t), \quad q_{\mathrm{sc}} = \sqrt{2I_1}e(\varphi_1) + \sqrt{2I_2}e(-\varphi_2).$$

Corollary 7.2: *(To Theorem 5.1). Let $\phi(t)=[\Phi]t/T$, $V=0$. The following limits are valid for any fixed initial condition:*

$$\frac{\pi B q^2(t)}{|\Phi(t)|} \to_{|\Phi(t)|\to\infty} 1,$$

$$\frac{\mathcal{H}(t,B)}{|\Phi(t)|} \to_{\Phi(t)\to-\infty} \frac{e\omega}{2\pi} \leftarrow_{\Phi(t)\to\infty} \frac{\mathcal{H}(t,-B)}{\Phi(t)},$$

$$\mathcal{H}(t,B) \to_{\Phi(t)\to\infty} \frac{\omega a_0^2}{4},$$

$$\mathcal{H}(t,-B)\to_{\Phi(t)\to-\infty}\frac{\omega\tilde{a}_0^2}{4},$$

$$\frac{\pi B}{\sqrt{|\Phi(t)|}}q(t)\to_{t\to\infty}e\left(\frac{a_0^2}{4}-K\right),$$

$$\sqrt{\frac{\pi B}{|\Phi(t)|}}q(t)\sim_{t\to-\infty}e(-\omega t).$$

Remark also that for the rescaled center it holds

$$\mathcal{H}(t,-B)=\frac{1}{2m\omega^2}c^2$$

so

$$\pi Bc^2(t)\sim_{t\to\infty}\Phi(t).$$

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