

ON THE DYNAMICS CREATED BY A TIME-DEPENDENT AHARONOV–BOHM FLUX

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We study the dynamics of classical and quantum particles moving in a punctured plane under the influence of a homogeneous magnetic field and driven by a time-dependent singular flux tube through the hole.

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1. Introduction

The model under consideration was introduced by physicists in order to understand the integer quantum Hall effect and much investigated by mathematical physicists who introduced topological indices in order to explain the quantization of charge transport observed in the experiments; consult [5] for an access to the literature.

Let $m > 0$, $e > 0$, $\hbar > 0$ be physical parameters, $q \in \mathbb{R}^2 \setminus \{0\}$, $q^\perp := (-q_2, q_1)$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function. The time-dependent Hamiltonian is

$$\frac{1}{2m} (p - eA(t, q))^2, \quad A(t, q) = \left(\frac{B}{2} - \frac{\Phi(t)}{2\pi|q|^2} \right) q^\perp,$$

where in the classical case $p \in \mathbb{R}^2$ and the Hamiltonian is a function on the phase space and where in the quantum case $p = (-i\hbar\partial_x, -i\hbar\partial_y)$ and the Hamiltonian is the Friedrichs extension of $(1/2m)(p - eA(t, q))^2$ defined on $C_0^\infty \setminus \{0\}$.

In the quantum case we discuss the meaning of the propagator and show that an adiabatic approximation is valid. To this end we introduce the notion of a propagator

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weakly associated to a time-dependent Hamiltonian. A detailed presentation is given in [2].

For the classical case we show: in the past the center is bound and the particles spiral inward towards the flux line, their motion being accompanied by energy loss; after hitting the puncture they become “conducting”, i.e. the motion becomes a cycloid around an outward drifting center orthogonal to the induced electric field. The outgoing drift is without energy loss.

The latter results have not been published yet but can be found in preprint [3]. Finally let us note that the dynamics of the classical system without magnetic field was discussed in [1].

2. The quantum case

2.1. Existence and adiabatic approximation

We discuss the case $\partial_t \Phi = \text{const}$. After rescaling the physical parameters and restricting ourselves to a sector of fixed angular momentum we consider in $L^2((0, \infty), r dr)$ the operator

$$H(s) = -\frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \left(s + \frac{r^2}{2} \right)^2.$$

which is essentially self-adjoint on $C_0^\infty(0, \infty)$ iff $|s| \geq 1$, and defined by the regular boundary condition at $r \rightarrow 0$ for $|s| < 1$. We study the “adiabatic” limit ($\varepsilon \rightarrow 0$) of the evolution equation

$$i \varepsilon \partial_s U(s, s_0) \psi = H(s) U(s, s_0) \psi$$

for the propagator U . Now, $\text{Dom}(H(s))$ is time-dependent and so the existence of a unique solution of the evolution equation is not assured (cf. [6]); on the other hand $\partial_s H(s)$ is not relatively bounded and the gaps between the eigenvalues, $E_{n+1}(s) - E_n(s)$, are approximately constant in n and thus the known theorems (cf. [4]) do not assure the validity of the adiabatic approximation.

Our solution to these problems is the following: we use the explicit knowledge of the spectral measure of $H(s)$ to show the existence of an “adiabatic” propagator U_{ad} . U_{ad} in turn is used to define a unique propagator U_w weakly related to $H(s)$. Then we show that U_{ad} is an approximation of U_w (see Section 2.2 for the weak relationship).

The spectrum of $H(s)$ is discrete. Denote by $E_n(s)$, $\psi_n(s)$, $P_n(s)$ the eigenvalues, eigenfunctions (chosen real) and eigenprojections respectively; let $\mathcal{P}(s) := i \sum_{\mathbb{N}} (\partial_s P_n) P_n(s)$. Define $H_{\text{ad}}(s)$ and its propagator U_{ad} by

$$H_{\text{ad}}(s) := H(s) + \varepsilon \mathcal{P}(s) \quad \text{and} \quad U_{\text{ad}}(s) \psi_n(0) := \exp\left(-\frac{i}{\varepsilon} \int_0^s E_n\right) \psi_n(s).$$

THEOREM 1. For $s \geq 0$,

(1) $\|\mathcal{P}(s)\| \leq M(s)$ where $M(s)$ is a positive increasing function on \mathbb{R}_+ ,

- (2) $\exists \Gamma(s)$ differentiable such that $\mathcal{P} = i[H, \Gamma]$ and $\|\Gamma(s)\| + \|\partial_s \Gamma(s)\| \leq \text{const}$,
 (3)

$$\left\| \int_0^s U_{\text{ad}}^{-1} \mathcal{P} U_{\text{ad}} \right\| \leq \text{const } \varepsilon s,$$

- (4) For $C(s)$ defined by $i \partial_s C(s) = -(U_{\text{ad}}^{-1} \mathcal{P} U_{\text{ad}})(s)C(s)$, $C(0) = \text{id}$, it holds
 $\|C(s) - \text{id}\| \leq \text{const } \varepsilon M(s) \exp(sM(s)).$

Comments on the proof. The main problem is to control the operator bound on $l^2(\mathbb{N})$ of the matrix

$$\langle \psi_m, \dot{\psi}_n \rangle \sim \frac{\langle \psi_m, \dot{H} \psi_n \rangle}{E_n - E_m} \sim \frac{1}{n - m} \left(\frac{m + 1}{n + 1} \right)^{s/2}.$$

This is done in a number of steps. As an illustration, the first step is to find a bound on $L^2((0, \infty), dx)$ for the self-adjoint integral operator with the kernel

$$K(x, y) = -\frac{i}{y} \left(\frac{x}{y} \right)^s \quad \text{for } x < y, \quad K(x, y) = \frac{i}{x} \left(\frac{y}{x} \right)^s \quad \text{for } x > y.$$

The bound reads $\|K\| \leq (s + \frac{1}{2})^{-1}$. But more steps of similar nature are needed to complete the proof. □

REMARKS 1.

1. $\text{Dom}(H_{\text{ad}}(s)) = \text{Dom}(H(s))$, $U_{\text{ad}}(s)(\text{Dom}(H_{\text{ad}}(0))) = \text{Dom}(H_{\text{ad}}(s))$,
 $i \varepsilon \partial_s U_{\text{ad}}(s) \psi = H_{\text{ad}}(s) U_{\text{ad}}(s) \psi, \quad \forall \psi \in \text{Dom}(H(0)).$
2. $C(s)$ is well defined by the Dyson formula.
3. $U_w(s) := U_{\text{ad}}(s)C(s)$ is a propagator and the candidate to be generated by $H(s)$. Further it holds

$$\|U_w(s) - U_{\text{ad}}(s)\| \leq \varepsilon M(s) \exp(sM(s)).$$

4. It is an open question whether $C(s)$ preserves $\text{Dom}(H(0))$ and thus whether $U(s) \text{Dom}(H(0)) \subset \text{Dom}(H(s))$.

2.2. Weakly associated propagator

While we cannot show that the propagator U_w is the propagator of $H(s)$ we can show that it is the unique propagator weakly associated to $\{H(s)\}$; so if the propagator for $H(s)$ exists, it equals U_w .

The definition of weak association relies heavily on the notion of the quasi-energy operator which is directly related to the propagator: $K = \mathfrak{U}(-i \partial_s) \mathfrak{U}^*$, where $\mathfrak{U} = \int^{\oplus} U_w(s, 0) ds$. We say that a propagator U_w is weakly associated to $H(s)$ iff

$$K = \overline{-i \partial_s + \mathfrak{H}} \quad \text{where } \mathfrak{H} = \int_{\mathbb{R}}^{\oplus} H(s) ds.$$

One can actually prove that in this way introduced notion of weak association generalizes the standard relationship between a propagator and a Hamiltonian as well as that at most one propagator can be weakly associated to a Hamiltonian. For details see [2].

3. The classical case

We again discuss the linear case $\Phi(t) = \Phi_0 t$. After a rescaling one is lead to consider the Hamiltonian flow of

$$H(s) = \frac{1}{2} (p - a(q))^2, \quad \text{with } a(q) := \left(\frac{1}{2} - \phi \frac{t}{q^2} \right) q^\perp,$$

for $\phi := e\Phi_0/(2\pi\omega)$, $\omega := eB/m$. Because of the cycloid-type nature of the trajectories $q(t)$ around a moving center c we use the natural splitting $q = c + v^\perp$ where $v := p - a(q)$, $c := q - v^\perp$. Let us denote

$$\mathbf{e}(\varphi) := (\cos \varphi, \sin \varphi).$$

An appropriate canonical coordinate system is then defined so that

$$q = |c|\mathbf{e}(\varphi_1) + |v|\mathbf{e}(-\varphi_2).$$

The action–angle coordinates read $I_1 = |c|^2/2$, $I_2 = H$, φ_1 , φ_2 ; the transformed Hamiltonian is an integral of motion

$$K(\varphi, I) = I_2 - \phi \arg(\sqrt{2I_1} \mathbf{e}(\varphi_1) + \sqrt{2I_2} \mathbf{e}(-\varphi_2)).$$

The fundamental relation between the center c and the energy is

$$\frac{|c(s)|^2}{2} = H(s) + \phi(s - s_0),$$

where s_0 is a constant depending on the trajectory. The asymptotic behaviour described below is illustrated by Fig. 1 depicting a typical trajectory.

THEOREM 2. *For any fixed initial condition there exists a constant $a_0 > 0$ such that*

$$\begin{aligned} \frac{q(s)}{\sqrt{s}} &\xrightarrow{s \rightarrow \infty} \sqrt{2\phi} \mathbf{e} \left(\frac{a_0^2}{4\phi^2} - \frac{K}{\phi} \right), & \frac{q(s)}{\sqrt{|s|}} &\underset{s \rightarrow -\infty}{\sim} \sqrt{2\phi} \mathbf{e}(-s), \\ H(s) &\xrightarrow{s \rightarrow \infty} \frac{a_0^2}{4\phi}, & \frac{H(s)}{|s|} &\xrightarrow{s \rightarrow -\infty} \phi. \end{aligned}$$

Proof: The problem can be reduced to a two-dimensional system with coordinates $J := I_1 + I_2$ and $\psi := \varphi_1 + \varphi_2$. After a change of variables one arrives at a system

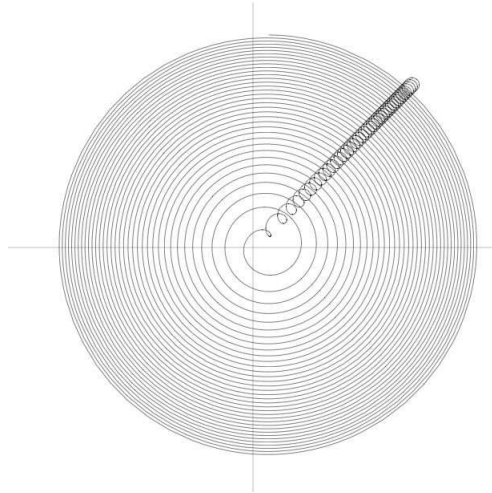


Fig. 1. A typical trajectory of the Hamiltonian $H(s)$.

of differential equations which is equivalent to the integral equations

$$x_j(s) = c_1 s J_{j-1}(s) + c_2 s Y_{j-1}(s) - \frac{\pi s}{2} \int_s^\infty (Y_{j-1}(s) J_1(\tau) - J_{j-1}(s) Y_1(\tau)) F(\tau, x_1(\tau), x_2(\tau)) d\tau,$$

$j = 1, 2$, where the numbers c_1, c_2 involve initial conditions and

$$F(s, x_1, x_2) := \phi - \frac{x_1}{s} - \frac{\phi^2 s}{\sqrt{x_1^2 + (x_2 - \phi)^2 + \phi^2 s^2 + x_1}}.$$

The integral equations allow for iterative solution and are well suited for asymptotic analysis. □

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