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Random inelasticity and velocity fluctuations in a driven granular gas

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Abstract. We analyze the deviations from Maxwell-Boltzmann statistics found in recent experiments studying velocity distributions in two-dimensional granular gases driven into a non-equilibrium stationary state by a strong vertical vibration. We show that in its simplest version, the "stochastic thermostat" model of heated inelastic hard spheres, contrary to what has been hitherto stated, is incompatible with the experimental data, although predicting a reminiscent high-velocity stretched-exponential behavior with an exponent 3/2. The experimental observations lead to refine a recently proposed random restitution coefficient model. Very good agreement is then found with experimental velocity distributions within this framework, which appears self-consistent and further provides relevant probes to investigate the universality of the velocity statistics.

PACS. 45.70.-n Granular systems – 05.20.Gg Classical ensemble theory – 51.10.+y Kinetic and transport theory of gases

1 Introduction

Whereas equilibrium statistical mechanics has reached a rather mature phase, the understanding of non-equilibrium processes is far from complete. In particular, granular (and thus inelastic) gases [1] driven into a non-equilibrium steady state by a suitable injection of energy define a stimulating research field where theoretical predictions can be confronted against model experiments, with the aim to understand the possible deviations from equilibrium behavior. A good probe to quantify these deviations is the velocity distribution of the grains, $P(v)$, which has focused sustained attention recently, and has been shown to exhibit pronounced differences from Maxwell-Boltzmann statistics [2–6]. Several authors reported a stretched-exponential law (on the whole range of velocities available, which covers an accuracy of 5 to 6 orders of magnitude for $P(v)$)

$$P(v) \propto \exp[-(v/v_0)^\nu] \,, \tag{1}$$

with an exponent ν close to 3/2 [3,4,6] (here v_0 is the "thermal" r.m.s. velocity). This behavior was observed for the horizontal velocity components of a vertically vibrated 2D system of steel beads in a wide range of driving frequencies and densities [4], but also in a three-dimensional electrostatically driven granular gas [6].

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At this point, some questions naturally arise, that will be addressed below: i) Is it possible to find a consistent model where this velocity distribution would emerge? ii) What physical ingredients are required?

One possible approach consists in performing "realistic" molecular-dynamics simulations. The model of inelastic hard spheres (IHS) with binary momentum-conserving collisions, and a "reasonable" restitution coefficient [1] provides the simplest candidate. The energy loss in a collision is proportional to the inelasticity parameter $1-\alpha_0^2$, where α_0 is the coefficient of normal restitution ($0 < \alpha_0 \le 1$), which in the simplest and efficient approximation is a constant independent of the relative velocity of colliding partners. Such an approach has been presented in [7,8], and allows to reproduce the experimental velocity statistics with a good accuracy. The possible lack of universality has also been addressed in [8].

Another route, which contrary to the previous numerical one has the merit to allow an analytical derivation of ν in some cases [9], consists in formulating an *effective* modeling of the energy injection, considering idealized homogeneous systems of inelastic hard spheres (given the experiments reported in [4], the assumption of homogeneity is well founded, see below). From this point of view, a simple and popular model consists in IHS with constant inelasticity, with a *homogeneous* forcing described by a "stochastic thermostat" [10,11,9,12–16]. This model has attracted attention, in particular because it has been shown analytically [9] that $P(v)$ exhibits a high-energy tail of the form of equation (1) with $\nu = 3/2$, independent of dimension and restitution coefficient, in apparent agreement with the experiments. This result holds at the mean-field level for a homogeneous system. The above model, where an external white-noise driving force acts on the particles and thus injects energy through random "kicks" between the collisions, is therefore considered to provide a relevant theoretical framework to quantify the non-Gaussian character of velocity distributions.

However, we shall see below that this uniformly heated model —in its simplest version— is unable to reproduce the experimental data: if simulated in dimension 2 or higher with an experimentally relevant value of the restitution coefficient (α_0 between 0.7 and 1), the obtained distribution $P(v)$ is *indistinguishable from a Gaussian within the experimental accuracy*[1]; in fact, the range of velocities for which the high-energy behavior $\exp[-(v/v_0)^{3/2}]$ may be observed is even beyond reach of precise numerical procedures and corresponds to a regime where $P(v)$ is practically vanishing (lower than $10^{-6}P(0)$). The predictions of this model are consequently incompatible with the experimental velocity distributions, that show important non-Gaussian features already at thermal velocity scale [3, 4,6]. We emphasize, however, that generalizations of the aforementioned heated model have been proposed [17,18]. With a convenient choice of the extra parameters introduced, one may obtain a velocity distribution close to that measured in the experiments (we will come back to this point in Sect. 4). However, in such approaches, energy is injected in between the collisions whereas in the experiments we are interested in, the transfer of horizontal momentum takes place at every inter-particle collision (see Sect. 3). We will therefore focus on this feature for the 2D experiment reported in [4] and investigate in detail the collision dynamics in the horizontal direction (Sect. 2). The vibrated system under study there shows important density and granular temperature gradients, especially close to the boundaries which inject energy, but since the shaking is violent, there is a region where both gradients are very small simultaneously. The velocity acquisition in [4] has been restricted to this region, where the system, although open, may be considered as homogeneous. In this article, we thus consider the following question: remaining at the level of a homogeneous system, what ingredients are required for a self-consistent *effective* description of the horizontal degrees of freedom, that exhibit the stretched-exponential law (1)?

2 Effective restitution coefficients

In order to characterize the collision process in the horizontally projected system, we have measured directly the effective 1D restitution coefficient from the experimental data provided by K. Feitosa and N. Menon, for a gas of stainless-steel spheres (the system investigated in [4]) but

[1] If this model is simulated in dimension 1 with $\alpha_0 \in [0.7; 1]$, the obtained non-Gaussian behavior at thermal velocities corresponds to $\nu > 2$ instead of $\nu \simeq 3/2$ in the experiments.

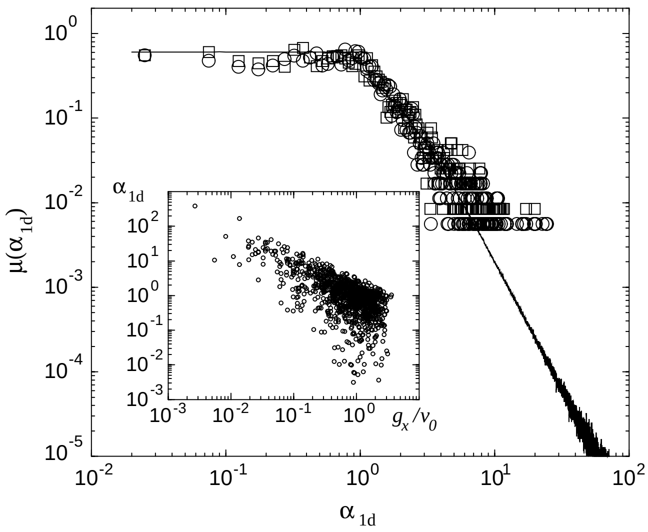

Fig. 1. Experimental distribution of α_{1d} for steel beads (circles) and glass beads (squares). Line: $\mu(\alpha_{1d})$ obtained in the RRC model as selected by the collisional dynamics (see text). Inset: experimental scatter plot of α_{1d} *versus* relative precollisional horizontal velocity g_x, for glass beads. Data from [4] and [19].

also for glass, brass and aluminum beads [19], which allow to sample a wide range of nominal inelasticities. Let us recall briefly the experimental set-up. The balls (diameter: $d = 1.600 \pm 0.002$ mm) are confined to a vertical, rectangular cage ($32d$ high $\times$ $48d$ wide $\times$ $1.1d$ thick) sandwiched between two parallel plates of Plexiglas. The cage is vibrated vertically at a frequency of 60 Hz and amplitudes up to $2.4d$, producing maximum accelerations, Γ, and velocities, v_0, of $56g$ and 1.45 m/s, respectively. The motions of the balls are recorded with a high-speed camera which allows a location of each ball with a precision of $0.03d$. The results we discuss here are taken in a rectangular ($10d \times 20d$) window around the geometrical center of the cell, where, as mentioned above, density and granular temperature are almost homogeneous [4]. Moreover, the measured velocity distributions do not vary with height nor with the phase of the vibration cycle. The experimental data can thus be considered as obtained in the bulk of a two-dimensional homogeneous (but open) system, reasonably far from the boundaries.

The horizontal component of relative velocities are computed before (g_x) and after (g_x^*) each collision, from which we deduce the effective restitution coefficient

$$\alpha_{1d} = \frac{|g_x^*|}{|g_x|}. \qquad (2)$$

Figure 1 displays the histogram $\mu_{\exp}(\alpha_{1d})$ obtained from the experimental data for different materials. At large α_{1d}, a power law tail is evidenced. Note that values $\alpha_{1d} > 1$ are expected, due to the transfer from vertical to horizontal translational kinetic energy [20].

The strong correlations between relative horizontal velocity g_x and α_{1d} are clearly seen in the scatter plot (inset of Fig. 1), with a very sharp cutoff above the second bisector $\alpha_{1d} \propto 1/g_x$. This cutoff follows from the definition of α_{1d}: since the post-collision velocity is finite, large values of α_{1d} may only be obtained for small values of g_x, in which case equation (2) implies that the maximum α_{1d} is of order $1/g_x$.

These features are qualitatively the same for all materials investigated. Moreover, three different densities have been investigated for steel beads, and the same distributions have been obtained [19]. Similar distributions have also been measured in a different experimental set-up with rolling beads [21]. The relatively small number of collisions investigated does not, however, allow us to get accurate histograms for the joint distributions $\mu(\alpha_{1d}, g_x)$ nor the *conditional* $\mu(\alpha_{1d}|g_x)$. The correlations evidenced in Figure 1 nevertheless play a crucial role, as will be shown below.

More insight into the conditional distributions $\mu(\alpha_{1d}|g_x)$ has been obtained in molecular dynamics of two-dimensional IHS driven by vibrating walls in reference [7]. Histograms of $\mu(\alpha_{1d})$ similar to the experimental results have been obtained. It turns out that the global $\mu(\alpha_{1d})$ is almost insensitive to the details of the system (density, velocity of the vibrating walls...), while this dependence exists for $\mu(\alpha_{1d}|g_x)$ (and also for $P(v)$). The following characteristics of $\mu(\alpha_{1d}|g_x)$ have been obtained: at constant g_x, $\mu(\alpha_{1d}|g_x)$ is almost constant for $\alpha_{1d} \in [0, \alpha_0]$, has a small peak at α_0, and decreases as $\exp(-A(g_x)\alpha_{1d}^2)$ for $\alpha_{1d} > \alpha_0$, with $A(g_x) \propto g_x^2$.

3 Random restitution coefficient model

We have considered the possibility to mimic the experimental distributions by including randomness in the restitution coefficient, following the approach of [20], where a random restitution coefficient (RRC) model was introduced to account for the fact that, in vertically shaken granular gases, the energy is transferred to the vertical degrees of freedom by the moving piston, and then to the horizontal ones through grain/grain collisions only. The heating of horizontal degrees of freedom thus occurs through the inter-particle collisions, and not in between as in the "stochastic thermostat" approach. Moreover, a globally dissipative collision may correspond to an energy gain for the horizontal components of the velocities. This leads to the study of horizontally projected collisions with an effective restitution coefficient that can be either smaller or larger than 1, as the experimental data of Figure 1 indeed show. In our case, the RRC model is therefore an effective approach in 1 dimension (since the original collisions are two-dimensional), in which IHS undergo binary, momentum-conserving collisions with a restitution coefficient α_{1d} drawn randomly at each collision from a given distribution $\mu(\alpha_{1d})$ which should mimic $\mu_{\exp}(\alpha_{1d})$. Even if the original collisions are not random, the projected ones may be considered as such.

Note that the energy injection is here given solely by the values of α_{1d} larger than 1. The simulations are performed at the mean-field level of the homogeneous non-linear Boltzmann equation for point particles, solved by the direct simulation Monte Carlo method (DSMC) [22]. The velocity statistics is computed in the non-equilibrium steady state which is reached after a transient.

We have first considered distributions with a large tail in order to reproduce the experimental $\mu(\alpha_{1d})$, but without any correlations with the relative velocities of the colliding particles; in this case, it turns out that $P(v)$ has a power law decay at large v, in marked contrast with experimental results [2]. This approach consequently needs to be refined and the next crucial step is to take into account the correlations between α_{1d} and g_x, with the insight given by the experimental scatter plot (inset of Fig. 1) and by the molecular-dynamics results [7].

We have thus used distributions decaying as $\mu(\alpha_{1d}|\tilde{g}_x) \sim \exp[-(\alpha_{1d}\tilde{g}_x)^2/R]$ at large α_{1d}, where $\tilde{g}_x = g_x/v_0$ is the rescaled velocity defined from the total kinetic energy of the system ($v_0^2 = \langle v^2 \rangle$), and the parameter R can be varied with values of order 1. The function $\mu(\alpha_{1d}|g_x)$ is the only input needed to simulate the RRC model. For the consistency of the approach, the distribution $\mu(\alpha_{1d})$ measured in the simulation needs to be close to its experimental counterpart. This comparison is displayed in Figure 1 and justifies *a posteriori* the choice made for $\mu(\alpha_{1d}|g_x)$. Both experimental and numerical distributions $\mu(\alpha_{1d})$ display a power law tail of the form α_{1d}^{-n} with $n \simeq 3$.

The velocity distribution obtained from the RRC model is compared to the experimental measure in Figure 2. The agreement is satisfactory over the whole range of velocities; in particular, the RRC distributions is compatible with the stretched-exponential behavior reported in [4], with an exponent ν close to 1.5. Note that, since no precise experimental data is available for the distributions of restitution parameters conditioned by relative precollisional velocity, the parameters R are tunable (as long as the global $\mu(\alpha_{1d})$ coincides with the experimental one), and the value giving the best agreement has been chosen ($2 \leq R \leq 4$). The agreement remains satisfactory upon changing R, provided that the resulting large α cutoff remains sharp (*i.e.* R should not be too large).

The RRC model therefore provides a self-consistent framework which allows to reproduce the experimental $P(v)$ if implemented with the correct distribution of effective coefficients. Moreover, the velocity statistics depends on the distribution of effective restitution coefficients: a broader $\mu(\alpha_{1d}|g_x)$ leads to a broader $P(v)$, consistently with the numerical study of [7] which showed both broader $P(v)$ and $\mu(\alpha_{1d}|g_x)$ as, *e.g.*, the density is increased. Both distributions P and μ are equally sensitive to a possible non-universality (dependence on material properties). As a consequence, an accurate experimental measure of μ appears as complementary to the direct computation of $P(v)$, in order to assess the experimentally difficult question of the velocity statistics universality.

[2] It is noteworthy that a large v power law is actually obtained for *any* distribution $\mu(\alpha_{1d})$, see [20].

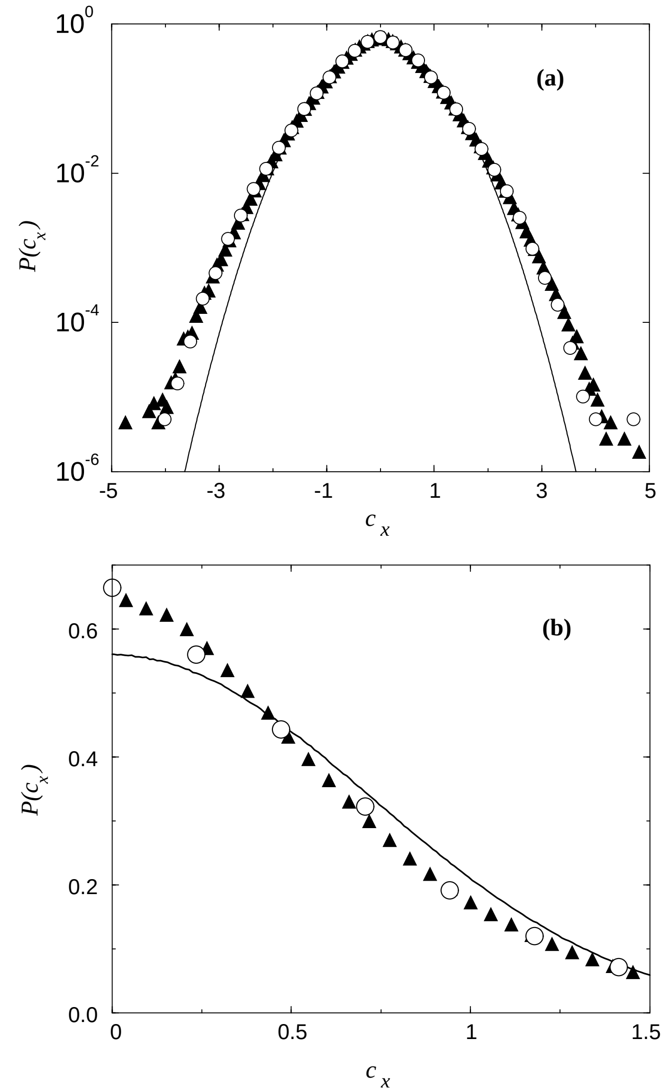

Fig. 2. Rescaled distribution $P(c_x)$ of horizontal velocities, on a linear-log scale (a) and linear scale (b). All distributions have the same variance $\langle c_x^2 \rangle = 1/2$. Circles represent the experimental data for steel beads [4]; filled triangles correspond to the Monte Carlo simulation of the RRC model, with $\alpha_{1\mathrm{d}}/g_x$ correlations.

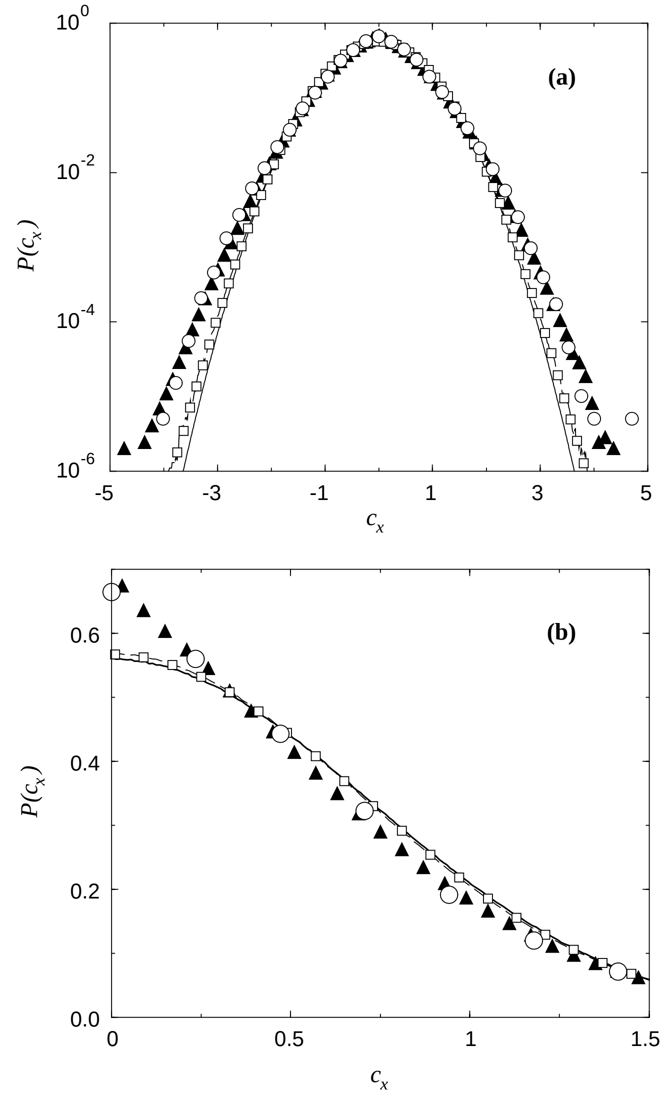

Fig. 3. Rescaled distribution $P(c_x)$ of horizontal velocities, on a linear-log scale (a) and linear scale (b). All distributions have the same variance $\langle c_x^2 \rangle = 1/2$. Circles represent the experimental data for steel beads [4]; squares correspond to a simulation of the "stochastic thermostat" with $\alpha_0 = 0.6$, whereas for $\alpha_0 > 0.7$, the corresponding $P(c_x)$ is indistinguishable from the Gaussian shown by the full line. The dashed line (very close to the squares) corresponds to a simulation of the three parameters model [23–25] with $\alpha_\mathrm{n} = 0.7$, $\alpha_\mathrm{t} = 0.5$, $\mu = 0.5$. Filled triangles show the results for the multiplicative driving stochastic thermostat with $\delta = 0.6$ and $\alpha = 0.9$ [17].

4 Stochastic thermostat model

For comparison, we have also considered heating of inelastic hard discs (2D) through the "stochastic thermostat", in the framework of the non-linear homogeneous Boltzmann equation, where the large velocity tail has been shown to behave like $\exp(-v^{3/2})$ [9]: this result has subsequently often been considered as an agreement with the experimental results. In this model, the energy injection is achieved through a random force $\eta(t)$ acting on each particle,

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} + \eta(t), \quad \langle \eta_i(t)\eta_j(t') \rangle = 2D\delta(t-t')\delta_{ij}, \qquad (3)$$

where D is the amplitude of the injected power and $\mathbf{F}$ the systematic force due to inelastic collisions. The variance of η determines the granular temperature in the non-equilibrium steady state, but has no influence on the form of the rescaled distribution function $P(c_x)$.

With the accuracy of Figure 3 (the current experimental resolution), the corresponding numerical velocity distributions are then found indistinguishable from a Gaussian, for physically relevant inelasticities in the range $0.7 \leq \alpha_0 \leq 1$. Departure from Maxwell-Boltzmann behavior becomes manifest below $\alpha_0 = 0.6$ (squares in Fig. 3), which is unphysically low, but the velocity distribution is still incompatible with its experimental counterpart. We also investigated the possibility to describe the effective horizontal dynamics with the 1D stochastic thermostat: for $0.7 < \alpha_0 < 1$ the velocity distributions are incompatible with the experimental $P(v)$ displayed in Figure 2, with opposite non-Maxwellian features (underpopulated both at vanishing and high energies [26]). More precisely, within the stochastic thermostat approach the kurtosis $\langle v_x^4 \rangle / \langle v_x^2 \rangle^2 - 3$ of the distribution is negative for $\alpha_0 > 1/\sqrt{2} \simeq 0.71$ [9], irrespective of dimension, which corresponds to an underpopulated low-velocity behavior at variance with the experimental data shown in Figure 2.

We have also considered the stochastic thermostat model for two-dimensional IHS with both tangential $\alpha_{\rm t}$ and normal $\alpha_{\rm n}(= \alpha_0)$ restitution coefficients [27]. No numerical studies of $P(v)$ can indeed be found in the literature in this case, although an investigation into the non-equipartition between translational and rotational kinetic energies has been performed in [27]. The resulting velocity distributions $P(v)$ remain very close to a Gaussian for $\alpha_{\rm n} \geq 0.7$ and arbitrary $\alpha_{\rm t}$ (where $-1 \leq \alpha_{\rm t} \leq 1$), as for smooth spheres (corresponding to $\alpha_{\rm t} = -1$). However, such a two parameter model may be too schematic compared to the experiments [23] and we have also considered a more realistic approach with Coulomb friction along the simplifications discussed in [24]: a friction coefficient μ, is introduced in addition to $(\alpha_{\rm t}, \alpha_{\rm n})$ [25]. This does not change significantly $P(c_x)$ (see the squares and dashed line in Fig. 3). This seems to discard the relevance of such an approach for the comparison of the velocity distributions with experimental data.

The above analysis shows that the stochastic thermostat in its original formulation (including some possible extensions) does not provide a relevant model of energy injection as far as the velocity distribution is concerned, although it may be useful to investigate other features such as kinetic energy non-equipartition in granular mixtures [28]. However, a variant of this model may improve the picture. In particular, a multiplicative driving (corresponding to a velocity-dependent amplitude $D \propto |\mathbf{v}|^{2\delta}$ in Eq. (3)) has been studied in references [17,18]. We have performed DSMC simulations for this model, choosing the value of δ that, for a given reasonable inelasticity parameter ($\alpha = 0.9$), gives the best agreement with the experimental $P(v)$. We obtained $\delta \simeq 0.6$, which leads to the distribution shown by the triangles in Figure 3. The agreement with the experimental data is satisfactory, and displays a similar accuracy as obtained within the RRC model. Consequently, models with homogeneous energy injection may also describe quite accurately the experimental $P(v)$, with the problem of predicting the values of the various parameters involved.

5 Conclusion

In conclusion, the random restitution coefficient model with point particles captures the essential features responsible for the observed non-Gaussian character of the velocity distribution $P(v)$ in vibrated granular gases experiments [4], and represents therefore a self-consistent framework. The conditional distribution $\mu(\alpha_{\rm 1d}|g_x)$ defining the appropriate collision rule has been shown to encode the relevant dynamic information and provides an alternative route to characterize the non-equilibrium steady state, complementary to the direct measure of $P(v)$. An interesting point would be to obtain an analytical prediction for $\mu(\alpha_{\rm 1d}|g)$. It is noteworthy that our approach is mean-field (Boltzmann) like, the only correlations considered being in the collision law. Its self-consistency, which was not an obvious point *a priori*, has been established by comparison with experiments.

While we have restricted our analysis to the two-dimensional case, such investigations can be extended to three-dimensional systems, for which, however, experimental measures of effective restitution coefficients seem more difficult. As implemented here, without an analytical knowledge of $\mu(\alpha_{\rm 1d}|g_x)$, the RRC model is not predictive since an experimental input is required to obtain the correct velocity statistics. Our results however suggest to assess experimentally the question of the universality of $P(v)$ from the direct measure of the distribution of effective restitution coefficients. These characteristics are indeed linked within the RRC model: the exponent ν, and the whole shape of $P(v)$, depend on the functional form of $\mu(\alpha_{\rm 1d}|g_x)$. At this point, the fact that with a rather poor accuracy, similar P and μ have been obtained for the case investigated in [4,19] simply confirms the RRC picture, and calls for experiments with widely different collisional properties, such as hollow spheres.

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