Real-Space Application of the Mean-Field Description of Spin-Glass Dynamics

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(Received 8 February 2001; published 1 August 2001)

The out of equilibrium dynamics of finite dimensional spin glasses is considered from a point of view going beyond the standard "mean-field theory" versus "droplet picture" debate of the past decades. The main predictions of both theories concerning the spin-glass dynamics are discussed. It is shown, in particular, that predictions originating from mean-field ideas concerning the violations of the fluctuation-dissipation theorem apply quantitatively, provided one properly takes into account the role of a spin-glass coherence length, which plays a central role in the droplet picture. Dynamics in a uniform magnetic field is also briefly discussed.

DOI: 10.1103/PhysRevLett.87.087204 PACS numbers: 75.10.Nr, 05.70.Ln, 75.40.Mg

Spin glasses have played a major and inspiring role in the rapidly developing field of the dynamics of glassy systems [1]. In particular, important developments have been achieved in the study of aging phenomena [2], which are encountered in several microscopic systems. In this context, two very different dynamic descriptions have emerged. The "mean-field theory" consists in the exact solution of the dynamics of fully connected (or equivalently infinite-dimensional) spin-glass models [2]. The "droplet picture" is more phenomenological but directly addresses the problem of a real space (as opposed to the configurational space) description [3].

It is a recurrent theme in the field to interpret numerical or experimental data as validating one description at the expense of the other [4]. Technical tools of the dynamic mean-field theory have been argued to be necessary for a complete understanding of the aging of finite dimensional spin glasses [5]. In this Letter, we show that their range of potential validity is in fact wider than previously thought (and can apply even when no replica symmetry breaking is present), provided the theory is supplemented with the idea of a growing length scale, which we call the "coherence length." A growing equilibration length, the "domain size," is at the heart of the description of aging phenomena by the droplet picture. Although we cannot give a rigorous proof of their identity, we argue that these two length scales have the same physical content. We propose a construction for the coherence length and show numerically that it is proportional to the domain size. Last, we argue that the use of dynamical concepts coming from mean-field ideas does not necessarily imply the existence of a static replica symmetry broken phase.

The mean-field description of spin-glass dynamics stems from the asymptotic solution of the dynamical equations for models which are statically solved by the Parisi replica symmetry breaking scheme [2]. The behavior of the system is encoded in the autocorrelation function $C(t, t_w)$ and the conjugated response function $R(t, t_w)$. It is shown that the decay of the correlation involves a complex pattern of time scales organized in a hierarchical way. This "dynamic ultrametricity" is a direct counterpart of the static Parisi solution for these models [6]. However, this feature is absent from all known experimental and numerical data in three dimensions [1,7] which show instead that the slow decay of the correlation [or the thermoremanent magnetization $M(t, t_w)$ in experiments] is a one-time-scale process, $C(t, t_w) \simeq \mathcal{C}(t/t_w)$ [or $M(t, t_w) \simeq \mathcal{M}(t/t_w)$], for times $t \gg t_w$.

Nontrivial predictions are also made concerning the relation between R and C which satisfy at equilibrium the fluctuation-dissipation theorem (FDT), $TR(t, t_w) = \partial_{t_w} C(t, t_w)$. A generalization of the FDT is obtained by introducing the function $X(t, t_w)$ through [6]

$$X(t, t_w) \equiv TR(t, t_w) \left(\frac{\partial C(t, t_w)}{\partial t_w} \right)^{-1}, \tag{1}$$

with $X(t, t_w) = 1$ at equilibrium. In mean-field models, it can be shown that $X(t, t_w)$ becomes at long times a single argument function, allowing the definition of the "fluctuation-dissipation ratio" (FDR) through [6]

$$x(q) \equiv \lim_{t, t_w \to \infty} X(t, t_w)|_{C(t, t_w) = q} \,. \tag{2}$$

Moreover, this purely *out of equilibrium* quantity is related to the spin-glass order parameter $P(q)$, which measures the *equilibrium* distribution function of overlaps [6,8],

$$x(q) = \int_0^q dq' \, P(q') \,. \tag{3}$$

It has been further argued that Eqs. (2) and (3) are true for finite dimensional glassy systems [8]. The existence of a FDR of the form (2) has been numerically investigated in finite dimensional models [9] through the study of the physically accessible quantities $C(t, t_w)$ and $\chi(t, t_w) \equiv \int_{t_w}^t dt' \, R(t, t')$. Equation (2) is then graphically checked by representing the variations of χ as a function of C parametrized by the time difference $t - t_w$, since Eq. (2) implies at large times

$$\chi(t,t_w) = \frac{1}{T}\int_{C(t,t_w)}^{1} dq\, x(q)\,, \tag{4}$$

i.e., the obtained $\chi(C)$ relation is independent of the time [6]. At equilibrium, $x = 1$ and thus $\chi = (1 - C)/T$. Numerically [9], a $\chi(C)$ parametric curve is obtained for a large t_w and compared to $S(C,L) \equiv \int_C^1 dq \int_0^q dq' P(q',L)$, where $P(q,L)$ is the Parisi function computed in a system of linear size L as large as possible. The coincidence of these curves was used to argue that both quantities were close to their limit, $t_w, L \to \infty$, and to deduce information on the nature of the low-temperature phase [9].

A somewhat different picture has been put forward in Ref. [10] and checked in the case of the 2D *XY* model: A real space view of the aging behavior as an equilibration process taking place on a growing coherence length $\ell(t_w)$ leads to a generalization of relations (3) and (4) at finite times and sizes:

$$T\chi(t,t_w) = S[C(t,t_w), \ell(t_w)]\,. \tag{5}$$

Equation (5) states that the *off-equilibrium properties* of the infinite aging system at *finite-time* t_w are connected to the *equilibrium properties* of a system of *finite-size* $\ell(t_w)$ [11], independently of the $t_w, L \to \infty$ limits. In other words, the system is quasiequilibrated up to a length scale $\ell(t_w)$. For the 2D *XY* model, the coherence length is thus proportional to the dynamic correlation length $\ell(t_w) \approx \xi(t_w)$ [10,11].

Inspired by dynamical mean-field theory, Eq. (5) *postulates* the existence of a growing coherence length $\ell(t_w)$ which represents the spatial extent to which the system is equilibrated at time t_w. Interestingly, this is the basic assumption of the droplet picture that the spin-glass dynamics is governed by large-scale excitations, whose size ξ increases with time t_w during the aging as $\xi(t_w) \sim (\ln t_w)^{1/\psi}$ [3]. Thus, we expect $\ell(t_w) \sim \xi(t_w)$.

Numerically, $\xi(t_w)$ can be extracted from a four-point correlation function [9,12–14]. Its relevance for the scaling properties of various physical quantities has been demonstrated in Refs. [14]. Experimentally, $\xi(t_w)$ was extracted through the scaling behavior of the Zeeman energy [15]. All these 3D investigations indicate a power law growth, $\xi(t_w) \sim t_w^{\alpha(T)}$, with $\alpha(T) \simeq 0.2T/T_g$.

We now investigate numerically the validity of Eq. (5) and then discuss its important consequences for the theoretical description of spin glasses. Our aim is to demonstrate its wide range of applicability, independently of the existence of replica symmetry breaking, in the context of which it was first proposed [6,8]. For this purpose, we consider the low (but finite) temperature dynamics of the *two-dimensional* Edwards-Anderson model, for which the spin-glass transition is at $T_c = 0$ [16]. At low temperature, the relaxation time becomes so large that the aging dynamics is very similar to the 3D case, including the power law growth $\xi(t_w) \simeq t_w^{0.2T}$ [13]. The model is defined by $H = \sum_{\langle i,j\rangle} J_{ij} s_i s_j$, where s_i ($i = 1,\dots,N$) are $N = L \times L$ Ising spins located on the sites of a square lattice of linear size L. The sum $\langle i,j\rangle$ runs over pairs of nearest neighbors. The J_{ij} are random Gaussian variables of mean 0 and variance 1.

To compute the autocorrelation function $C(t,t_w) = N^{-1}\sum_i \overline{\langle s_i(t) s_i(t_w)\rangle}$ ($\langle\cdots\rangle$ indicates an average over initial conditions and $\overline{\cdots}$ over realizations of the disorder), and the susceptibility $\chi(t,t_w)$, a very large system, $L = 400$, is quenched at the initial time $t_w = 0$ from a disordered state to $T \in [0.2, 1.0]$. The susceptibility $\chi(t,t_w)$ is measured after applying a random magnetic field h_i (taken from a Gaussian distribution of mean 0 and variance h_0^2) between times t_w and t in each site. In the linear response regime (we used $0.02 \le h_0 \le 0.05$), one gets $\chi(t,t_w) = h_0^{-2} N^{-1}\sum_i \overline{\langle h_i s_i(t)\rangle}$. The equilibrium $P(q,L)$ is computed independently by equilibrating (using parallel tempering [17]) samples of sizes $L \in [6,24]$ and temperatures $T \in [0.2, 1.2]$. By definition, $P(q,L)$ is the disorder-averaged histogram of the overlap $q = N^{-1}\sum_i s_i^a s_i^b$ between two equilibrated copies (a,b) of the system. For the sizes and temperatures investigated, $P(q,L)$ has its common nontrivial structure [5], with a peak around an L-dependent value of the "Edwards-Anderson parameter," and a tail extending towards $q \simeq 0$ values, although we know that $\lim_{L\to\infty} P(q,L) = \delta(q)$ at all temperatures $T > 0$.

We are now in position to compare the $\chi(C)$ curves obtained in the aging situation, with $S(C,L)$. Our results are summarized in Fig. 1. We show first in Fig. 1a the parametric curves for the same large waiting time, $t_w = 10^4$, and different temperatures. The dynamic curves are qualitatively similar to the 3D case (to our knowledge, no such data are available in 2D). At short times ($1 - C \ll 1$), the curves follow the equilibrium FDT relation, which they smoothly leave at longer times.

The validity of Eq. (5) is demonstrated in two steps. Figure 1a shows first that the dynamic curves follow, within our numerical precision, the curves of $S(C,L)$ obtained by the double integration of $P(q,L)$, for a given value of L. Note that the coincidence of these two independently computed functions on their whole support is a very strong requirement which unambiguously defines the coherence length, $\ell(t_w) = L$. The time evolution of $\ell(t_w)$ is then followed for $T = 0.4$ in Fig. 1b which shows that dynamic curves for increasing t_w coincide with static curves for increasing sizes. For all temperatures investigated, we find a growth law for $\ell(t_w)$ consistent with the 2D growth laws reported in Ref. [13] for $\xi(t_w)$, showing that $\ell(t_w) \propto \xi(t_w)$. Physically, the coincidence between dynamic and static data means that, at time t_w, the system is locally equilibrated up to a coherence length $\ell(t_w)$. The precise link between $\ell(t_w)$ and $\xi(t_w)$ is, however, a tricky point since the static $P[q,\ell(t_w)]$ is sensitive to the boundary conditions [5]. We use here (as is usual [9]) periodic boundary conditions which lead to the estimate $\ell(t_w) \simeq 2\xi(t_w)$. The equality $\ell(t_w) = \xi(t_w)$ would probably be obtained computing $P(q,L)$ in a box of size L

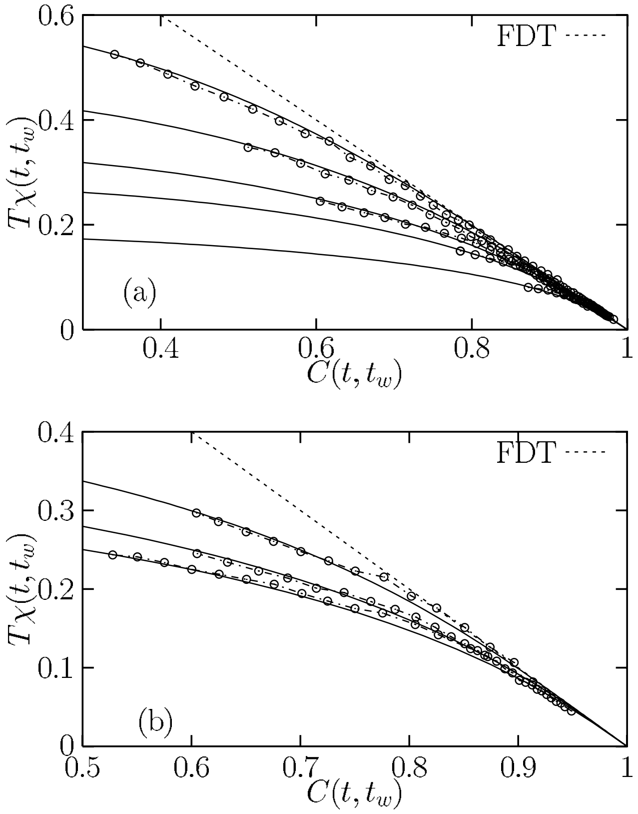

FIG. 1. Susceptibility-correlation parametric curves obtained in the aging regime (circles) and from the static $S(C, L)$ (solid lines). The dashed line is the equilibrium FDT. (a) Constant waiting time, $t_w = 10^4$, $T = 0.6, 0.5, 0.4, 0.3$, and 0.2 (from top to bottom). (b) Constant temperature, $T = 0.4$, and $t_w = 10^2$, 3×10^3, and 3×10^4 (from bottom to top). Equilibrium data: $L = 8$, 10, and 18 (from bottom to top).

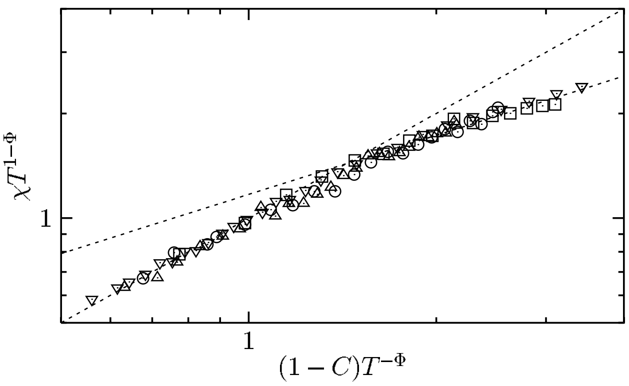

FIG. 2. Scaling behavior of the dynamic curves of Fig. 1a with $\Phi \simeq 2.3$, guessed from the Parisi-Toulouse approximation. The dashed lines are the relations $B = A$ and $B \sim A^{0.565}$.

inside a much larger system, as proposed in Ref. [18]. This is, of course, much more computationally demanding.

At very low temperature ($T \lesssim 0.3$), we find that the agreement between statics and dynamics is not as good, with slightly different shapes for static and dynamic data. This is probably because, even at very large times, the coherence length is very small ($\ell \simeq 1$–3), so that the regime where Eq. (5) becomes valid is not reached. However, a similar trend has been observed in Ref. [10], attributed to the role of topological defects. No data are available yet in 3D or 4D spin glasses at very low temperature [9], and this point should be checked.

The dynamic data of Fig. 1a obtained at different temperatures can be collapsed using the scaling variables $A \equiv \chi T^{1-\Phi}$ and $B \equiv (1 - C)T^{-\Phi}$, with Φ a free parameter chosen to get the best collapse of the curves in Fig. 2. For $1 - C \ll 1$ (short times), the FDT implies $B = A$, while we obtain at large times the power law $B \sim A^{1-1/\Phi} \simeq A^{0.565}$, i.e., $\Phi \simeq 2.3$. This scaling behavior has been proposed in Ref. [9] as a dynamic analog for the so-called Parisi-Toulouse approximation used in a mean-field static context. The relation $B \sim A^{0.41}$ has been numerically found in 3D and 4D spin glasses, and compared to an expected mean-field behavior $B \sim A^{0.5}$ [9]. That the scaling works also in 2D explicitly shows that it is not necessarily connected to an underlying replica symmetry breaking, and weakens the evidence presented in other numerical works.

We discuss now different important implications of the finite-time/finite-size connection described by Eq. (5). From a pragmatic point of view, first, this relation implies that numerical studies of large aging systems or small equilibrated systems potentially contain the same information. In our opinion, this fact has been largely underestimated in the spin-glass literature, which often tries to overcome the difficulty of obtaining thermalized samples of large sizes by using large times in dynamical simulations. We exemplify this point by discussing the behavior of spin glasses in a uniform magnetic field. Various static tests of the existence of a spin-glass phase give inconclusive results [$P(q, L)$ in a field differs from its mean-field shape, Binder cumulants do not cross in a field [5]], while the dynamic behavior in a field has been claimed to clearly demonstrate the existence of a replica symmetry broken phase, in 3D and 4D [19]. This intriguing fact led us to perform in 2D the simulations of Refs. [19]. We have computed different values of the overlap, $q_{\min}$ and $\langle q \rangle \equiv \int dq'\, P(q', L) q'$ which, according to the mean-field shape of $P(q, L = \infty)$ in a field, should be different [19]; $q_{\min}$ is obtained dynamically as the infinite time extrapolation of the overlap between two independently aging copies of the system [19]. Figure 3 shows that, as in 3D and 4D, the two values differ at low enough T. Taken at face value, this result would lead to the erroneous conclusion that a replica symmetry breaking transition occurs at $T_c(h = 0.4) \simeq 0.5$ [20]. This shows that dynamic studies, using the same computer resources, are in fact probing the same length scales as equilibrium ones and therefore have to be taken with the same care. Much larger times (in dynamics) and sizes (in statics) should be used to show that there is no transition in a field in 2D.

The validity of Eq. (5) implies that the same excitations of length $\ell(t_w)$ are governing the dynamics, i.e., the decay

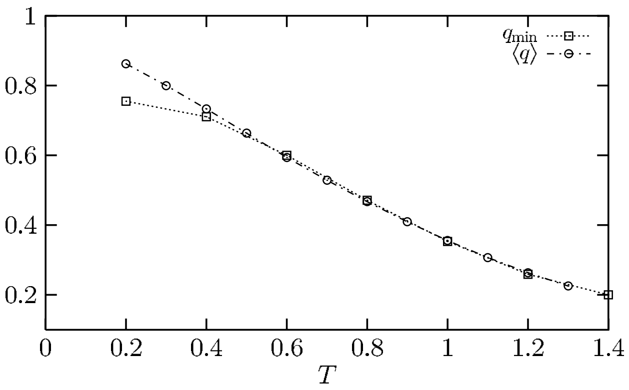

FIG. 3. Temperature dependence of the overlaps $\langle q \rangle$ and $q_{\rm min}$, in a uniform magnetic field $h = 0.4$.

of the correlation as $C(t, t_w) \simeq C(\ell(t)/\ell(t_w))$, and are contributing to the low-q part of $P(q, L)$, i.e., to the non-FDT part of the parametric curves in Fig. 1. We emphasize that Eq. (5) would not be satisfied at finite time during the coarsening in a pure ferromagnet (below T_c) where the domain walls governing the aging are absent in the static situation with usual boundary conditions. (See the related discussion of the role of vortices in Ref. [10].) Hence, we preferred the name coherence length to domain size, since these two names refer to qualitatively different physical situations. Our results give no indication on the structure of the relevant excitations in spin glasses, but emphasize the relevance (and the need) of finite-T equilibrium studies of large-scale excitations for the description of aging. It is not established, in particular, that the description [3] in terms of compact droplets is correct [21].

From a fundamental point of view, our results give support to the picture of aging in spin glasses as the successive equilibration of excitations of increasing length scales, recently put forward to interpret temperature-cycling experiments [22]. We speculate that the absence of $\ell(t_w)$ in mean-field theory and, hence, of the resulting multi-length-scale dynamics, is "compensated" by the dynamic ultrametricity [6]. This results in the inability of the theory of correctly predicting the simple t/t_w scaling of dynamic functions. Note that the reported growth laws are inconsistent with the logarithmic law predicted by scaling arguments [3,22], which would anyway lead to the unobserved $\log t/\log t_w$ scaling. A thorough investigation of this law seems necessary, and current experiments and simulations [23] should clarify this point.

Note, finally, that the extremely slow growth of the coherence length implies that the length scales involved in the dynamics are relatively small, *even at experimental times*. The provocative idea that the thermodynamic limit might be of no *practical* importance directly follows. This also implies that Eq. (5) should apply in experiments, leading to results quantitatively similar to Fig. 1.

We thank J.-P. Bouchaud, C. De Dominicis, P. Holdsworth, J. Kurchan, M. Mézard, M. Moore, F. Ricci-Tersenghi, and F. Zuliani for discussions. This work was supported by the PSMN at ENS-Lyon.

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