Contemporary Mathematics

The classification of subgroups of quantum $SU(N)$

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1. The classification and structure of subgroups of quantum groups

1.1. The classification and structure of subgroups of quantum $SU(2)$. Drinfeld and Jimbo have shown that the simple Lie groups have quantum deformations. At roots of unity, the semisimple quotient of the quantum groups has only finitely many irreducibles, in a phenomenon called the Wess-Zumino-Witten cutoff. The irreducible representations of $SU(2)$ are labeled by the half-line $\{0, 1, 2, 3, \dots\}$, and the quantum cutoff $SU(2)_l$ at the $l+2$-nd root of unity (here l is the level and $l+2$ the Coxeter number) has irreducible representations labeled by the graph A_{l+1} with vertices $\{0, 1, 2, \dots, l\}$; the level l is the highest degree of a representation which survives the cutoff. The graph D_k is obtained by folding in 2 the graph A_{2k-1} by a $\mathbf{Z}_2$ action which splits the irreducible in the middle into $\pm$ components. Thus the D_n series consists of orbifolds of the A_n series. There are also 3 exceptional graphs associated to $SU(2)$ namely E_6, E_7, E_8 at levels $10, 16$ and 28 respectively. Each of these graphs has eigenvalues among the eigenvalues of the graph A_n on the same level; these eigenvalues are labeled by the exponents of the graph.

In [**1**], we showed a new phenomenon: that in a way similar to the subgroups of the classical $SU(2)$ described by Felix Klein in his book "Das Ikosahedron", **the quantum cutoff $SU(2)_l$ has subgroups** as well. The irreducible representations of the Kleinian subgroups live, as observed by J. McKay, on the vertices of the affine ADE graphs. We have described subgroups of the quantum cutoffs $SU(2)_l$ for which the irreducibles live on the graphs A_n, D_{2n}, E_6 and E_8 respectively. The subfactors of Jones index < 4, which we had described earlier, correspond to these subgroups. In addition to subgroups there are also modules, for which the irreducibles can be tensored with irreducibles of $SU(2)_l$, but not tensored among themselves. The modules which are not subgroups correspond to the graphs D_{2n+1} and E_7. Each module is canonically obtained from a subgroup with an antiautomorphism, which we called the ambichiral twist; thus D_{2n+1} comes from A_{4n-1} and E_7 comes from an exceptional twist of the ambichiral part D_{10}^{even} of D_{10}. We also connected these subgroups and modules to the modular invariants for $SU(2)$ classified by Capelli, Itzykson and Zuber, and we showed that the modular invariants appeared in several different roles in the theory. A modular invariant for $SU(k)_l$ is a matrix M with entries $M_{i,j} \in \{0, 1, 2, \dots\}$ labeled by pairs (i, j) of Young diagrams with $\leq k-1$ rows and $\leq l$ columns. The modular invariance requirement is that the matrix M be a self-intertwiner of a representation of the modular group $SL(2, \mathbf{Z})$ associated to $SU(k)_l$.

We have shown that the entries M_{ij} of the modular invariant corresponding to a graph G describe the following mathematical objects.

(i) the number of essential paths on a pair of chiral graphs with common vertices, corresponding to the common components of the irreducibles i and j of $SU(k)_l$ restricted to the subgroup. In an equivalent form, for $SU(2)$ this description is the following. There are 2 Hecke algebra connections for an ADE graph introduced by V. Jones, differing by a choice of $\sqrt{-1}$. Iterate the first one i times and the second one j times and decompose each into irreducible components. The number of common irreducible pairs is M_{ij}.

(ii) M_{ij} is the number of Kleinian invariants of degree (i,j). The original Kleinian invariants were polynomials left invariant by the action of a subgroup of $SU(2)$. Here there are 2 copies of the subgroup, complex conjugate to each other, and each invariant has a bidegree (i,j).

(iii) The fusion algebra of the connections between two copies of G can be block diagonalized and the numbers M_{ij} are the dimensions of the blocks.

1.2. Zuber's higher Coxeter graphs problem. The theoretical physicists at the Centre d'Energie Atomique, Saclay, Paris, most notably Zuber and di Francesco have discovered what they called higher Coxeter graphs about 15 years ago.

The irreducible representations of $SU(3)$ (very important in physics, since they appear in the standard model) are labeled by a planar triangular lattice inside a Weyl chamber, and the level l cutoff $SU(3)_l$ has irreducibles labeled by the lattice points in an equilateral triangle, corresponding to Young diagrams with ≤ 2 rows and $\leq l$ columns; the edges correspond to tensoring with the generator of $\mathrm{Irr}\, SU(3)_l$. The triangular graph with these vertices will be called the A_l graph for $SU(3)$; it is the analog of the usual A_l graph for $SU(2)$.

One can obtain an orbifold series $A_l/3$, analogous to the D_n series for $SU(2)$. There are 2 more conjugate orbifold series, due to the fact that the graph A_2 from which $SU(3)$ itself is built has a symmetry called conjugation. Finally there are several exceptional graphs, among the list found by Zuber and collaborators empirically, with what they called computer aided flair. Zuber imposed spectral conditions analogous to the properties of the ADE graphs for $SU(2)$, such as the fact that their eigenvalues be among the eigenvalues of the A_l graph as described above. He also asked that the adjacency matrix of the graph be normal, due to the fact that with the Drinfeld-Jimbo braiding, the fusion algebra of $\mathrm{Irr}\, SU(3)_l$ is commutative.

A parallel list classifying $SU(3)$ modular invariants was due to T. Gannon, but the precise correspondence between it and the list of Zuber's graphs was unclear. Zuber stated about 10 years ago the problem of classifying the higher analogs of the ADE graphs and in an effort toward private support of research offered 1,2 and respectively 3 bottles of champagne for the classifications corresponding to $SU(3)$, $SU(4)$ and $SU(5)$ respectively.

1.3. The classification and structure of quantum subgroups of $SU(3)$. The natural reformulation of Zuber's higher Coxeter graphs problem is the following. The quantum group $SU(k)_l$ has subgroups and modules. Each subgroup and module G gives raise to a family of graphs, having as common vertices $\operatorname{Irr} G$ and edges corresponding to tensoring with the $k-1$ generators of $\operatorname{Irr} SU(k)_l$. The proper way to look at the usual ADE graphs is to view them as the graphs of subgroups and modules of $SU(2)_l$.

As a result the higher Coxeter graphs problem is to be reformulated as the problem of classifying the subgroups and modules of $SU(k)_l$ (and more generally of any semisimple quantum Lie group at roots of unity), a version adopted by Zuber.

For the classification, we had to develop conditions which were possible to check on any given graph, and which insured that the graph corresponded to a subgroup of $SU(k)_l$. The problem here is that any computation which involves explicitly intertwiners of $SU(k)$ is completely out of reach.

The graph of a subgroup of $SU(3)$ or $SU(3)_l$ is made of triangles, corresponding to the fact that the generator $\sigma = \sigma_1 \in \operatorname{Irr} SU(3)$ satisfies $\sigma^{\otimes 3} \ni 1$. We have shown that in each triangle there is a complex number, which we call a **cell**, coming from the explicit composition of morphisms corresponding to the edges of the triangle. Together, the cells on a graph form what we call an **internal connection**, defined up to a gauge which comes from the choice of morphisms for edges. These cells satisfy quadratic and quartic equations of a cohomological nature, which are local, i.e. involve only cells in a small neighborhood of the cell. This is done in the same spirit in which checking that weights on the vertices of a graph form a Perron-Frobenius eigenvector for a given eigenvalue involves checking only neighboring vertices. One of the exceptional graphs in the empirical list proposed by Zuber fails this test and has to be eliminated, since according to our theory it is not related at a deep level with $SU(3)$. Later, Zuber and collaborators started to notice that its behavior is indeed aberrant from other points of view, which further justifies our reformulation of Zuber's problem.

The connection has been adopted rapidly by the physics community, where several papers on what are now called as "Ocneanu cells" have already appeared. The cells solve immediately and explicitly one of the major goals of the initial program of Zuber, the construction of new solutions of the QYB equation corresponding to each graph.

We then developed methods for an exhaustive description of all possible subgroups and modules of $SU(3)_l$ at all levels l. The starting point is the classification of modular invariants of $SU(3)$ by Gannon. We showed that given a modular invariant matrix M it is possible to construct a matrix

$$\mathcal{M}_{(i,j),(i',j')} = \sum_{i''} \sum_{j''} N_{i,i'}^{i''} N_{j,j'}^{j''} M_{i'',j''}$$

where $N_{i,i'}^{i''}$ are fusion numbers for $i, i', i'' \in \operatorname{Irr} SU(3)_l$. The matrix $\mathcal{M}_{(i,i'),(j,j')}$ decomposes as a product

$$\mathcal{M}_{(i,j),(i',j')} = \sum_x \widetilde{M}_{i,j}^x \widetilde{M}_{i',j'}^x$$

of matrices $\widetilde{M}^x$ indexed by a label x of a vertex of the graph of the subgroup. The matrices $\widetilde{M}^x = (M_{i,j}^x)$, which we called the torus spectrum of the vertex x, generalize the modular matrix M and have natural numbers as entries; they describe for $SU(2)_q$ the dual asymptotic graph for the inclusion described by Goodman, Jones, and de la Harpe of the subalgebra generated by Jones projections into the algebra constructed from an ADE graph.

For $x = 1 \in \operatorname{Irr} G$ we have $\widetilde{M}^1 = M$. Let us now regard each $\widetilde{M}^x_{i',j'}$ as a vector and construct a matrix $\widetilde{M} = (\widetilde{M}^x_{i',j'})_{(i,j),x}$ with nonnegative integer entries. We have to solve thus the equation $\mathcal{M} = \widetilde{M}\widetilde{M}^*$. A further reduction comes from the fact that the vertices x of the chiral left graph that we are looking for are obtained by solving the above equations with $j, j', j'' = 0$, i.e.

$$\sum_{i''} N^{i''}_{i,i'} M_{i'',0} = \mathcal{M}_{(i,0),(i',0)} = \sum_x \widetilde{M}^x_{i,0} \widetilde{M}^x_{i',0}$$

We called the above decomposition, in which the matrix $\mathcal{M}_{(i,0),(i',0)}$ is constructed from the first line $M_{i'',0}$ of the modular invariant M, and is then split into a product, the **chiral modular splitting**. This phenomenon has been, subsequent to our work, interpreted in conformal field theory by Petkova and Zuber.

If there is an upper triangular solution $\widetilde{M}$ with 1 on the diagonal, then the solution is unique; since $\widetilde{M}$ is easily computed from a given graph it becomes easy to check that a given graph is the unique solution corresponding to a given modular invariant M. This allows in the case of $SU(3)$ to check that there is precisely one graph of a subgroup (what the physicists call type I graph) for each first line of the modular invariants in Gannon's classification. There follows the classification of all ambichiral twists, and checking that there are, in the list of graphs that passed the necessary and sufficient cell test, enough graphs to account for each ambichiral twist. This completes the classification of the subgroups of $SU(3)_l$, which does not require machine help.

There are 4 series of orbifolds $A_l, A_l/3, (A_l)^c$ and $(A_l/3)^c = 3(A_l)^c$, where c is the conjugation coming from the symmetry of the Coxeter graph A_2 on which the Lie group $SU(3)$ is built; the subscript l denotes the level. Among these the graphs A_{3n} are flat, or type I, i.e. correspond to subgroups and the rest are modules, or type II. There are also 3 exceptional subgroups E_5, E_9 and E_{21} which come from conformal inclusions; E_5 and E_9 have a module-orbifold each, denoted by $E_5/3$ and $E_9/3$. There is also an exceptional twist $(A_9/3)^t$ of $A_9/3$ and its conjugate $(A_9/3)^{tc}$; these are analogous to the graph $E_7 = (D_{10})^t$ for $SU(2)$.

A remarkable fact is that, while the correspondence between the classical, Kleinian, subgroups of $SU(2)$ labeled by the affine ADE graphs and the quantum subgroups of $SU(2)_l$ indexed by the non-affine ADE graphs was bijective, the classical and quantum subgroups are quite far away from each other for $SU(3)$; about half of the quantum series and exceptionals have no classical correspondent, and several classical subgroups have no quantum correspondent. In fact going from $SU(3)$ to $SU(4)$ and $SU(5)$ increases the number of subgroups dramatically; in the quantum case, where subgroups of $SU(k)$ are not, in general, subgroups of $SU(k+1)$, the number of exceptional subgroups appears to decrease from $SU(3)$ to $SU(5)$.

1.4. The quantum subgroups of $SU(4)$: the non-conformal exceptional. We have almost completed the classification of quantum subgroups of $SU(4)_l$ at all levels l. We have found 6 orbifold series, 3 exceptional subgroups and 3 exceptional modules. The nature of the problem was very different from $SU(3)$: no classification of modular invariants existed and no candidates for the subgroups existed either. The classification required new theoretical methods and very intensive machine computation. Among the surprises was the subgroup labeled E_8 (= the exceptional on level 8) of $SU(4)_8$. This is the first exceptional subgroup which does not come from a conformal inclusion, thus contradicting the running conjecture among physicists that the conformal inclusion method is exhaustive. The cell method was extended to $SU(4)$; the construction method is described in the next paragraph. We do not have effective general methods to construct the modules, or type II graphs, which correspond to a type I graph although the methods that we do have were sufficient for $SU(4)$.

1.5. Rigidity, effective bounds and construction algorithms for subgroups of quantum groups. We have described in a previous work (NSF proposal) our main rigidity results concerning subfactors and systems of bimodules.

For any fusion algebra there are finitely many 6j-symbols up to equivalence. This has as a consequence the fact that there are countably many finite depth subfactors of the hyperfinite II_1 factor up to conjugacy.

For any finite system of bimodules there are finitely many possibilities for a braiding.

The maximal atlas of any finite system of bimodules is finite. This latter result has as consequence that $SU(k)_l$ at any given rank k and level l has finitely many subgroups.

We can now show that for any given rank k the subgroups of $SU(k)_l$ at all levels l fit into a finite number of series which are orbifolds, and a finite number of exceptionals. We show that above a certain level there are no more exceptional subgroups. The maximal level of exceptionals is 28 for $SU(2)$ (the level of the E_8 subgroup), 21 for $SU(3)$, 8 for $SU(4)$ and probably 7 for $SU(5)$ (note the unexpected decrease in the maximal exceptional level with the rank in this range). The algorithms of T. Gannon allow the exhaustive computations of modular invariants up to very high levels (e.g. 10000 for $SU(4)$ in a few seconds on a current PC). However he found no exceptionals above very small levels. What his methods are lacking, except in the $SU(3)$ case, is a stopping point, and this is precisely what our methods provide.

We start from the modular splitting identity, which is nonlinear, and show that in the first row of the modular invariant the gap up to the first nonzero entry in the modular invariant is bounded by a universal bound. The only exception is for the case in which all nonzero entries in the first row of the modular invariant correspond to bimodules of index 1, or corners, of $SU(k)_l$; the latter case corresponds to the orbifold series analogous to D_n.

For the subgroup E_8 of $SU(2)$, the first of the modular invariants is $\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}$, the gap to the first nontrivial entry is thus 10 and this gap manifests itself by the fact that the graph E_8 is identical to an A_n graph for the first 5=gap/2 vertices. Afterwards the 6th vertex of A_n splits into 2 vertices of E_8. We can show that the modular splitting identity gives a sharp bound on the gap, via the following **fundamental inequality**.

For a Young diagram λ denote by $|\lambda|$ its quantum dimension and by $\overline{\lambda}$ its conjugate. Then if the first row of the modular invariant has a nonzero coefficient for λ and if we denote by I the set of all the Young diagrams of level less than half the gap, we have

$$\sum_{\mu \in I} |\mu| N^{\mu}_{\lambda, \overline{\lambda}} \leq |\lambda|$$

Note that $\sum_\mu |\mu| N^{\mu}_{\lambda,\overline{\lambda}} = |\lambda|^2$. Thus for a large gap, the left member is proportional to $|\lambda|^2$, and the only possibility is $|\lambda| = 1$, which produces the orbifold series.

On the other hand the first entry in the diagonal T matrix equal to 1 after the 0th entry arises on a level proportional to the level l of the cutoff, since the cutoff level appears in the denominator of the exponent. Thus the gap must be large if the level is large. Together with the previous absolute bound on the gap, this gives a bound on the maximal level l of exceptionals for $SU(k)_l$.

1.6. The internal mechanism of subfactors and QFT: the coefficients of the quantum symmetrizers. The theory of the irreducible representations of $SU(2)$ and its quantum cutoffs $SU(2)_l$ at the $l+2$-nd root of unity appears to be well understood; the 6j-symbols allow in principle the computation of the invariants of knots links and 3-manifolds. The irreducible representations σ_n of each degree n arise on symmetric tensors in this case homogeneous polynomials of degree n in 2 variables. The projection p_n from the space of all tensors of degree n corresponding to $(\sigma_1)^{\otimes n}$ to the symmetric tensors corresponding to the highest weight irreducible σ_n is the symmetrizer, which sums over all permutations of the n variables and then divides all by $n!$; in the quantum case there is an analogous formula, with braiding instead of permutations and with each term multiplied by a root of 1.

The trace τ_n of the symmetrizers $p_n = 1 - e_1 \vee \cdots \vee e_{n-1}$ for $SU(2)_l$, where e_i are the Jones projections satisfying $e_i e_{i\pm1} e_i = [2]^{-2} e_i$ and $e_i e_j = e_j e_i$ if $|i-j| > 1$, has been shown by V.Jones to satisfy a recurrence relation $\tau_n = \tau_{n-1} - [2]^{-2}\tau_{n-2}$ and thus $\tau_n = [n+1]/[2]^n$, where $[k]$ is the quantum number k given by $[k] = (q^{k/2} - q^{-k/2})/(q^{1/2} - q^{-1/2}) = \sin(k\pi/(l+2))/\sin(\pi/(l+2))$ with $q = e^{2\pi i/(l+2)}$. This relation allowed him to prove that l must be an integer, i.e. the remarkable rigidity of the index $[2]^2 = 4\cos^2(\pi/(l+2))$, l integer, when the index is < 4.

Hans Wenzl has proved the recurrence relation

$$p_n = p_{n-1} - [2][n-1]/[n] p_{n-1} e_{n-1} p_{n-1}.$$

From this

$$p_1 = 1 \ , \qquad p_2 = 1 - e_1 \ ,$$

and

$$p_3 = 1 - [2]^2/[3]e_1 - [2]^2/[3]e_2 + [2]^2/[3]e_1e_2 + [2]^2/[3]e_2e_1 \ .$$

In general a repeated application of the Wenzl inductive formula allows one to express the symmetrizer p_n in terms of the linear basis consisting of monomials $(e_{i_1} e_{i_1+1} \ldots e_{j_1})(e_{i_2} e_{i_2+1} \ldots e_{j_2}) \ldots (e_{i_m} e_{i_m+1} \ldots e_{j_m})$ where $i_1 > i_2 > \cdots > i_m$ and $j_1 > j_2 > \cdots > j_m$. These monomials correspond, up to a normalization, to all planar diagrams without closed loops in a rectangle with n entrances and n exits. The complexity of the computation and the number of such monomials grows exponentially with n, however.

The problem, asked by Vaughan Jones, was to find **a closed formula for the coefficient of each monomial term in the expression of the symmetrizer**. The general opinion among mathematicians and physicists, who had been searching for such a formula for applications in quantum field theory, appeared to be that such a closed formula might not exist in general.

We have obtained the following closed formula for the coefficient ρ_B of a planar diagram without closed loops in a rectangle with n entrances and n exits. The formula is unexpected in the sense that it has no analog in the representation theory of $SU(2)_l$ among 6j-symbols, knot and 3-manifold invariants. It points out that there is a "micro level" structure, inside the symmetrizers, different from the "macro level" structure of intertwiners. It lead further to an unexpected discrete 1-dimensional QFT model with long distance interactions.

Let B be a planar box corresponding to a monomial in the e_i, with n entrances labeled from left to right $1, \ldots, n$ on the upper row and n exits labeled similarly on the lower row. The wires in B are divided into through wires connecting an entrance with an exit, cups connecting two entrances and caps connecting two exits. Through wires are grouped into maximal contiguous blocks as follows, from left to right. For $k = 1, \ldots, m$ there are sets of r'_k through wires called the k-th wall followed by the k-th room with p'_k cups and q'_k caps, where $r'_k > 0$ for $1 < k \leq m$ and $\max(p'_k, q'_k) > 0$ for $1 \leq k < m$. Cumulatively from the left we define $r_k = r'_1 + \cdots + r'_k$, $p_k = p'_1 + \cdots + p'_k$, $q_k = q'_1 + \cdots + q'_k$.

For a cap or cup with end points at positions k, l with $k < l$ we define the jump as $(l-k-1)/2$ and we let J denote the set of strictly positive jumps of all cups and caps. Let $S = \{(s_1, \ldots, s_m) \in \mathbf{N}^m : 0 = s_1 \leq s_k \leq s_{k+1} \leq \min(p_k, q_k)$ for all $k = 1, \ldots, m-1\}$. Define the monomials

$$\begin{aligned} f_0(r) &= \frac{1}{[r]!} \\ f_1(p, r; s) &= \frac{[p-s]!^2 [p+r+s]!^2}{[2p+r]!} \\ f(p, q, r, r_1; s, s_1) &= \frac{[p-s]![q-s]![p+r+s]![q+r+s]!}{[p-s_1]![q-s_1]![p+r_1+s_1]![q+r_1+s_1]!} \cdot \\ &\quad \cdot \frac{[r_1-r+s_1-s-1]![r_1-1+s+s_1]![r_1+2s_1]}{[r_1-r-1]![r+s+s_1]![s_1-s]!} \end{aligned}$$

Let

$$\varepsilon_B = (-1)^{p_m + \sum_{k=1}^{m-1} (p_k - q_k)(r_{k+1} - r_k) + \sum_{j \in J} j}$$

and $\iota_B = \prod_{j \in J} [j+1]^{-1}$. We have the following expression for the weight of the planar box B:

$$\rho_B = \varepsilon_B \iota_B f_0(r_1) \sum_{(s_1, \ldots, s_m) \in S} \left(\prod_{k=1}^{m-1} f(p_k, q_k, r_k, r_{k+1}; s_k, s_{k+1}) \right) f_1(p_m, r_m; s_m) \ .$$

The formula, which is a sum of quotients of factorials, is a discrete analog of a partition function in quantum field theory (QFT) and can be given the following physical interpretation. Consider n identical particles, say bosons,in a 1-dimensional model. Their partition function is symmetric and the symmetrizer acts by interchanging the particles and averaging over all such interchanges. However the process of interchanging particles is not physical. Instead, what is physical is the process of creation and annihilation. A pair of particles annihilate with the release of a photon followed by the creation of a new pair from that photon. Such a contiguous cup-cap pair corresponds to a Jones projection e_i. The problem then is to find in the symmetrizer the amplitude corresponding to each possible creation – annihilation mechanism (labeled by a planar diagram B or by a monomial in the e_i).

The formula given above for the coefficient ρ_B is additive, in the sense that all terms have the same sign. As a consequence, all basis elements have nonzero coefficients. The global sign is precisely the parity of the number of Jones projections in the monomial.

The above formula is directional, i.e. it gives a different expression when computed from right to left and when computed from left to right. The equality between the two expressions could lead to interesting and natural identities for basic (or $q-$)hypergeometric functions of higher type.

From the above formula we have developed a discrete analog of QFT, with -bra and -ket states which are half boxes and operators which are the middle part of a box, modulo symmetrizer relations, and we have found natural bases for each of these. The inner product is in this theory precisely the coefficient of the box computed above. It is remarkable, and unexpected, that the discrete operator product expansion of a product of two base elements has coefficients which are monomials. We have also extended the formula from the symmetrizer boxes as above, which are 2-gons, to general polygons.

We propose to develop the above theory in several natural directions. It should be possible to find global manifestations of this formula, i.e. to understand surface spaces, knot and 3-manifold invariants, as well as the 6j-symbols of $SU(2)_l$, in terms of the monomial coefficient expressions and the corresponding discrete form of QFT. In a different direction, there is a similar canonical basis for the projections corresponding to the projection onto the highest weight irreducible in a tensor product of generators of $\mathrm{Irr}\, SU(k)_l$, with planar diagrams consisting of hexagonal cells. The analogous coefficients of the highest weight projection of $SU(k)_l$ in the hexnet base are likely to have a very interesting mathematical structure.

1.7. The geometrization of quantum subgroups: a construction of weight lattices and roots from quantum subgroups. Subfactors have been linked to many different branches of mathematics; however no geometrical structure of subfactors has been previously found. In particular links between subfactors connected to Coxeter ADE graphs on one side and the Lie algebras, Lie groups and quantum Lie groups built from the same Coxeter ADE graphs were missing. The ADE subfactors had their structure centered around the real valued Jones index, tensoring bimodules, composition of homomorphisms, and knot and 3-manifold invariants. The main structure of the ADE, i.e. simple unimodular, Lie groups is very geometrical with a weight lattice, roots, reflections and the Weyl group, and the deformation quantization.

We have found the natural link between the subgroups of quantum $SU(2)$ and the classical and quantum Lie groups, showing that the information for building a simple Lie group from copies of $SU(2)$ put together in a combinatorial way using the root lattice comes naturally from the fusion structure on representations of a quantum subgroup of $SU(2)$. The bridge between these two areas of research is a hitherto unobserved crystallographic property of homology theory: commuting squares and 6 term exact sequences are squares and hexagons in a lattice, i.e $A_1 \times A_1$ and A_2 root systems, and the 12 terms in the snake lemma form precisely an A_3 root system.

As a consequence, we have obtained a very elementary and natural construction of a canonical basis in the sense of Lusztig from an ADE graph. This elementary construction does not use in an over way quantum subgroups, although the mechanism behind it is based on the ADE subfactor and quantum subgroup.

The fact which shows that this link is indeed of a very general nature which transcends the quiver methods of Ringel and Lusztig is that the process works for our newly discovered quantum subgroups of $SU(3)$. More generally for any subgroup of a quantum semisimple Lie group at a root of unity we can associate in a natural way a finite dimensional weight lattice and a finite set of unimodular roots in it. The problem of associating roots and weights to the generalized Coxeter graphs had been set 15 years ago by Zuber and collaborators, although none of the attempts to make the vertices of such a graph into analogs of simple roots worked.

In our approach, even for the classical Lie groups, we obtain from the ADE graph all the roots at once, with no simple roots distinguished. It is precisely this fact that makes the process work in a very general framework.

For an ADE graph G with Coxeter number N consider the Cartesian product graded over $\mathbf{Z}_2$ of G (on the horizontal) with $\mathbf{Z}_{2N}$ (on the vertical), i.e. divide the vertices of G into even and odd vertices and retain from $\mathrm{Vert}G \times \mathbf{Z}_{2N}$ only the pairs with the same parity. Equivalently, construct a Bratelli diagram on the whole ADE graph G and identify the bottom to the top after $2N$ levels, making the product graph $\mathcal{G}$ lie on the surface of a cylindrical band.

The graph $\mathcal{G}$ has now $|G|\ N$ vertices, exactly the number of roots (according to a theorem of Kostant) in the root system associated to the ADE graph G. We shall in fact **define the roots as the vertices of** $\mathcal{G}$. We now define the inner product between roots as follows. Denote by $\mathrm{Path}^{(n)}_{(x,i),(y,j)}\,\mathcal{G}$ the (downward moving) paths between the vertices $(x,i),(y,j) \in \mathrm{Vert}\mathcal{G} \subset \mathrm{Vert}G \times \mathbf{Z}_{2N}$, of length $n \in \{0,\dots,2N-1\}, n \equiv j-i \mod 2N$. Let $\mathrm{HPath}^{(n)}_{(x,i),(y,j)}\,\mathcal{G}$ denote the Hilbert space with orthonormal basis $\mathrm{Path}^{(n)}_{(x,i),(y,j)}\,\mathcal{G}$. Let μ denote the Perron Frobenius eigenvector of G, extended by $\mu((x,i)) = \mu(x)$ to $\mathcal{G}$ and let $\beta = 2\cos(\pi/N) = [2]$ be its eigenvalue. For $k < n$ denote by $c_k : \mathrm{HPath}^{(n)}\,\mathcal{G} \to \mathrm{HPath}^{(n-2)}\,\mathcal{G}$ the contraction operator, defined for a path $\xi = (\xi(1),\dots,\xi(n))$ by

$$c_k(\xi) = \mathrm{coef}\,\delta_{\xi(k),\xi(k+1)^{-1}}(\xi(1),\dots,\widehat{\xi(k)},\widehat{\xi(k+1)},\dots,\xi(n)),$$

in which $\mathrm{coef} = \mu(s(\xi(k))^{-1/2}\mu(r(\xi(k))^{1/2}$, i.e. c_k cancels the edges $\xi(k),\xi(k+1)$ if they are inverse to each other, and is 0 otherwise. Then $\beta^{-1/2}c_k$ is a coisometry and $e_k = \beta^{-1}c_k^* c_k$ is a Jones projection. Define the **essential path subspace**

$$\mathrm{EssPath}^{(n)}\,\mathcal{G} = \{\rho \in \mathrm{HPath}^{(n)}\,\mathcal{G} : c_k(\rho) = 0 \text{ for any } k = 1,\dots,n-1\}$$

consisting of the “non-repetitive” linear combinations of paths.

If we view the vertices of G as irreducibles of a subgroup or module S of $\mathrm{Irr}\,SU(2)_{N-2}$ then paths of length n correspond to repeated tensor products n times with the generator σ_1 of $\mathrm{Irr}\,SU(2)_{N-2}$ while essential paths correspond to tensor products with the irreducible $\sigma_n \subset \sigma_1^{\otimes n}$. The maximal length of essential paths is thus the level $l = N-2$, since $\sigma_{N-1} = 0 \in \mathrm{Irr}\,SU(2)_{N-2}$.

Let $N^{(y,j)}_{n,(x,i)} = \dim \mathrm{EssPath}^{(n)}_{(x,i),(y,j)}\,\mathcal{G}$. Then $N^{(y,j)}_{n,(x,i)} = \dim \mathrm{Hom}[\sigma_n \otimes x, y]$ is the fusion number corresponding to the module structure of the vertices of G over $\mathrm{Irr}\,SU(2)_{N-2}$. Recall that $\sigma_{N-1} \in \mathrm{Irr}\,SU(2)_{N-2}$ is killed by the cutoff. We shall continue counting the irreducibles of $SU(2)_{N-2}$ by reflection around the killed level $N-1$, i.e. we take $\sigma_{N-1} = 0, \sigma_N = -\sigma_{N-2}, \sigma_{N+1} = -\sigma_{N-3}$, etc. Accordingly we let $N^{(y,j)}_{N-1+k,(x,i)} = -N^{(y,j)}_{N-1-k,(x,i)}$ and we further extend the fusion number $N^{(y,j)}_{n,(x,i)}$ to arbitrary $n \in \mathbf{Z}$ by replacing n with $n \mod 2N$. We have then $N^{(y,j)}_{-1+k,(x,i)} = -N^{(y,j)}_{-1-k,(x,i)}$.

We define the **inner product between roots** $(x,i),(y,j) \in \mathrm{Vert}\mathcal{G}$ by

$$\langle (x,i),(y,j)\rangle = N^{(y,j)}_{j-i,(x,i)} + N^{(y,j)}_{i-j,(x,i)} = N^{(y,j)}_{j-i,(x,i)} - N^{(y,j)}_{j-i-2,(x,i)}$$

The above inner product extends by linearity to an inner product on the linear span of the roots; the nondegenerate quotient is positive definite and has dimension $|G|$. Although the fusion numbers $N^{(y,j)}_{j-i,(x,i)}$ can be as big as 6 for the graph E_8, the inner product $\langle (x,i),(y,j)\rangle$ always takes values in $\{-2,-1,0,1,2\}$.

We have defined this way all the roots at once, entirely in terms of fusion on the subgroup or module S of $SU(2)_{N-2}$ (or elementarily in terms of essential paths on the graph.) In this setting the vertical shift by 2 is a Coxeter element of the Weyl group acting on roots.

A similarly simple construction gives the weight lattice as follows. For scalar valued functions on the vertices of the graph G define the Laplacian Δ_G as the sum of neighbors, i.e. the matrix of Δ_G is the adjacency matrix of G. Similarly $\Delta_{\mathbf{Z}_{2N}}$ is the Laplacian on $\mathbf{Z}_{2N}$. Call a $\mathbf{Z}$-valued function f on $\mathrm{Vert}\mathcal{G} \subset \mathrm{Vert}G \times \mathbf{Z}_{2N}$ **harmonic** if $\Delta_G(f) = \Delta_{\mathbf{Z}_{2N}}(f)$, i.e. if the sum of neighbors taken vertically equals the sum of neighbors taken horizontally. For instance for any fixed $(x,i) \in \mathrm{Vert}\mathcal{G}$ the function $(y,j) \mapsto N^{(y,j)}_{j-i,(x,i)}$ is harmonic. The **weights are the integer valued harmonic functions** on $\mathrm{Vert}\mathcal{G}$.

The remarkable fact about the above formulae is that they extend naturally to any subgroup or module S of any quantum Lie group $\mathbf{G}_l$ at a root of unity, such as the ones we have classified fro $SU(3)_l$ and $SU(4)_l$. Instead of having $\mathcal{G} \subset G \times_{\mathbf{Z}_2} \mathbf{Z} = G \times_{\mathbf{Z}_2} P_{SU(2)}$ where $P_{SU(2)}$ is the weight lattice of $SU(2)$ we let $\mathcal{G} \subset G \times_Z P_{\mathbf{G}}$ where the Cartesian product is graded over the center Z of $\mathbf{G}$.

The formula for the inner product between roots becomes, in the spirit of the Weyl character formula,

$$\langle (x,i),(y,j)\rangle = \sum_{w \in W} \varepsilon(w) N^{(y,j)}_{j-i+w\rho-\rho,(x,i)}$$

where W is the Weyl group and ρ the Weyl vector of the Lie group $\mathbf{G}$. All roots have square length $|W|$, i.e. the square length 2 for the classical unimodular lattice vectors comes from the fact that the Weyl group of $SU(2)$ has 2 elements.

This inner product is periodic, i.e. the points of $G \times_Z P_{\mathbf{G}}$ are naturally quotiented by a sublattice into a finite set of roots lying on a torus.

Thus for example from the first exceptional subgroup of $SU(3)_5$ one obtains 256 roots of square length 6 in a weight lattice of dimension 24.

The first nontrivial new root lattice corresponds to $SU(3)_1$, a graph with 3 points analogous to the graph $A_2 = SU(2)_1$ for $SU(2)$. The construction above yields from $SU(3)_1$ 16 roots of square length 6 in a lattice of dimension 6. A number theoretical formula for the theta function of this lattice has helped us identify it, using the online database of Conway, Sloane and Nebe as the lattice D_6^+, the union between D_6 and a translated copy of D_6. This lattice had not appeared before in representation theory (as opposed to D_8^+ which is the lattice E_8). The lattice D_6^+ exhibits in its new role a hexagonal symmetry not noticed before. This is a coincidence of small numbers (or rather small lattices); all the other lattices produced by our method from subgroups appear to be new.

For dimensions ≤ 8 the best sphere packing is provided by the classical root lattices, which from our point of view correspond to exceptional subgroups of quantum $SU(2)$. In higher dimensions, where good candidates are missing, the exceptional subgroups of quantum $SU(k)$, $k \geq 3$ could be candidates for optimal packing, a direction which we intend to investigate.

1.8. Homology and crystallography. In the previous paragraph we have described the way in which the roots and weight lattices of a simple Lie group arise out of fusion numbers for the irreducibles of a subgroup. It is in fact possible to go beyond counting and construct a natural basis of the universal enveloping algebra of a quantum simple Lie group.

Essential paths have a natural product, obtained by projecting onto the essential paths the concatenation of two paths. One extends this product to $V \otimes \mathrm{EssPath}^{(n)}_{(x,i),(y,j)}\, \mathcal{G} \otimes \overline{W} \,=\, \mathrm{EssPath}^{(n)}_{(x,i),(y,j)}\, \mathcal{G} \otimes \mathrm{Hom}[V,W]$ where V, W are finite dimensional multiplicity vector spaces attached respectively to $(x,i),(y,j)$. We can now take kernels of maps; due to the presence of the essential path factor kernels have kernels again, and in 6 steps any exact sequence moves around the cylinder $2N$ levels and returns to the starting point.

Unimodular root systems are characterized by the fact that any 2 dimensional section is a square or a hexagon, and for us the squares correspond to commuting squares and the hexagons correspond to 6 term exact sequences. This points to the fact that in general homology theory appears to have a crystallographic aspect, manifested by the ubiquitous presence of 6-term exact sequences, but not of 5 or 7 terms exact sequences. The snake lemma, which connects 12 terms by means of 4 exact sequences with 6 terms each, is precisely the root system of type A_3, i.e. if we join the top to the bottom of the snake, the 4 exact sequences become the 4 hexagons joining the mid-edges of a cube. These observations appear to be new and deserve further investigation. This crystallographic aspect is essential for us as a bridge between an ADE subgroup of $SU(2)_l$ and the corresponding ADE Lie group.

An element of the canonical basis of the off diagonal universal enveloping algebra $\mathbf{U}^+ \cup \mathbf{U}^-$ is a formal exponential e^f where f is a function $f : \mathrm{Vert}\mathcal{G} \to \mathbf{N}$. The product $e^f \circ e^g$ is defined as follows. Define vector spaces $F(x,i)$ of dimension $f((x,i))$ at every vertex (x,i) of $\mathcal{G}$, define similarly spaces $G(x,i)$ from g. Define

$$\mathrm{Hom}[F,G] = \bigoplus_{(x,i),(y,j)} F(x,i) \otimes \mathrm{EssPath}^{(n)}_{(x,i),(y,j)}\, \mathcal{G} \otimes \overline{G(y,j)}\ .$$

Construct now extensions H of G by F, i.e. find spaces $H(x,i)$ of dimension $h((x,i))$ at every vertex (x,i) of $\mathcal{G}$, and maps $\alpha \in \mathrm{Hom}[F,H]$ and $\beta \in \mathrm{Hom}[H,G]$ which form an exact sequence. Count the number of such extensions when the vector spaces are taken over a field with q elements and obtain (after a suitable normalization) a number $c^h_{f,g}(q)$ which is a polynomial in q. Then the product is

$$e^f \circ e^g = \sum_h c^h_{f,g}(q)\ e^h$$

This way we obtain directly the quantum deformation of the simple Lie groups. The snake lemma then shows that this product is associative. Another similar term takes care of the diagonal part $\mathbf{U^0}$.

2. New directions of research

We have described in a very simple and natural way the construction of the simple Lie groups and of their quantum deformations from the quantum subgroups of $SU(2)$, and have developed a general theory of subgroups of quantum groups to a level which allowed the first classification results for $SU(3)$ and $SU(4)$, as well as general bounds on the level of exceptionals.

A first direction that should be investigated in this context is the construction of the irreducible representations of quantum groups.

The main question which follows is the **existence of higher analogs of the simple Lie groups**, constructed from the above lattices. The classical and quantum simple Lie groups, constructed by the above procedure from $SU(2)_l$ have a binary composition law related to the fact that the main building block of the classical root lattices, the hexagon $SU(2)_1$, is 2 dimensional. The higher analogs of the simple Lie groups obtained this way would likely model $SU(k)_1$ and the corresponding composition laws would be natural many-to-one laws. We have recently obtained experimental evidence for this mechanism.

The importance of such multi-nary laws is the following. Spaces of homomorphisms provide natural and universal models for quantum field theoretic Hilbert spaces, such as the ones which appear in conformal field theory. The main property is the **tensoriality**. The sections of ordinary vector bundles over a disjoint union of base sets are the direct sum of the sections over each set. In Quantum Field Theory, the direct sums must be replaced by tensor products. Tensoring over a common algebra is the gluing mechanism for overlapping base sets.

The model of a tensorial functor is $\mathrm{Hom}[\alpha \otimes \beta, \gamma]$ for $\alpha, \beta, \gamma \in \mathrm{Irr}\, G$ e.g. with G a group or quantum group, naturally associated to a triangle with edges labeled α, β, γ. This yields naturally 2-dimensional models, e.g. 2-dimensional conformal field theory (CFT). String theory then maps such 2-surfaces into higher dimensional spaces.

In order to make such models of QFT physical, which is a crucial problem in theoretical physics, a stronger approach would be to construct natural mathematical objects with tensorial behavior, which are attached, as functors, to the 3 or preferably 4 dimensional manifolds in general relativity. This problem is the **dimension barrier** in QFT. In our view, the impact of the last century's quantum mechanics has been strong enough to draw attention upon noncommutative mathematics, such as operator algebra, but the study of the higher dimensional tensorial objects required by quantum field theory is only beginning.

To break the dimension barrier, one must find for instance natural examples of $\mathrm{Hom}[\alpha \otimes \beta \otimes \gamma, \delta]$, where now $\alpha, \beta, \gamma, \delta$ live on the faces of a tetrahedron and exhibit a tetrahedron symmetry and behavior.

This is in our view part of the potential of the higher analogs of the simple Lie groups for which we have now roots and weight lattices, built as described in these notes. The next step after the understanding of their structure would be the construction of higher dimensional QFT models, invariants for higher dimensional PL manifolds, etc.

3. Conclusions

It is our opinion that **the study of the group-like invariants of finite depth subfactors and of quantum subgroups is now in a situation similar to the study of simple Lie algebras a century ago**. While general rigidity results are now known, the discrete structure behind their classification is only beginning to appear in a few concrete cases. The further study of these structures, with strong motivation coming from the physics of quantum field theory (via topological quantum field theory), quantum gravity (from the spin models originated by Regge and Penrose to current spin foam models) and quantum chromodynamics (with the apparition of quantum SU(3) in Connes's model based on noncommutative geometry) is likely to be an important direction of research in the coming century.

The inner structure of quantum symmetrizers, i.e. the closed form for coefficients of the Wenzl projectors, shows that unexpected new forms of discrete QFT appear inside mathematical objects which were considered well understood, and deserves further study.

A topic of particular interest would be the discovery of the higher analogs of the classical and quantum Lie groups, since these structures could impact constructive quantum field theory in a physical number of dimensions. This is also the most difficult of the topics to study, since the structures we are looking for, although very natural, are very different from any existing mathematical objects, but we have begun to obtain encouraging experimental data on it.

4. Figures

The following figures describe our classification of the modules and subgroups of quantum $SU(N)$, $N = 2, 3, 4$ at the k-th roots of 1, for which the Young diagrams of the irreducible representations are constrained to less than N (=rank) rows and k (=level) columns. The graphs show the fusion with the generators of $SU(N)_k$, in a picture analogous to the McKay fusion graphs for the finite subgroups of $SU(2)$. They answer problems of Zuber concerning the classification of higher analogs of the Coxeter ADE graphs. In general we have shown that any WZW theory has a finite number of exceptionals.

Subgroups and modules. The modules of $SU(N)_k4$ have a natural fusion with the irreducibles of $SU(N)_k$. Among modules, the subgroups have a naturally defined self-fusion; subgroups are titled bold. Each subgroup has a unit, which is starred. The subscript in a graph name denotes the level, except for $SU(2)_k$, for which the level notation is given in parentheses under the traditional name. Small level duplicates in the series are bracketed. Modules are interpreted in topological quantum field theories (TQFT) as new types of vertices, and also as boundary extensions of the theory. In the operator algebras bimodule picture they are new types of algebras and in 2-dimensional conformal field theory (CFT) they are boundary extensions.

Vertices. The vertices of the graphs are irreducible representations. The shades of gray from white to black indicate the grading. In CFT, vertices are primary operators, while in operator algebras the vertices are irreducible bimodules; in TQFT they label 1-dimensional edges.

Graphs. The red graph is the graph of tensoring with the standard irreducible σ_1 of $SU(N)_k$. The blue graph for $SU(4)_k$ is the graph of tensoring with $\sigma_2 = \sigma_1\Lambda\sigma_1$. Where necessary the edges of the graphs are oriented explicitly.

Self-connections. A necessary and sufficient condition for the existence of a module is given by a self-connection, which consists of a system of complex numbers called cells, satisfying certain local equations of cohomological nature. Cells reside in the triangles of the graphs and describe the composition of the intertwiners represented by edges. Cells are chosen up to a unitary gauge coming from the choice of intertwiners for the edges. The graphs and the self-connection determine a module completely. For $SU(2)_k$ the cells reside in the 2-gons of the graphs and the existence of the self-connection excludes the tadpoles. In the classical case, the existence of the self-connection on an $SU(N)$ graph is necessary and sufficient for the existence of a subgroup of $SU(N)$ with the corresponding irreducibles. The cells provide the data for a Hecke algebra representation yielding a solvable statistical mechanical plaques model for each graph.

Chirality. To each module M corresponds a subgroup G^+ called the chiral positive (or geometrically flat) part of M, a closed subystem S of G^+ called the ambichiral set, and an automorphism θ called the ambichiral twist on S. The quantum automorphisms of the module M are a product of the flat part G^+ of M with the conjugate G^- of G^+, fibered over the ambichirals $S = G^+ \cap G^-$ with the twist θ. Among the ambichiral automorphisms of $SU(N)_k$, $N > 2$, the conjugation c replaces the generators σ_i of $SU(N)_k$ by their conjugates σ_{N-i}.

Exceptional twists. For each N there is an exceptional ambichiral twist, denoted by t, on an orbifold of $SU(N)_k$: on $SU(2)_{16}$ it gives $(SU(2)_{16})^t = E_7$; on $SU(3)_9$ it gives $(SU(3)_9)^t$, and on $SU(4)_8$ it gives $(SU(4)_8)^t$.

Orbifold series and exceptionals. $SU(N)_k$ is denoted by A_k. Its series subgroups are, for $SU(2)_k$: $A_k/2$ for $k = 0 \mod 4$; for $SU(3)_k$: $A_k/3$ for $k = 0 \mod 3$ and for $SU(4)_k$: $A_k/2$ for $k = 0 \mod 2$, $A_k/4$ for $k = 0 \mod 8$. The exceptional subgroups are denoted by E_k.

For $SU(2)_k$ the series $A_k/2 = D_{k/2+2}$ modules exist only for $k = 0 \mod 2$, and for $SU(4)_k$ the series $A_k/4$ modules and their conjugates $(A_k/4)^c = 2(A^c/2)_k$ exist only for $k = 0, 2, 6 \mod 8$.

The modules of $SU(N)_k$ arise from (i.e., have the chiral part G^+ equal to) the following subgroups: for $SU(3)_k$ and $k = 0 \mod 3$, $3{A^c}_k$ from $A_k/3$; for $SU(4)_k$ and $k = 0 \mod 8$, $2(A^c/2)_k$ from $A_k/4$; for $k = 0 \mod 2$, $2{A^c}_k$ from $A_k/2$; all other series modules from A_k. The notation of exceptional modules indicates the chiral part.

Conformal inclusions. All exceptional subgroups of $SU(2)_k$ and $SU(3)_k$ as well as some small orbifolds arise from conformal inclusions. It was conjectured that all exceptional subgroups and modular invariants come from conformal inclusions. For $SU(4)_k$ the exceptional $E8$ is not a conformal inclusion. The following subgroups arise from conformal inclusions: for $SU(2)_k$, D_4 from $SU(2)_4$ in $SU(3)_1$, E_6 from $SU(2)_{10}$ in $SO(5)_1$, and E_8 from $SU(2)_{28}$ in $(G_2)_1$; for $SU(3)_k$, $A_3/3$ from $SU(3)_3$ in $SO(8)_1$, E_5 from $SU(3)_5$ in $SU(6)_1$, E_9 from $SU(3)_9$ in $(E_6)_1$, and E_{21} from $SU(3)_{21}$ in $(E_7)_1$; for $SU(4)_k$, $2{A^c}_2 = A_2/2$ from $SU(4)_2$ in $SU(6)_1$, E_4 from $SU(4)_4$ in $SO(15)_1$, and E_6 from $SU(4)_6$ in $SU(10)_1$.

Modular invariants. Each module produces a modular invariant, which is a positive integer valued matrix, intertwiner of the modular group representation on the affine characters of $SU(N)_k$. The modular invariant is a manifestation of the fact that the corresponding TQFT and CFT are defined on the torus. The modular invariants for $SU(2)_k$ were classified by Capelli, Itzykson, and Zuber. The graphs for $SU(2)_k$ arose in our classification of small index subfactors. The $SU(3)_k$ modular invariants were classified by Gannon. The graphs for $SU(3)_k$ are a subset of the list proposed empirically by di Francesco and Zuber; the precise connection between graphs and modular invariants was an open problem. The natural interpretation of the graphs is part of our maximal atlas theory, which is a 3-dimensional TQFT analog of the Morita equivalence theory for 2-dimensional TQFT. The $SU(3)_k$ classification showed that different graphs can share the same modular invariant, and the exceptional E_8 of the $SU(4)_k$ classification showed that not all exceptional graphs come from conformal inclusions.

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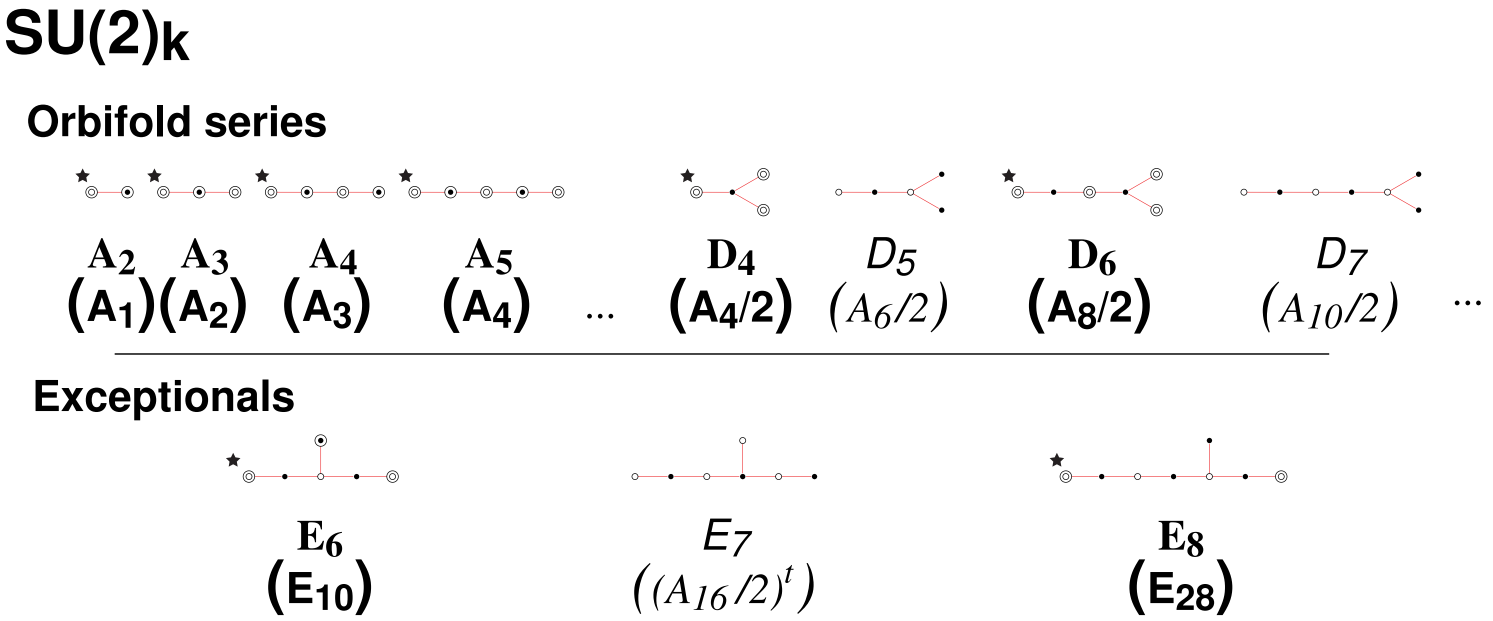


FIGURE 1. Classification of modules and subgroups of quantum $SU(2)$.

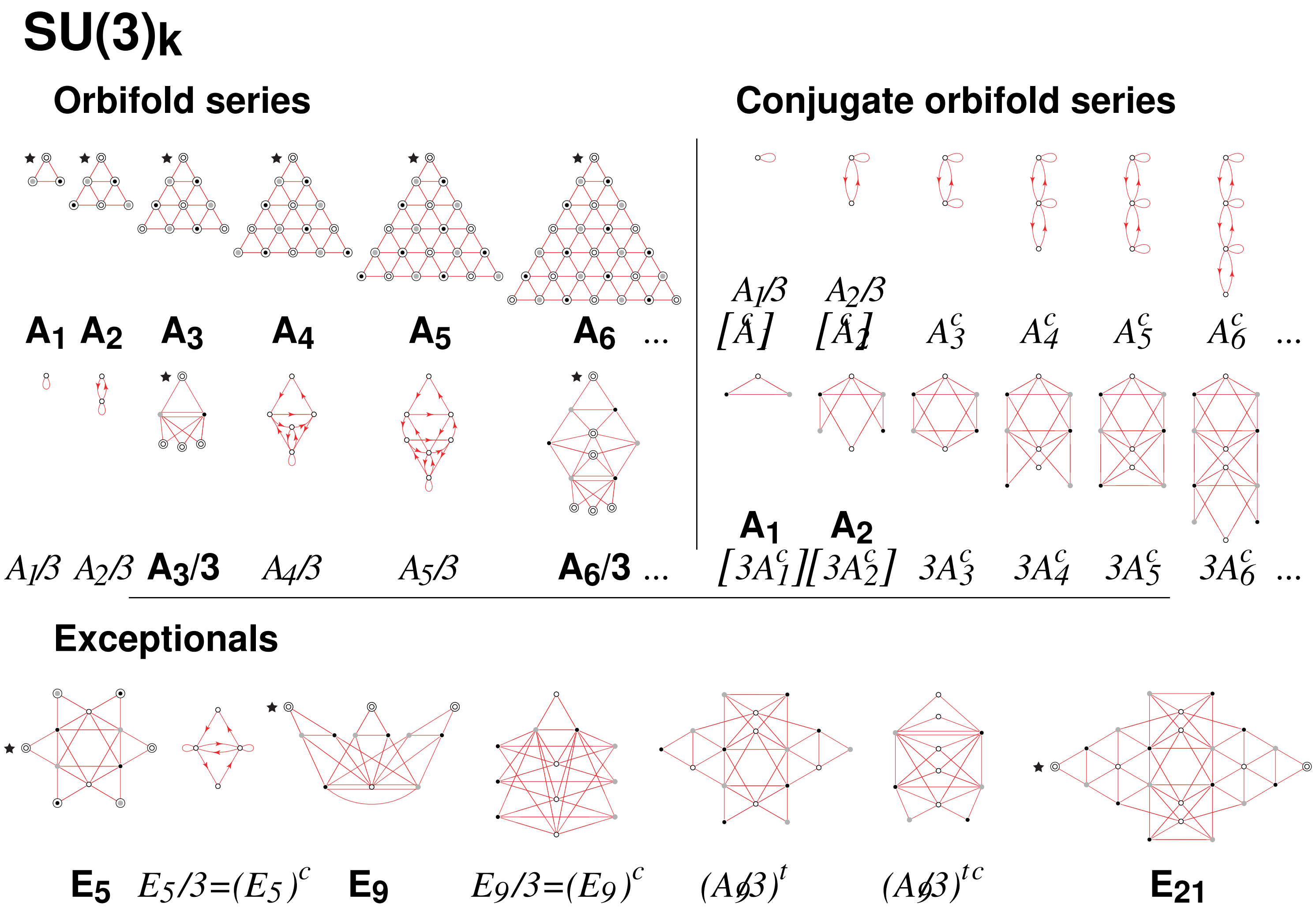


FIGURE 2. Classification of modules and subgroups of quantum $SU(3)$.

$SU(4)_k$

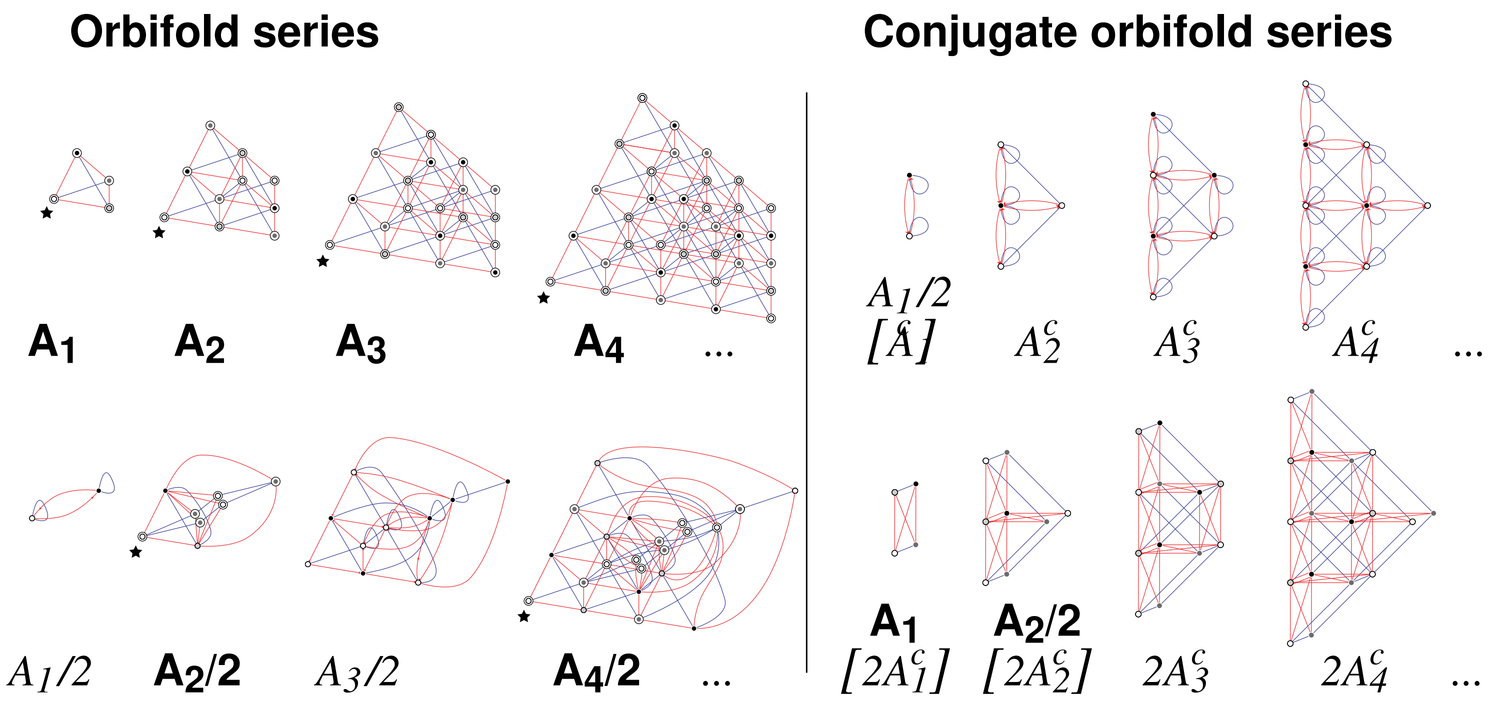


FIGURE 3

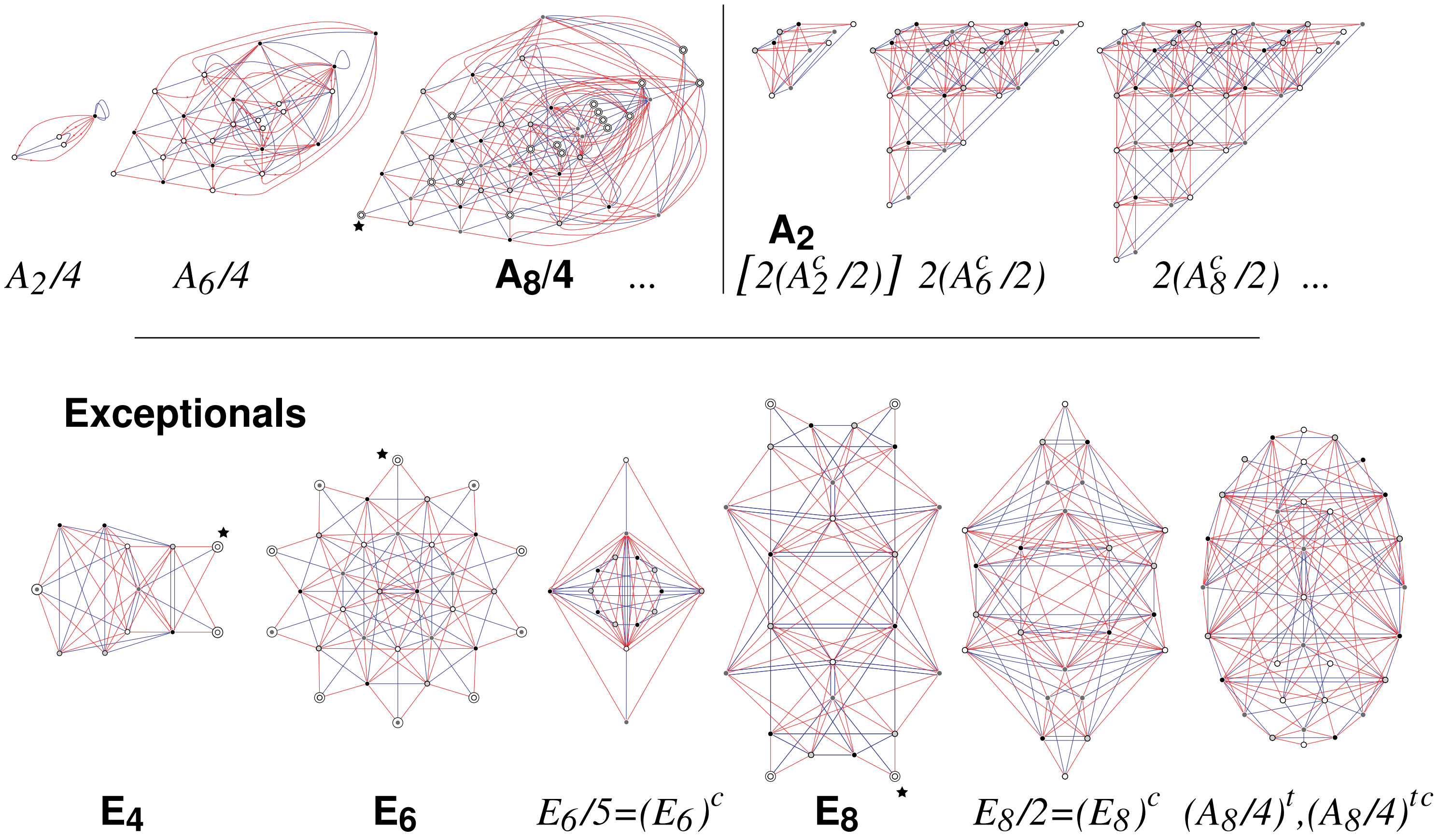


FIGURE 4. Classification of modules and subgroups of quantum $SU(4)$.

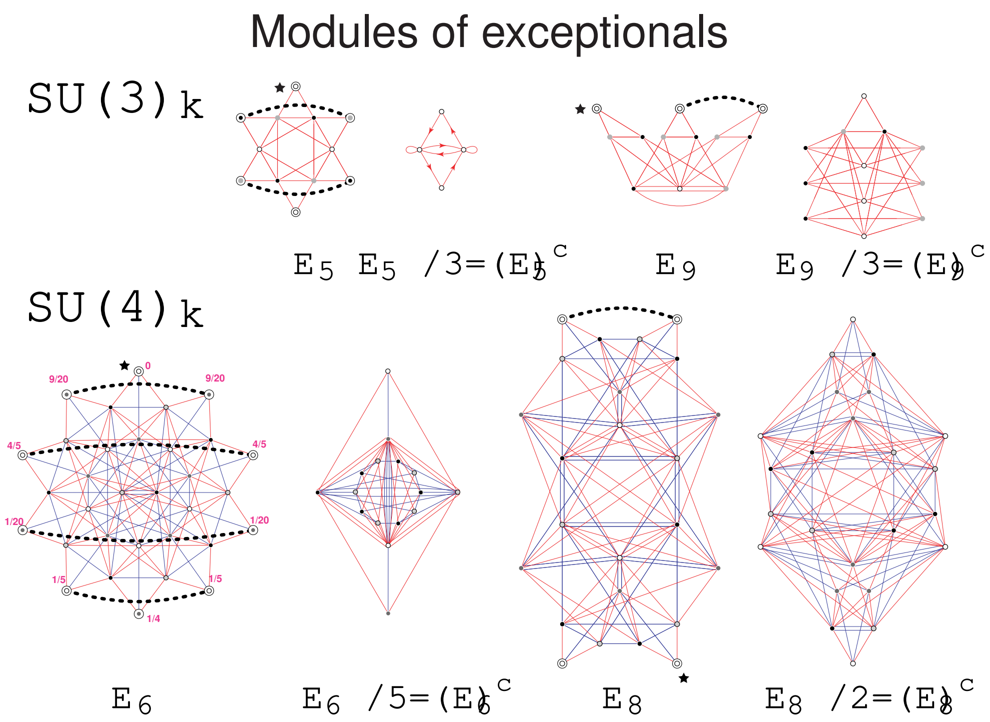


FIGURE 5. Modules of exceptionals.

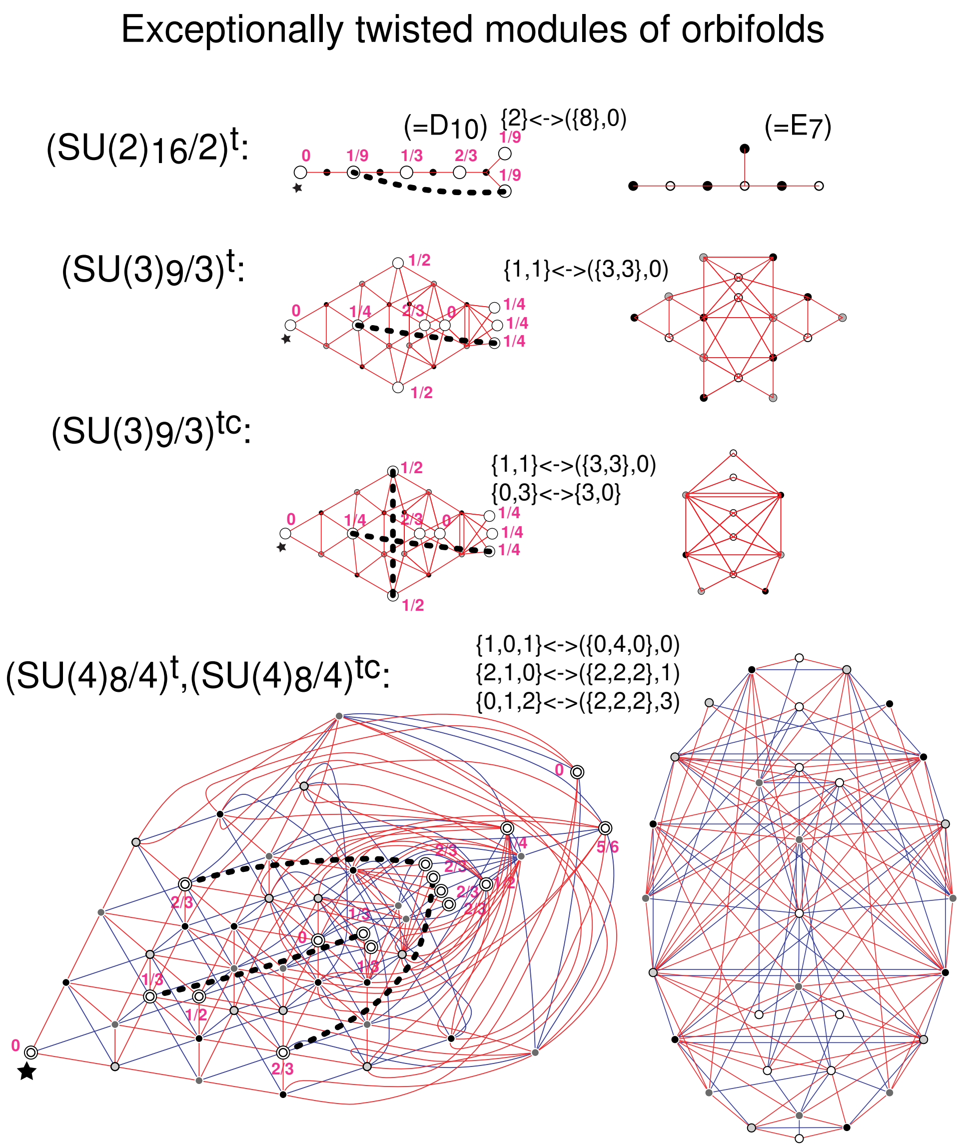
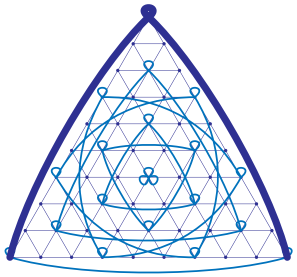
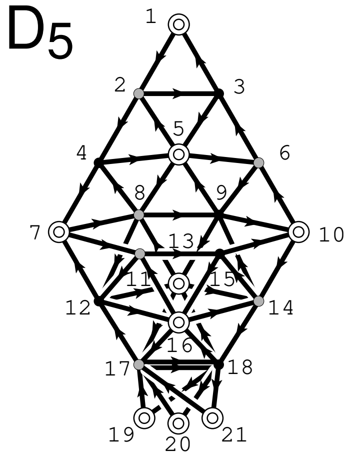


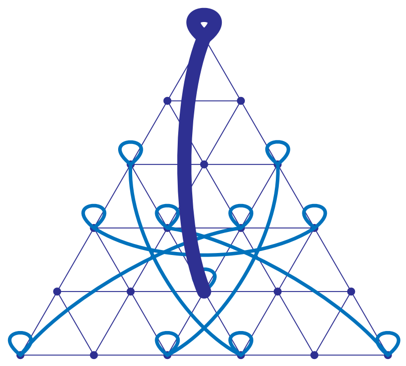
FIGURE 6. Exceptionally twisted modules of orbifolds.

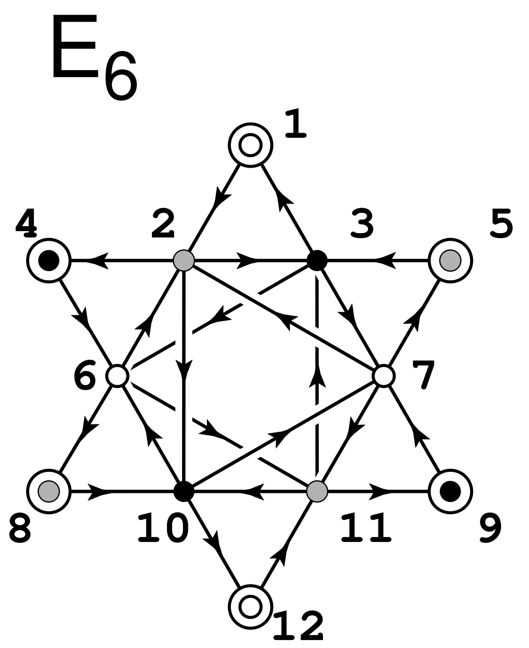




		0	1	2	3	4	5	6	7	8	9
1	(0,0)	1	..	...	....	.....	......	....	....	....	1........1
2	(1,0)	.	1.	...	....	.....	......	....	....	1	.1........
3	(0,1)	.	.1	...	....	.....	......	....	....	1........	........1.
4	(2,0)	.	..	1..	....	.....	......	....	1	....	..1.......
5	(1,1)	.	..	.1.	....	.....	......	....	....	.1.....1.	....
6	(0,2)	.	..	..1	....	.....	......	....	1.......	....	.......1..
7	(3,0)	.	..	...	1...	.....	......	1	....	....	...1......
8	(2,1)	.	..	...	.1..	.....	......	....	......1.	..1......	....
9	(1,2)	.	..	...	..1.	.....	......	....	.1......	......1..	....
10	(0,3)	.	..	...	...1	.....	......	1......	....	....	......1...
11	(4,0)	.	..	...	....	1....	1	....	....	....	....1.....
12	(3,1)	.	..	...	....	.1...	......	.....1.	....	...1.....	....
13	(2,2)	.	..	...	....	..1..	......	....	..1..1..	....	....
14	(1,3)	.	..	...	....	...1.	......	.1.....	....	.....1...	....
15	(0,4)	.	..	...	....	1	1.....	....	....	....	.....1....
16	(4,1)	.	..	...	....	.....	.1..1.	....	....	....1....	....
17	(3,2)	.	..	...	....	.....	..1...	....1..	...1....	....	....
18	(2,3)	.	..	...	....	.....	...1..	..1....	....1...	....	....
19	(3,3)	.	..	...	....	.....	......	...1...	....	....	....
20	(3,3)	.	..	...	....	.....	......	...1...	....	....	....
21	(3,3)	.	..	...	....	.....	......	...1...	....	....	....

FIGURE 7. Modular ladder for the D_5 graph.





		0	1	2	3	4	5
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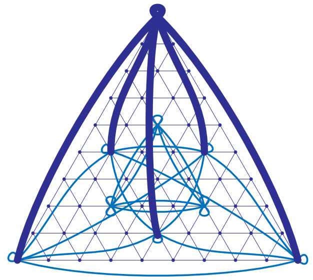
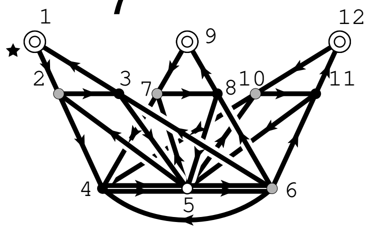
2	(1,0)	.	1.	...	.1..	...1.	..1...
3	(0,1)	.	.1	...	..1.	.1...	...1..
4	(2,3)	.	..	1..	....	.....	...1..
5	(3,2)	.	..	..1	....	.....	..1...
6	(1,4)	.	..	.1.	1...	..1..	....1.
7	(4,1)	.	..	.1.	...1	..1..	.1....
8	(0,5)	.	..	...	.1..	.....	1
9	(5,0)	.	..	...	..1.	.....	1.....
10	(0,2)	.	..	1..	..1.	.1..1	......
11	(2,0)	.	..	..1	.1..	1..1.	......
12	(3,0)	.	..	...	1..1	.....	......

FIGURE 8. Modular ladder for the E_6 graph.



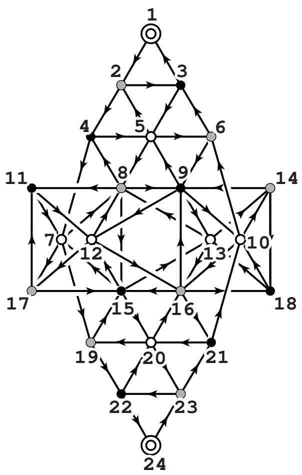
E_7^S



		0	1	2	3	4	5	6	7	8	9
1	(0,0)	1	..	...	....	.....	.1..1.	....	....	....1....	1........1
2	(1,0)	.	1.	...	....	1..1.	..1..1	.1..1..	...1....	.....1..1	.1..1.....
3	(0,1)	.	.1	...	....	.1..1	1..1..	..1..1.	....1...	1..1.....	.....1..1.
4	(2,0)	.	..	1..	..1.	.1..1	1..2..	..2..1.	.1..2..1	...1..1..	..1..1....
5	(1,1)	.	..	.1.	1..1	..2..	.2..2.	1..3..1	..2..2..	.1..2..1.	...1..1...
6	(0,2)	.	..	..1	.1..	1..1.	..2..1	.1..2..	1..2..1.	..1..1...	....1..1..
7	(2,1)	.	..	...	.1..	...1.	..1...	.1..1..	...1..1.	..1..1...	....
8	(1,2)	.	..	...	..1.	.1...	...1..	..1..1.	.1..1...	...1..1..	....
9	(2,2)	.	..	...	....	..1..	......	....	..1..1..	....	....
10	(2,1)	.	..	...	.1..	...1.	..1...	.1..1..	...1..1.	..1..1...	....
11	(1,2)	.	..	...	..1.	.1...	...1..	..1..1.	.1..1...	...1..1..	....
12	(2,2)	.	..	...	....	..1..	......	....	..1..1..	....	....

FIGURE 9. Modular ladder for the E_7^s graph.

E8



		0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	(0,0)	1	..	...	....	.....	......	....	....	....1....	....	....	.1........1.	......1......	....
2	(1,0)	.	1.	...	....	.....	......	....	...1....	.....1...	....1.....	1........1.	..1..1.....1	.1.....1..1..	......1.......
3	(0,1)	.	.1	...	....	.....	......	....	....1...	...1.....	.....1....	.1........1	1.....1..1..	..1..1.....1.	.......1......
4	(2,0)	.	..	1..	....	.....	......	..1....	....1...	...1..1..	.....1..1.	.1..2.....1	1..1..1..1..	..1..1..1..1.	.1..1..2..1...
5	(1,1)	.	..	.1.	....	.....	......	...1...	..1..1..	....2....	1..1..1..1	..1..2..1..	.2..1..1..2.	1..1..2..1..1	..1..2..2..1..
6	(0,2)	.	..	..1	....	.....	......	....1..	...1....	..1..1...	.1..1.....	1.....2..1.	..1..1..1..1	.1..1..1..1..	...1..2..1..1.
7	(3,0)	.	..	...	1...	.....	.1....	...1...	..1..1..	....1..2.	...2..1..1	..1..2..1..	.1..3..1..1.	1..2..2..2...	..1..2..2..1..
8	(2,1)	.	..	...	.1..	.....	..1...	.1..1..	...2..1.	..1..2..1	.1..3..2..	1..3..2..2.	..2..3..2..1	.2..3..3..2..	1..2..4..3..1.
9	(1,2)	.	..	...	..1.	.....	...1..	..1..1.	.1..2...	1..2..1..	..2..3..1.	.2..2..3..1	1..2..3..2..	..2..3..3..2.	.1..3..4..2..1
10	(0,3)	.	..	...	...1	.....	....1.	...1...	..1..1..	.2..1....	1..1..2...	..1..2..1..	.1..1..3..1.	...2..2..2..1	..1..2..2..1..
11	(11,0)	.	..	...	....	.1...	......	..1....	....1..1	...1..1..	..1..1..1.	....2..1..1	1..2..1..1..	..1..2..1....	.1..1..2..2...
12	(12,0)	.	..	...	....	..1..	.1....	...1..1	..1..1..	.1..2..1.	...2..2..1	..2..2..2..	.1..3..2..1.	1..2..3..2...	..2..3..3..2..
13	(9,0)	.	..	...	....	..1..	....1.	1..1...	..1..1..	.1..2..1.	1..2..2...	..2..2..2..	.1..2..3..1.	...2..3..2..1	..2..3..3..2..
14	(10,0)	.	..	...	....	...1.	......	....1..	1..1....	..1..1...	.1..1..1..	1..1..2....	..1..1..2..1	....1..2..1..	...2..2..1..1.
15	(1,17)	.	..	...	....	.1...	1..1..	..1..1.	.1..1..1	...2..2..	..2..2..2.	.1..3..2...	...3..3..2..	..2..4..3..1.	.2..3..3..3...
16	(17,1)	.	..	...	....	...1.	..1..1	.1..1..	1..1..1.	..2..2...	.2..2..2..	...2..3..1.	..2..3..3...	.1..3..4..2..	...3..3..3..2.
17	(2,16)	.	..	...	....	1....	..1...	....	...1..1.	..1..1..1	....1..1..	...2..1..1.	..1..2..1...	.1..2..1..1..	1..1..2..2....
18	(16,2)	.	..	...	....	1	...1..	....	.1..1...	1..1..1..	..1..1....	.1..1..2...	...1..2..1..	..1..1..2..1.	....2..2..1..1
19	(4,0)	.	..	...	....	1....	......	.1..1..	......1.	..1.....1	....1..1..	...1..1....	..1..2..1...	....2..1..1..	1..1..1..2....
20	(4,1)	.	..	...	....	.....	.1..1.	1.....1	..1..1..	.1.....1.	...1..1...	..1..2..1..	....2..2....	...2..2..2...	..1..2..2..1..
21	(0,4)	.	..	...	....	1	......	..1..1.	.1......	1.....1..	..1..1....	....1..1...	...1..2..1..	..1..1..2....	....2..1..1..1
22	(5,0)	.	..	...	....	.....	1.....	.....1.	.1.....1	....	....	....1..1...	..1..1......	.....1..1....	....1.....1...
23	(0,5)	.	..	...	....	.....	1	.1.....	1.....1.	....	....	...1..1....	.....1..1...	....1..1.....	...1.....1....
24	(6,0)	.	..	...	....	.....	......	1.....1	....	....	....	....	....1..1....	....	....

FIGURE 10

		14	15	16	17	18	19	20	21
1	(0,0)	....	......1..1......	....	....1........1....	....	....	..........1..........	1....................1
2	(1,0)	.....1..1......	.......1..1.....	...1..1..1..1....	.....1........1...	....1........1.....	..........1.........	...........1........1	.1........1...........
3	(0,1)	......1..1.....	.....1..1.......	....1..1..1..1...	...1........1.....	.....1........1....	...........1........	1........1...........	...........1........1.
4	(2,0)	......2..1.....	..1..1..2..2....	....1..1..1..1...	...1..2..1..1..1..	.....1..1.....1....	....1.....1..1.....1	.........1..1........	..1........1..........
5	(1,1)	....1..3..1....	...1..2..2..1...	..1..2..2..2..1..	....2..1..1..2....	...1..1..1..1..1...	......1..1..1..1....	.1........2........1.	.........1..1.........
6	(0,2)	.....1..2......	....2..2..1..1..	...1..1..1..1....	..1..1..1..2..1...	....1.....1..1.....	1.....1..1.....1....	........1..1.........	..........1........1..
7	(3,0)	.2..1..3..3....	...1..2..2..2...	..1..2..2..2..1..	....1..3..1..1..1.	...1..2..2..1..1..1	......1..1..1..1....	....1.....1..2.......	...1........1.........
8	(2,1)	..2..3..4..3...	.1..2..4..3..1..	...2..3..3..3..1.	..1..3..3..2..2...	....2..3..2..2..1..	...1..1..2..2..1..1.	..1..1..1..2..1......	..........1..1........
9	(1,2)	...3..4..3..2..	..1..3..4..2..1.	.1..3..3..3..2...	...2..2..3..3..1..	..1..2..2..3..2....	.1..1..2..2..1..1...	.......1..2..1..1..1.	........1..1..........
10	(0,3)	....3..3..1..2.	...2..2..2..1...	..1..2..2..2..1..	.1..1..1..3..1....	1..1..1..2..2..1...	......1..1..1..1....	........2..1.....1...	.........1........1...
11	(11,0)	1..1..2..2..1..	..1..1..2..2....	....1..2..1..1...	...1..2..1..1..1..	.....1..2..1..1..1.	....1..1..1..1......	...1........1........	...........1..1.......
12	(12,0)	.1..2..3..3..1.	1..2..3..3..2...	..1..2..3..2..1..	....2..3..2..2..1.	...1..2..2..2..1...	..1..1..2..2..1..1..	....1..1..1..1.......	............1..1......
13	(9,0)	.1..3..3..2..1.	...2..3..3..2..1	..1..2..3..2..1..	.1..2..2..3..2....	...1..2..2..2..1...	..1..1..2..2..1..1..	.......1..1..1..1....	......1..1............
14	(10,0)	..1..2..2..1..1	....2..2..1..1..	...1..1..2..1....	..1..1..1..2..1...	.1..1..1..2..1.....	.......1..1..1..1...	.........1........1..	.......1..1...........
15	(1,17)	1..2..3..4..2..	..2..3..3..3..1.	.1..2..4..3..2..1	...1..3..3..2..1..	..1..2..3..2..1..1.	....1..2..2..2..1...	...1..1..1..2..1.....	.....1........1.......
16	(17,1)	..2..4..3..2..1	.1..3..3..3..2..	1..2..3..4..2..1.	..1..2..3..3..1...	1..1..2..3..2..1...	...1..2..2..2..1....	......1..2..1..1..1..	.......1........1.....
17	(2,16)	..1..1..2..2...	.1..1..2..2..1..	...1..2..1..1....	.....1..2..1..1..1	....1..2..1..1..1..	...1..1..1..1.......	...........1..1......	....1........1........
18	(16,2)	...2..2..1..1..	..1..2..2..1..1.	....1..1..2..1...	1..1..1..2..1.....	..1..1..1..2..1....	........1..1..1..1..	......1..1...........	........1........1....
19	(4,0)	..1..1..1..1...	.1..1..1..2..1..	......2..1..1..1.	..1..1..2..1.....1	........1..1.......	.......1..1..1..1...	.....1........1......	....1........1........
20	(4,1)	.1..2.....2..1.	1..1..2..2..1..1	..1..2..2..2..1..	.1.....2..2.....1.	.......1..2..1.....	......1..1..1..1....	....1..1.....1..1....	......1........1......
21	(0,4)	...1..1..1..1..	..1..2..1..1..1.	.1..1..1..2......	1.....1..2..1..1..	.........1..1......	....1..1..1..1......	......1........1.....	........1........1....
22	(5,0)	1..1.....1.....	.....1.....1..1.	.1.....1..1.....1	......1..1........	.........1..1......	....	......1........1.....	.....1........1.......
23	(0,5)	.....1.....1..1	.1..1.....1.....	1.....1..1.....1.	........1..1......	........1..1.......	....	.....1........1......	.......1........1.....
24	(6,0)	....1.....1....	1..............1	....	.......1..1.......	....	....	....	......1........1......

FIGURE 11. Modular ladder for the E_8 graph.