Uses of Quantum Spaces

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Contents

1991 *Mathematics Subject Classification.* 16W30, 16W35, 16W50, 81R50, 17B37, 20G42.
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1. Introduction

Quite often, a group appears as a set of symmetries of some object - a set equipped with geometrical, algebraic or combinatorial data. The theory of quantum groups enlarges the notion of symmetry; a quantum group (often) describes "generalized symmetries" of an object. In the case of a linear (orthogonal, symplectic) quantum group, this object is a linear (orthogonal, symplectic) quantum space - an algebra with certain quadratic relations. A study of these underlying objects, the quantum spaces, helps to understand the structure of the quantum groups. In these lectures I will illustrate the role of the quantum spaces on two examples: non-perturbative effects in the theory of Yang-Baxter operators and real forms of quantum groups.

To talk about non-perturbative effects, one should explain first, what means "perturbative" or "deformational". This is the subject of the subsection 2.1. The initial data for a quantum deformation of a Lie algebra $\mathcal{L}$ is conveniently encoded in terms of another Lie algebra $D(\mathcal{L})$, the Drinfeld double of $\mathcal{L}$. The Lie algebra $D(\mathcal{L})$ has an invariant scalar product and I have included a subsection 2.2 on the structure of Lie algebras with an invariant scalar product.

For a semi-simple Lie algebra $\mathcal{L}$, the most important deformations are those which are called quasitriangular. They are classified by Belavin-Drinfeld triples. The subsection 2.3 contains some information about the combinatorics of the Belavin-Drinfeld triples.

In section 3, after a geometrical interpretation of the quantum deformations of Lie groups, we introduce an algebra of functions on a quantum group; a definition of GL-type quantum groups and quantum spaces is given in subsection 3.1. In subsection 3.2 we explain how to use a differential calculus on a GL-type quantum space for calculating the Poincaré series.

Subsection 3.3 is devoted to 3-dimensional quantum spaces. We exhibit an unexpected appearance of Yang-Baxter operators and give an example of a non-perturbative Yang-Baxter operator. We prove the Poincaré-Birkhoff-Witt theorem for the quantum space defined by this Yang-Baxter operator.

Subsection 3.4 deals with effects specific to quantum groups at roots of unity. We introduce a terminology of formatted matrix algebras over local graded rings, which is useful in the study of non semi-simple algebras. We describe the matrix structure of the reduced quantum enveloping algebra and the reduced function algebra for $sl_q(2)$.

Subsection 4.1 contains a summary of the theory of quasi-triangular Hopf algebras. In subsection 4.2 we classify Yang-Baxter matrices, which can have non-zero entries only at places where the pair of lower indices is a permutation of the pair of the upper ones. Subsection 4.3 gives a construction of the Yang-Baxter matrices for orthogonal and symplectic groups from the Yang-Baxter matrices for GL.

In section 5 we describe a method of classification of real forms of quantum groups. The method is based on the study of the corresponding quantum spaces.

Throughout the text, a sum over repeated indices is assumed. If $X = \{X^i_j\}$ and $Y = \{Y^i_j\}$ are two operators, the indices are summed as $(XY)^i_j = X^i_k Y^k_j$ in their product.

2. Lie bialgebras

A Hopf algebra H is a collection of data $\{H, m, \Delta, S, \epsilon\}$, where H is a vector space over a ground field k; $m : H \otimes H \to H$ a multiplication; $\Delta : H \to H \otimes H$ a comultiplication; $\epsilon : H \to k$ is a counit and $S : H \to H$ an antipode. For a precise formulation of various relations between these maps see *e.g.* [**1**]. Let me just remind that for a Hopf algebra H one knows how to build tensor products of representations and it is given universally by Δ; the counit gives rise to a trivial representation; the antipode is needed to build contragredient representations.

The classical examples of Hopf algebras are group algebras $k[G]$ of finite groups G and universal enveloping algebras $\mathcal{U}(\mathcal{L})$ of Lie algebras $\mathcal{L}$.

2.1. Deformation of the coproduct. Let $\mathcal{L}$ be a Lie algebra over $\mathbb{C}$ and $\mathcal{U}$ its universal enveloping algebra. Denote by $\{X_i\}$ a basis of $\mathcal{L}$. The classical coproduct $\Delta_0 : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ is given on generators X_i by $\Delta_0 X_i = X_i \otimes 1 + 1 \otimes X_i$. The map Δ_0 is a coassociative homomorphism (coassociativity means $(\Delta_0 \otimes \mathrm{I})\Delta_0 = (\mathrm{I} \otimes \Delta_0)\Delta_0$ for the maps $\mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$; here I is the identity map). In this subsection we shall study deformations of the coproduct Δ_0. A deformation of Δ_0 is, by definition, a coassociative homomorphism $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$,

$$\Delta(a) = \Delta_0(a) + \alpha\phi_1(a) + \alpha^2\phi_2(a) + \dots \ . \tag{2.1.1}$$

The right hand side is a formal power series in the parameter α, which is called a deformation parameter. The coefficients $\phi_k(a)$ are elements of $\mathcal{U} \otimes \mathcal{U}$.

Our task is to understand which deformations are "essential", in the sense that they cannot be removed by some redefinition of generators. Here is the answer modulo α^2.

Theorem 1. Any deformation of Δ_0, after a change of generators, takes a form (in the first order in α)

$$\Delta X_i = \Delta_0 X_i + \alpha \mu_i^{jk} X_j \otimes X_k. \tag{2.1.2}$$

The antisymmetric tensor μ_i^{jk} ($\mu_i^{jk} = -\mu_i^{kj}$) is a 1-cocycle with values in $\Lambda^2\mathcal{L}$, $\mu \in Z^1(\mathcal{L}, \Lambda^2\mathcal{L})$; explicitly:

$$N_{[ij]}^{[ab]} = \Gamma_{ij}^s \mu_s^{ab} \ , \tag{2.1.3}$$

where $N_{ij}^{ab} = \Gamma_{i\nu}^a \mu_j^{\nu b}$ and $[ab]$ means antisymmetrization in indices a and b, $t^{[ab]} = t^{ab} - t^{ba}$ for a tensor t^{ab}. Here Γ_{ij}^k are the structure constants of the Lie algebra $\mathcal{L}$, $[X_i, X_j] = \Gamma_{ij}^k X_k$.

Proof. Assume that Δ is a deformation of the classical coproduct Δ_0. On the generators X_i we have

$$\Delta(X_i) = \Delta_0(X_i) + \alpha\phi_i + \dots \tag{2.1.4}$$

with some $\phi_i \in \mathcal{U} \otimes \mathcal{U}$, where dots denote higher powers in α.

The coassociativity, in order α^1, is equivalent to a following equation on ϕ_i in $\mathcal{U}^{\otimes 3}$

$$\phi_i \otimes 1 + (\Delta_0 \otimes \mathrm{I})\phi_i = 1 \otimes \phi_i + (\mathrm{I} \otimes \Delta_0)\phi_i \ , \tag{2.1.5}$$

I is the identity operator. The algebra $\mathcal{U}^{\otimes 3}$ is the enveloping algebra of $\mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}$. Let X_i, Y_i and Z_i be the generators of the first, second and third copies of $\mathcal{L}$, respectively. Then the equation (2.1.5) can be rewritten as

$$\phi_i(X,Y)+\phi_i(X+Y,Z)=\phi_i(Y,Z)+\phi_i(X,Y+Z)\ . \tag{2.1.6}$$

The statement that Δ is a homomorphism reads, in terms of ϕ_i, as

$$[X_i+Y_i,\phi_j]-[X_j+Y_j,\phi_i]=\Gamma^k_{ij}\phi_k \tag{2.1.7}$$

(the algebra $\mathcal{U}\otimes\mathcal{U}$ is the enveloping algebra of $\mathcal{L}\oplus\mathcal{L}$; X_i and Y_i are the generators of the first and second copies of $\mathcal{L}$).

Let $\sigma:\mathcal{U}\otimes\mathcal{U}\to\mathcal{U}\otimes\mathcal{U}$ be the flip, $\sigma(x\otimes y)=y\otimes x$. Decompose ϕ_i into symmetric and antisymmetric parts with respect to σ,

$$\phi_i=s_i+a_i \tag{2.1.8}$$

with $\sigma(s_i)=s_i$ and $\sigma(a_i)=-a_i$.

Proposition 2. If ϕ_i satisfies (2.1.5) and (2.1.7) then both s_i and a_i satisfy (2.1.5) and (2.1.7).

Proof. We have $\Delta'(X_i)=\Delta_0(X_i)+\alpha\phi'_i+\dots$, where $\phi'_i=\sigma(\phi_i)=s_i-a_i$.

If Δ is a coproduct then $\Delta'=\sigma\circ\Delta$ is a coproduct as well, so ϕ'_i satisfies (2.1.7),

$$[X_i+Y_i,\phi'_j]+[X_j+Y_j,\phi'_i]=\Gamma^k_{ij}\phi'_k\ , \tag{2.1.9}$$

and (2.1.5),

$$\phi'_i\otimes 1+(\Delta_0\otimes\ \mathrm{I})\phi'_i=1\otimes\phi'_i+(\ \mathrm{I}\otimes\Delta_0)\phi'_i\ . \tag{2.1.10}$$

Take the sum and difference of (2.1.5) and (2.1.10) (respectively, (2.1.7) and (2.1.9)) to finish the proof. □

In particular, each part (symmetric or antisymmetric) of ϕ_i alone defines a coproduct in order α^1.

Clearly, a redefinition of generators can change only the symmetric part of ϕ_i. We start by analyzing this case (the case of symmetric ϕ_i).

Proposition 3. Assume that Δ is symmetric in order α^1, $\phi'_i=\phi_i$. Then the α^1 terms can be removed by a redefinition of generators.

Proof. $\mathcal{U}$ is the algebra of polynomials in the generators X_i. It is filtered by the degree of polynomials, $F_k\mathcal{U}$ are polynomials of degree $\le k$. The associated graded term $F_k\mathcal{U}/F_{k-1}\mathcal{U}$ is isomorphic to $S^k\mathcal{L}$, the symmetric power of $\mathcal{L}$. Any element $u\in\mathcal{U}$ has a well-defined "highest symbol": if $u\in F_k\mathcal{U}\backslash F_{k-1}\mathcal{U}$ ($\backslash$ is the set-theoretic complement) then its highest symbol is the image of u in $S^k\mathcal{L}$. Denote by x_i the basis of commuting variables corresponding to generators X_i. The highest symbol is a homogeneous polynomial in a set of commuting variables x_i.

The algebra $\mathcal{U}\otimes\mathcal{U}$ is the enveloping algebra of $\mathcal{L}\oplus\mathcal{L}$, the highest symbols are homogeneous polynomials in two sets of variables, x_i and y_i.

Let f_i be the symbol of ϕ_i. Then f_i is a polynomial in two sets of variables, $f_i=f_i(x,y)$. The symmetry condition implies that $f_i(x,y)=f_i(y,x)$.

The coassociativity implies, in order α^1, an equation

$$f(x+y,z)+f(x,y)=f(x,y+z)+f(y,z) \tag{2.1.11}$$

for each f_i.

Lemma 4. Let $f(x,y)$ be a homogeneous polynomial, symmetric with respect to the flip $x \leftrightarrow y$. The polynomial f satisfies (2.1.11) if and only if there exists a homogeneous polynomial $g(x)$ (a polynomial in only one set of variables x_i) such that

$$f(x,y) = g(x+y) - g(x) - g(y) \ . \tag{2.1.12}$$

Proof. It is straightforward to see that $f(x,y) = g(x+y) - g(x) - g(y)$ satisfies (2.1.11).

Assume now that f satisfies (2.1.11). Let M be a total degree of f. If $M = 0$ then $f = c$ is a constant and it is enough to take $g = -c$. Assume that $M > 0$.

Applying $\frac{\partial}{\partial x_i}$ to (2.1.11) and evaluating at $x = 0$, we obtain an equation (after replacing $y \to x$ and $z \to y$)

$$\partial_1^i f|_{x,y} = \partial_1^i f|_{0,x+y} - \partial_1^i f|_{0,x} \quad , \tag{2.1.13}$$

where ∂_1^i are the partial derivatives in the first set of variables.

Applying $\frac{\partial}{\partial z_i}$ to (2.1.11) and evaluating at $z = 0$, we obtain an equation

$$\partial_2^i f|_{x,y} = \partial_2^i f|_{x+y,0} - \partial_2^i f|_{y,0} \quad , \tag{2.1.14}$$

where ∂_2^i are the partial derivatives in the second set of variables.

Since f is homogeneous of degree M, we have $(x_i\partial_1^i + y_i\partial_2^i)f = Mf$, which, together with (2.1.13) and (2.1.14), gives

$$Mf = x_i\partial_1^i f|_{0,x+y} + y_i\partial_2^i f|_{x+y,0} - x_i\partial_1^i f|_{0,x} - y_i\partial_2^i f|_{y,0} \quad . \tag{2.1.15}$$

The symmetry of f, $f(x,y) = f(y,x)$ implies that $\partial_i^1 f|_{x,y} = \partial_i^2 f|_{y,x}$. Therefore we can rewrite (2.1.15) in the form (2.1.11) with $g(x) = \frac{1}{M}x^i\partial_i^1 f|_{0,x}$. The proof of the Lemma 4 is finished. □

We proved that for each i there exists $g_i(x)$ such that

$$f_i(x,y) = g_i(x+y) - g_i(x) - g_i(y) \ . \tag{2.1.16}$$

Let $g_i^\vee(X)$ be an element whose highest symbol is $g_i(x)$. The combination $g_i^\vee(X+Y) - g_i^\vee(X) - g_i^\vee(Y)$ satisfies the equation (2.1.6). Therefore, an element $\phi_i(X,Y) - g_i^\vee(X+Y) + g_i^\vee(X) + g_i^\vee(Y)$, which has the filtration degree smaller than the degree of $\phi_i(X,Y)$, satisfies (2.1.6) as well, and we can apply the Lemma 4 again.

Repeating this process a needed number of times, we shall finally build a set of elements $\gamma_i \in \mathcal{U}$ such that

$$\phi_i(X,Y) = \gamma_i(X+Y) - \gamma_i(X) - \gamma_i(Y) \ . \tag{2.1.17}$$

Let $X_i^\gamma = X_i - \alpha\gamma_i(X)$. It is straightforward to see that in the order α^1 the coproduct for the generators X_i^γ is classical, $\Delta(X_i^\gamma) = X_i^\gamma \otimes 1 + 1 \otimes X_i^\gamma$.

It is left to show that one can choose γ_i in such a way that the generators X_i^γ satisfy the same Lie algebraic relations as the original generator X_i. It will be so if and only if $[X_i, \gamma_j] - [X_j, \gamma_i] - \Gamma_{ij}^k\gamma_k = 0$ for all i and j.

Since Δ is a homomorphism, it follows immediately that elements $\gamma_{ij} = [X_i, \gamma_j] - [X_j, \gamma_i] - \Gamma_{ij}^k\gamma_k$ satisfy relations

$$\gamma_{ij}(X+Y) = \gamma_{ij}(X) + \gamma_{ij}(Y) \ . \tag{2.1.18}$$

Equation (2.1.18) implies that the functions γ_{ij} are linear, $\gamma_{ij}(X) = \gamma_{ij}^k X_k$.

We shall need a short digression into the general theory of universal enveloping algebras (see, *e.g.* [**2**]).

An element $u \in \mathcal{U}$ can be uniquely decomposed into a sum

$$u = \text{ symb}^0(u) + \text{ symb}^1(u) + \cdots + \text{ symb}^d(u) \ , \tag{2.1.19}$$

where d is the filtration degree of u and $\text{symb}^A(u) = c^{i_1 \cdots i_A} X_{i_1} \ldots X_{i_A}$ for some completely symmetric tensor $c^{i_1 \cdots i_A}$. Elements of the form $c^{i_1 \cdots i_A} X_{i_1} \ldots X_{i_A}$ with a completely symmetric $c^{i_1 \cdots i_A}$ form a subspace $\mathcal{U}^A \subset \mathcal{U}$ and the above decomposition of u implies that $\mathcal{U}$ is a direct sum of $\mathcal{U}^A$, $\mathcal{U} = \oplus_{A=0}^{\infty} \mathcal{U}^A$. Each $\mathcal{U}^A$ is a $\mathcal{L}$-module (that is, commutators of generators X_i with $\text{symb}^A(u)$ are again in $\mathcal{U}^A$); in other words,

$$\text{symb}^A([X_i, u]) = [X_i, \ \text{symb}^A(u)] \ . \tag{2.1.20}$$

The module $\mathcal{U}^A$ is isomorphic to the symmetric power $S^A \mathcal{L}$.

Let $\gamma_j = \sum \text{ symb}^A(\gamma_j)$ be a decomposition of the form (2.1.19) of the element γ_j. We have seen that γ_{ij} is in $\mathcal{U}^1$ for each i and j. It follows then from (2.1.20) that $\gamma_{ij} = [X_i, \ \text{symb}^1(\gamma_j)] - [X_j, \ \text{symb}^1(\gamma_i)] - \Gamma^k_{ij} \ \text{symb}^1(\gamma_k)$. Therefore, $[X_i, \tilde{\gamma}_j] - [X_j, \tilde{\gamma}_i] - \Gamma^k_{ij} \tilde{\gamma}_k = 0$ for all i and j, where $\tilde{\gamma}_i = \gamma_i - \text{ symb}^1(\gamma_i)$. Therefore, the elements $\tilde{X}^{\gamma}_i = X^{\gamma}_i - \alpha \tilde{\gamma}_i(X)$ satisfy the same Lie algebraic relations as the original generators X_i, $[\tilde{X}_i, \tilde{X}_j] = \Gamma^k_{ij} \tilde{X}_k$.

Moreover, since for an element $g^1 \in \mathcal{U}^1$, the combination $g^1(X+Y) - g^1(X) - g^1(Y)$ vanishes, the elements $\tilde{\gamma}_i = \gamma_i - \text{ symb}^1(\gamma_i)$ still verify (2.1.16). Therefore, as before, the coproduct for the elements $\tilde{X}_i$ is classical.

Thus, the elements $\tilde{X}_i$ provide the needed redefinition of the generators X_i. The proof of the Proposition 3 is finished. □

Using, if necessary, the redefinition of the Proposition 3, we get rid of the symmetric part of ϕ_i. Assume therefore that ϕ_i is antisymmetric. Again, let f_i be the highest symbol of ϕ_i. The symmetry condition is now $f_i(x, y) = -f_i(y, x)$. As before, the coassociativity implies, in order α^1, the equation (2.1.11) for each i.

Proposition 5. Let $f(x, y)$ be a homogeneous polynomial, antisymmetric with respect to the flip $x \leftrightarrow y$. Assume that the polynomial f satisfies (2.1.11). Then

$$f(x, y) = \nu^{jk} x_j y_k \tag{2.1.21}$$

for some antisymmetric tensor ν, $\nu^{jk} = -\nu^{kj}$.

Proof. The derivatives of f satisfy equations (2.1.13) and (2.1.14). There is one more equation which we didn't need for the Lemma 4. It is obtained by applying $\frac{\partial}{\partial y_i}$ to (2.1.11) and evaluating at $y = 0$ (we change variables, $z \to y$)

$$\partial^i_1 f|_{x,y} + \partial^i_2 f|_{x,0} = \partial^i_2 f|_{x,y} + \partial^i_1 f|_{0,y} \quad . \tag{2.1.22}$$

The antisymmetry of f, $f(x, y) = -f(y, x)$, implies $\partial^1_i f|_{x,y} = -\partial^2_i f|_{y,x}$. Substituting $\partial^i_1 f|_{x,y}$ and $\partial^i_2 f|_{x,y}$ from (2.1.13) and (2.1.14) into (2.1.22) and using the antisymmetry, we find

$$\partial^i_1 f|_{0,x+y} = \partial^i_1 f|_{0,x} + \partial^i_1 f|_{0,y} \ . \tag{2.1.23}$$

Thus, $\partial_1^i f|_{0,x}$ is a linear function. Substituting (2.1.23) into (2.1.13), we find $\partial_1^i f|_{0,x+y} = \partial_1^i f|_{0,y}$. Thus, $\partial_1^i f|_{x,y}$ is a linear function which depends on the second set of variables only. In other words, $\partial_1^i f|_{x,y} = \nu_1^{ij} y_j$.

Similarly, $\partial_2^i f|_{x,y}$ is a linear function which depends on the first set of variables only, $\partial_2^i f|_{x,y} = \nu_2^{ij} x_j$.

The antisymmetry, $\partial_i^1 f|_{x,y} = -\partial_i^2 f|_{y,x}$, implies that $\nu_1^{ij} = -\nu_2^{ij}$. Let $\nu^{ij} = \nu_1^{ij}$. Then

$$\partial_1^i f|_{x,y} = \nu^{ij} y_j \quad \text{and} \quad \partial_2^i f|_{x,y} = -\nu^{ij} x_j \ . \tag{2.1.24}$$

Since the derivatives of f are homogeneous of degree 1, the function f itself is homogeneous of degree 2. So $2f = (x_i \partial_1^i + y_i \partial_2^i) f$. Substituting expressions (2.1.24), we find

$$2f = x_i \nu^{ij} y_j - y_i \nu^{ij} x_j = \nu^{[ij]} x_i y_j \ , \tag{2.1.25}$$

where the square brackets mean antisymmetrization, $\nu^{[ij]} = \nu^{ij} - \nu^{ji}$ and the assertion of the Proposition 5 follows. □

After the Propositions 2, 3 and 5 it is only left to check a condition that Δ is a homomorphism in the first order in α. A straightforward calculation gives the cocycle condition (2.1.3). The proof of the Theorem 1 is finished. □

Remark. It is not necessary to assume that the ground field is $\mathbb{C}$. The Theorem 1 holds for an arbitrary field of characteristic 0 (and it is not true if the characteristic is different from 0).

Repeating the proof of the Proposition 3 consecutively in powers of α, one obtains a version of the Milnor-Moore theorem (for its general formulation see, *e. g.* [**3**]):

Corollary 6. A formal (i.e. given by a formal power series in α) cocommutative deformation of the coproduct on a universal enveloping algebra is always trivial, that is, it can be removed by a formal redefinition of generators.

In the rest of this subsection we explain what happens in the next order in α, in the α^2-terms.

It turns out that the consistency in the α^2 terms imposes new conditions on μ_i^{jk}.

Assume that we can extend the deformation (2.1.2) to α^2-terms,

$$\Delta X_i = \Delta_0 X_i + \alpha \mu_i^{jk} X_j \otimes X_k + \alpha^2 \psi_i \ , \tag{2.1.26}$$

and Δ is coassociative up to α^3. Coassociativity implies

$$\begin{aligned} &\psi_i \otimes 1 + (\Delta_0 \otimes \ \mathrm{id}) \psi_i + \mu_i^{abc} X_a \otimes X_b \otimes X_c \\ &= 1 \otimes \psi_i + (\ \mathrm{id} \otimes \Delta_0) \psi_i + \mu_i^{cba} X_a \otimes X_b \otimes X_c \ , \end{aligned} \tag{2.1.27}$$

with a notation: $\mu_i^{abc} = \mu_i^{jc} \mu_j^{ab}$ for a tensor μ_i^{jk}.

To cancel the μ_i^{abc}-terms, one has (in order to get trilinear in X expressions) to choose ψ_i in the form

$$\psi_i = A_i^{abc} X_a X_b \otimes X_c + B_i^{abc} X_c \otimes X_a X_b \ . \tag{2.1.28}$$

with tensors A_i^{abc} and B_i^{abc} symmetric in indices $\{a,b\}$. In the next formulas, the lower index is omitted.

Coassociativity (2.1.27) gives

$$2(B^{bca} - A^{abc}) = \mu^{abc} + \mu^{bca}\ . \tag{2.1.29}$$

Exchange b and c in (2.1.29) and subtract from (2.1.29), taking into account the symmetry of B^{abc} in $\{a,b\}$:

$$2(A^{acb} - A^{abc}) = \mu^{bca} + J^{abc}\ , \tag{2.1.30}$$

where

$$J^{abc} = \mu^{abc} + \mu^{bca} + \mu^{cab}\ . \tag{2.1.31}$$

The tensor J^{abc} is totally antisymmetric. Under the exchange $a \leftrightarrow b$, eqn. (2.1.30) becomes

$$2(A^{bca} - A^{bac}) = \mu^{acb} + J^{bac}\ . \tag{2.1.32}$$

Under the exchange $a \leftrightarrow c$, eqn. (2.1.30) becomes

$$2(A^{cab} - A^{cba}) = \mu^{bac} + J^{cba}\ . \tag{2.1.33}$$

The combination (2.1.30) - (2.1.32) - (2.1.33) gives (due to the symmetry of A^{abc} in $\{a,b\}$)

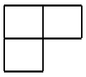
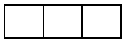
$$0 = 4J^{abc}\ . \tag{2.1.34}$$

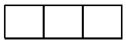
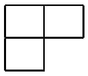
This is the Jacobi identity (2.1.40) for μ.

Now from 6 permutations of $\{a,b,c\}$ one gets only two equations

$$2(A^{acb} - A^{abc}) = \mu^{bca}\ , \tag{2.1.35}$$

$$2(A^{bca} - A^{bac}) = \mu^{acb}\ . \tag{2.1.36}$$

The tensor A^{abc}, being already symmetric in the first two indices, can have two types of symmetry, corresponding to Young diagrams  and  .

The totally symmetric part (the diagram ) of A cannot be defined by (2.1.35)-(2.1.36), it is arbitrary. The part, corresponding to the diagram  satisfies

$$A^{abc} + A^{bca} + A^{cab} = 0\ . \tag{2.1.37}$$

Together, eqs. (2.1.35), (2.1.36) and (2.1.37) can be easily solved and we conclude (taking into account the totally symmetric part) that the general solution for A is

$$A^{abc} = \frac{1}{6}(\mu^{cba} + \mu^{cab}) + \chi^{abc} \tag{2.1.38}$$

with totally symmetric χ^{abc}.

From (2.1.29) it follows then that

$$B^{bca} = \frac{1}{6}(\mu^{abc} + \mu^{acb}) + \chi^{abc}\ . \tag{2.1.39}$$

In particular, $A^{abc} = B^{abc}$.

The totally symmetric part χ^{abc} can be removed by a redefinition (solve for g the equation (2.1.12) with $f(x,y) = \chi^{abc}(x_a x_b y_c + x_c y_a y_b)$).

We conclude that the coassociative extension of Δ to the α^2-terms is possible only if the Jacobi identity for μ is satisfied,

(2.1.40) $$\mu_i^{abc} + \text{ cycle } in\ (a,b,c) = 0\ .$$

This extension has the form

(2.1.41)
$$\Delta X_i = \Delta_0 X_i + \alpha\mu_i^{jk} X_j \otimes X_k + \frac{\alpha^2}{6}(\mu_i^{cba} + \mu_i^{cab})(X_a X_b \otimes X_c + X_c \otimes X_a X_b)\ .$$

In general, Δ does not preserve the original commutation relations $[X_i, X_j] = \Gamma_{ij}^k X_k$, the multiplication structure of $\mathcal{U}$ has also to be deformed in the order α^2.

Exercise. The map Δ preserves the following relations [**4**]

(2.1.42) $$[X_i, X_j] = \Gamma_{ij}^k X_k + \frac{\alpha^2}{18}\mu_i^{sa}\mu_j^{tb}\Gamma_{st}^c X_{(a} X_b X_{c)}\ ,$$

with round brackets in $X_{(a} X_b X_{c)}$ denoting the symmetrization, $X_{(i_1} X_{i_2} X_{i_3)} = \sum X_{i_{\sigma(1)}} X_{i_{\sigma(2)}} X_{i_{\sigma(3)}}$, the sum is over all permutations $\sigma \in S_3$.

Given the multiplication (2.1.42) and the comultiplication (2.1.41), one needs to know the counit and the antipode to complete the Hopf algebra structure.

Exercise. The counit ϵ stays undeformed,

(2.1.43) $$\epsilon(X_i) \equiv 0 \mod \alpha^3\ ;$$

for the antipode mod α^3 one has

(2.1.44) $$S(X_i) = -X_i + \frac{1}{2}\alpha M_i^a X_a + \frac{1}{4}\alpha^2 \mu_i^{bl} M_l^a \Gamma_{ab}^c X_c\ ,$$

where $M_a^v = -\mu_a^{vc}\Gamma_{ck}^k - \mu_k^{kj}\Gamma_{ja}^v \equiv \mu_a^{st}\Gamma_{st}^v$. (Note that S stays linear in generators.)

It is known today that in higher orders in α no further restriction on μ appears; in other words, if μ_i^{jk} satisfies the Jacobi and cocycle conditions, there exist, as formal power series in α, the multiplication, which begins as (2.1.42), and the comultiplication, which begins as (2.1.41) (and the counit and antipode). Moreover, there exists such deformation that each term in the formal power series (for the multiplication, comultiplication and the antipode) is expressible in terms of the tensors μ_i^{jk} and Γ_{ij}^k only.

2.1.1. *Discrete groups.* The situation with discrete groups is different. It is an easy exercise to analyze the formal deformations of the coproduct for the group algebras of discrete groups. In contrast to the case of universal enveloping algebras, the result is trivial.

Let G be a discrete group, $U = \mathbb{C}[G]$ its group algebra over complex numbers.

Theorem 7. *U does not admit a nontrivial deformation of the standard coproduct.*

Proof. Assume that there is a first order deformation of a coproduct,

(2.1.45) $$\Delta g = g \otimes g + \alpha \sum_{k,l} C_g^{k,l} k \otimes l\ ,$$

where $\alpha^2 = 0$.

(i) The coassociativity condition in the first order in α gives

$$\sum_{k,l} C_g^{k,l}(k\otimes l\otimes g + k\otimes k\otimes l - g\otimes k\otimes l - k\otimes l\otimes l) = 0\ . \tag{2.1.46}$$

Collecting terms $a\otimes b\otimes c$ with fixed a and c, $a\neq g$ and $c\neq g$, one finds

$$C_g^{a,c} a = C_g^{a,c} c\ . \tag{2.1.47}$$

This holds for all a and c different from g. Therefore, only $C_g^{g,g}$, $C_g^{k,g}$, $C_g^{g,k}$ and $C_g^{k,k}$ might differ from 0.

Now the condition (2.1.46) becomes

$$\sum_{k\neq g} \{C_g^{g,k}(g\otimes k\otimes g - g\otimes k\otimes k) + C_g^{k,g}(k\otimes k\otimes g - g\otimes k\otimes g)$$

$$+\ C_g^{k,k}(k\otimes k\otimes g - g\otimes k\otimes k)\} = 0\ . \tag{2.1.48}$$

This implies that there is a set of constants B_g^k for $k\neq g$, s. t.

$$C_g^{g,k} = C_g^{k,g} = B_g^k\ ,\ C_g^{k,k} = -B_g^k\ ,\quad k\neq g\ . \tag{2.1.49}$$

This solves the coassociativity condition. Thus, we have

$$\Delta g = (1+\alpha c_g) g\otimes g + \alpha\sum_{k\neq g} B_g^k (g\otimes k + k\otimes g - k\otimes k)\ . \tag{2.1.50}$$

(ii) The condition that Δ is a homomorphism implies (in the first order in α):

$$B_h^{g^{-1}k} + B_g^{kh^{-1}} = B_{gh}^k \ \text{ and }\ c_{gh} = c_g + c_h\ . \tag{2.1.51}$$

Let

$$g' = (1+\alpha c_g) g + \alpha\sum_{k\neq g} B_g^k k\ . \tag{2.1.52}$$

A direct calculation shows that (2.1.51) is exactly the condition saying that $g\to g'$ is an algebra homomorphism, $g'h' = (gh)'$.

Again a straightforward calculation shows that

$$\Delta g' = g'\otimes g'\ . \tag{2.1.53}$$

Therefore, given a deformation of the standard coproduct, we can explicitly construct an isomorphism with the original bialgebra. The proof is finished. □

In the same way as the Corollary 6 followed from the Proposition 3, we obtain the information about the formal deformations in this case.

Corollary 8. At formal level, all deformations of the coproduct for the group algebras of discrete groups are trivial.

2.2. Lie algebras with an invariant scalar product. We have seen in the previous subsection that the essential role in the theory of deformations of the coproduct on universal enveloping algebras is played by a tensor μ_i^{jk}. All the conditions on the tensor μ are expressed in terms of the Lie algebra itself, without any reference to the deformation theory. The relevant classical notion is a "Lie bialgebra".

Definition. A Lie bialgebra is a Lie algebra $\mathcal{L}$ equipped with a map $\delta : \mathcal{L} \to \Lambda^2\mathcal{L}$, $\delta X_i = \mu_i^{jk} X_j \otimes X_k$, where the tensor μ (antisymmetric in the upper indices) satisfies the Jacobi identity and belongs to $Z^1(\mathcal{L}, \Lambda^2\mathcal{L})$.

Both Γ and μ satisfy the Jacobi identity. The condition $\mu \in Z^1$, written explicitly as (2.1.3), is symmetric in $\mu \leftrightarrow \Gamma$. So the notion of the Lie bialgebra is self-dual (like the notion of the Hopf algebra). In other words, if $\mathcal{L}$ is a Lie bialgebra then there is a Lie bialgebra structure on the dual space $\mathcal{L}^*$, the roles of Γ and μ being interchanged. There is an object which explicitly realizes this symmetry between μ and Γ. It turns out (*Exercise*: verify it) that all the data for a Lie bialgebra can be conveniently expressed as the Jacobi identity for a larger Lie algebra with generators X_i and X^i, satisfying

$$[X_i, X_j] = \Gamma_{ij}^k X_k \quad , \quad \text{for generators of } \mathcal{L} \ , \tag{2.2.1}$$

$$[X^i, X^j] = \mu_k^{ij} X^k \quad , \quad \text{for generators of } \mathcal{L}^* \ , \tag{2.2.2}$$

$$[X_i, X^j] = -\Gamma_{ik}^j X^k + \mu_i^{jk} X_k \ . \tag{2.2.3}$$

This Lie algebra is called a Drinfeld double of the Lie bialgebra $\mathcal{L}$ and denoted $D\mathcal{L}$. As a vector space, $D\mathcal{L}$ is isomorphic to $\mathcal{L} \oplus \mathcal{L}^*$.

Definition. A scalar product $\langle x, y\rangle$ on a Lie algebra $\mathcal{L}$ (i.e. a nondegenerate symmetric pairing $\mathcal{L} \otimes \mathcal{L} \to k$) is called invariant if $\langle [x, y], z\rangle = \langle x, [y, z]\rangle$ for all $x, y, z \in \mathcal{L}$.

Example: The Killing form on a semi-simple Lie algebra is invariant.

The natural pairing between $\mathcal{L}$ and $\mathcal{L}^*$, given by

$$\langle X^i, X^j\rangle = 0 \ , \ \langle X_i, X_j\rangle = 0 \ , \ \langle X_i, X^j\rangle = \delta_i^j \tag{2.2.4}$$

is an invariant scalar product on $D\mathcal{L}$. Moreover, the commutation relations between X_i and X^j can be reconstructed (with relations (2.2.1) and (2.2.2) being given) from the demand of invariance of the natural pairing. Indeed, let $[X_i, X^j] = A_{ik}^j X^k + B_i^{jk} X_k$. Then

$$-A_{ia}^j = \langle [X^j, X_i], X_a\rangle = \langle X^j, [X_i, X_a]\rangle = \langle X^j, \Gamma_{ia}^b X_b\rangle = \Gamma_{ia}^j$$

and similarly for B_i^{jk}.

Definition. A set of data $\{\mathfrak{g}, \mathcal{L}_1, \mathcal{L}_2\}$ where $\mathfrak{g}$ is a Lie algebra with an invariant scalar product, $\mathcal{L}_1$ and $\mathcal{L}_2$ are isotropic Lie subalgebras of dimension $= \dfrac{\dim \mathcal{L}}{2}$ and $\mathfrak{g} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is called a Manin triple.

A Lie bialgebra $\mathcal{L}$ defines a Manin triple $\{D\mathcal{L}, \mathcal{L}, \mathcal{L}^*\}$. Conversely, a Manin triple $\{\mathfrak{g}, \mathcal{L}_1, \mathcal{L}_2\}$ defines a Lie bialgebra $\mathcal{L}_1$. From this perspective, the study of Lie bialgebras splits into two parts: Lie algebras with an invariant scalar product; their maximal isotropic subalgebras. I will shortly comment on the first part.

Denote by $\{\mathfrak{g}, \phi\}$ a Lie algebra $\mathfrak{g}$ with an invariant scalar product ϕ ($\phi(x, y) = \langle x, y\rangle$). The pair $\{\mathfrak{g}, \phi\}$ is called indecomposable if it cannot be represented as a direct sum $\{\mathfrak{g}_1, \phi_1\} \oplus \{\mathfrak{g}_2, \phi_2\}$.

Example of $\{\mathfrak{g},\phi\}$. Let $\mathcal{M}$ be a Lie algebra with generators X_i. Let $\mathfrak{g} = \mathcal{M} \ltimes \mathcal{M}^*$ (the semi-direct product with respect to the coadjoint action). Then the scalar product

$$\langle X^i, X^j\rangle = 0\ ,\ \langle X_i, X_j\rangle = \xi_{ij}\ ,\ \langle X_i, X^j\rangle = \delta_i^j\ , \tag{2.2.5}$$

where $\xi(X_i, X_j) = \xi_{ij}$ is an arbitrary bilinear symmetric form, is an invariant scalar product.

Generalization of this example. Let $\{W,\phi\}$ be a Lie algebra with an invariant scalar product. Suppose that a Lie algebra $\mathfrak{g}$ acts on W by derivations, $T_a[x,y] = [T_a x, y] + [x, T_a y]$, where T is the action, $T : \mathfrak{g}\otimes W \to W$, $a\otimes w \mapsto T_a(w)$; suppose that the operators T_a, $a\in\mathfrak{g}$, are antisymmetric with respect to the scalar product on W, $\phi(T_a x, y) = -\phi(x, T_a y)$ for all $x, y \in W$ and $a\in\mathfrak{g}$.

Exercise. Show that the map $\beta : \Lambda^2 W \to \mathfrak{g}^*$ defined by $\langle a, \beta(x,y)\rangle = \phi(T_a x, y)$, where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, is a 2-cocycle, $\beta\in Z^2(W,\mathfrak{g}^*)$ ($\mathfrak{g}^*$ is considered here as a trivial W-module).

As a 2-cocycle, β defines a central extension of W by $\mathfrak{g}^*$. In other words, the bracket

$$[x,y] = [x,y]_W + \beta(x,y) \tag{2.2.6}$$

where $[x,y]_W$ is the commutator of x and y in the Lie algebra W, defines a Lie algebra structure on $W\oplus\mathfrak{g}^*$. Denote this Lie algebra by $\tilde{W}$.

Exercise. For $a\in\mathfrak{g}$ $x\in W$ and $f\in\mathfrak{g}^*$ let

$$\tilde{T}_a(x+f) = T_a x + \mathrm{ad}^*_a f\ , \tag{2.2.7}$$

where ad^* is the coadjoint action. Show that the formula (2.2.7) defines an action of $\mathfrak{g}$ on $\tilde{W}$.

We have therefore a Lie algebra structure on the space $A = \mathfrak{g}\oplus W\oplus\mathfrak{g}^*$: a semi-direct product $\mathfrak{g}\ltimes\tilde{W}$ with respect to the action (2.2.7).

Define a scalar product ϕ_A on A: the pairings between the generators of $\mathfrak{g}$ and $\mathfrak{g}^*$ are given by (2.2.5); the restriction of ϕ_A on W is ϕ; all the other pairings are 0.

Exercise. The scalar product ϕ_A is invariant.

The Lie algebra A with the scalar product ϕ_A is called the double extension of $\{W,\phi\}$ by S (and the action of S on W).

Theorem ([5]). If a Lie algebra with an invariant scalar product is not simple or 1-dimensional then it is either decomposable or a double extension. Moreover, one can always choose $\mathfrak{g}$ to be simple or 1-dimensional.

This theorem gives a way to construct higher-dimensional Lie algebras with an invariant scalar product from lower-dimensional ones. However, this is not a classification.

Example of a nontrivial double extension: $\mathfrak{g} = so(n)$, W is the n-dimensional fundamental representation of $\mathfrak{g}$; consider W as an abelian Lie algebra. The cocycle, giving a bracket on $W\oplus\mathfrak{g}^*$ is given by the natural map $\beta : W\wedge W\to\mathfrak{g}^*$, and $A = \mathfrak{g}\ltimes(V\oplus\mathfrak{g}^*)$.

Exercises.

In dimension 2 there is only one non abelian Lie algebra; choose a basis $\{x, y\}$ in such a way that the commutation relation is $[x, y] = y$. Denote this Lie algebra by L_2.

1. Show that any bialgebra structure on L_2 can be written (after possible redefinitions) in one of two forms:

$$\delta x = 0\ ,\ \delta y = x \wedge y \tag{2.2.8}$$

or

$$\delta x = x \wedge y\ ,\ \delta y = 0\ . \tag{2.2.9}$$

2. Show that for the bialgebra structure (2.2.8) the double is $\mathfrak{gl}_2$; for (2.2.9) the double is a semi-direct product $\mathbb{C} \ltimes \mathcal{N}$ of a one-dimensional Lie algebra (with a generator W) and the three-dimensional Heisenberg algebra $\mathcal{N}$ (with generators X, Y, Z and relations $[X, Y] = Z$, $[Z, X] = [Z, Y] = 0$); the action of W on $\mathcal{N}$ is given by $[W, X] = 2X$, $[W, Y] = -2Y$ and $[W, Z] = 0$.

3. Show that operations

$$\Delta x = x \otimes 1 + 1 \otimes x\ ,\ \Delta y = y \otimes 1 + e^{\alpha x} \otimes y \tag{2.2.10}$$

and

$$\Delta x = x \otimes (1 - 2\alpha y)^{-1} + 1 \otimes x\ ,\ \Delta y = y \otimes 1 + (1 - 2\alpha y) \otimes y \tag{2.2.11}$$

provide Hopf algebra structures on corresponding completions of $\mathcal{U}L_2$.

4. Show that the Hopf algebra structure, defined by (2.2.10) (respectively, (2.2.11)) is a quantization of the Lie bialgebra structure (2.2.8) (respectively, (2.2.9)). Note that the terms of order 1 in the deformation parameter α are not antisymmetric, but, as you remember, the symmetric part can be removed by redefinitions.

5. Let $L = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Show that any invariant scalar product on L has a form $\nu \oplus c\nu$, where ν is the Killing form on $\mathfrak{sl}_2$ and c is a constant.

6. Let $c = -1$. Show that the diagonal $\mathfrak{g}_1 = \mathfrak{sl}_2$ is isotropic. A subalgebra $\mathfrak{g}_2$ with a basis $\{(e_+, 0), (0, e_-), (h, -h)\}$ is a complementary isotropic subalgebra ($\{h, e_+, e_-\}$ is a standard basis in $\mathfrak{sl}_2$, $[h, e_\pm] = \pm 2e_\pm$, $[e_+, e_-] = h$). Thus, this Manin triple provides a Lie bialgebra structure on $\mathfrak{sl}_2$.

7. Classify all Manin triples on L.

8. Classify 3-, 4- and 5-dimensional Lie algebras with an invariant scalar product.

2.3. Belavin-Drinfeld triples. Let $\mathcal{L}$ be a simple Lie algebra over $\mathbb{C}$. In this case, every 2-cocycle is a coboundary (see any textbook on Lie algebras, *e.g.*, [**6**]) so one can solve the cocycle condition for μ: $\mu = \partial\rho$ or, explicitly, $\mu_i^{jk} X_j \otimes X_k = [\Delta_0 X_i, \rho]$, with $\rho = \rho^{ab} X_a \otimes X_b$ an element of the wedge square of $\mathcal{L}$ (that is, $\rho^{ab} = -\rho^{ba}$).

Now the Jacobi identity for μ can be rewritten as a non-linear equation for the element ρ.

Notation: for an element $A \in \mathcal{U} \otimes \mathcal{U}$, $A = \sum_\alpha x_\alpha \otimes y_\alpha$ let $A_{12} = \sum_\alpha x_\alpha \otimes y_\alpha \otimes 1$, $A_{13} = \sum_\alpha x_\alpha \otimes 1 \otimes y_\alpha$ and $A_{23} = \sum_\alpha 1 \otimes x_\alpha \otimes y_\alpha$; the elements A_{12}, A_{13} and A_{23} are from $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$:

Exercise. Show that the Jacobi identity for μ can be rewritten in terms of ρ as

$$[[\rho_{12},\rho_{13}]+[\rho_{12},\rho_{23}]+[\rho_{13},\rho_{23}], X_i\otimes 1\otimes 1+1\otimes X_i\otimes 1+1\otimes 1\otimes X_i]=0 \tag{2.3.1}$$

for all i.

The element $[\rho_{12},\rho_{13}]+[\rho_{12},\rho_{23}]+[\rho_{13},\rho_{23}]$ belongs to the third wedge power of $\mathcal{L}$, i.e., it has a form $A^{ijk}X_i\otimes X_j\otimes X_k$ with totally antisymmetric A^{ijk}. The space of invariant elements in $\Lambda^3\mathcal{L}$, for the simple $\mathcal{L}$, is known (see, *e.g.*, [**6**]) to be one-dimensional; it is generated by an element $\gamma=\Gamma^{ijk}X_i\otimes X_j\otimes X_k$, where $\Gamma^{ijk}=\Gamma^k_{ab}B^{ai}B^{bj}$, $B_{ij}=\langle X_i,X_j\rangle$ for the Killing form $\langle\cdot,\cdot\rangle$ and B^{ij} is inverse to B_{ij}, $B^{ij}B_{jk}=\delta^i_k$ (δ^i_k is the Kronecker delta). We conclude that the Jacobi identity for $\mu=\partial\rho$ is satisfied iff $[\rho_{12},\rho_{13}]+[\rho_{12},\rho_{23}]+[\rho_{13},\rho_{23}]$ is proportional to γ.

Let $C=B^{ij}X_i\otimes X_j$.

Exercise. Show that $[C_{12},C_{13}]+[C_{12},C_{23}]+[C_{13},C_{23}]$ is proportional to γ.

Therefore we can find a combination $r=\rho+\text{ const}\cdot C$ for which

$$[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0\ . \tag{2.3.2}$$

Note that we still have $\mu_i^{jk}X_j\otimes X_k=[\Delta X_i,r]$ since B commutes with Δ_0X_i for all i. The equation (2.3.2) is called the classical Yang-Baxter equation (cYBe). We explained that for a simple Lie algebra $\mathcal{L}$ the problem of finding the Lie bialgebra structures on $\mathcal{L}$ reduces to cYBe for r which satisfies: $r+r'$ is proportional to C, $r+r'=xC$ with $x\in\mathbb{C}$ (r' is the flip of r, $r'=r^{ij}X_j\otimes X_i$ for $r'=r^{ij}X_i\otimes X_j$). If $x\neq 0$ one can set $x=1$ by rescaling r.

The Yang-Baxter equation (which reduces to the cYBe in the classical limit) is

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\ . \tag{2.3.3}$$

Solutions of the cYBe for which $x\neq 0$ are the most interesting - their quantizations find lots of applications in statistical models, knot theory, representation theory *etc.*

Exercise. In the situation of the exercise 6 from the previous subsection, show that the corresponding coproduct on $\mathfrak{sl}_2$ arises from an r-matrix, $r=\frac{1}{4}h\otimes h+e_-\otimes e_+$. Verify the cYBe for this r.

We shall now explain how the solutions of cYBe with $r+r'=C$ are classified in terms of so called Belavin-Drinfeld triples. A procedure of quantizing these solutions is known today [**7, 8**]. It is however interesting to enumerate the Belavin-Drinfeld triples, which is a combinatorial question; in the end of this subsection we shall discuss and partly answer it.

Classification of solutions.

Fix a Cartan subalgebra $\mathfrak{h}$. Let R be the set of roots, $R=R_+\cup R_-$, and Γ the set of simple positive roots.

Definition. A Belavin-Drinfeld triple (Γ_1,Γ_2,τ) consists of the following data: Γ_1 and Γ_2 are subsets in Γ and $\tau:\Gamma_1\to\Gamma_2$ is a one-to-one mapping which satisfies properties:

(i) τ preserves the scalar product, that is, $\langle\tau(\alpha),\tau(\beta)\rangle=\langle\alpha,\beta\rangle$ for all α and β from Γ_1.

(ii) τ is "nilpotent". It means the following. Assume that $\tau(\alpha)$, which is an element from Γ_2 is still in Γ_1. Then $\tau^2(\alpha)$ is defined. If again $\tau^2(\alpha)\in\Gamma_1$ then there is $\tau^3(\alpha)$. Nilpotency means that the sequence must terminate, that is, for some $k\in\mathbb{N}$, an element $\tau^k(\alpha)$ is not any more in Γ_1 for any $\alpha\in\Gamma_1$.

Given a Belavin-Drinfeld triple, consider a system of equations for a tensor $r_0\in\mathfrak{h}\otimes\mathfrak{h}$,

$$\begin{aligned} &r_0+r_0'=t_0\ ,\\ &(\tau(\alpha)\otimes\ \mathrm{id}+\ \mathrm{id}\otimes\alpha)(r_0)=0\quad \text{for all}\ \ \alpha\in\Gamma_1\ . \end{aligned}\tag{2.3.4}$$

Here t_0 is the "Cartan part" of t: for a basis H_μ of $\mathfrak{h}$ let $B^o_{\mu\nu}=\langle H_\mu,H_\nu\rangle$; then $t_0=B^{o\,\mu\nu}H_\mu H_\nu$ where $B^{o\,\mu\nu}$ is the inverse to $B_{\mu\nu}$, $B^{o\,\mu\nu}B_{\nu\rho}=\delta^\mu_\rho$.

The system (2.3.4) is compatible [**9**].

Recall that $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$, where $[h,x]=\alpha(h)x$ for $x\in\mathfrak{g}_\alpha$, $\dim\mathfrak{g}_\alpha=1$.

Let $\mathcal{A}_i$ be a Lie subalgebra generated by e_α with $\alpha\in\Gamma_i$, $i=1,2$. Then $\mathcal{A}$ is the direct sum of those $\mathfrak{g}_\alpha$ for which the expansion of α in terms of simple roots contains simple roots from Γ_i only.

The map $\tau:\Gamma_1\to\Gamma_2$ extends to an isomorphism $\tau:\mathcal{A}_1\to\mathcal{A}_2$ (denoted also by τ), by the formula $e_\alpha\mapsto e_{\tau(\alpha)}$. It is an isomorphism because the only relations in $\mathcal{A}_i$ are Serre relations which depend on the scalar product $\langle .,.\rangle$ only and τ respects the scalar product.

For each $\alpha\in R$ choose e_α in such a way that

(i) $\langle e_{-\alpha},e_\alpha\rangle=1$,

(ii) $e_{\tau(\alpha)}=\tau(e_\alpha)$ whenever $\tau(e_\alpha)$ is defined ($e_\alpha\in\mathcal{A}_1$).

Define a partial order: $\alpha<\beta$ for $\alpha,\beta\in R$ means that there exists a natural k such that $\tau^k(\alpha)=\beta$.

Theorem [**9**]**.** Let

$$r=r_0+\sum_{\alpha\in R_+}e_{-\alpha}\otimes e_\alpha+\sum_{\alpha,\beta\in R;\alpha<\beta}(e_{-\alpha}\otimes e_\beta-e_\beta\otimes e_{-\alpha})\ ,\tag{2.3.5}$$

where r_0 is a solution of (2.3.4).

Then

(i) the tensor r satisfies $\mathrm{cYBE}(r)=0$ and $r+r'=t$;

(ii) any solution of equations $\mathrm{cYBE}(r)=0$ and $r+r'=t$, after a suitable change of the basis, is of the form (2.3.5).

The r corresponding to the trivial Belavin-Drinfeld triple (Γ_1 and Γ_2 are empty sets), with $r_0=\frac{1}{2}t_0$, is called the "standard" r.

2.3.1. *Maximal triples.* I shall say several words about the combinatorics of Belavin-Drinfeld triples. The whole information about scalar products is contained in the Dynkin diagram for the algebra $\mathfrak{g}$. We shall consider the most interesting case of the Lie algebras of the type A (that is, Lie algebras $sl(n)$), for which the Dynkin diagram is

Fig. 1

Given a Belavin-Drinfeld triple, it is useful to draw a diagram, corresponding to it, like:

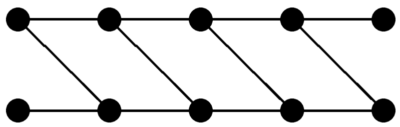


Fig. 2

The upper and lower rows are two copies of the Dynkin diagram A_5, the lines between the rows carry an information about the triple; the lines should be thought as going from the upper low to the lower one; the roots from Γ_1 are the roots at the upper row from which the lines start; the roots from the lower row are those at which the lines end; they are from Γ_2. The angles between the roots are determined by the number of edges connecting the corresponding vertices of the Dynkin diagram; it is therefore easy to understand, looking at the picture, whether the map τ preserves scalar products. To check the nilpotency one needs to draw more than two rows - depicting the powers of τ. For example, for the diagram on Fig. 2 one draws:

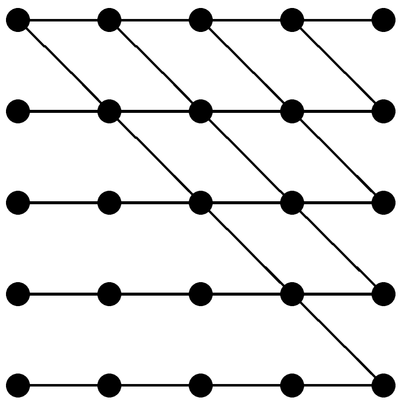


Fig. 3

The meaning of Fig. 3 is clear: the lines going from the first row to the third one represent τ^2, from the first row to the fourth one represent τ^3, *etc.*

There are two types of natural equivalences for triples:

(*i*) $T_\updownarrow : \ (\Gamma_1, \Gamma_2, \tau) \mapsto (\Gamma_2, \Gamma_1, \tau^{-1})$; this corresponds to a reflection of the picture of the triple in the horizontal mirror, $T_\updownarrow^2 = \ \mathrm{id}$;

(*ii*) if a Dynkin diagram has a symmetry κ then $(\kappa(G_1), \kappa(G_2), \kappa\tau\kappa^{-1})$ is a triple and the equivalence is $T_\kappa : (G_1, G_2, \tau) \mapsto (\kappa(G_1), \kappa(G_2), \kappa\tau\kappa^{-1})$.

For A_n-diagram there is a symmetry: a reflection of Fig. 1 in the vertical mirror; let $T_\leftrightarrow$ be the corresponding equivalence; we have $T_\leftrightarrow^2 = \ \mathrm{id}$.

If $\tilde{\Gamma}_1$ is any subset of Γ_1 then $(\tilde{\Gamma}_1, \tau(\tilde{\Gamma}_1), \tau|_{\tilde{\Gamma}_1})$ is clearly a triple. So it is interesting to look only for "maximal" triples, i. e. those to which one cannot add any more vertices.

For example, the only nontrivial triple for A_2 is

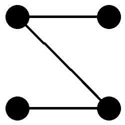


Fig. 4

Exercises. 1. Show that for A_3 there are, up to equivalences, two maximal triples:

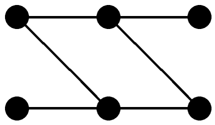


Fig. 5

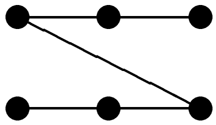


Fig. 6

2. Show that for A_4 there are, up to equivalences, four maximal triples:

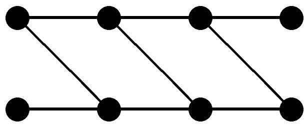


Fig. 7

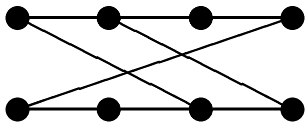


Fig. 8

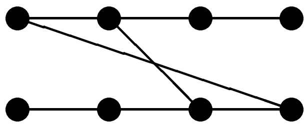


Fig. 9

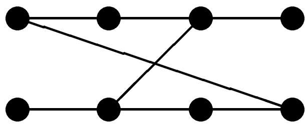


Fig. 10

With the growth of rank it becomes more and more difficult to decide if a given triple is maximal. For example, the triple on Fig. 11

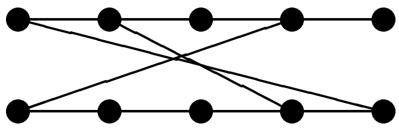


Fig. 11

is maximal, but one has to draw several rows (like on Fig. 3) to see that loops appear when one adds one more vertex.

If $\Gamma \setminus \Gamma_1$ consists of only one vertex (or $\#\Gamma_1 = \#\Gamma - 1$) then the triple is certainly maximal. We shall enumerate triples with $\#\Gamma_1 = \#\Gamma - 1$.

Proposition 9. For the Dynkin diagram A_l, the number of triples with $\#\Gamma_1 = l-1$ is $\frac{1}{2}\Phi(l+1)$, Φ is the Euler function, $\Phi(n) = \#\{j \in \{1,\dots,n\} | j$ is coprime to $n\}$.

Proof.
(i) $\Phi(l+1)$ is the number of primitive roots of unity of order $(l+1)$.

We shall first associate a Belavin-Drinfeld triple to any primitive root of unity of order $(l+1)$. Let $\zeta = \exp(\frac{2\pi i}{l+1})$. Label the vertices of the Dynkin diagram A_l as shown on Fig. 12:

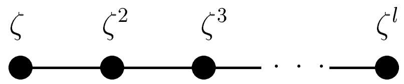


Fig. 12

If a and b are labels of two vertices then a is connected by an edge to b if and only if $a = \zeta^{\pm} b$.

Fix a primitive root q. Let $\Gamma_1 = \{q, q^2, \dots, q^{l-1}\}$ and $\Gamma_2 = \{q^2, q^3, \dots, q^l\}$ (more precisely, Γ_i, $i = 1, 2$, are the sets of vertices of A_l labeled by the corresponding roots of unity). Since q is primitive, each of the sets Γ_1 and Γ_2 contain $(l-1)$ distinct elements.

Let $\tau : \Gamma_1 \to \Gamma_2$ be the multiplication by q. Multiplying a label q^i by q, we obtain a sequence $q^i \to q^{i+1} \to \dots \to q^l$, and the sequence terminates since $q^{l+1} = 1$ is not a label of any vertex. Thus, the map τ is nilpotent.

The condition of being neighbors, $q^i = \zeta^{\pm} q^j$ is stable under the multiplication by q, therefore τ preserves scalar products.

Thus, $(\Gamma_1, \Gamma_2, \tau)$ is a Belavin-Drinfeld triple. Call it $\mathcal{T}_q$.

Consider an arbitrary Belavin-Drinfeld triple $\mathcal{T} = (\Gamma_1, \Gamma_2, \tau)$ with $\#\Gamma_1 = l-1$. We shall prove that it coincides with one of $\mathcal{T}_q$'s.

(ii) Denote the vertex omitted from Γ_1 by q^{-1}. It divides the row of the diagram A_l (as on Fig. 1) into two segments I_1 and I_2:

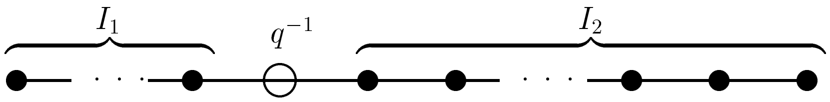


Fig. 13

We have $q^{-1} = \zeta^a$ for some a. Making, if necessary, a vertical reflection, we can, without loss of generality, assume that $\#I_1 \leq \#I_2$.

Let q' be a label of a vertex omitted from Γ_2. The lower row of the picture corresponding to $\mathcal{T}$ is also divided by q' into two segments J_1 and J_2 (J_1 to the left of q', J_2 to the right of it). The map τ preserves neighbors and it follows that either $\tau : I_1 \to J_1$, $I_2 \to J_2$ or $\tau : I_1 \to J_2$, $I_2 \to J_1$. The former case is excluded since otherwise $I_1 = J_1$, $I_2 = J_2$ and restrictions of τ on the sets I_i, $i = 1, 2$, are permutation of these sets and therefore τ cannot be nilpotent.

Thus, $\tau : I_1 \to J_2$, $I_2 \to J_1$ and $q' = q$.

We cannot have $\#I_1 = \#I_2$ – then restrictions of τ^2 on I_i, $i = 1, 2$, would be permutations of these sets. Therefore, $\#I_1 < \#I_2$.

(iii) Consider the restriction of τ on the set I_2, $\tau : I_2 \to J_1$. There are two possibilities: τ preserves the order or reverses it. We shall prove that τ cannot reverse the order. Indeed, if τ reverses the order then τ maps q to q^{-1} (it is useful to draw a picture here). Then τ induces a permutation on the set $\Gamma \setminus (q \cup q^{-1})$ and cannot be nilpotent.

Let us collect obtained information about the triple $\mathcal{T}$ in a picture:

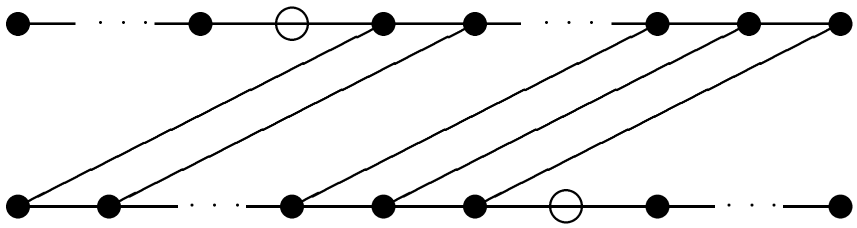


Fig. 14

(iv) For the restriction of τ on the set I_1 we have again two possibilities, the order is reversed or preserved. We shall prove that it is preserved. If the cardinality of I_1 is 0 or 1, there is nothing to prove, so, without a loss of generality we assume that $\#I_1 > 1$, in other words, $a \geq 3$. Thus we have $l \geq 6$ since $\#I_2 > \#I_1$.

Extend τ to a map $\tilde{\tau} : \Gamma \to \Gamma$ by $\tilde{\tau} : q^{-1} \mapsto q$ (and $\tilde{\tau} = \tau$ on Γ_1). The map $\tilde{\tau}$ is a permutation of Γ. Decompose $\tilde{\tau}$ into a product of cycles. Since q^{-1} maps to q, the decomposition contains a cycle $c = (\dots q^{-1} q \dots)$. If there are other cycles, $\tilde{\tau} = c \cdot c_1 \cdot c_2 \dots$ then the product $c_1 \cdot c_2 \dots$ is a permutation of some set S. This permutation is the restriction of τ on S, thus τ cannot be nilpotent. We conclude that $\tilde{\tau}$ is a cycle.

Explicitly, the action of τ is

$$\left\{ \begin{array}{l} \zeta^{a+i} \mapsto \zeta^i \ , \ i = 1, \dots, n-a \ , \\ \\ \zeta^i \mapsto \zeta^{-i} \ , \ i = 1, \dots, a \ . \end{array} \right. \tag{2.3.6}$$

We shall follow a sequence $\tilde{\tau}^n(q^{-1})$. First, $q^{-1} = \zeta^a$ maps to $q^{-1} = \zeta^{l+1-a}$. Then it goes back, $\zeta^{l+1-a} \mapsto \zeta^{l+1-2a} \mapsto \dots \mapsto \zeta^{l+1-ka}$, where $l + 1 - ka \leq a$ but $l + 1 - (k-1)a > a$ or $l + 1 \leq (k+1)a$ but $l + 1 > ka$. This requires k steps (i.e. this is the result of the action of $\tilde{\tau}^k$ on q^{-1}). At the next step, ζ^{l+1-ka} maps to ζ^{ka} and then again goes back, $\zeta^{ka} \mapsto \zeta^{(k-1)a} \mapsto \dots \mapsto \zeta^a$. This takes k more steps. Thus, $\tilde{\tau}^{2k}(q^{-1}) = q^{-1}$.

But $3k \leq ak < l + 1$. Therefore, $2k < \frac{2l+2}{3} < l$ because l is at least 6.

Therefore, the permutation $\tilde{\tau}^{2k}$ has a fixed point and $2k < l$. Thus $\tilde{\tau}$ cannot be a cycle.

We are left with only one possibility: the restriction of τ on I_1 preserves the order.

(v) The map τ preserves the order on I_2 by (iii) and on I_1 by (iv). Written explicitly, it means that the map τ is the multiplication by q. It will not be nilpotent if q is not primitive, therefore, the triple $\mathcal{T}$ coincides with one of $\mathcal{T}_q$'s.

(vi) The group, generated by flips $T_\updownarrow$ and $T_\leftrightarrow$, is $\mathbb{Z}_2\times\mathbb{Z}_2$. But on triples $\mathcal{T}_q$ the operations $T_\updownarrow$ and $T_\leftrightarrow$ coincide: each of them is $\mathcal{T}_q \mapsto \mathcal{T}_{q^{-1}}$ (in general, it is not so, even for maximal triples: consider the triple corresponding to Fig. 11). Therefore, only $\mathbb{Z}_2$ acts on the space of triples $\mathcal{T}_q$. This action does not have fixed points and we conclude that the number of triples, under equivalences, is $\frac{1}{2}\Phi(l+1)$ as stated. The proof of the Theorem 9 is finished. $\square$

3. Quantum spaces

We shall briefly, without going into details, give geometrical motivations which lead to the notion of quantum spaces.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. In the section 2, Lie bialgebras appeared in the study of deformations of the coproduct on the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. Geometrically, Lie algebra $\mathfrak{g}$ is the Lie algebra of left-invariant vector fields on G. The universal enveloping algebra of the Lie algebra $\mathfrak{g}$ can therefore be realized as the algebra of left invariant differential operators on G. Up to topological and functional analytic considerations (convergence, etc.), a function on $\mathbb{R}^n$ can be reconstructed, as a Taylor series, from the knowledge of its derivatives at the origin. For a Lie group G, the knowledge of derivatives of a function f at the origin is replaced by the knowledge of values on f of all left invariant differential operators at the unity of G. The elements of $\mathcal{U}\mathfrak{g}$ are linear functionals on the space $\mathcal{F}G$ of functions on G, so, up to topological considerations, the spaces $\mathcal{U}\mathfrak{g}$ and $\mathcal{F}G$ are dual to each other, the pairing between $X\in\mathcal{U}\mathfrak{g}$ and $f\in\mathcal{F}G$ is given by $\langle X,f\rangle = X(f)|_e$, where $e\in G$ is the unity element. It follows then that the coproduct on $\mathcal{U}\mathfrak{g}$ corresponds to the product on $\mathcal{F}G$ - the usual product of functions. Thus, deformations of the coproduct on $\mathcal{U}\mathfrak{g}$ correspond to deformations of the commutative algebra $\mathcal{F}G$ of functions. Infinitesimal deformations from Section 1 correspond to particular Poisson brackets on G – Poisson brackets which are compatible with the group structure. One says that Poisson brackets are compatible with the group structure if the multiplication $m: G\times G\to G$ is a Poisson map. In other words: define, for a given function f on G, a function $f\tilde{}$ on $G\times G$ by the rule $f\tilde{}(x,y) = f(x\cdot y)$; the compatibility of the Poisson brackets $\{.,.\}$ means $\{f,g\}\tilde{} = \{f\tilde{},g\tilde{}\}$, where the Poisson brackets on $G\times G$ are Poisson brackets of the direct product of two Poisson manifolds. Groups with compatible Poisson brackets are called Poisson-Lie groups.

In the other direction, it is not difficult to check that if G carries compatible Poisson brackets then its Lie algebra $\mathfrak{g}$ gets a Lie bialgebra structure.

Compatible Poisson brackets are of a very special form. We shall illustrate it on an example of a matrix group G (a subgroup of the group of invertible matrices). Let a^i_j be matrix elements. Assume that $\{a^i_k, a^m_s\} = \Phi^{im}_{ks}(a)$ are compatible Poisson brackets. Then $\{a^i_j b^j_k, a^m_n b^n_s\} = \Phi^{im}_{ks}(ab)$ (this is the equality $\{f\tilde{}, g\tilde{}\} = \{f,g\}\tilde{}$ for $f = a^i_k$ and $g = a^m_s$). On the other hand, $\{a^i_j b^j_k, a^m_n b^n_s\} = \{a^i_j, a^m_n\} b^j_k b^n_s + \{b^j_k, b^n_s\} a^i_j a^m_n = \Phi^{im}_{jn}(a) b^i_k b^n_s + \Phi^{jn}_{ks}(b) a^i_j a^m_n$ because $G\times G$ is equipped with the Poisson structure of the direct product. Therefore Φ must be homogeneous of degree 2 (this reflects the fact that μ_i in the deformation of the coproduct on $\mathcal{U}\mathfrak{g}$ belongs to $\mathfrak{g}\otimes\mathfrak{g}$). Thus, the Poisson brackets are quadratic. To quantize constant or linear Poisson brackets, one simply replaces the Poisson brackets by the commutator. However, it is not obvious how to quantize quadratic Poisson brackets - we cannot replace Poisson brackets by the commutator because we don't know how to order consistently the quadratic right hand side.

Exercises. 1. Show that formulas

$$\begin{aligned} &\{a,b\}=ab\ ,\ \{a,c\}=ac\ ,\ \{b,d\}=bd\ , \\ &\{c,d\}=cd\ ,\ \{b,c\}=0\ ,\ \{a,d\}=2bc \end{aligned} \tag{3.0.1}$$

are Poisson brackets for four variables a, b, c and d.

2. Show that if a, b, c and d are matrix elements of a matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the Poisson brackets (3.0.1) provide $GL(2)$ with a Poisson-Lie structure.

3. Show that the Poisson brackets (3.0.1) of the determinant of A, $\det A$ with all matrix elements vanish.

The main interest about the most commonly appearing groups, like GL, SO, Sp, ... - is that they arise as groups of symmetry (of a vector space, of a vector space with a bilinear form, ...). The Poisson-Lie groups one can interpret in this way too. One says that a Poisson-Lie group G acts on a Poisson manifold $\mathcal{M}$ in a Poisson way if

(i) G acts on M;

(ii) the action $G\times\mathcal{M}\to\mathcal{M}$ is a Poisson map, where $G\times\mathcal{M}$ is equipped with a Poisson structure of the direct product.

Again, for a matrix group G acting on a vector space V^2, $x^i\mapsto a^i_jx^j$, a similar calculation shows that the Poisson brackets of coordinates, $\{x^i,x^j\}$ must be quadratic in x^i.

Exercise. For a two dimensional vector space with coordinates x^1 and x^2 let

$$\{x^1,x^2\}=x^1x^2\ . \tag{3.0.2}$$

Show that $GL(2)$ with the Poisson structure given by (3.0.1) acts in a Poisson way on this Poisson vector space.

It turns out that from the point of view of the theory of quantum groups, the appropriate way to quantize the Poisson brackets (3.0.2) is provided by the following commutation relations:

$$x^1x^2=qx^2x^1\ . \tag{3.0.3}$$

This is the first example of a quantum vector space. Denote this quantum vector space (that is, the algebra of polynomials in x^1 and x^2 subjected to the relation (3.0.3)) by V^2_q.

The linear group of transformations preserving the relation (3.0.3) is poor, it consists only of rescalings $x^i\mapsto c_ix^i$ with some constants c_i. It is the quantum group which is the right analogue of the symmetry group of the quantum vector space.

General picture.

Let $\mathcal{U}$ be a quasitriangular Hopf algebra with a universal R-matrix $\mathcal{R}$. Let $\mathcal{U}^*$ be a dual Hopf algebra. The pairing between $\mathcal{U}$ and $\mathcal{U}^*$ satisfies $\langle\Delta a,x\otimes y\rangle=\langle a,xy\rangle$ and $\langle\alpha\otimes b,\Delta x\rangle=\langle ab,x\rangle$.

We shall think of $\mathcal{U}$ as of analogue of the universal enveloping algebra of a semi-simple Lie algebra. Ideologically (in the spirit of the Peter-Weyl theorem) the dual Hopf algebra is generated by matrix elements of representations of $\mathcal{U}$.

Let ρ be a representation of $\mathcal{U}$ on a vector space V, ρ maps an element $x \in \mathcal{U}$ to a matrix $T^i_j(x)$.

The coproduct Δ on matrix elements T^i_j looks especially simple:

$$\begin{aligned} \langle \Delta T^i_j, x\otimes y\rangle &= \langle T^i_j, xy\rangle = T^i_j(xy) = T^i_k(x)T^k_j(y) \\ &= \langle T^i_k, x\rangle\langle T^k_j, y\rangle = \langle T^i_k\otimes T^k_j, x\otimes y\rangle \end{aligned} \tag{3.0.4}$$

and therefore

$$\Delta T^i_j = T^i_k \otimes T^k_j \ . \tag{3.0.5}$$

The commutation relations between the matrix elements T^i_j are expressed in terms of a numerical R-matrix R, the image of the universal R-matrix $\mathcal{R} = \sum_i a_i \otimes b_i$ in the representation, $R = \sum_i \rho(a_i)\otimes\rho(b_i)$. Let $\Delta x = \sum_\alpha u_\alpha \otimes v_\alpha$.

$$\langle T_1T_2, x\rangle = \langle T_1\otimes T_2, \sum u_\alpha\otimes v_\alpha\rangle = \sum T_1(u_\alpha)T_2(v_\alpha) \ . \tag{3.0.6}$$

By the definition of $\mathcal{R}$ we have $\sum u_\alpha\otimes v_\alpha = \mathcal{R}^{-1}\sum v_\alpha\otimes u_\alpha\mathcal{R}$. Therefore

$$\begin{aligned} \langle T_1\otimes T_2, \textstyle\sum u_\alpha\otimes v_\alpha\rangle &= \langle T_1\otimes T_2, \mathcal{R}^{-1}\textstyle\sum v_\alpha\otimes u_\alpha\mathcal{R}\rangle \\ &= T_1\otimes T_2(\mathcal{R}^{-1}\textstyle\sum v_\alpha\otimes u_\alpha\mathcal{R}) \\ &= T_1\otimes T_2(\mathcal{R}^{-1})\cdot T_1\otimes T_2(\textstyle\sum v_\alpha\otimes u_\alpha)\cdot T_1\otimes T_2(\mathcal{R}) \\ &= R^{-1}\textstyle\sum T_1(v_\alpha)T_2(u_\alpha)R = R^{-1}\textstyle\sum T_2(u_\alpha)T_1(v_\alpha)R \ . \end{aligned} \tag{3.0.7}$$

The "$\cdot$" means matrix multiplication.

Arguments u_α and v_α are now in the same order as in (3.0.6). Therefore, $T_1T_2 = R^{-1}T_2T_1R$ or

$$RT_1T_2 = T_2T_1R \ . \tag{3.0.8}$$

Because of the form of this relation, this algebra is often called the "RTT"-algebra.

The algebra $\mathcal{U}^*$ was first written in the form (3.0.8) and (3.0.5) in [**10**].

We shall write the relations (3.0.8) in a different way. Let P be a permutation of factors in $V\otimes V$, $P(x\otimes y) = y\otimes x$. Let $\hat{R} = PR$. Then (3.0.8) is equivalent to

$$\hat{R}T_1T_2 = T_1T_2\hat{R} \ . \tag{3.0.9}$$

A motivation to use $\hat{R}$ instead of R: the eigenvalues of $\hat{R}$ have a representation-theoretic meaning. A theorem, due to Drinfeld [**11**], says that there exists an element $\mathcal{F}$ such that $\mathcal{R} = \mathcal{F}_{21}q^t\mathcal{F}^{-1}$, where $q = \exp(\alpha)$ (α is the deformation parameter), t is the invariant tensor $B^{ij}X_i\otimes X_j$. Let $C = B^{ij}X_iX_j$. Now let $V\otimes V = \sum W_\gamma$ be a decomposition of the tensor square of the space V into irreducible representations. We have $\Delta_0 C = C\otimes 1 + 1\otimes C + 2t$, where Δ_0 is the classical coproduct. Therefore, $t|_{W\gamma} = \frac{1}{2}C|_{W_\gamma} - C|_V$. Denote this quantity by t_γ. We have $R = PFPq^{\rho\otimes\rho(t)}F^{-1}$, therefore, $\hat{R} = FPq^{\rho\otimes\rho(t)}F^{-1}$.

First of all, $V\otimes V$ decomposes into the symmetric and antisymmetric parts, $V\otimes V = S^2V\oplus\Lambda^2V$. The operator P takes the value $(+1)$ on S^2V and (-1) on Λ^2V. Every W_γ is either is either in S^2V or in Λ^2V so the "sign" of W_γ, depending on whether W_γ is S^2V or Λ^2V, $\operatorname{sign}(W_\gamma) := P|_{W_\gamma}$ is well defined.

Since $q^{\rho\otimes\rho(t)}|_{W_\gamma} = q^{t_\gamma}$ we have

$$\hat{R}|_{FW_\gamma} = FPq^{\rho\otimes\rho(t)}|_{W_\gamma} = q^{t_\gamma}\ \operatorname{sign}(W_\gamma)\ . \tag{3.0.10}$$

Thus, the projector decomposition of $\hat{R}$ reflects the decomposition of the tensor product of representations.

Polynomials in the matrix elements T^i_j are "quantized" functions on the group. We also had the Poisson brackets on the coordinates x^i and $x^i \mapsto T^i_j x^j$ was a Poisson map. On the quantum level we have the $\hat{R}TT$ relations for T^i_j. Which relations can we impose on x's in such a way that the map $x^i \mapsto T^i_j x^j$ is an algebra homomorphism? These relations should also quantize the Poisson brackets for x's. Since the quantization of the Poisson brackets for T^i_j produced quadratic relations, we expect to have quadratic relations for the algebra of x's as well. Impose a set of quadratic relations for x's, $E^\alpha_{ij}x^ix^j = 0$, α labels relations. Then for Tx we have $E^\alpha_{ij}T^i_a x^a T^j_b x^b = E^\alpha_{ij}T^i_a T^j_b x^a x^b$, so we have to understand which tensors we can move through $T^i_a T^j_b$.

The defining relation (3.0.9) shows that we can move $\hat{R}$ and therefore any function of $\hat{R}$. As we have seen, $\hat{R} = \sum \nu_\gamma \Pi_\gamma$, where $\nu_\gamma = q^{t_\gamma}\ \operatorname{sign}(W_\gamma)$ and Π_γ is the projector on the space FW_γ. So essentially, all functions of $\hat{R}$ are linear combinations of projectors.

Conclusion: covariant algebras are given by relations $(\Pi_\gamma)^{ij}_{kl}x^kx^l = 0$ for some γ (one or several). Denote by A^l_γ (l for "left") the quadratic algebra defined by one projector Π_γ. Equally well, there is a covariant "right" algebra A^r_γ (the algebra of covectors x_i), defined by $x_ix_j(\Pi_\gamma)^{ij}_{kl} = 0$, the covariance is $x_i \mapsto x_a T^a_i$.

An important fact is that the RTT-relations can be reconstructed from the requirement that all the algebras A^l_γ (or all the algebras A^r_γ) are covariant. Indeed, A^l_γ is covariant means

$$(\Pi_\gamma)^{ij}_{kl}T^k_a T^l_b\, x^a x^b = 0\ , \tag{3.0.11}$$

therefore, $(\Pi_\gamma)^{ij}_{kl}T^k_aT^l_b$ must be proportional to Π_γ whose lower indices are a, b:

$$(\Pi_\gamma)^{ij}_{kl}T^k_aT^l_b = \Sigma^{ij}_{uv}(\Pi_\gamma)^{uv}_{ab}\ . \tag{3.0.12}$$

Multiplying (3.0.12) by $(\Pi_\tau)^{ab}_{cd}$ with $\tau \neq \gamma$, we find

$$\Pi_\gamma T_1 T_2 \Pi_\tau = 0 \tag{3.0.13}$$

for all pairs $\{\tau, \gamma \,|\, \tau \neq \gamma\}$.

Lemma 10. The system (3.0.13) of relations for the matrix elements T^i_j is equivalent to the RTT-relations $\hat{R}T_1T_2 = T_1T_2\hat{R}$.

Proof.

(i) (3.0.9) implies (3.0.13).

We have $\hat{R} = \sum \nu_\gamma \Pi_\gamma$; moreover, $\sum \Pi_g = 1$ is a decomposition of unity. Multiplying $\hat{R}T_1T_2 = T_1T_2\hat{R}$ by Π_γ from the left and Π_τ from the right, we find $\nu_\gamma \Pi_\gamma T_1T_2\Pi_\tau = \nu_\tau \Pi_\gamma T_1 T_2 \Pi_\tau$. If $\gamma \neq \tau$ then $\nu_\gamma \neq \nu_\tau$ and it follows that the relation $\Pi_\gamma T_1T_2\Pi_\tau = 0$ is satisfied.

(ii) (3.0.13) implies (3.0.9).

We have $\hat{R}T_1T_2 = \hat{R}T_1T_2 \cdot 1 = \sum_{\gamma,\tau} \nu_\gamma \Pi_\gamma T_1T_2\Pi_\tau$. The last expression, due to the relations (3.0.13), can be rewritten as

$$\sum_\gamma \nu_\gamma \Pi_\gamma T_1 T_2 \Pi_\gamma \ . \tag{3.0.14}$$

Similarly, $T_1T_2\hat{R} = 1 \cdot T_1T_2\hat{R} = \sum_{\gamma,\tau} \nu_\tau \Pi_\gamma T_1T_2\Pi_\tau$. Again, due to (3.0.13), this equals $\sum_\gamma \nu_\gamma \Pi_\gamma T_1T_2\Pi_\gamma$, which coincides with (3.0.14). Thus, (3.0.9) holds. □

If $\hat{R}$ appears in the process of a deformation then there is a candidate for an especially nice quantum space. Again, $V \otimes V = \bigoplus W_\gamma$ classically; denote the set of $\{\gamma\}$ by J, $J = J_+ \cup J_-$ where $J_\pm = \{\gamma \mid \operatorname{sign} W_\gamma = \pm 1\}$. Then projectors $\Pi_\pm = \sum_{\gamma \in J_\pm} \Pi_\gamma$ have ranks $\operatorname{rk}\Pi_\pm = \frac{N(N\pm 1)}{2}$. Therefore, a set $(\Pi_-)^{ij}_{kl} x^k x^l = 0$ contains $\frac{N(N-1)}{2}$ relations - exactly the number of relations which we have classically for commuting variables. The quantum space defined by relations $(\Pi_-)^{ij}_{kl}x^kx^l = 0$ is the only reasonable candidate for the quantization of $\mathbb{C}^N$ on which a group G acted in a Poisson way. Similarly, the quantum space defined by relations $(\Pi_+)^{ij}_{kl}x^kx^l = 0$ is the quantization of the algebra of odd (grassmanian) variables.

For $GL(N)$, in the decomposition $V \otimes V = S^2V \oplus \Lambda^2 V$, the summands S^2V and $\Lambda^2 V$ are irreducible. It is natural therefore to call an $\hat{R}$, which contains only two projectors, an $\hat{R}$-matrix of GL type. One usually rescales $\hat{R}$ to have $\hat{R} = q\Pi_+ - q^{-1}\Pi_-$, where Π_+ and Π_- are projectors, which are called, due to their origin, the q-symmetrizer and the q-antisymmetrizer respectively (and we shall often denote Π_+ by S and Π_- by A).

To conclude:

- Geometrically and physically meaningful $\hat{R}$-matrices decompose into projectors

$$\underbrace{\alpha_1\Pi_1^+ + \dots \alpha_k\Pi_k^+}_{S} + \underbrace{\beta_1\Pi_1^- + \dots \beta_l\Pi_l^-}_{A} \tag{3.0.15}$$

  and we know which projectors constitute the q-symmetrizer S and the q-antisymmetrizer A, respectively (as shown by underbracing in (3.0.15)).

  The ranks of the projectors $S = \Pi_+ = \Pi_1^+ + \dots \Pi_k^+$ and $A = \Pi_- = \Pi_1^- + \dots + \Pi_l^-$ are classical, $\operatorname{rk}\Pi_\pm = \frac{N(N\pm 1)}{2}$.
- Covariant algebras are quadratic algebras of the form $(\Pi_\gamma)^{ij}_{kl}x^kx^l = 0$ or $x_ix_j(\Pi_\gamma)^{ij}_{kl} = 0$ where Π_γ is one of the projectors in the decomposition (3.0.15).
- The algebra of functions on a quantum group is given by the relations (3.0.9) and these relations are equivalent to the condition that all the algebras A^l_γ are covariant; in other words, the RTT-algebra can be reconstructed from the A^l_γ algebras. The same holds if one replaces left algebras by the right ones.

3.1. *GL* **type.** For GL-type algebras we have only two projectors[1], $\hat{R} = qS - q^{-1}A$ In this case, to reconstruct the RTT-algebra, it is enough to require covariance of two algebras, A^l_+ and A^l_- defined by relations $S^{ij}_{kl}x^kx^l = 0$ and $A^{ij}_{kl}x^kx^l = 0$ respectively [**12**].

[1] When $\hat{R}$ has two eigenvalues, one says that $\hat{R}$ is of Hecke type. We require additionally that the ranks of the projectors are fixed $\operatorname{rk} S = \frac{N(N+1)}{2}$, $\operatorname{rk} A = \frac{N(N-1)}{2}$.

As we have seen, covariance of the algebra A^l_γ implies the condition $\Pi_\gamma T_1 T_2 \Pi_\tau = 0$ for all τ different from γ. Thus, in the GL case, the covariance of the algebra A^l_+ implies

$$ST_1T_2A = 0 \tag{3.1.1}$$

while the covariance of A^l_- implies

$$AT_1T_2S = 0\ . \tag{3.1.2}$$

On the other hand, the covariance of the algebra A^r_+ implies the same relation (3.1.2). Together, relations (3.1.1) and (3.1.2), are equivalent to the RTT-relations. This shows that in the GL case, one can interpret the RTT-relations in two ways: either as the condition of the covariance of the algebras A^l_+ and A^l_- or as the condition of the covariance of the algebras A^l_+ and A^r_+. We shall use the latter interpretation in the sequel.

The algebras $A^{l,r}_+$ are the left and right quantum spaces. If they are good deformations then the dimension $d(N,k)$ of the space of polynomials of degree k coincides with the dimension of the space of polynomials in N commuting variables,

$$d(N,k) = \begin{pmatrix} N+k-1 \\ k \end{pmatrix}\ . \tag{3.1.3}$$

So, quantum spaces are quadratic algebras with correct Poincaré series.

As we shall see below, the behavior of Poincaré series is intimately related to the theory of quantum groups.

Definition. Given a set of tensors $\mathcal{E} = \{E^\alpha = E^\alpha_{ij}\}$, $i,j = 1,\ldots,n$, define an algebra $A_\mathcal{E}$ with generators x^i and relations

$$E^\alpha_{ij} x^i x^j = 0 \quad \text{for all} \quad \alpha\ . \tag{3.1.4}$$

Let $d(N,k)$ be the dimension of the space of polynomials of degree k in x^i. We say that $A_\mathcal{E}$ has a Poincaré-Birkhoff-Witt property (or that $A_\mathcal{E}$ is a PBW-algebra) if (3.1.3) is satisfied. In particular the range of the index α is $\{1,\ldots,\frac{N(N-1)}{2}\}$.

Relation $\Pi^{ij}_{kl} x^k x^l = 0$ (with Π a projector) is an example of (3.1.4) but in general the tensors E^α are not organized in a projector.

Requirement that x^i are covariant, that is, that the relations (3.1.4) are satisfied by $T^i_j x^j$ implies some relations between T^i_j's.

Assume that we are given two quantum spaces, $A^l_\mathcal{E}$ with generators x^i and relations $E^\alpha_{ij} x^i x^j = 0$ and $A^r_\mathcal{F}$ with generators x_i and relations $x_i x_j F^{ij}_\beta = 0$.

Definition. We say that the quantum spaces $A^l_\mathcal{E}$ and $A^r_\mathcal{F}$ are compatible if the covariance algebra of T^i_j has the PBW property (that is, its Poincaré series coincide with the Poincaré series for N^2 variables).

Next subsection is a digression on the Poincaré series; then we shall continue with a discussion of compatible quantum spaces in dimension 3.

3.2. Technics of checking the Poincaré series. Consider an algebra with generators x^i and relations (3.1.4). Sometimes it is useful to try to apply the diamond lemma [**13, 14**]. In its easiest form it says: assume that there is a basis $\{x^1,\ldots,x^N\}$ in which relations look: for $j > i$, $x^j x^i$ is a sum of monomials $x^a x^b$

with $a < b$ and all the monomials in the sum are lexicographically smaller or equal $x^j x^i$. Take these relations as instructions: replace $x^j x^i$ by the right hand side. Apply these instructions to $x^k x^j x^i$, where $k > j > i$, in two ways, starting from $x^k x^j$ or from $x^j x^i$. If both ways will eventually produce the same result, to which no more instructions can be applied then the ordered monomials $x_1^{\mu_1} \dots x_N^{\mu_N}$ form a linear basis in the algebra, which implies that the algebra has the PBW property.

Note that whether one can apply this procedure depends on a choice of a basis in the algebra. Such a basis might not exist.

We shall now describe another way of checking the Poincaré series which one can apply in the GL case. It uses a differential calculus on quantum planes, developed in [**15**].

3.2.1. *Differential calculus and Poincaré series.* Assume that the quadratic relations are given by

$$A^{ij}_{kl} x^k x^l = 0 \ , \tag{3.2.1}$$

where A is the q-antisymmetrizer in the Hecke $\hat{R}$-matrix, $\hat{R} = qS - q^{-1}A$.

Let ξ^i be generators of the odd quantum space for $\hat{R}$, that is, the relations for ξ^i are

$$S^{ij}_{kl} \xi^k \xi^l = 0 \ . \tag{3.2.2}$$

One can unify generators x^i and ξ^i into one quadratic algebra by requiring that

$$x^i \xi^j = \hat{R}^{ij}_{kl} \xi^k x^l \ . \tag{3.2.3}$$

Exercise. Verify that relations (3.2.3) are compatible with (3.2.1) and (3.2.2). The compatibility here means the following. Let ϕ be a combination of quadratic expressions $S^{ij}_{kl}\xi^k\xi^l$. Then $\phi = 0$ in the algebra with generators ξ^i. Take an element $x^i\phi$ with some i and use (3.2.3) to move x to the right. We obtain an element of the form $\sum_j \psi_j x^j$, with some quadratic (in ξ's) elements ψ_j. Since ϕ was 0, we must have $\psi_j = 0$. In other words, for each j, the element ψ_j must be a combination of expressions $S^{ij}_{kl}\xi^k\xi^l$. In the same manner, there is a compatibility check when one moves ξ^i to the left through relations (3.2.1) for x's.

At the next step, one adds "q-derivatives" ∂_i in the generators x^i. An algebra of the derivatives ∂_i is the algebra with generators ∂_i and relations

$$\partial_i \partial_j A^{ji}_{kl} = 0 \tag{3.2.4}$$

(note the order of i, j).

To add ∂_i, one needs cross-commutation relations with the already existing generators x^i and ξ^i. These relations are:

$$\partial_i x^j = 1 + q\hat{R}^{js}_{it} x^t \partial_s \ , \tag{3.2.5}$$

$$\partial_i \xi^j = (\hat{R}^{-1})^{js}_{it} \xi^t \partial_s \ . \tag{3.2.6}$$

Exercise. Verify that relations (3.2.5) and (3.2.6) are compatible with (3.2.1), (3.2.2) and (3.2.3) (in the above sense of compatibility).

Finally, one adds derivatives δ_i in ξ^i. An algebra of the derivatives δ_i is the algebra with generators δ_i and relations

$$\delta_i\delta_j S^{ji}_{kl} = 0 \ . \tag{3.2.7}$$

The cross-commutation relations between δ_i and the generators x^i, ξ^i and ∂_i are:

$$\delta_i x^j = \hat{R}^{js}_{it} x^t \delta_s \ , \tag{3.2.8}$$

$$\delta_i \xi^j = 1 - q\hat{R}^{js}_{it} \xi^t \delta_s \ , \tag{3.2.9}$$

$$\delta_i \partial_j = (\hat{R}^{-1})^{kl}_{ji} \partial_l \delta_k \ . \tag{3.2.10}$$

.

Exercise. Verify that relations (3.2.8), (3.2.9) and (3.2.10) are compatible with (3.2.1)- (3.2.7).

We shall need the following singlets, the Euler operators E_e and E_o and the differentials d and δ:

$$\begin{gathered} E_e = x^i\partial_i ,\ E_o = \xi^i\delta_i \ , \\ d = \xi^i\partial_i \ ,\ \delta = x^i\delta_i \ . \end{gathered} \tag{3.2.11}$$

Their relations with the generators of the algebra are (*Exercise*: verify the relations):

$$\begin{gathered} E_e x^i = x^i(1+q^2E_e) \ ,\ \partial_i E_e = (1+q^2E_e)\partial_i \ , \\ E_e \xi^i = \xi^i E_e \ ,\ \delta_i E_o = E_o \delta_i \ , \end{gathered} \tag{3.2.12}$$

$$\begin{gathered} E_0 x^i = x^i E_o \ ,\ \partial_i E_o = E_o \partial_i \ , \\ E_o \xi^i = \xi^i(1+q^2E_o) \ ,\ \delta_i E_o = (1+q^2E_o)\delta_i \ , \end{gathered} \tag{3.2.13}$$

$$\begin{gathered} dx^i = \xi^i + qx^i d \ ,\ \partial_i d = qd\partial_i \ , \\ d\xi^i = -q\xi^i d \ ,\ \delta_i d = (1+(q^2-1)E_o)\partial_i - qd\delta_i \ , \end{gathered} \tag{3.2.14}$$

$$\begin{gathered} \delta x^i = qx^i\delta \ ,\ \partial_i\delta = (1+(q^2-1)E_e)\delta_i + q\delta\partial_i \ , \\ \delta\xi^i = x^i - q\xi^i\delta \ ,\ \delta_i\delta = -q^{-1}\delta\delta_i \ . \end{gathered} \tag{3.2.15}$$

Using operators $N_e = 1 + (q^2-1)E_e$ and $N_o = 1 + (q^2-1)E_o$, appearing in the right hand sides of (3.2.14) and (3.2.15), one can rewrite (3.2.12) and (3.2.13) in a form

$$\begin{gathered} N_e x_{\langle 1|} = q^2 x_{\langle 1|} N_e \ ,\ N_e \partial_{|1\rangle} = q^{-2}\partial_{|1\rangle} N_e \ , \\ N_e \xi_{\langle 1|} = \xi_{\langle 1|} N_e \ ,\ N_e \delta_{|1\rangle} = \delta_{|1\rangle} N_e \ , \end{gathered} \tag{3.2.16}$$

$$\begin{gathered} N_o x_{\langle 1|} = x_{\langle 1|} N_o \ ,\ N_o \partial_{|1\rangle} = \partial_{|1\rangle} N_o \ , \\ N_0 \xi_{\langle 1|} = q^2 \xi_{\langle 1|} N_o \ ,\ N_o \delta_{|1\rangle} = q^{-2}\delta_{|1\rangle} N_o \ . \end{gathered} \tag{3.2.17}$$

Exercise. Verify:

1. The Euler operators commute,

$$E_e E_o = E_o E_e \ . \tag{3.2.18}$$

2. Commutation relations between the Euler operators and the differentials are

$$\begin{aligned} dE_e &= (1+q^2E_e)d \quad \text{or} \quad N_e d = q^{-2} dN_e d \ , \\ E_o d &= d(1+q^2E_o) \quad \text{or} \quad N_o d = q^2 dN_o \ , \\ E_e\delta &= \delta(1+q^2E_e) \quad \text{or} \quad N_e\delta = q^2\delta N_e \ , \\ \delta E_o &= (1+q^2E_o)\delta \quad \text{or} \quad N_o\delta = q^{-2}\delta N_o \ . \end{aligned} \tag{3.2.19}$$

3. The differentials square to zero,

$$d^2 = 0 \ \ , \ \ \delta^2 = 0 \ \ . \tag{3.2.20}$$

4. For the anticommutator of d and δ we have

$$\Lambda := d\delta + \delta d = E_e + E_o + (q^2-1)E_oE_e = \frac{1}{q^2-1}(N_oN_e - 1) \ . \tag{3.2.21}$$

The last exercise implies that

$$\Lambda x^i = x^i(q^2\Lambda + 1) \ , \ \Lambda\xi^i = \xi^i(q^2\Lambda + 1) \ . \tag{3.2.22}$$

Let $M_{a,b}$ be a space of polynomials in x and ξ, of degree a in x and of degree b in ξ. For $\phi \in M_{a,b}$ one finds, by induction,

$$\Lambda\phi = \phi((a+b)_{q^2} + q^{2(a+b)}\Lambda) \ , \tag{3.2.23}$$

where the q-number $(n)_q$ is defined by $(n)_q = \dfrac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$.

Let $M_n = \oplus_{a+b=n} M_{a,b}$ and $M = \oplus_{n=0}^{\infty} M_n$. The space M is a $\mathbb{Z}_2$-graded vector space, the grading is given by the degree of a monomial in ξ's.

One can consider ∂_i and δ_i as operators acting on the space M. To this end, one introduces a vacuum Vac, which satisfies ∂_i Vac $= 0$ and δ_i Vac $= 0$. Let X be an expression in x^i, ξ^i, ∂_i and δ_i. To evaluate it on an element $\phi \in M$, take an element $X\phi$. Using the commutation relations, we move all ∂_i and δ_i to the right and evaluate on the vacuum. This gives an element of M which we denote $X(\phi)$. The consistency requires only that the vacuum is a representation of the algebra of ∂_i and δ_i which is clearly true.

For instance, we have $\Lambda(\phi) = (a+b)_{q^2}\phi$ for $\phi \in M_{a,b}$.

For each n, the space $M_n = \oplus_{a+b=n} M_{a,b}$ is finite dimensional. We have $\dim M_{a,b} = d_e(a)d_o(b)$, where $d_e(a)$ and $d_o(b)$ are dimensions of the spaces of polynomials in x of degree a and in ξ of degree b, respectively. The grading of $M_{a,b}$ is $(-1)^b$.

The space M_n is closed under the action of d and δ. Therefore, the supertrace of their anticommutator (of the operator Λ) vanishes, $\operatorname{Str}\Lambda = 0$, which implies

$$\sum_{a+b=n} d_e(a)d_o(b)(-1)^b(a+b)_{q^2} = 0 \tag{3.2.24}$$

for each n.

One can write the set (labeled by n) of identities (3.2.24) in a compact form. Let t be an indeterminate. Denote by P_e and P_o the Poincaré series for even and odd variables, respectively; that is, P_e and P_o are the generating functions for the dimensions d_e and d_o, $P_e(t)=\sum_a d_e(a)t^a$ and $P_o(t)=\sum_a d_o(a)t^a$. We have

$$P_e(t)P_o(-t)=\sum_n \sum_{a+b=n} d_e(a)d_o(b)(-1)^b \ . \tag{3.2.25}$$

Introduce a q-derivative in t. It satisfies, by definition, a relation $\partial_t t = 1+q^2 t\partial_t$. By induction,

$$\partial_t t^n = (n)_{q^2} t^{n-1} + q^{2n} t^n \partial_t \ . \tag{3.2.26}$$

As above, ∂_t becomes an operator after we define a vacuum - a one dimensional representation of the algebra of polynomials in ∂_t, Vac_t, by $\partial_t\ \mathrm{Vac}_t=0$. In particular, $\partial_t(t^n)=(n)_{q^2}t^{n-1}$.

The formula (3.2.26) shows that the action of ∂_t on the formal power series in t is well defined.

Now, (3.2.24) implies

$$\partial_t(P_e(t)P_o(-t))=0 \ . \tag{3.2.27}$$

Note that the series P_e and P_o start with 1, $P_e(t)=1+\mathcal{O}(t)$, $P_o(t)=1+\mathcal{O}(t)$. Therefore,

$$P_e(t)P_o(-t)=1+\mathcal{O}(t) \tag{3.2.28}$$

as well. Classically ($q=1$), equations (3.2.27) and (3.2.28) imply that

$$P_e(t)P_o(-t)=1 \ . \tag{3.2.29}$$

For a generic q the same conclusion (3.2.29) holds. Here "generic" means that q is not a root of unity. However, if q^2 is a primitive root of unity of order l one can conclude only that $P_e(t)P_o(-t)=1+t^lF(t^l)$ for some power series F. By a different method, without using the differential operators, the formula (3.2.29) for generic q was obtained in [**16**].

The advantage of having a formula like (3.2.29) is that in the GL case the relations for the odd generators ξ are strong enough to force the space of polynomials in ξ to be finite-dimensional. Then $P_o(t)$ is a polynomial and instead of checking the infinite number of coefficients in $P_e(t)$ one has only finite number of checks for $P_o(t)$.

3.3. Geometry of 3-dimensional quantum spaces. In dimension 2, a quantum vector space is a quadratic algebra with two generators and one relation. This situation can be quickly analyzed [**17**] and we shall not stop at it here.

For a 3-dimensional quantum space we need 3 generators and 3 relations. Let

$$E^\alpha_{ij}x^ix^j=0 \ ,\alpha=1,2,3 \ , \tag{3.3.1}$$

be the three relations. The number of independent monomials of degree k in dimension 3 is $\begin{pmatrix} k+2 \\ k \end{pmatrix}$, so we need to have $\begin{pmatrix} 3+2 \\ 2 \end{pmatrix}=10$ cubics.

In the free associative algebra with three generators there are 27 cubics. Thus we need 17 relations for cubics. How many relations can we deduce from (3.3.1)? We have 9 relations of the form $(E^\alpha_{ij}x^ix^j)x^k=0$ and 9 relations of the form $x^k(E^\alpha_{ij}x^ix^j)=0$, altogether 18 relations. Therefore they cannot be independent, there should 18-17=1 linear combination of them which vanishes. Therefore,

$$e_{i\alpha}E^\alpha_{jk}=E^\beta_{ij}f_{\beta k} \tag{3.3.2}$$

for some tensors $e_{i\alpha}$ and $f_{\beta k}$. We shall assume that the tensors $e_{i\alpha}$ and $f_{\beta k}$ are nondegenerate.

Let $E_{ijk} = e_{i\alpha} E^{\alpha}_{jk}$ (all the indices of E_{ijk} are now of the same nature; before, α, β labeled relations, while i, j, k labeled variables). The equation (3.3.2) now becomes

$$E_{ijk} = Q^l_k E_{lij} \ , \tag{3.3.3}$$

where $Q^i_j = f_{\alpha j}(e^{-1})^{\alpha i}$ (e^{-1} is inverse to e,

$$e_{\alpha i}(e^{-1})^{\alpha j} = \delta^j_i \quad \text{and} \quad e_{\alpha i}(e^{-1})^{\beta i} = \delta^{\beta}_{\alpha} \ , \tag{3.3.4}$$

δ^j_i and δ^{β}_{α} are Kronecker delta's; the fact that two relations (3.3.4) hold is because in dimension 3, both indices i and α run from 1 to 3).

A direct inspection shows that classically (for commuting variables) E_{ijk} is the ϵ-tensor. The ϵ-tensor has a good behavior under all permutations of indices. The moral is that for the PBW-algebras, it is enough that the E-tensor behaves well only under cyclic permutations of indices - the effect of a cyclic permutation is a rotation in one index by an operator Q.

This simple behavior under cyclic permutations makes possible a classification of PBW-algebras in dimension 3: go to a basis in which Q has a normal form then solve the cyclicity equation (3.3.3) for the E-tensor and select nondegenerate solutions (which give exactly three relations for quadrics). The article [**18**] contains the result of the classification. The list of PBW-algebras is quite large; for us it is the beginning of the work: one has to classify compatible pairs of quantum spaces.

We have now two tensors, E_{ijk} and F^{ijk}. The analysis is quite lengthy - because one has to work with the Poincaré series of nine variables T^i_j. But the final result [**19**] is surprisingly simple.

It turns our that $E_{jmn}F^{mni} = x\delta^i_j$ where x is a number (in fact, this relation describes a little more narrow $SL(3)$-case, when the quantum group has a central determinant and one can define a corresponding special linear quantum group; for the general situation, see [**19**]).

Define $A^{ij}_{kl} = xE_{klm}F^{mij}$. Then the resulting equations say that A is a projector, $A^2 = A$ and

$$(1+\kappa)\ \mathrm{tr}_3(A_{13}A_{23}Q^{-1}_3) = x^{-1}P_{12}Q^{-1}_1 + 1 \ , \tag{3.3.5}$$

where P_{12} is a permutation of spaces 1 and 2 and $\kappa = x\ \ \mathrm{tr}\, Q$.

Surprise: the equations for A imply that $\hat{R} = 1 + (1-q)A$ satisfies the Yang-Baxter equation, where q is a solution of $q^2 + (1-\kappa)q + 1 = 0$ (the other root defines $\hat{R}^{-1}$). Classically, $\kappa = 3$, $q = 1$ and $\hat{R} = P$.

This $\hat{R}$-matrix is of $GL(3)$-type and the relations for T^i_j ensuring that the compatible left and right spaces are covariant are nothing else but the RTT-relations.

In the beginning there was no demand to have a solution of the Yang-Baxter equations. The demands were to have PBW-algebras and the compatibility between them. So, unexpectedly, the study of the correct Poincaré series is a machine to produce $\hat{R}$-matrices and quantum groups.

In the list of $\hat{R}$-matrices found in this way in [**19**] there is an example which stands out for several reasons. The left quantum space is defined by relations

$$\begin{aligned} &\zeta zx + \zeta^5 y^2 + xz = 0 \ , \\ &\zeta^2 z^2 + yx + \zeta^4 xy = 0 \ , \\ &zy + \zeta^7 yz + \zeta^8 x^2 = 0 \ . \end{aligned} \tag{3.3.6}$$

Here ζ is a primitive 9-th root of unity; the operator Q has the form

$$Q = \begin{pmatrix} \zeta & & \\ & \zeta^4 & \\ & & \zeta^7 \end{pmatrix} \ . \tag{3.3.7}$$

The left quantum space (3.3.6) is compatible with an (isomorphic as an algebra) right quantum space; one can take $x = 1$. Thus we have a quantum group and an $\hat{R}$-matrix.

The $\hat{R}$-matrix is given by

$$\hat{R} = \zeta^6 + D\hat{\Sigma} \ , \tag{3.3.8}$$

where

$$\hat{\Sigma} = \begin{pmatrix} 1 & & & & & \zeta^4 & & \zeta^2 & \\ & \zeta^8 & & 1 & & & & & \zeta \\ & & \zeta^4 & & \zeta^8 & & 1 & & \\ & 1 & & \zeta & & & & & \zeta^2 \\ & & \zeta^5 & & 1 & & \zeta & & \\ \zeta^7 & & & & & \zeta^2 & & 1 & \\ & & 1 & & \zeta^4 & & \zeta^5 & & \\ \zeta^5 & & & & & 1 & & \zeta^7 & \\ & \zeta^7 & & \zeta^8 & & & & & 1 \end{pmatrix} , \tag{3.3.9}$$

and

$$D = \mathrm{diag}(\sigma_2, \sigma_2, \sigma_1, \sigma_2, \sigma_3, \sigma_3, \sigma_1, \sigma_3, \sigma_1) \tag{3.3.10}$$

with

$$\begin{aligned} 3\sigma_1 &= (\zeta^4 + \zeta^{-4}) - (\zeta^2 + \zeta^{-2}) \ , \\ 3\sigma_2 &= (\zeta^1 + \zeta^{-1}) - (\zeta^4 + \zeta^{-4}) \ , \\ 3\sigma_3 &= (\zeta^2 + \zeta^{-2}) - (\zeta^1 + \zeta^{-1}) \ . \end{aligned} \tag{3.3.11}$$

We have $q = \zeta^3$.

For the standard Drinfeld-Jimbo deformation, the left quantum space is given by $x^i x^j = q x^j x^i$ for $i < j$. When q is a primitive root of unity of order l, then the left quantum space has a center generated by elements $(x^i)^l$. If one requires that the covariance algebra of T^i_j preserves relations $(x^i)^l = c_i$, one obtains additional relations for T^i_j. The quotient algebra of the algebra of T^i_j by these relations is called a "small quantum group" [**20**].

The center of the algebra (3.3.6) is a polynomial ring generated by three elements of degrees 3,6 and 9. The algebra (3.3.6) is finite-dimensional over its center, the dimension equals 162 ([**18**]). Therefore, the quantum group defined by the $\hat{R}$-matrix (3.3.8) has finite-dimensional quotients as well.

The algebra (3.3.6) does not admit ordering. In other words, in any basis, the defining relations are not ordering relations, ordering of $x^3x^2x^1$ will always produce loops. The algebra (3.3.6) is the first example of a PBW-algebra with this property. Therefore, the $\hat{R}$-matrix (3.3.8) is a very particular point in the moduli space of solutions of the Yang-Baxter equation in dimension 3. Another peculiarity is that the $\hat{R}$-matrix (3.3.8) is an isolated point in the space of solutions of the Yang-Baxter equation; it cannot be obtained as a deformation of any other solution; in particular, one cannot reach it starting from the classical solution (the permutation). In this sense, this $\hat{R}$-matrix is non-perturbative.

Call $\mathcal{E}$ the algebra (3.3.6). In the next subsection we prove some of its properties mentioned above, in particular, the PBW property.

3.3.1. *Gröbner base for $\mathcal{E}$.* For a homogeneous element f of a free associative algebra $\mathcal{A}$ with generators $\{x^1,\dots,x^N\}$, let $\hat{f}$ be a "highest symbol" of f, the lexicographically highest word in f.

Let $\mathcal{B}$ be a quotient algebra of the algebra $\mathcal{A}$ by some homogeneous relations $\mathcal{S}_0=\{r_1,\dots,r_M\}$. Every relation r we write in the form $\hat{r}=$ terms, smaller than $\hat{r}$; we understand it as an instruction to replace $\hat{r}$ by the right hand side. Taking, if necessary, linear combinations of relations, we always assume that all $\hat{r}$ are different.

Let $\hat{\mathcal{S}}_0=\{\hat{r}_1,\dots,\hat{r}_M\}$.

A word can contain several entries of the form $\hat{r}_\alpha$ for some α. Comparing different ways of applying instructions to this word, we may obtain new instructions - relations, whose highest symbols do not belong to $\hat{\mathcal{S}}_0$. We add these relations to $\mathcal{S}_0$ and obtain a new set $\mathcal{S}_1$. Let again $\hat{\mathcal{S}}_1$ be the set of highest symbols.

Continuing the process, we shall build an (eventually infinite) set $\mathcal{S}=\cup_{i=0}^{\infty}\mathcal{S}_i$, which is called a Gröbner base for the algebra $\mathcal{A}$ (it depends on a choice of generators $\{x^1,\dots,x^N\}$ and on a choice of an order). Let $\hat{\mathcal{S}}$ be the corresponding set of highest symbols.

Now the basis of $\mathcal{A}$, as a vector space, consists of "normal" words - words, which do not have subwords belonging to $\hat{\mathcal{S}}$. This gives sometimes a way to estimate the Poincaré series of the algebra. See, *e.g.*, **[21]** for further information about Gröbner bases.

For the algebra $\mathcal{E}$, written in generators $\{x,y,z\}$, as in (3.3.6), the Gröbner base seems to be infinite ant non analyzable.

There are several other nice sets of generators and one of them leads to a finite Gröbner base. Let

$$\begin{aligned} x&=\gamma\zeta^2(A+\zeta^3B+\zeta^6C)\ ,\\ y&=-\tfrac{1}{1+\zeta}(A+B+C)\ ,\\ z&=\gamma^2(A+\zeta^6B+\zeta^3C)\ , \end{aligned}\tag{3.3.12}$$

where γ satisfies $\gamma^3=\zeta\dfrac{1+\zeta^2}{1+\zeta}$.

In terms of new generators A, B and C the relations are

$$\begin{aligned} \zeta A^2+\zeta^5AB+B^2&=0\ ,\\ \zeta C^2+\zeta^5CA+A^2&=0\ ,\\ \zeta B^2+\zeta^5BC+C^2&=0\ . \end{aligned}\tag{3.3.13}$$

Choose the order $A > B > C$. Then the set (3.3.13) of relations gives the following set of instructions

$$
\begin{aligned}
&A^2 \rightsquigarrow -\zeta C^2 - \zeta^5 CA \ ,\\
&B^2 \rightsquigarrow -\zeta^4 BC - \zeta^8 C^2 \ ,\\
&AB \rightsquigarrow -C^2 + \zeta CA + \zeta^8 BC \ .
\end{aligned}
\tag{3.3.14}
$$

Possible overlaps are A^2B, AB^2, A^3 and B^3. This leads to new instructions

$$
\begin{aligned}
&ACA \rightsquigarrow BC^2 + \zeta^2 CAC + \zeta^6 CBC + \zeta^8 C^2 A + \zeta^3 C^2 B - \zeta C^3 \ ,\\
&BCB \rightsquigarrow -\zeta^4 C^2 B - \zeta^8 C^3 \ ,\\
&AC^2 \rightsquigarrow -\zeta^4 BC^2 - \zeta^6 CAC - \zeta CBC - \zeta^3 C^2 A - \zeta^7 C^2 B \ .
\end{aligned}
\tag{3.3.15}
$$

One has new overlaps and they, in turn, lead to new instructions with highest symbols BC^3 and $ACBC$. We shall not give more details, but it turns out that now overlaps are all compatible, so the construction of the Gröbner base is completed and we have

$$
\hat{\mathcal{S}} = \{A^2, B^2, AB, ACA, BCB, AC^2, BC^3, ACBC\} \ . \tag{3.3.16}
$$

For such $\hat{\mathcal{S}}$ it is possible to explicitly describe the normal form.

For a word $w = x_1 x_2 \ldots x_k$, let $\beta_j(w)$ be the beginning, of the length j (that is, first j symbols), of a word $www\ldots w$ (the word w repeated sufficiently many times). For example, $\beta_4(ACB) = ACBA$, $\beta_5(ACB) = ACBAC$.

Lemma. For $\hat{\mathcal{S}}$, as in (3.3.16), the normal words have a form

$$
C^i \beta_j(BCC) \beta_k(ACB) \ . \tag{3.3.17}
$$

Corollary. The algebra $\mathcal{E}$ has the PBW property.

Proof. Normal words (3.3.17) are characterized by ordered triples of numbers $\{i, j, k\}$, as for monomials $x_1^i x_2^j x_3^k$. $\square$.

3.4. $sl_q(2)$ **at roots of unity.** The simplest example of a quantum space is the algebra V_q^2 with two generators x^1 and x^2 subjected to the relation (3.0.3). If one chooses for a left quantum space $(V^*)_q^2$ an algebra with two generators x_1 and x_2 and the same relation

$$
x_1 x_2 = q x_2 x_1 \ , \tag{3.4.1}
$$

then the quantum group (rather, bialgebra, for the moment we will not talk about invertibility of quantum matrices) which preserves the relations (3.0.3) and (3.4.1) is the standard $Mat_q(2)$, the matrix elements of $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy relations

$$
\begin{aligned}
&ab = qba \ , \ ac = qca \ , \ bd = qdb \ , \ cd = qdc \ ,\\
&bc = cb \ , \ ad = da + (q - q^{-1}) bc
\end{aligned}
\tag{3.4.2}
$$

(this is a correct quantization of the Poisson brackets (3.0.1)).

If q is a primitive l-th root of unity, the bialgebra $Mat_q(2)$ has a finite dimensional quotient $\overline{Mat}_q(2)$, one adds

$$a^l = d^l = 1 \ , b^l = c^l = 0 \tag{3.4.3}$$

to the relations (3.4.2).

The bialgebra $\overline{Mat}_q(2)$ has a symmetry interpretation as well. The elements x_1^l and x_2^l lie in the center of V_q^2. Let $V^2_{q\mu_1\mu_2}$ be a quotient of V_q^2 by relations $(x^1)^l = \mu_1$ and $(x^2)^l = \mu_2$ for some constants μ_1 and μ_2. If one requires that all the algebras $V^2_{q\mu_1\mu_2}$ are preserved by the coaction, one finds the extra relations (3.4.3); the same relations (3.4.3) one finds from a demand that all the left algebras $(V^*)^2_{q\mu_1\mu_2}$ are preserved.

In this subsection we shall illustrate, on this simple 2-dimensional example, some phenomena, pertinent to a situation when a non-commutative quantum space has a large center: loss of quasi-triangularity, loss of semi-simplicity, appearance of finite-dimensional Hopf quotients *etc.*

We shall give a description of the reduced universal enveloping algebra and of the reduced function algebra in terms of matrix algebras over local rings. This language seems to be quite appropriate to talk about such algebraic concepts as Ext-groups, a scheme of an algebra, its Cartan matrix *etc.* The material on the matrix structure is partly taken from [**22**].

Notation: n_q is a q-number, $n_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ and $n_q! = 1_q 2_q \dots n_q$ is a q-factorial. Let q be a l-th primitive root of unity, $l > 2$ (so $q^2 \neq 1$). Denote

$$\tilde{l} = \begin{cases} l & , \quad l \equiv 1 \ (mod\ 2) \\ l/2 & , \quad l \equiv 0 \ (mod\ 2) \end{cases} \tag{3.4.4}$$

Thus, $q^{2n} = 1 \iff 2n \equiv 0 \ (mod\ l) \iff n \equiv 0 \ (mod\ \tilde{l})$; $n_q = 0 \iff n \equiv 0 \ (mod\ \tilde{l})$. Denote $\nu = l/\tilde{l}$ and $\tilde{q} = q^\nu$; $\tilde{q}^{\tilde{l}} = 1$, $\tilde{q}$ is a primitive $\tilde{l}$-th root of unity.

3.4.1. *Preliminaries.* The Hopf algebra which gives rise (as in the section 3) to the quantum space V_q^2 is an algebra $\mathcal{U} = \mathcal{U}_q(sl_2)$, generated by elements K, K^{-1}, E and F and relations

$$\begin{gathered} KK^{-1} = K^{-1}K = 1 \ , \ KE = q^2 EK \ , \\ KF = q^{-2} FK \ , \ [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \ ; \end{gathered} \tag{3.4.5}$$

the coproduct is defined on the generators by

$$\Delta K = K \otimes K \ , \ \Delta E = E \otimes K + 1 \otimes E \ , \ \Delta F = F \otimes 1 + K^{-1} \otimes F \ ; \tag{3.4.6}$$

the counit ε and the antipode S are defined on the generators by

$$\varepsilon(K) = 1 \ , \ \varepsilon(E) = 0 \ , \ \varepsilon(F) = 0 \ , \tag{3.4.7}$$

$$S(K) = K^{-1} \ , \ S(E) = -EK^{-1} \ , \ S(F) = -KF \ . \tag{3.4.8}$$

The algebra $\mathcal{U}_q(sl_2)$ has a central element, a q-deformed Casimir operator:

$$C = qK + q^{-1}K^{-1} + (q - q^{-1})^2 FE \ . \tag{3.4.9}$$

If $q = \exp(\alpha)$ and $K = \exp(\alpha H)$, the combination $\frac{C-2}{(q-q^{-1})^2} - \frac{1}{4}$ tends to the standard Casimir operator $\frac{H^2}{4} + \frac{H}{2} + FE$ in the classical limit $\alpha \to 0$.

Consider a vector space $V(\sigma,j)$, $j\in \mathbf{Z}/2$, $j=0,\frac{1}{2},\ldots,\frac{\tilde{l}-1}{2}$ and $\sigma=\pm 1$ with a basis $\{e_j^m\ ,\ m=j,j-1,\ldots,-j\}$. Denote by $K(\sigma,j)$, $E(\sigma,j)$ and $F(\sigma,j)$ the operators

$$\begin{aligned} K(\sigma,j)e_j^m &= \sigma q^{2m} e_j^m\ ,\\ E(\sigma,j)e_j^m &= e_j^{m+1}\ ,\\ F(\sigma,j)e_j^m &= \sigma\,(j+m)_q(j-m+1)_q e_j^{m-1}\ . \end{aligned} \tag{3.4.10}$$

In these formulas, the right hand side should be replaced by 0 if $m\pm 1$ runs out of the allowed range.

The operators (3.4.10) realize standard representations of U_q. When q is not a root of unity, the representations $V(\sigma,j)$ exhaust the list of all irreducible representations.

The expression

$$\mathcal{R}=e^{\alpha\frac{H\otimes H}{2}}\sum_{m\geq 0}\frac{(q-q^{-1})^m}{m_q!}q^{\frac{m(m-1)}{2}}(E\otimes F)^m\ , \tag{3.4.11}$$

being understood informally, intertwines the coproduct with the opposite coproduct. However, because of the denominators, the expression (3.4.11) does not make sense when q is a root of unity. One may ask whether it is possible to redefine $\mathcal{R}$ at these values of q. The answer is negative. A standard argument goes as follows. If $\mathcal{R}$ existed, we would have an isomorphism $V\otimes W\simeq W\otimes V$ for any two representations of $\mathcal{U}$, for which $\mathcal{R}$ is defined ($\mathcal{R}$ would intertwine the tensor products).

Elements $x=E^{\tilde{l}}$, $y=F^{\tilde{l}}$ and $z=K^{\tilde{l}}$ are central; we have

$$\Delta z=z\otimes z\ ,\Delta x=x\otimes z+1\otimes x\ ,\ \Delta y=y\otimes 1+z^{-1}\otimes y\ . \tag{3.4.12}$$

There is a family of representations $W_{\mu ab}$ of dimension $\tilde{l}$ (the index j runs from 0 to $\tilde{l}-1$):

$$\begin{aligned} &K: v_j\mapsto \mu q^{-2j}v_j\ ,\\ &F: v_j\mapsto v_{j+1}\quad \text{for } j<\tilde{l}-1\ ,\\ &E: v_j\mapsto \left(\tfrac{\mu q^{1-j}-\mu^{-1}q^{j-1}}{q-q^{-1}}j_q+ab\right)v_{j-1}\ ,\quad \text{for } j>0\ ,\\ &F: v_{\tilde{l}-1}\mapsto bv_0\ ,\ E: v_0\mapsto av_{\tilde{l}-1}\ . \end{aligned} \tag{3.4.13}$$

The values of the parameters μ, a and b are not restricted (one only needs $\mu\neq 0$).

In the representation $W_{\mu ab}$, the value of the element y is b, the value of the element z is $\mu^{\tilde{l}}$.

Assume that $V\otimes W\simeq W\otimes V$. Then, applying the formula (3.4.12) for the coproduct of the element y, we find

$$y_V+z_V^{-1}y_W=y_W+z_W^{-1}y_V\ , \tag{3.4.14}$$

where y_V and z_V are the operators, corresponding to the elements y and z in the representation V (the same for W).

Take $V=W_{\mu ab}$ and $W=W_{\nu cd}$. Then (3.4.14) implies a relation between μ, ν, b and d, a contradiction.

1. Hopf ideals

Here we collect some information about Hopf ideals of a finite codimension in $\mathcal{U}$.

The Hopf subalgebra of Laurent polynomials in K coincides with the group algebra $\mathbf{C}[\mathbf{Z}]$ of the additive group $\mathbf{Z}$ of integers.

Lemma. Let I be a proper Hopf ideal in $\mathbf{C}[\mathbf{Z}]$. Then I is generated by (K^j-1) for some j.

Proof. Any ideal I in $\mathbf{C}[\mathbf{Z}]$ is a principal ideal, $I=(f)$, where $f(t)=t^j+a_{j-1}t^{j-1}+\cdots+a_0$ is a polynomial with $a_0\neq 0$.

The element

$$\Delta f(K)=K^j\otimes K^j+a_{j-1}K^{j-1}\otimes K^{j-1}+\cdots+a_0 \tag{3.4.15}$$

equals

$$(a_{j-1}K^{j-1}+\cdots+a_0)\otimes(a_{j-1}K^{j-1}+\cdots+a_0)+a_{j-1}K^{j-1}\otimes K^{j-1}+\cdots+a_0$$

in the algebra $\mathbf{C}[\mathbf{Z}]/I\otimes\mathbf{C}[\mathbf{Z}]/I$. If I is a Hopf ideal then the element (3.4.15) must be zero. In particular, in the expression above, the coefficient in $1\otimes K^i$ for $1\le i\le j-1$ must vanish, which gives $a_0a_i=0$. Therefore $a_i=0$. Vanishing of the coefficient in $1\otimes 1$ gives $a_0(a_0+1)=0$, therefore $a_0=-1$. Thus, $f(K)=K^j-1$, and it is straightforward to check that (f) is a Hopf ideal for such f. $\square$

Consider a Hopf ideal I of a finite codimension. If $E\in I$ then $K^2-1=(q-q^{-1})K[E,F]\in I$, therefore

$$(K^2-1)F=F(q^{-2}K^2-1)\equiv F(q^{-2}-1)(mod\ I)\in I\ ,$$

so $F\in I$. Thus the factor-algebra is $\mathbf{C}[\mathbf{Z}_2]$.

Assume now that $E\notin I$. Let $\bar{\mathcal{U}}$ be the factor-algebra of $\mathcal{U}$ by I.

According to the Lemma, $(K^j-1)\in I$ for some j. Therefore,

$$(K^j-1)E=E(q^{2j}K^j-1)\equiv E(q^{2j}-1)(mod\ I)\in I\ ,$$

which implies $j=m\tilde{l}$, so $\bar{z}^m=1$ in the factor-algebra $\bar{\mathcal{U}}$ ($\bar{z}$ is the image in $\bar{\mathcal{U}}$ of the central element $z=K^{\tilde{l}}\in\mathcal{U}$).

Lemma. The central elements $x=E^{\tilde{l}}$ and $y=F^{\tilde{l}}$ belong to I.

Proof. Let $\bar{x}$ be the image of x in $\bar{\mathcal{U}}$. Let $f(t)=t^i+b_{i-1}t^{i-1}+\cdots+b_0$ be a characteristic polynomial of $\bar{x}$ in $\bar{\mathcal{U}}$. We have

$$\Delta f(x)=(x_1z_2+x_2)^i+b_{i-1}(x_1z_2+x_2)^{i-1}+\cdots+b_0\ ,$$

where $x_1=x\otimes 1$, $x_2=1\otimes x$ and $z_2=1\otimes z$. Thus, in $\bar{\mathcal{U}}\otimes\bar{\mathcal{U}}$ one has

$$\begin{aligned}0=\Delta f(\bar{x})&=-(b_{i-1}(\bar{x}_1\bar{z}_2)^{i-1}+\cdots+b_0)-(b_{i-1}\bar{x}_2^{i-1}+\cdots+b_0)\\&+\textstyle\sum_{s=1}^{i-1}\begin{pmatrix} i\\ s\end{pmatrix}(\bar{x}_1\bar{z}_2)^s\bar{x}_2^{i-s}+b_{i-1}(\bar{x}_1\bar{z}_2+\bar{x}_2)^{i-1}+\cdots+b_0\ .\end{aligned} \tag{3.4.16}$$

where $\bar{x}_1=\bar{x}\otimes 1$, $\bar{x}_2=1\otimes\bar{x}$ and $\bar{z}_2=1\otimes\bar{z}$. The coefficient in, for example, $(\bar{x}_1\bar{x}_2^{i-1})$ is $\bar{z}_2 i$. Thus, $i=0$, therefore, $\bar{x}=0$ or $x\in I$. Similarly, $y\in I$. $\square$

Denote by I_m the ideal

$$I_m = \{E^{\tilde{l}}, F^{\tilde{l}}, K^{m\tilde{l}} - 1\} \ . \tag{3.4.17}$$

We shall call it a congruence ideal, and the number m - level.

We have shown that each Hopf ideal of a finite codimension contains a congruence ideal I_m for some m. The minimal m for which it happens, we shall call the level of the ideal.

Denote $\mathcal{U}_q/I_m$ by $\bar{\mathcal{U}}_{q,m}$ and the images of the elements E, F and K by $\bar{E}$, $\bar{F}$ and $\bar{K}$.

We shall give a complete description of $\bar{\mathcal{U}}_{q,1}$ (and of $\bar{\mathcal{U}}_{q,2}$) as an algebra[2].

2. Equation for C

We shall find a polynomial $\chi(x)$ such that $\chi(\bar{C}) = 0$ ($\bar{C}$ is the image of the Casimir operator C (3.4.9)). Later we shall prove that χ is a minimal polynomial for $\bar{C}$.

One has

$$(q - q^{-1})^2 FE = C - (qK + q^{-1}K^{-1}) \ . \tag{3.4.19}$$

Lemma. The following relation holds in U_q:

$$(q - q^{-1})^{2i} F^i E^i = \prod_{a=0}^{i-1} (C - (q^{1+2a}K + q^{-1-2a}K^{-1})) \ . \tag{3.4.20}$$

Proof. For $i = 1$ this is (3.4.19). Induction in i:

(3.4.21)

$$(q - q^{-1})^{2i+2} F^{i+1} E^{i+1} = (q - q^{-1})^2 F \prod_{a=0}^{i-1} (C - (q^{1+2a}K + q^{-1-2a}K^{-1}))E$$
$$= (q - q^{-1})^2 FE \prod_{a=0}^{i-1} (C - (q^{1+2a+2}K + q^{-1-2a-2}K^{-1})) \ .$$

Use (3.4.19) to finish the proof. □

Corollary. In $\bar{\mathcal{U}}_{q,m}$ one has

$$\prod_{a=0}^{\tilde{l}-1} (\bar{C} - (q^{1+2a}\bar{K} + q^{-1-2a}\bar{K}^{-1})) = 0 \ . \tag{3.4.22}$$

Proof. For $i = \tilde{l}$, the lhs of (3.4.20) belongs to the ideal I_m. □

Denote by $p(x)$ a polynomial $p(x) = 1 + x + \dots x^{\tilde{l}-1}$ and let

$$p_a = \frac{1}{\tilde{l}} p(\tilde{q}^a \bar{K}) \ , \ a = 0, \dots, \tilde{l} - 1 \ . \tag{3.4.23}$$

---

[2] We shall not use it, but it is known (see, *e.g.* [**23**]) that $\bar{\mathcal{U}}_{q,1}$ and $\bar{\mathcal{U}}_{q,2}$ are quasitriangular, say, for $\bar{\mathcal{U}}_{q,1}$ the universal $\mathcal{R}$-matrix is

$$\mathcal{R} = \frac{1}{\tilde{l}} \sum_{i,j=0}^{\tilde{l}} q^{-2ij} K^i \otimes K^j \sum_{s=0}^{\tilde{l}} \frac{(q - q^{-1})^s}{s_q!} q^{\frac{s(s-1)}{2}} \bar{E}^s \otimes \bar{F}^s \ . \tag{3.4.18}$$

Then

$$\bar{K} p_a = \tilde{q}^{-a} p_a \ . \tag{3.4.24}$$

The elements p_a are the usual idempotents, decomposing 1:

$$p_a p_b = \delta_{ab} p_b \ , \ 1 = p_0 + \cdots + p_{\tilde{l}-1} \ . \tag{3.4.25}$$

Define a polynomial $\chi(x)$,

$$\chi(x) = \prod_{a=0}^{\tilde{l}-1} (x - q^{1+2a} - q^{-1-2a}) \ . \tag{3.4.26}$$

More precisely,

$$\chi(x) = \begin{cases} \prod_{a=0}^{l-1} (x - q^a - q^{-a}) \ , \ \nu = 1 \\ \prod_{a=0}^{\tilde{l}-1} (x - q\tilde{q}^a - q^{-1}\tilde{q}^{-a}) \ , \ \nu = 2 \end{cases} \tag{3.4.27}$$

Lemma. In $\bar{\mathcal{U}}_{q,1}$, one has

$$\chi(\bar{C}) = 0 \ . \tag{3.4.28}$$

Proof. We have (using (3.4.24), the Corollary above and the fact that $\bar{C}$ and $\bar{K}$ commute)

$$\begin{aligned} 0 &= \textstyle\prod_{a=0}^{\tilde{l}-1} (\bar{C} - (q^{1+2a}\bar{K} + q^{-1-2a}\bar{K}^{-1})) p_{-b} \\ &= \textstyle\prod_{a=0}^{\tilde{l}-1} (\bar{C} - (q^{1+2a}\tilde{q}^b + q^{-1-2a}\tilde{q}^{-b})) p_{-b} = \chi(\bar{C}) p_{-b} \end{aligned} \tag{3.4.29}$$

for all b. Summing over b and using (3.4.25) we conclude that $\chi(\bar{C}) = 0$. □

Remarks. 1. For odd l the eigenvalue in (3.4.27) corresponding to $a = 0$ is simple, the others have the multiplicity 2 - pairs $(a, l - a)$. For l even, $\tilde{l}$ odd: the eigenvalue corresponding to $a = (\tilde{l} - 1)/2$ is simple, the others have the multiplicity 2 - pairs $(a, \tilde{l} - 1 - a)$. For l even, $\tilde{l}$ even: all eigenvalues have the multiplicity 2 - pairs $(a, \tilde{l} - 1 - a)$.

2. If we knew that χ is a minimal polynomial, we could immediately state that there are indecomposable but not irreducible representations: the center of a semisimple algebra is semisimple.

3.4.2. *Formatted matrix algebras over graded rings.* Let Γ be a finite abelian group, $\hat{\Gamma}$ its dual.

Let A be a Γ-graded ring over $\mathbf{C}$, that is, $A = \oplus_{\chi \in \hat{\Gamma}} A_\chi$ and if $a \in A_\chi$, $b \in A_{\chi'}$ then $ab \in A_{\chi\chi'}$.

Let $\mathcal{I}$ be a set. A couple ϕ, consisting of the set $\mathcal{I}$ and a map $\mathcal{I} \to \hat{\Gamma}$ we shall call a "format".

Definition. A set of matrices $X = \{X^i_j\}$ with indices belonging to the set $\mathcal{I}$ and with entries in A will be called a matrix algebra of format ϕ over A (and denoted by $M_\phi(A)$) if $X^i_j \in A_{\phi(i)\phi(j)^{-1}}$.

Clearly, $M_\phi(A)$ is an algebra.

In our examples the ring A will satisfy two conditions:

C1 A is local, that is, $A/\operatorname{rad} A$ is isomorphic to $\mathbf{C}$.

C2 The part of A which corresponds to the trivial representation of Γ is $\mathbf{C}$ itself; in other words, $\operatorname{rad} A = \oplus_{\text{nontrivial } \chi} A_\chi$.

The simplest example is the algebra $M_n(\mathbf{C})$ of matrices of size $n \times n$. Here the group Γ is trivial. The algebra $M_n(\mathbf{C})$ has only one representation - a column of the matrix.

Similarly, for any algebra $M_\phi(A)$, the columns provide representation spaces. Now there might be several types of columns, corresponding to the chosen format ϕ.

An advantage of the introduced terminology is summarized in the following lemma, which generalizes the properties of $M_n(\mathbf{C})$.

Lemma. Assume that A satisfies conditions C1 and C2. Then the columns realize principal projective modules of $M_\phi(A)$. The set of principal projective modules is in 1-1 correspondence with types of columns (that is, with the image of ϕ). The classes of isomorphism of the quotients of the principal projective modules of each type are in 1-1 correspondence with the graded quotients of A.

Therefore, a knowledge that some algebra is isomorphic to $M_\phi(A)$ gives a complete information about the representation theory for this algebra; there is no need to study first irreducible representations, then their extensions *etc.*

Let Λ_2 be the Grassmann algebra in two variables ξ and η. It is graded by the parity; the group Γ is $\mathbf{Z}_2$. Essentially, the format is specified by two numbers, m and n; we shall write $\phi = m|n$. The algebra $M_{m|n}(\Lambda_2)$ is the algebra of matrices

$$\left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) , \tag{3.4.30}$$

the entries of the matrices X and W are even, the entries of the matrices Y and Z are odd elements of Λ_2.

Remark. Let B be an algebra. Suppose that we know that it belongs to the class of formatted algebras over graded rings, *i. e.* it can be represented as $M_\phi(A)$ for some choice of Γ, A, $\mathcal{I}$ and ϕ. One may ask, how intrinsic the ring A is, whether it is defined by the algebra B. It turns out that for different rings A and A', the formatted matrix algebras over them can be isomorphic. In this case we shall say that A and A' are GM-equivalent (GM stands for "Graded Matrices").

Example. Let A' be a ring over $\mathbf{C}$ generated by two elements θ_1 and θ_2, satisfying $\theta_1^2 = \theta_2^2 = 0$ and $\theta_1\theta_2 = \theta_2\theta_1$. The ring A' is graded by $\mathbf{Z}_2$, the grading is given by a degree in the variables θ_i, $\deg \theta_i = 1$, $i = 1, 2$. The format is again specified by two numbers, $\phi = m|n$. The algebras $M_{1|1}(\Lambda_2)$ and $M_{1|1}(A')$ are isomorphic, the isomorphism is given by

$$\left(\begin{array}{c|c} a_1 + a_2\xi\eta & b_1\xi + b_2\eta \\ \hline c_1\xi + c_2\eta & d_1 + d_2\xi\eta \end{array}\right) \mapsto \left(\begin{array}{c|c} a_1 + a_2\theta_1\theta_2 & b_1\theta_1 - b_2\theta_2 \\ \hline c_1\theta_1 + c_2\theta_2 & d_1 - d_2\theta_1\theta_2 \end{array}\right) , \tag{3.4.31}$$

where $a_i, b_i, c_i, d_i \in \mathbf{C}$, $i = 1, 2$.

Thus, the rings Λ_2 and A' are GM-equivalent.

3.4.3. *Matrix structure.* In [**24**], after a description of irreducible and some indecomposable representations of $\bar{\mathcal{U}}_{q,2}$, a regular representation of $\bar{\mathcal{U}}_{q,2}$ was decomposed into a direct sum of indecomposable representations. As a consequence, the algebra $\bar{\mathcal{U}}_{q,2}$ decomposes into a direct sum of ideals. It was noticed in [**24**] that each of these ideals is isomorphic to a subalgebra in the matrix algebra whose matrix elements belong to a Grassmann algebra in two variables.

We shall adopt an opposite point of view and start by establishing homomorphisms into the matrix algebras with Grassmanian entries. Then we shall prove that the reduced enveloping algebras are direct sums of formatted matrix algebras over the local ring Λ_2. As explained above, this immediately provides an entire information about all modules, in particular, the principal projective modules.

Some homomorphisms into matrix algebras, odd l

Here we shall consider the case when the number l is odd.

Let $i = 2j+1$. Let $K(i)$, $E(i)$ and $F(i)$ be operators corresponding to $\sigma = 1$ in formulas (3.4.10) (pay attention to the order of the basis vectors: $m = j, j-1, \dots, -j$; say, the matrix of the operator $E(i)$ is upper-triangular).

We shall also use a matrix $M(i)$, defined by

$$M(i)e_j^m = e_j^{m-1} \tag{3.4.32}$$

on the same basis as in (3.4.10).

Let ξ and η be two Grassmann variables, $\xi^2 = \eta^2 = \xi\eta + \eta\xi = 0$.

Let

$$\mathcal{K}(i) = \left(\begin{array}{c|c} K(i) & \bullet \\ \hline \bullet & K(\frac{l}{2}-i-1) \end{array}\right) , \tag{3.4.33}$$

$$\mathcal{E}(i) = \left(\begin{array}{c|c} E(i) & \begin{matrix} \vdots & \bullet \\ \xi & \cdots \end{matrix} \\ \hline \begin{matrix} \vdots & \bullet \\ \xi & \cdots \end{matrix} & E(\frac{l}{2}-i-1) \end{array}\right) , \tag{3.4.34}$$

$$\mathcal{F}(i) = \left(\begin{array}{c|c} F(i) & \begin{matrix} \cdots & -\eta \\ \bullet & \vdots \end{matrix} \\ \hline \begin{matrix} \cdots & \eta \\ \bullet & \vdots \end{matrix} & F(\frac{l}{2}-i-1) \end{array}\right) + \xi\eta \left(\begin{array}{c|c} M(i) & \bullet \\ \hline \bullet & -M(\frac{l}{2}-i-1) \end{array}\right) . \tag{3.4.35}$$

Dots mean that the corresponding entries are zero.

The diagonal entries of the operator $\mathcal{K}(i)$ form a sequence $\{a_n\}$, $n \in \mathbf{Z}/l\mathbf{Z}$,

$$\{a_n\} = \{q^{2i}, q^{2(i-1)}, \dots, q^{-2i}, q^{2(\frac{l}{2}-i-1)}, \dots, q^{-2(\frac{l}{2}-i-1)}\} .$$

Since $q^l = 1$, we have $a_{n+1} = q^{-2}a_n$ for all $n \in \mathbf{Z}/l\mathbf{Z}$. The non-zero entries of the operators $\mathcal{E}(i)$ and $\mathcal{F}(i)$ are exactly on those places which are allowed by relations

$KE = q^2 EK$ and $KF = q^{-2} FK$. Next, one finds

$$\mathcal{E}(i)\mathcal{F}(i) = \left(\begin{array}{c|c} E(i)F(i) & \\ \hline & E(\frac{l}{2}-i-1)F(\frac{l}{2}-i-1) \end{array}\right) + \xi\eta\mathcal{P} \tag{3.4.36}$$

and

$$\mathcal{F}(i)\mathcal{E}(i) = \left(\begin{array}{c|c} F(i)E(i) & \\ \hline & (F\frac{l}{2}-i-1)E(\frac{l}{2}-i-1) \end{array}\right) + \xi\eta\mathcal{P} \ , \tag{3.4.37}$$

where

$$\mathcal{P} = \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right) \ . \tag{3.4.38}$$

We have $[E(k), F(k)] = \dfrac{K(k) - K(k)^{-1}}{q - q^{-1}}$ for all k, so

$$[\mathcal{E}(i), \mathcal{F}(i)] = \frac{\mathcal{K}(i) - \mathcal{K}(i)^{-1}}{q - q^{-1}} \ .$$

Therefore, the operators $\mathcal{E}(i)$, $\mathcal{F}(i)$ and $\mathcal{K}(i)$ provide a representation of the algebra $\mathcal{U}$.

It is easy to verify that $\mathcal{E}(i)^l = \mathcal{F}(i)^l = 0$ due to nilpotency of the Grassmann variables. The relation $\mathcal{K}(i)^l = 1$ is evident. Thus, the matrices $\mathcal{E}(i)$, $\mathcal{F}(i)$ and $\mathcal{K}(i)$ realize a representation of $\bar{\mathcal{U}}_{q,1}$.

Matrix structure of $\bar{\mathcal{U}}_{q,1}$, odd l

Formulas (3.4.33)-(3.4.35) provide homomorphisms ($j = 2i + 1$)

$$\rho_j : \bar{\mathcal{U}}_{q,1} \to M_{j|l-j}(\Lambda_2) \tag{3.4.39}$$

for $j = 1, \ldots, l-1$ and a homomorphism

$$\rho_0 : \bar{\mathcal{U}}_{q,1} \to M_l(\mathbf{C}) \ , \tag{3.4.40}$$

corresponding to $j = \frac{l-1}{2}$.

All the eigenvalues of the operator $\mathcal{K}(i)$ are different, so the diagonal matrices $\mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ are polynomials in $\mathcal{K}(i)$ (projectors on the eigenspaces of $\mathcal{K}(i)$) and belong to the image of ρ_i. Now, looking at the matrices for the operators $\mathcal{E}(i)$ and $\mathcal{F}(i)$, one concludes immediately that ρ_j is an epimorphism for all $j = 1, \ldots, l$.

For the Casimir element $\bar{C}$ one computes

$$\rho_j(\bar{C}) = (q^j + q^{-j})I + (q - q^{-1})^2 \xi\eta\mathcal{P} \ , \tag{3.4.41}$$

(I is the identity operator) for $j = 1, \ldots, l-1$ and $\rho_0(\bar{C}) = 2$.

Because of the $\xi\eta$-term in (3.4.41), $\chi(t) = \prod_{a=0}^{l-1}(x - q^a - q^{-a})$ is indeed the minimal polynomial for $\bar{C}$ in $\bar{\mathcal{U}}_{q,1}$.

Let $P_j \in \bar{\mathcal{U}}_{q,1}$, $j=0,\dots,[\frac{l}{2}]$ ($[z]$ is the integer part of z) be central idempotents corresponding to the eigenvalues q^j+q^{-j} of the semi-simple part of $\bar{C}$; $P_0+\dots+P_{[\frac{l}{2}]}=1$ is the central decomposition of unity. We have $\bar{\mathcal{U}}_{q,1}=\oplus_j P_j\bar{\mathcal{U}}_{q,1}$.

Let $Y_j=P_j\bar{\mathcal{U}}_{q,1}$, $j=0,\dots,[\frac{l}{2}]$ and let B_j be the matrix algebras, $B_0=M_l(\mathbf{C})$ and $B_a=M_{a|l-a}(\Lambda_2)$, $a=1,\dots,[\frac{l}{2}]$. Then $\rho_j:\bar{\mathcal{U}}_{q,1}\to B_j$ vanishes on Y_k for $k\neq j$ because of the value of $\bar{C}$. Thus, we have a collection of epimorphisms $Y_j\to B_j$, so their direct sum is an epimorphism

$$\rho:\bar{\mathcal{U}}_{q,1}\to\oplus_{j=0}^{[\frac{l}{2}]}B_j\ . \tag{3.4.42}$$

Let $B=\oplus_{j=0}^{[\frac{l}{2}]}B_j$. We have $\dim(B_0)=l^2$ and $\dim(B_a)=2l^2$, $a=1,\dots,[\frac{l}{2}]$. Therefore, $\dim(B)=l^2+\frac{l-1}{2}\cdot 2l^2=l^3$.

On the other hand, relations (3.4.5) clearly allow an ordering: we can rewrite any expression as, say, a linear combination of monomials $\bar{K}^a\bar{F}^b\bar{E}^c$. Therefore, $\dim(\bar{\mathcal{U}}_{q,1})\leq l^3$. But (3.4.42) is the epimorphism, so $\dim(\bar{\mathcal{U}}_{q,1})=l^3$ and (3.4.42) is an isomorphism. We proved:

Proposition. For odd l, the algebra $\bar{\mathcal{U}}_{q,1}$ is isomorphic to a direct sum of formatted matrix algebras,

$$\bar{\mathcal{U}}_{q,1}\simeq M_l(\mathbf{C})\oplus\bigoplus_{a=1}^{[\frac{l}{2}]}M_{a|l-a}(\Lambda_2)\ . \tag{3.4.43}$$

As a byproduct, we saw that the monomials $\bar{K}^a\bar{F}^b\bar{E}^c$ are linearly independent. This is a version of the Poincaré-Birkhoff-Witt theorem for $\bar{\mathcal{U}}_{q,1}$: the monomials $\bar{K}^a\bar{F}^b\bar{E}^c$, with $a,b,c=1,\dots,l$ form a basis.

Exercise. Describe the matrix structure of $\bar{\mathcal{U}}_{q,2}$ (that is, $K^{2l}=1$): replace the operators $K(i)$, $E(i)$ and $F(i)$ in formulas (3.4.33)-(3.4.35) by the operators corresponding to $\sigma=-1$ in (3.4.10), verify the defining relations for $\bar{\mathcal{U}}_{q,2}$ and show

$$\bar{\mathcal{U}}_{q,2}\simeq M_l(\mathbf{C})\oplus M_l(\mathbf{C})\oplus\bigoplus_{a=1}^{l-1}M_{a|l-a}(\Lambda_2)\ . \tag{3.4.44}$$

Remark. The algebra $\bar{\mathcal{U}}_{q,1}$ (or $\bar{\mathcal{U}}_{q,2}$) is unimodular, that is, the left and the right integrals coincide, $\int=\int_L=\int_R$ (they are defined by $x\int_R=\varepsilon(x)\int_R$ and $\int_L x=\varepsilon(x)\int_L$). The location of the integral inside the matrix blocks is very natural. In the direct sum, describing the matrix structure of the algebra, there is exactly one block $M_{1|\tilde{l}-1}(\Lambda_2)$, for which the 1×1 sub-block realizes the trivial representation (the same holds for even l). The integral is

$$\int=\left(\begin{array}{c|c}\xi\eta & \cdot\\ \hline \cdot & \cdot\end{array}\right) \tag{3.4.45}$$

(so the evaluation on the integral might remind to someone a true integration over Grassmann variables).

Example: $l=3$

For $q^3 = 1$ we have $(q-q^{-1})^2 = -3$ and $2_q = -1$. The Casimir element satisfies

$$\bar{C}^3 - 3\bar{C} - 2 = (\bar{C}+1)^2(\bar{C}-2) = 0 \ . \tag{3.4.46}$$

For the block M_3 (the value of the Casimir element is 2) we have:

$$\rho_0(\bar{K}) = \begin{pmatrix} q^2 & . & . \\ . & 1 & . \\ . & . & q^{-2} \end{pmatrix} , \tag{3.4.47}$$

$$\rho_0(\bar{E}) = \begin{pmatrix} . & 1 & . \\ . & . & 1 \\ . & . & . \end{pmatrix} , \tag{3.4.48}$$

$$\rho_0(\bar{F}) = \begin{pmatrix} . & . & . \\ -1 & . & . \\ . & -1 & . \end{pmatrix} . \tag{3.4.49}$$

Irreducible representations of dimensions 1 and 2 have the same value (-1) of the Casimir element, they can be glued indecomposably into a block $M_{1|2}(\Lambda_2)$:

$$\rho_1(\bar{K}) = \left(\begin{array}{c|cc} 1 & . & . \\ \hline . & q & . \\ . & . & q^{-1} \end{array}\right) , \tag{3.4.50}$$

$$\rho_1(\bar{E}) = \left(\begin{array}{c|cc} . & \xi & . \\ \hline . & . & 1 \\ \xi & . & . \end{array}\right) , \tag{3.4.51}$$

$$\rho_1(\bar{F}) = \left(\begin{array}{c|cc} . & . & -\eta \\ \hline \eta & . & . \\ . & 1-\xi\eta & . \end{array}\right) . \tag{3.4.52}$$

The algebra $\bar{\mathcal{U}}_{q,1}$ has two blocks, $\bar{\mathcal{U}}_{q,1} \simeq M(3) \oplus M_{1|2}(\Lambda_2)$.

Case when l is even, $\tilde{l}$ is odd

Now $l = 2\tilde{l}$ and $\tilde{l} = 2s+1$. We have $q^{2s+1} = -1$, so $q' = -q$ is a primitive $\tilde{l}$-th root of unity. A substitution

$$\bar{E}' = -\bar{E} \ , \ \bar{K}' = \bar{K} \ , \ \bar{F}' = \bar{F} \ , \ q' = -q \tag{3.4.53}$$

establishes an isomorphism of the algebra $\bar{\mathcal{U}}_{q,1}$ and the algebra $\bar{\mathcal{U}}_{q',1}$ whose matrix structure we know already.

Matrix structure for even $\tilde{l}$

We have $l = 4s$, $\tilde{l} = 2s$ and $q^{2s} = -1$.

We shall describe simultaneously the matrix structures of $\bar{\mathcal{U}}_{q,1}$ $(K^{2s} = 1)$ and $\bar{\mathcal{U}}_{q,2}$ $(K^{4s} = 1)$.

Let $K(\sigma, j)$, $E(\sigma, j)$ and $F(\sigma, j)$ be the operators as in (3.4.10), $j = 0, \frac{1}{2}, \ldots, s-1, s-\frac{1}{2}$. Note that (3.4.10) gives a representation of $\bar{\mathcal{U}}_{q,1}$ only when j (the "spin") is integer.

Let $j' = s-1-j$ for $j = 0, \frac{1}{2}, \ldots, s-1$. Let $C(\sigma, j)$ be the value of the Casimir element in the representation $V(\sigma, j)$. We have $C(\sigma, j) = \sigma(q^{1+2j} + q^{-1-2j})$. Thus, $C(-1, j') = C(1, j)$.

On the representation $V(\sigma, s-\frac{1}{2})$, the Casimir element takes a value (-2σ).

Now the assignment

$$\bar{K} \mapsto \left(\begin{array}{c|c} K(1,j) & \bullet \\ \hline \bullet & K(-1,j') \end{array}\right) \ , \tag{3.4.54}$$

$$\bar{E} \mapsto \left(\begin{array}{c|c} E(1,j) & \begin{matrix} \vdots & \bullet \\ \xi & \cdots \end{matrix} \\ \hline \begin{matrix} \vdots & \bullet \\ \xi & \cdots \end{matrix} & E(-1,j') \end{array}\right) \ , \tag{3.4.55}$$

$$\bar{F} \mapsto \left(\begin{array}{c|c} F(1,j) & \begin{matrix} \cdots & -\eta \\ \bullet & \vdots \end{matrix} \\ \hline \begin{matrix} \cdots & \eta \\ \bullet & \vdots \end{matrix} & F(-1,j') \end{array}\right) + \xi\eta \left(\begin{array}{c|c} M(2j+1) & \bullet \\ \hline \bullet & -M(2j'+1) \end{array}\right) \tag{3.4.56}$$

(dots mean that the corresponding entries are zero) establishes homomorphisms of $\bar{\mathcal{U}}_{q,2}$ into graded matrix algebras over Λ_2.

There are also two homomorphisms

$$\bar{\mathcal{U}}_{q,2} \to M_{2s}(\mathbf{C}) \ , \tag{3.4.57}$$

corresponding to the representations $V(\sigma, s-\frac{1}{2})$.

We have a collection $\mathcal{C}$ of homomorphisms (3.4.54)-(3.4.56) and (3.4.57). Parallelly to the case of odd l, one shows that these are epimorphisms and then, by counting dimensions, that the direct sum of the homomorphisms from $\mathcal{C}$ is an isomorphism. This proves:

Proposition. For even $\tilde{l} = 2s$, the algebra $\bar{\mathcal{U}}_{q,2}$ is isomorphic to a direct sum of formatted matrix algebras,

$$\bar{\mathcal{U}}_{q,2} \simeq M_{2s}(\mathbf{C}) \oplus M_{2s}(\mathbf{C}) \oplus \bigoplus_{a=1}^{2s-1} M_{a|2s-a}(\Lambda_2) \ . \tag{3.4.58}$$

The algebra $\bar{\mathcal{U}}_{q,1}$ is a direct sum of those terms in (3.4.58) for which a is odd,

$$\bar{\mathcal{U}}_{q,1} \simeq \bigoplus_{b=1}^{s} M_{2b-1|2s-2b+1}(\Lambda_2) \ . \tag{3.4.59}$$

As for odd l, the matrix description implies the Poincaré-Birkhoff-Witt theorem: The monomials $\bar{K}^a \bar{F}^b \bar{E}^c$, $a, b, c = 1, \ldots, l$ for $\bar{\mathcal{U}}_{q,1}$ ($a = 1, \ldots, 2l$ for $\bar{\mathcal{U}}_{q,2}$), are linearly independent and hence form a basis.

Remark. The appearance of the sign σ in the formulas (3.4.10) is related to the existence of the following involution ϕ in the case when $\tilde{l}=2s$ is even:

$$\phi : \bar{K} \mapsto -\bar{K} \ , \ \bar{E} \mapsto \bar{E} \ , \ \bar{F} \mapsto -\bar{F} \ . \tag{3.4.60}$$

The subalgebra $\bar{\mathcal{U}}_{q,2,\phi}$ of fixed points of the involution ϕ consists of polynomials in $\bar{K}^2$, $\bar{F}\bar{K}$ and $\bar{E}$.

To describe the matrix structure of the algebra $\bar{\mathcal{U}}_{q,2,\phi}$, let $Q_s(\Lambda_2)$ be an algebra of matrices

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \ , \tag{3.4.61}$$

A and B are $s\times s$ matrices, the entries of A are even, the entries of B are odd elements of the ring Λ_2.

Then

$$\bar{\mathcal{U}}_{q,2,\phi} \simeq M_{2s}(\mathbf{C}) \oplus \bigoplus_{a=1}^{s-1} M_{a|2s-a}(\Lambda_2) \ \oplus Q_s(\Lambda_2) \tag{3.4.62}$$

As for the algebra $\bar{\mathcal{U}}_{q,1,\phi}$, one keeps those terms in the direct sum (3.4.62) which correspond to an integer spin. Now the answer depends on the parity of s (the appearance of the algebra Q), that is, on the residue of l modulo 8.

Example: $l=4$

The algebra $\bar{\mathcal{U}}_{q,1}$:

$$\bar{K}\bar{E} = -\bar{E}\bar{K} \ , \ \bar{K}\bar{F} = -FK \ , \ [\bar{E},\bar{F}] = 0 \ , \tag{3.4.63}$$

and

$$\bar{K}^2 = 1 \ , \ \bar{E}^2 = \bar{F}^2 = 0 \ . \tag{3.4.64}$$

The Casimir operator is $\bar{C}=-4\bar{F}\bar{E}$; it satisfies $\bar{C}^2=0$.

The realization:

$$\bar{K} \mapsto \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right) \ , \ \bar{E} \mapsto \left(\begin{array}{c|c} & \xi \\ \hline \xi & \end{array}\right) \ , \ \bar{F} \mapsto \left(\begin{array}{c|c} & -\eta \\ \hline \eta & \end{array}\right) \ . \tag{3.4.65}$$

This realization is faithful, the algebra $\bar{\mathcal{U}}_{q,1}$ has only one block, $\bar{\mathcal{U}}_{q,1}\simeq M_{1|1}(\Lambda_2)$.

We have $\bar{E}\bar{F}'+\bar{F}'\bar{E}=0$, where $\bar{F}'=\bar{F}\bar{K}$, so the algebra $\bar{\mathcal{U}}_{q,1,\phi}$ is isomorphic to the ring Λ_2 itself.

3.4.4. *Reduced function algebra.* A reduced function algebra $\bar{\mathcal{F}}_q$ on $SL_q(2)$ at roots of unity is the algebra with generators a, b, c and d, subjected to relations (3.4.2), (3.4.3) and

$$ad - qbc = 1 \ . \tag{3.4.66}$$

This last relation, together with $d^l=1$, allows to express a in terms of d, b and c; the algebra $\bar{\mathcal{F}}_q$ is generated by d, b and c only.

The algebra $\bar{\mathcal{F}}_q$ also has a formatted matrix structure. Let ξ_1 and ξ_2 be two variables which satisfy

$$\xi_1^l = \xi_2^l = 0 \ , \ \xi_1\xi_2 = \xi_2\xi_1 \ . \tag{3.4.67}$$

The algebra $\mathbf{C}[\xi_1,\xi_2]$ is graded by the degree in the variables ξ_i, $\deg\xi_1 = \deg\xi_2 = 1$. The group Γ is the cyclic group $\mathbf{Z}/l\mathbf{Z}$. The format ϕ is specified by a set of

l numbers, $\phi = n_0|\dots|n_{l-1}$, the number n_j corresponds to the character $z \mapsto \exp(\frac{2\pi i}{l} j)$, where z is a given generator of Γ.

A map

$$b \mapsto \xi_1 \begin{pmatrix} . & 1 & & & \\ & . & 1 & & \\ & & . & . & \\ & & & . & 1 \\ 1 & & & & . \end{pmatrix}, \quad c \mapsto \xi_2 \begin{pmatrix} . & 1 & & & \\ & . & 1 & & \\ & & . & . & \\ & & & . & 1 \\ 1 & & & & . \end{pmatrix},$$

$$d \mapsto \begin{pmatrix} 1 & & & & \\ & q & & & \\ & & . & & \\ & & & . & \\ & & & & q^{l-1} \end{pmatrix}$$

establishes an isomorphism

$$\bar{\mathcal{F}}_q \xrightarrow{\sim} M_{1|1\dots|1}(\mathbf{C}[\xi_1, \xi_2]) \tag{3.4.68}$$

(all the numbers in the format equal 1). As for reduced enveloping algebras, this isomorphism implies the Poincaré-Birkhoff-Witt theorem.

3.4.5. *Centre.* We conclude the subsection by several remarks concerning the centers of the algebras $\bar{\mathcal{U}}_{q,m}$.

1. The center of the formatted matrix algebra $M_{m|n}(\Lambda_2)$ consists of matrices

$$\left(\begin{array}{c|c} \alpha + \beta\xi\eta & . \\ \hline . & \alpha + \gamma\xi\eta \end{array}\right), \tag{3.4.69}$$

with some constants α, β and γ. It is 3-dimensional.

There is a conjecture by Kaplansky: "A Hopf algebra of characteristic zero has no non-zero central idempotents" (the citation is according to [**25**]).

This conjecture is false, the algebras $\bar{\mathcal{U}}_{q,m}$ provide a counter-example.

2. We have seen (eq. (3.4.41)) that the image of the Casimir element is of the form

$$\left(\begin{array}{c|c} \alpha + \beta\xi\eta & . \\ \hline . & \alpha - \beta\xi\eta \end{array}\right). \tag{3.4.70}$$

Therefore, the Casimir element does not generate the whole center.

For the algebra $\mathcal{U}$, when q is a primitive l-th root of unity, a theorem (see [**26**]) states that the center of $\mathcal{U}$ is generated by the elements $E^{\tilde{l}}$, $F^{\tilde{l}}$, $K^{\tilde{l}}$ and C (and that there is a polynomial relation between these elements, which is eq. (3.4.20) at $i = \tilde{l}$; for $i = \tilde{l}$, the r.h.s. of (3.4.20) depends only on C and $K^{\tilde{l}}$). The image (in $\bar{\mathcal{U}}_{q,1}$) of the algebra, generated by $E^{\tilde{l}}$, $F^{\tilde{l}}$, $K^{\tilde{l}}$ and C, is the algebra of polynomials in $\bar{C}$. As we saw above, it is a strict subalgebra in the center.

3. Let $C(K)$ be a centralizer of K in $\mathcal{U}$. One has $C(K) = \bigoplus_{i=0}^{\tilde{l}} A_i$ where A_i is spanned by elements $F^i K^a E^i$. The subspace $A_{>0} = \bigoplus_{i=1}^{\tilde{l}} A_i$ is an ideal in $C(K)$ and A_0 is a complementary subalgebra, $A = A_0 \oplus A_{>0}$.

We have a well-defined projection $\pi : C(K) \to A_0$.

Let $\mathcal{Z}$ be the center of $\mathcal{U}$, it is a subalgebra in $C(K)$. The restriction $\phi = \pi_{\mathcal{Z}}$ of the projection ϕ to the center $\mathcal{Z}$ is called a Harish-Chandra homomorphism. It is known to be injective when q is not a root of unity.

For the reduced algebras $\bar{\mathcal{U}}_{q,1}$ (or $\bar{\mathcal{U}}_{q,2}$) the Harish-Chandra homomorphism is defined in the same way. However, the injectivity is lost, because the center $\bar{\mathcal{Z}}$ is not semi-simple while the algebra $\bar{A}_0$ is. One verifies that the kernel of the Harish-Chandra homomorphism coincides with the $\mathrm{Rad}\mathcal{Z}$. It is natural to conjecture that this holds for quantum deformations for all semi-simple Lie algebras.

4. $\hat{R}$-matrices

The first subsection is a summary of some essential facts from the theory of quasi-triangular Hopf algebras and their representations.

The $\hat{R}$ matrix for the standard quantum group $GL_q(N)$ is [**27, 28**],

$$\hat{R}^{ij}_{kl} = q^{\delta_{ij}}\,\delta^i_l\delta^j_k + (q-q^{-1})\Theta(l-k)\delta^i_k\delta^j_l \ , \tag{4.0.1}$$

where $\Theta(i) = 1$ for $i > 0$ and $\Theta(i) = 0$ otherwise. The indices run from 1 to N.

The $\hat{R}$-matrix (4.0.1) belongs to a class of "ice" $\hat{R}$-matrices; the precise definition of the ice condition is in the second subsection. There we give a classification of ice $\hat{R}$-matrices. The main result is that they are all of GL type.

The final subsection establishes a way to build, starting from an arbitrary $\hat{R}$-matrix of GL-type, $\hat{R}$-matrices for orthogonal and symplectic quantum groups.

4.1. Skew-invertibility. The first part of this chapter is a short reminder on the general theory of quasi-triangular Hopf algebras, originating mostly from [**29**].

Then we discuss an important notion of "skew-invertibility" and explain how it arises in the context of the quasi-triangular Hopf algebras.

In the second part we derive, on a representation level, matrix analogues of some identities in Hopf algebras. These matrix identities will be needed for the discussion of the $\hat{R}$-matrices for orthogonal and symplectic quantum groups.

4.1.1. *Generalities on Hopf algebras.* Let A be a Hopf algebra.

We recall that

$$m(S\otimes \mathrm{id})\Delta(a) \;=\; \epsilon(a)\,1 \ , \tag{4.1.1}$$

$$m(\mathrm{id}\otimes S)\Delta(a) \;=\; \epsilon(a)\,1 \ , \tag{4.1.2}$$

$$(\epsilon\otimes \mathrm{id})\Delta \;=\; (\mathrm{id}\otimes\epsilon)\Delta = \mathrm{id} \ , \tag{4.1.3}$$

where S is the antipode and ϵ is a counit.

We use a standard notation omitting a summation index, for example, instead of writing $\Delta(x) = \sum_i x^i_1 \otimes x^i_2$ we shall simply write $\Delta(x) = x_{(1)}\otimes x_{(2)}$.

1. The Hopf algebra A is called almost cocommutative if there exists an invertible element $\mathcal{R}\in A\otimes A$ such that

$$\Delta'(x)\mathcal{R} = \mathcal{R}\Delta(x) \tag{4.1.4}$$

for any element $x \in A$. Here Δ' is the flipped coproduct, $\Delta'(x) = x_{(2)}\otimes x_{(1)}$ for $\Delta(x) = x_{(1)}\otimes x_{(2)}$.

We symbolically write $\mathcal{R} = a\otimes b$ instead of $\mathcal{R} = \sum_i a_i\otimes b_i$.

Let $\Delta^2(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ $(\Delta^2 = (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta)$. By (4.1.4), we have

$$x_{(2)} a \otimes x_{(1)} b \otimes x_{(3)} = a x_{(1)} \otimes b x_{(2)} \otimes x_{(3)} \ , \tag{4.1.5}$$

$$x_{(1)} \otimes x_{(3)} a \otimes x_{(2)} b = x_{(1)} \otimes a x_{(2)} \otimes b x_{(3)} \ . \tag{4.1.6}$$

Let $u = S(b)a$. Applying $\mathrm{id} \otimes S \otimes S^2$ to (4.1.5) and multiplying terms in the inverse order, one obtains

$$S^2(x) u = u x \ . \tag{4.1.7}$$

Applying $\mathrm{id} \otimes S \otimes S^2$ to (4.1.6) and multiplying terms, one obtains

$$x S(u) = S(u) S^2(x) \ . \tag{4.1.8}$$

Eqs. (4.1.7) and (4.1.8) hold for an arbitrary $x \in A$ so the element $S(u)u$ is central.

Exercises.

1. Take a flip of (4.1.4): $\Delta(x)\mathcal{R}_{21} = \mathcal{R}_{21}\Delta'(x)$, and derive, parallelly to (4.1.7) and (4.1.8), identities

$$x v = v S^2(x) \ , \tag{4.1.9}$$

$$S^2(x) S(v) = S(v) x \ , \tag{4.1.10}$$

where $v = aS(b)$.

2. This exercise is taken from [**30**].

Let A be a Hopf algebra (not necessarily almost cocommutative). Let T be an operator on $A \otimes A$ defined by

$$T(a \otimes b) = a S(b_{(1)}) b_{(3)} \otimes b_{(2)} \ . \tag{4.1.11}$$

Show that T satisfies the Yang-Baxter equation, $T_{12}T_{13}T_{23} = T_{23}T_{13}T_{12}$. Show that for a cocommutative A, the operator T reduces to the identity operator.

2. The Hopf algebra A is called quasi-triangular if

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \ , \tag{4.1.12}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{4.1.13}$$

Exercise. Show that any of these formulas, together with (4.1.4), implies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{4.1.14}$$

Applying $\epsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to the formula (4.1.12) gives $\mathcal{R} = (\epsilon \otimes \mathrm{id})(\mathcal{R})\mathcal{R}$ or, upon canceling by $\mathcal{R}$,

$$(\epsilon \otimes \mathrm{id})\mathcal{R} = 1 \ . \tag{4.1.15}$$

Similarly, an application of $\mathrm{id} \otimes \mathrm{id} \otimes \epsilon$ to (4.1.13) gives

$$(\mathrm{id} \otimes \epsilon)(\mathcal{R}) = 1 \ . \tag{4.1.16}$$

Applying S to the first tensor argument of (4.1.12), multiplying the first two arguments and using (4.1.15), one obtains

(4.1.17) $$(S\otimes \text{ id})\mathcal{R} = \mathcal{R}^{-1} \ .$$

Similarly,

(4.1.18) $$(\text{ id}\otimes S)\mathcal{R}^{-1} = \mathcal{R} \ .$$

Together, eqs. (4.1.17) and (4.1.18) imply

(4.1.19) $$(S\otimes S)\mathcal{R} = \mathcal{R} \ .$$

3. Some properties of the element u

An immediate consequence of (4.1.19) is

(4.1.20) $$v = S(u) \ .$$

Let $\mathcal{R}^{-1} = c\otimes d$. We have $1 = (\mathrm{id}\otimes S)(\mathcal{R}\mathcal{R}^{-1}) = (\mathrm{id}\otimes S)(ac\otimes bd) = ac\otimes S(d)S(b)$. Using (4.1.18), one can rewrite it in the form

(4.1.21) $$a'a\otimes bS(b') = 1 \ ,$$

where the prime means another copy, the full version of $a'a\otimes bS(b')$ is $\sum_{i,j} a_i a_j \otimes b_j S(b_i)$. Multiplying the tensor terms of (4.1.21) in the inverse order, we get $bua = 1$, or, by (4.1.7), $bS^2(a)u = 1$. On the other hand, $ubS^2(a) = S^2(b)uS^2(a)$, which, by (4.1.19), equals $bua = 1$. Thus, the element u is invertible,

(4.1.22) $$u^{-1} = bS^2(a) \ .$$

Exercise. Prove that the element u is invertible in the general almost cocommutative setting (i.e., without assuming the quasi-triangularity).

Using the invertibility of u, one can rewrite (4.1.7) in the form

(4.1.23) $$S^2(x) = uxu^{-1} \ .$$

In particular, the antipode S is invertible (since S^2 is invertible). Note that in the quasitriangular situation, eqs. (4.1.8), (4.1.9) and (4.1.10) follow from (4.1.7) because of (4.1.20) and the invertibility of S.

For $x = u$, eq. (4.1.23) gives

(4.1.24) $$S^2(u) = u \ .$$

For $x = S(u)$, eq. (4.1.7) gives $S^3(u)u = uS(u)$, which, in view of (4.1.24), implies

(4.1.25) $$uS(u) = S(u)u \ .$$

4. Coproduct of u

From quasi-triangularity properties (4.1.12) and (4.1.13) it follows that

(4.1.26) $$(\Delta\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{14}\mathcal{R}_{24}\mathcal{R}_{13}\mathcal{R}_{23} \ ,$$

or

(4.1.27) $$a_{(1)}\otimes a_{(2)}\otimes b_{(1)}\otimes b_{(2)} = aa''\otimes a'a'''\otimes b''b'''\otimes bb' \ ,$$

where, as usual, primes denote different copies.

Rewriting the Yang-Baxter equation (4.1.14) in the form

$$\mathcal{R}_{23}^{-1}\mathcal{R}_{12}\mathcal{R}_{13} = \mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1}$$

and using (4.1.17), we obtain

$$a'a'' \otimes S(a)b' \otimes bb'' = a''a' \otimes b'S(a) \otimes b''b \ . \tag{4.1.28}$$

Now

$$\begin{aligned}
\Delta(u) &= S(b_{(2)})a_{(1)} \otimes S(b_{(1)})a_{(2)} \overset{(4.1.27)}{=} S(bb') \otimes aa'' \otimes S(b''b''')a'a''' \\
&= S(b')ua'' \otimes S(b''b''')a'a''' \overset{(4.1.7)}{=} S(b')S^2(a'')u \otimes S(b''b''')a'a''' \\
&= S(S(a'')b')u \otimes S(b''b''')a'a''' \overset{(4.1.28)}{=} S(b'S(a''))u \otimes S(b'''b'')a'''a' \\
&= S^2(a'')S(b')u \otimes S(b'')ua' \overset{(4.1.19)}{=} S(a'')S(b')u \otimes b''ua' \\
&\overset{(4.1.17)}{=} \mathcal{R}^{-1} \cdot S(b')u \otimes ua' \overset{(4.1.7)}{=} \mathcal{R}^{-1} \cdot S(b')u \otimes S^2(a')u \\
&\overset{(4.1.19)}{=} \mathcal{R}^{-1} \cdot b'u \otimes S(a')u \overset{(4.1.17)}{=} \mathcal{R}^{-1}\mathcal{R}_{21}^{-1} \cdot u \otimes u \ .
\end{aligned} \tag{4.1.29}$$

A number over "=" refers to an equation which is used in the corresponding equality.

Denote the element $\mathcal{R}_{21}\mathcal{R} \in A \otimes A$ by ϕ, $\phi = \mathcal{R}_{21}\mathcal{R}$. We obtained

$$\Delta(u) = \phi^{-1} \cdot u \otimes u \ . \tag{4.1.30}$$

Obviously, $\phi\Delta(x) = \Delta(x)\phi$ for any $x \in A$. The element ϕ plays in important role in the theory of quasi-triangular Hopf algebras; a map from A^* (a dual Hopf algebra) to A, $f \mapsto \langle \phi, f \rangle_2$ (the pairing with the second argument of ϕ) is called a factorization map. The algebra A is called factorizable if the factorization map is not degenerate (and A is called triangular if $\phi = 1$).

For $x = u$, eq. (4.1.4) gives (using (4.1.30))

$$\mathcal{R} \cdot u \otimes u = u \otimes u \cdot \mathcal{R} \tag{4.1.31}$$

(note that this equality follows from eqs. (4.1.23) and (4.1.19) as well).

Using now that $\Delta(S(x)) = (S \otimes S)\Delta'(x)$ for any x, one obtains

$$\Delta(S(u)) = \phi^{-1} \cdot S(u) \otimes S(u) \ . \tag{4.1.32}$$

Therefore, the element $g = uS(u)^{-1}$ is group-like, $\Delta(g) = g \otimes g$; the fourth power of the antipode is given by the conjugation by g, $S^4(x) = gxg^{-1}$.

For the central element $uS(u)$ we have $\Delta(uS(u)) = \phi^{-2} \cdot uS(u) \otimes uS(u)$. If there exists a central element $\rho \in A$ such that $\rho^2 = uS(u)$ and $\Delta(\rho) = \phi^{-1} \cdot \rho \otimes \rho$, one says that A is a ribbon Hopf algebra; the element ρ is then called the ribbon element.

Exercise. Show that $\epsilon(\rho) = 1$, $S(\rho) = \rho$ and $\mathcal{R} \cdot \rho \otimes \rho = \rho \otimes \rho \cdot \mathcal{R}$.

4.1.2. *Matrix picture.* Let t be a representation of A in a vector space V. The numerical R-matrix is

$$R = (t \otimes t)(\mathcal{R}) \tag{4.1.33}$$

or, in some basis of V, $R^{ij}_{kl} = t(a)^i_k t(b)^j_l$. As usual, P will denote the permutation matrix.

Eq. (4.1.21) produces the following matrix equation

$$R^{ij}_{kl}\Psi^{sl}_{it} = \delta^s_k\delta^j_t \ , \tag{4.1.34}$$

where $\Psi = (t\otimes t)(a\otimes S(b))$, $\Psi^{ij}_{kl} = t(a)^i_k t(S(b))^j_l$.

Thus for $\hat{R} = PR$, $\hat{R}^{de}_{af} = R^{ed}_{af}$, and $\hat{\Psi} = P\Psi$, $\hat{\Psi}^{ba}_{cd} = \Psi^{ab}_{cd}$, we have

$$\hat{R}^{ba}_{cd}\hat{\Psi}^{de}_{af} = \delta^e_c\delta^b_f \ . \tag{4.1.35}$$

One can rewrite it without indices as

$$\mathrm{tr}_2(\hat{R}_{12}\hat{\Psi}_{23}) = P_{13} \ . \tag{4.1.36}$$

We could have used instead of (4.1.21) an equivalent relation

$$1 = (\mathrm{id}\otimes S)(\mathcal{R}^{-1}\mathcal{R}) = (\mathrm{id}\otimes S)(ca\otimes db) = ca\otimes S(b)S(d) \overset{(4.1.18)}{=} a'a\otimes S(b)b'$$

to obtain in the matrix form

$$\mathrm{tr}_2(\hat{\Psi}_{12}\hat{R}_{23}) = P_{13} \ . \tag{4.1.37}$$

Definition. Given an operator $\hat{R}$, a solution of eq, (4.1.36) (respectively, eq. (4.1.37)) is called a right (respectively, left) skew inverse of $\hat{R}$. The operator $\hat{R}$ is called skew-invertible if it has left and right skew inverses.

We are concerned only with a finite-dimensional case, in which the relations (4.1.36) and (4.1.37) are equivalent: $(A\hat{\odot}B)^{eb}_{cf} = A^{es}_{ct}B^{tb}_{sf}$ is an associative product on the space of tensors with two upper and two lower indices, the permutation $P^{ij}_{kl} = \delta^i_l\delta^j_k$ is a unit element for the operation $\hat{\odot}$ and eq. (4.1.36) (correspondingly, (4.1.37)) defines $\hat{\Psi}$ as the right (correspondingly, left) inverse of $\hat{R}$ with respect to $\hat{\odot}$. In a finite-dimensional algebra left and right inverses (when one of them exists) coincide.

This product reflects a product[3] $(\alpha\otimes\beta)\odot(\gamma\otimes\delta) = \gamma\alpha\otimes\beta\delta$ defined for elements of the tensor square of an arbitrary algebra: for $x, y\in A\otimes A$ and $z = x\odot y$ let $X = (t\otimes t)(x)$, $Y = (t\otimes t)(y)$ and $Z = (t\otimes t)(z)$ be their images for the representation t. Then $Z^{ij}_{kl} = X^{aj}_{kb}Y^{ib}_{al}$ or $\hat{Z} = \hat{X}\hat{\odot}\hat{Y}$.

Let $Q = t(u)$ be the image of the element u, $Q^i_j = t(S(b))^i_k t(a)^k_j = \Psi^{ki}_{jk}$, or

$$Q_1 = \ \mathrm{tr}_2(\hat{\Psi}_{12}) \ . \tag{4.1.38}$$

Similarly, for $\tilde{Q} = t(S(u))$ we have

$$\tilde{Q}_2 = \ \mathrm{tr}_1(\hat{\Psi}_{12}) \ . \tag{4.1.39}$$

Thus,

$$\mathrm{tr}_2(\hat{R}_{12}Q_2) = I_1 \qquad \text{and} \qquad \mathrm{tr}_1(\tilde{Q}_1\hat{R}_{12}) = I_2 \ , \tag{4.1.40}$$

where I stands for the identity operator in a corresponding space.

If the representation t is irreducible, the central element $uS(u)$ takes a constant value, the square of the value of the ribbon element. Thus, for an irreducible representation, the product $Q\tilde{Q}$ is proportional to unity.

[3]More generally, for an element $\mu = \mu_1\otimes\dots\mu_n\in A^{\otimes n}$ and an element $\xi = \alpha\otimes\beta\in A\otimes A$ one defines $\mu\odot\xi_{kl} := \mu_1\otimes\dots\otimes\alpha\mu_k\otimes\dots\otimes\mu_l\beta\otimes\dots\otimes\mu_n$ and $\xi_{kl}\odot\mu := \mu_1\otimes\dots\otimes\mu_k\alpha\otimes\dots\otimes\beta\mu_l\otimes\dots\otimes\mu_n$; then there are rules like $(x_{12}y_{13})\odot z_{23} = (y_{13}z_{23})\odot x_{12}$, $x_{12}x_{13}x_{23} = x_{13}\odot(x_{12}x_{23})$ *etc.*

Exercise. Show that the standard $\hat{R}$-matrix (4.0.1) is skew-invertible with

$$\hat{\Psi}^{ab}_{cd} = q^{-\delta_{ab}}\delta^a_d\delta^b_c - (q-q^{-1})\Theta(d-c)q^{2c-2d}\delta^a_c\delta^b_d \ . \tag{4.1.41}$$

Show that

$$Q^a_b = q^{-2N+2a-1}\delta^a_b \quad \text{and} \quad \tilde{Q}^a_b = q^{1-2a}\delta^a_b \ , \tag{4.1.42}$$

so the value of the square of the ribbon element is q^{-2N}.

We now adopt another point of view and forget that there was a quasi-triangular Hopf algebra behind. We shall leave, as a trace of quasi-triangularity, only the assumption that the numerical matrix $\hat{R}$ is skew-invertible, and derive, purely in the matrix language, some consequences (for $\hat{\Psi}$) of the Yang-Baxter equation.

Below we constantly use the following simple fact:

$$\mathrm{tr}_1(P_{12}) = I_2 \ . \tag{4.1.43}$$

Multiplying the Yang-Baxter equation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$ from the left by $\hat{\Psi}_{a1}$, from the right by $\hat{\Psi}_{3b}$ (a and b should be understood as numbers of some copies of the space V), taking traces in the spaces 1 and 3 and using (4.1.36) and (4.1.37), we obtain (after relabeling spaces - we do it in order to avoid a redundancy of unnecessary symbols; the result is formulated for the spaces with numbers 1, 2 and 3)

$$\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}_{02}\hat{R}_{03})P_{23} = P_{12}\ \mathrm{tr}_0(\hat{R}_{10}\hat{R}_{20}\hat{\Psi}_{03}) \ . \tag{4.1.44}$$

Exercise. The Yang-Baxter equation implies that $\hat{R}_{12}\hat{R}_{23}\hat{R}^n_{12} = \hat{R}^n_{23}\hat{R}_{12}\hat{R}_{23}$ and $\hat{R}^n_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}^n_{23}$ for an arbitrary integer n. Show that

$$\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}_{02}\hat{R}^n_{03})P_{23} = P_{12}\ \mathrm{tr}_0(\hat{R}^n_{10}\hat{R}_{20}\hat{\Psi}_{03}) \tag{4.1.45}$$

and

$$\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}^n_{02}\hat{R}_{03})P_{23} = P_{12}\ \mathrm{tr}_0(\hat{R}_{10}\hat{R}^n_{20}\hat{\Psi}_{03}) \ . \tag{4.1.46}$$

Deduce from (4.1.45) and (4.1.46) that

$$\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}^{n+1}_{02}) = P_{12}\ \mathrm{tr}_0(\hat{R}^n_{10}\hat{R}_{20}Q_0) \ , \tag{4.1.47}$$

$$\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}^{n+1}_{02}) = P_{12}\ \mathrm{tr}_0(\hat{R}_{10}\hat{R}^n_{20}Q_0) \tag{4.1.48}$$

and then

$$\mathrm{tr}_0(\tilde{Q}_0\hat{R}^{n+1}_{01}) = \mathrm{tr}_0(\hat{R}^{n+1}_{10}Q_0) \ . \tag{4.1.49}$$

Since the permutation matrix P squares to the identity, we can rewrite (4.1.44) as

$$P_{12}\ \mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}_{02}\hat{R}_{03}) = \mathrm{tr}_0(\hat{R}_{10}\hat{R}_{20}\hat{\Psi}_{03})P_{23} \ . \tag{4.1.50}$$

Multiplying (4.1.50) from the left by $\hat{\Psi}_{a1}$, from the right by $\hat{\Psi}_{3b}$, taking traces in the spaces 1 and 3 and using (4.1.36) and (4.1.37), we obtain

$$\mathrm{tr}_{01}(\hat{\Psi}_{a1}P_{12}\hat{\Psi}_{10}\hat{R}_{02}P_{0b}) = \mathrm{tr}_{03}(P_{a0}\hat{R}_{20}\hat{\Psi}_{03}P_{23}\hat{\Psi}_{3b}) \ . \tag{4.1.51}$$

This equation can be simplified. The expression under the trace in the l.h.s. can be rewritten as $\hat{\Psi}_{a1}P_{12}\hat{\Psi}_{10}\hat{R}_{02}P_{0b} = \hat{\Psi}_{a1}P_{12}P_{0b}\hat{\Psi}_{1b}\hat{R}_{b2}$. Now the trace in the space

0 can be taken (by (4.1.43)), so the l.h.s. of (4.1.51) is $\mathrm{tr}_1(\hat{\Psi}_{a1}P_{12}\hat{\Psi}_{1b}\hat{R}_{b2})$. We have

$$
\begin{aligned}
& \mathrm{tr}_1(\hat{\Psi}_{a1}P_{12}\hat{\Psi}_{1b}\hat{R}_{b2}) = \mathrm{tr}_1(\hat{\Psi}_{a1}P_{12}\hat{\Psi}_{1b})\,\hat{R}_{b2} \\
&= \mathrm{tr}_1(P_{12}\hat{\Psi}_{a2}\hat{\Psi}_{1b})\,\hat{R}_{b2} = \mathrm{tr}_1(P_{12}\hat{\Psi}_{1b}\hat{\Psi}_{a2})\,\hat{R}_{b2} \\
&= \mathrm{tr}_1(P_{12}\hat{\Psi}_{1b})\,\hat{\Psi}_{a2}\hat{R}_{b2} = \mathrm{tr}_1(\hat{\Psi}_{2b}P_{12})\,\hat{\Psi}_{a2}\hat{R}_{b2} \\
&= \hat{\Psi}_{2b}\hat{\Psi}_{a2}\hat{R}_{b2}\ .
\end{aligned}
\tag{4.1.52}
$$

In a similar way one simplifies the r.h.s. of (4.1.51) and obtains (after relabeling spaces)

$$\hat{\Psi}_{23}\hat{\Psi}_{12}\hat{R}_{32} = \hat{R}_{21}\hat{\Psi}_{23}\hat{\Psi}_{12}\ . \tag{4.1.53}$$

Assume that an operator B has a left skew inverse A, $\mathrm{tr}_2(A_{12}B_{23}) = P_{13}$ (or $A\hat{\odot}B = P$). Then for any operator X_1, which acts as the identity in the space 2, we have

$$
\begin{aligned}
& \mathrm{tr}_2(A_{12}X_1B_{21}) = \mathrm{tr}_{02}(A_{12}X_1P_{10}B_{21}) \\
&= \mathrm{tr}_{02}(A_{12}X_1B_{20}P_{10}) = \mathrm{tr}_{02}(A_{12}B_{20}X_1P_{10}) \\
&= \mathrm{tr}_0(P_{10}X_1P_{10}) = \mathrm{tr}(X)I_1\ ,
\end{aligned}
\tag{4.1.54}
$$

where I_1 is the identity operator in the second space.

Therefore, taking tr_3 of (4.1.53), one obtains

$$\hat{R}_{21}Q_2\hat{\Psi}_{12} = Q_1I_2\ ; \tag{4.1.55}$$

similarly, taking tr_1 of (4.1.53), one obtains

$$\hat{\Psi}_{12}\tilde{Q}_1\hat{R}_{21} = I_1\tilde{Q}_2\ . \tag{4.1.56}$$

Here Q and $\tilde{Q}$ are the operators defined in (4.1.38) and (4.1.39).

On the other hand, one can rewrite eq. (4.1.44) as $P_{23}\,\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}_{03}\hat{R}_{02}) = \mathrm{tr}_0(\hat{R}_{20}\hat{R}_{10}\hat{\Psi}_{03})P_{12}$ or

$$\mathrm{tr}_0(\hat{\Psi}_{10}\hat{R}_{03}\hat{R}_{02})P_{12} = P_{23}\,\mathrm{tr}_0(\hat{R}_{20}\hat{R}_{10}\hat{\Psi}_{03})\ . \tag{4.1.57}$$

Exercises.

1. Multiply (4.1.57) from the left by $\hat{\Psi}_{a1}$, from the right by $\hat{\Psi}_{3b}$, take traces in the spaces 1 and 3 and obtain

$$\hat{\Psi}_{12}\hat{\Psi}_{23}\hat{R}_{21} = \hat{R}_{32}\hat{\Psi}_{12}\hat{\Psi}_{23}\ . \tag{4.1.58}$$

2. Assume that an operator B has a left skew inverse A, $\mathrm{tr}_2(A_{12}B_{23}) = P_{13}$. Show, similarly to (4.1.54) that

$$\mathrm{tr}_2(A_{21}X_1B_{12}) = \mathrm{tr}(X)I_1\ . \tag{4.1.59}$$

3. Apply tr_1 or tr_3 to eq. (4.1.58) and deduce that

$$\hat{\Psi}_{12}Q_2\hat{R}_{21} = Q_1I_2 \tag{4.1.60}$$

and

$$\hat{R}_{21}\tilde{Q}_1\hat{\Psi}_{12} = \tilde{Q}_2 I_1 \ . \tag{4.1.61}$$

4. Let $\psi = a \otimes S(b) \in A \otimes A$ ($\mathcal{R} = a \otimes b$ is the universal R-matrix). Show that

$$\mathcal{R}_{23}\psi_{12}\psi_{13} = \psi_{13}\psi_{12}\mathcal{R}_{23} \ , \tag{4.1.62}$$

$$\mathcal{R}_{12}\psi_{23}\psi_{13} = \psi_{13}\psi_{23}\mathcal{R}_{12} \ . \tag{4.1.63}$$

Show that these equalities induce, on the level of a representation, equalities (4.1.53) and (4.1.58) respectively.

5. What are Hopf-algebraic counterparts of eqs. (4.1.55), (4.1.56), (4.1.60) and (4.1.61)?

Write equations (4.1.55), (4.1.56), (4.1.60) and (4.1.61) in the form

$$Q_2\hat{\Psi}_{12} = \hat{R}_{21}^{-1}Q_1 \ , \tag{4.1.64}$$

$$\hat{\Psi}_{12}\tilde{Q}_1 = \tilde{Q}_2\hat{R}_{21}^{-1} \ , \tag{4.1.65}$$

$$\hat{\Psi}_{12}Q_2 = Q_1\hat{R}_{21}^{-1} \ , \tag{4.1.66}$$

$$\tilde{Q}_1\hat{\Psi}_{12} = \hat{R}_{21}^{-1}\tilde{Q}_2 \ . \tag{4.1.67}$$

A compatibility of these equations provides new relations[4].

Comparing tr_1 of eqs. (4.1.64) and (4.1.66):

$$Q_2\tilde{Q}_2 = \mathrm{tr}_1(\hat{R}_{21}^{-1}Q_1) \qquad \text{and} \qquad \tilde{Q}_2 Q_2 = \mathrm{tr}_1(Q_1\hat{R}_{21}^{-1}) \tag{4.1.68}$$

and using the cyclic property of trace to move Q_1, we conclude that

$$Q\tilde{Q} = \tilde{Q}Q \ . \tag{4.1.69}$$

This is a matrix counterpart of eq. (4.1.25).

Using (4.1.64)-(4.1.67), we can express in two different ways combinations $Q_2\hat{\Psi}_{12}\tilde{Q}_1$, $Q_2\hat{\Psi}_{12}Q_2$, $Q_2\tilde{Q}_1\hat{\Psi}_{12}$, $\hat{\Psi}_{12}\tilde{Q}_1Q_2$, $\tilde{Q}_1\hat{\Psi}_{12}\tilde{Q}_1$ and $\tilde{Q}_1\hat{\Psi}_{12}Q_2$. This results in

$$\hat{R}_{21}^{-1}Q_1\tilde{Q}_1 = Q_2\tilde{Q}_2\hat{R}_{21}^{-1} \ , \tag{4.1.70}$$

$$\hat{R}_{21}^{-1}Q_1Q_2 = Q_1Q_2\hat{R}_{21}^{-1} \ , \tag{4.1.71}$$

$$\tilde{Q}_1\hat{R}_{21}^{-1}Q_1 = Q_2\hat{R}_{21}^{-1}\tilde{Q}_2 \ , \tag{4.1.72}$$

$$\tilde{Q}_2\hat{R}_{21}^{-1}Q_2 = Q_1\hat{R}_{21}^{-1}\tilde{Q}_1 \ , \tag{4.1.73}$$

$$\hat{R}_{21}^{-1}\tilde{Q}_1\tilde{Q}_2 = \tilde{Q}_1\tilde{Q}_2\hat{R}_{21}^{-1} \ , \tag{4.1.74}$$

$$\hat{R}_{21}^{-1}\tilde{Q}_2Q_2 = \tilde{Q}_1Q_1\hat{R}_{21}^{-1} \ . \tag{4.1.75}$$

[4] Eqs. (4.1.53) and (4.1.58) also have a nontrivial compatibility relation: $\hat{R}_{21}$ commutes with $\hat{\Psi}_{23}\hat{\Psi}_{12}^2\hat{\Psi}_{23}$.

Eqs. (4.1.71) and (4.1.74) reflect the fact that for a quasi-triangular Hopf algebra A, elements $u \otimes u$ and $S(u) \otimes S(u)$ commute with $\mathcal{R}$ (see eq. (4.1.31)).

It is interesting to compare eqs. (4.1.70) and (4.1.75) in the Hecke case, when the $\hat{R}$-matrix satisfies a quadratic equation $\hat{R}^2 = \lambda \hat{R} + 1$ with $\lambda \neq 0$. Rewriting eq. (4.1.75) as $\hat{R}_{21} Q_1 \tilde{Q}_1 = Q_2 \tilde{Q}_2 \hat{R}_{21}$ (we used that Q commutes with $\tilde{Q}$) and subtracting from (4.1.70) we obtain

$$Q\tilde{Q} = const \ . \tag{4.1.76}$$

So, even if a representation t is not irreducible but the $\hat{R}$-matrix is of Hecke type, the value of the square of the ribbon element on all subrepresentations of t is the same.

If $\lambda = 0$ (*i.e.* $\hat{R}$ is triangular, $\hat{R}^2 = 1$), eq. (4.1.68) implies immediately that $Q\tilde{Q} = I$.

Exercise. Suppose that operators Q and $\tilde{Q}$ are invertible. Show, without taking skew inverses, that eqs. (4.1.72) and (4.1.73) follow from eqs. (4.1.70), (4.1.71), (4.1.74) and (4.1.75).

Use (4.1.48) (or multiply (4.1.37) from the left by Q_3 and use (4.1.64)) to obtain

$$\mathrm{tr}_2(\hat{R}_{12}\hat{R}_{32}^{-1} Q_2) = Q_3 P_{13} \ . \tag{4.1.77}$$

Therefore, if Q is invertible then $\hat{R}^{-1}$ has a skew inverse $\hat{\Xi}$, $\hat{\Xi}_{12} = Q_1 \hat{R}_{21} Q_2^{-1}$.

On the other hand, assume that $\hat{R}^{-1}$ has a skew inverse $\hat{\Xi}$. Multiply (4.1.77) by $\hat{\Xi}_{03}$ and take tr_3 to obtain $\mathrm{tr}_0(\hat{\Xi}_{01}) Q_1 = I_1$.

Therefore, Q is invertible iff $\hat{R}^{-1}$ has a skew inverse. Similarly, $\tilde{Q}$ is invertible iff $\hat{R}^{-1}$ has a skew inverse.

It follows then that Q is invertible iff $\tilde{Q}$ is invertible.

There is also an implication: Q is invertible $\Rightarrow$ $\hat{\Psi}$ is invertible (it follows immediately from, for example, (4.1.64).

Assuming that the operator $\hat{\Psi}$ is invertible, one can rewrite the Yang-Baxter equation entirely in terms of $\hat{\Psi}$. To this end, rewrite eq. (4.1.53) in the form

$$\hat{\Psi}_{12} \hat{R}_{32} \hat{\Psi}_{12}^{-1} = \hat{\Psi}_{23}^{-1} \hat{R}_{21} \hat{\Psi}_{23} \ . \tag{4.1.78}$$

Multiplying (4.1.78) from the left by $\hat{\Psi}_{a3}$, from the right by $\hat{\Psi}_{1b}$, taking traces in the spaces 1 and 3 and using (4.1.36) and (4.1.37), we obtain

$$P_{12}\ \mathrm{tr}_0(\hat{\Psi}_{01} \hat{\Psi}_{02}^{-1} \hat{\Psi}_{03}) = \ \mathrm{tr}_0(\hat{\Psi}_{10} \hat{\Psi}_{20}^{-1} \hat{\Psi}_{30}) P_{23} \ . \tag{4.1.79}$$

Note that on the way from the Yang-Baxter to eq. (4.1.79) we were making only reversible transformations, so eq. (4.1.79) is equivalent (assuming the skew-invertibility of $\hat{\Psi}$) to the original Yang-Baxter equation.

We conclude by a remark that from the Hopf-algebraic point of view the invertibility of $\hat{\Psi}$ is natural. The element $\psi = a \otimes S(b) \in A \otimes A$ has an inverse,

$$\psi^{-1} = a \otimes S^2(b) \ . \tag{4.1.80}$$

Also, the element $\mathcal{R}^{-1}$ has a (left and right) skew-inverse ξ (that is, the inverse with respect to the multiplication $\odot$),

$$\xi = S^2(a) \otimes b \ . \tag{4.1.81}$$

From the Hopf-algebraic perspective the matrix identities which we derived are quite transparent. However, for the construction of orthogonal and symplectic $\hat{R}$-matrices one needs the matrix form of the identities, so it is important to understand how much one can derive using only matrices.

Exercises.

1. Verify (4.1.80) and (4.1.81); show that

$$\mathcal{R}_{12}\xi_{13}\xi_{23} = \xi_{23}\xi_{13}\mathcal{R}_{12} \ , \tag{4.1.82}$$

$$\mathcal{R}_{23}\xi_{13}\xi_{12} = \xi_{12}\xi_{13}\mathcal{R}_{23} \ , \tag{4.1.83}$$

$$\mathcal{R}_{13}\xi_{12}\psi_{23} = \psi_{23}\xi_{12}\mathcal{R}_{13} \ , \tag{4.1.84}$$

$$\mathcal{R}_{13}\xi_{23}\psi_{12} = \psi_{12}\xi_{23}\mathcal{R}_{13} \ . \tag{4.1.85}$$

2. What are Hopf-algebraic counterparts of eqs. (4.1.44) and (4.1.79)?

4.2. Ice $\hat{R}$-matrices. The standard $\hat{R}$-matrix (4.0.1) has two properties: it is of Hecke type (that is, it has two eigenvalues) and it satisfies the so-called "ice" condition which means that $\hat{R}^{ij}_{kl}$ can be different from zero only if the pair of the upper indices $\{i,j\}$ is a permutation of the pair of the lower ones, $\{i,j\}=\{k,l\}$ or $\{i,j\}=\{l,k\}$. Here we shall explain that these two properties (Hecke and ice) are not independent; we shall introduce the notion of indecomposable ice $\hat{R}$-matrix and demonstrate that such $\hat{R}$-matrices satisfy the Hecke condition[5]. Ideologically, this shows that the search of ice solutions of equations similar to the Yang-Baxter equation is justified only in the Hecke case (and then one imposes the Hecke condition first, as it is done in [**31**] for the dynamical Yang-Baxter equation).

Let $\hat{R}^{ij}_{kl}=a_{ij}\delta^i_l\delta^j_k+b_{ij}\delta^i_k\delta^j_l$ be an ice matrix. We fix $b_{ii}=0$ for uniqueness. Let also $a_i=a_{ii}$.

We suppose that the matrix $\hat{R}$ is invertible and skew-invertible. It follows then (an easy exercise) that $a_i\neq 0$ and $a_{ij}\neq 0$ for all i and j.

Assume that $\hat{R}$ satisfies the Yang-Baxter equation, $Y^{ikj}_{abc}=0$, where $Y^{ikj}_{abc}=(\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}-\hat{R}_{23}\hat{R}_{12}\hat{R}_{23})^{ijk}_{abc}$.

When two indices among $\{i,j,k\}$ are different, the equation $Y^{ikj}_{abc}=0$ gives (here $i\neq j$):

$$a_{ij}b_{ij}b_{ji}=0 \ , \tag{4.2.1}$$

$$b_{ij}(a_i^2-a_ib_{ij}-a_{ij}a_{ji})=0 \ , \tag{4.2.2}$$

$$b_{ij}(a_j^2-a_jb_{ij}-a_{ij}a_{ji})=0 \ . \tag{4.2.3}$$

[5]The opposite is not true: there are many Hecke $\hat{R}$-matrices which cannot be brought to an ice form by a change of a basis.

For all three indices $\{i, j, k\}$ different, $i \neq j \neq k \neq i$, equations are

$$(a_{ij}a_{ji} - a_{jk}a_{kj})b_{ik} + b_{ij}b_{jk}(b_{ij} - b_{jk}) = 0 \ , \tag{4.2.4}$$

$$a_{jk}(b_{ij}b_{ik} - b_{ij}b_{jk} - b_{ik}b_{kj}) = 0 \ , \tag{4.2.5}$$

$$a_{ij}(b_{jk}b_{ik} - b_{ij}b_{jk} - b_{ji}b_{ik}) = 0 \ . \tag{4.2.6}$$

Let Γ be a graph with vertices i. We draw an oriented edge $\vec{ij}$ from the vertex i to the vertex j if the number b_{ij} is not zero.

Since $a_{ij} \neq 0$, eq. (4.2.1) shows that two vertices can be joined by not more that one edge.

When the graph Γ is not connected, equations, corresponding to different connected components, do not notice each other. So, one has to study only the situation when the graph Γ is connected.

Definition. We say that the ice $\hat{R}$-matrix is indecomposable if its graph Γ is connected.

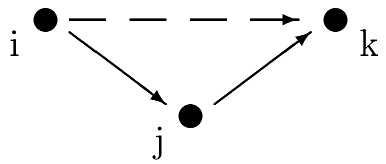
Proposition. Let $\hat{R}$ be an invertible and skew-invertible solution of the Yang-Baxter equation. Assume that $\hat{R}$ satisfies the ice condition and is indecomposable. Then $\hat{R}$ is of Hecke type (that is, it satisfies a quadratic equation).

Proof. Since $a_{ij} \neq 0$ for all i and j, eqs. (4.2.5) and (4.2.6) imply

$$b_{ij}b_{ik} - b_{ij}b_{jk} - b_{ik}b_{kj} = 0 \ , \tag{4.2.7}$$

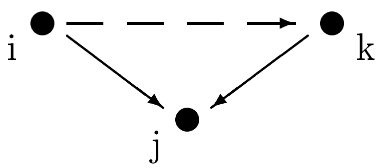
$$b_{jk}b_{ik} - b_{ij}b_{jk} - b_{ji}b_{ik} = 0 \ . \tag{4.2.8}$$

(i) Suppose that the graph Γ has edges $\vec{ij}$ and $\vec{jk}$. Then Γ has an edge $\vec{ik}$, as on the Figure:

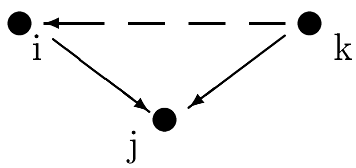


This is an immediate consequence of eq. (4.2.7): $b_{kj} = 0$ because, by assumption, $b_{jk} \neq 0$; therefore, $b_{ij}(b_{ik} - b_{jk}) = 0$ but, by assumption, $b_{ij} \neq 0$.

(ii) Suppose that the graph Γ has edges $\vec{ij}$ and $\vec{kj}$. Then Γ has either an edge $\vec{ik}$ or an edge $\vec{ki}$, as on the Figures:



or



To prove this, interchange j and k, $j \leftrightarrow k$, in eq. (4.2.8):

$$b_{kj}b_{ij} - b_{ik}b_{kj} - b_{ki}b_{ij} = 0 \tag{4.2.9}$$

and note that either b_{ik} or b_{ki} is 0.
(iii) Situations when Γ has edges $\vec{ji}$ and $\vec{kj}$ or edges $\vec{ji}$ and $\vec{jk}$ are considered similarly.

We conclude:

(a) If Γ contains two sides of a triangle, it contains the third side. This immediately implies (since Γ is connected) that Γ is a full graph, that is, every two vertices are joined. In other words, for each pair (i,j) at least one number, b_{ij} or b_{ji}, is not zero, $\{b_{ij}, b_{ji}\} \neq \{0,0\}$.

(b) An oriented triangle of Γ is never a cycle (see Figures above). By (a), Γ is the full graph; an easy exercise shows then that Γ has no cycles. Therefore, orientations of edges induce an order on the set of vertices and we can relabel vertices in such a way that Γ has an edge $\vec{ij} \in \Gamma$ if and only if $i<j$. In other words, $b_{ij} \neq 0$ if and only if $i<j$.

Consider a triangle with vertices i, j and k, $i<j<k$. Then the oriented edges are $\vec{ij}$, $\vec{ik}$ and $\vec{jk}$. We have $b_{ji} = b_{kj} = 0$; eq. (4.2.7) shows that $b_{ik} = b_{jk}$; eq. (4.2.8) shows that $b_{ik} = b_{ij}$. Therefore, for all i and j with $i<j$ the parameters b_{ij} take the same value, say b, $b_{ij} = b$.

At this stage, eqs. (4.2.1), (4.2.5) and (4.2.6) are solved and the $\hat{R}$-matrix has the form

$$\hat{R}^{ij}_{kl} = a_{ij}\delta^i_l\delta^j_k + b\Theta(l-k)\delta^i_k\delta^j_l \ . \tag{4.2.10}$$

(iv) Eq. (4.2.4) simplifies now; it implies that all the products $a_{ij}a_{ji}$ take the same value. Denote this value by a, $a_{ij}a_{ji} = a$ for all i and j with $i \neq j$.
(v) The remaining two equations, (4.2.2) and (4.2.3) imply that for all i the parameters a_i satisfy a quadratic equation

$$a_i^2 - ba_i - a = 0 \ . \tag{4.2.11}$$

Now it is immediate to verify that the matrix $\hat{R}$ satisfies the same quadratic equation

$$\hat{R}^2 = a + b\hat{R} \ . \tag{4.2.12}$$

The proof of the Proposition is finished. □

When $4a + b^2 = 0$, the matrix $\hat{R}$ has a nontrivial jordanian structure.

Assume that eq. $t^2 - bt - a = 0$ has two different roots μ_1 and μ_2. In this case the matrix $\hat{R}$ is diagonalizable and has two projectors. Let m (correspondingly n) be the number of those a_i which are equal to μ_1 (correspondingly μ_2). Then the ranks of the projectors are $\dfrac{m(m+1)}{2} + \dfrac{n(n-1)}{2} + mn$ and $\dfrac{m(m-1)}{2} + \dfrac{n(n+1)}{2} + mn$. These are exactly the ranks of the symmetrizer and the antisymmetrizer for the superspace of dimension $m|n$. The $\hat{R}$-matrices, constructed in the Proposition above, are called the multiparametric $\hat{R}$-matrices for the quantum supergroups $GL_q(m|n)$; we have shown that with the ice condition there are essentially no more solutions.

4.3. Construction of orthogonal and symplectic $\hat{R}$-matrices. Let V be a vector space and V^* its dual. The natural pairing between V and V^* can be used to define either a symmetric or antisymmetric scalar product on the space $V \oplus V^*$. This scalar product is invariant under the natural action of the group $GL(V)$ of general linear transformations of the space V. Therefore, $GL(V)$ gets imbedded into a corresponding orthogonal or symplectic group.

Such logic goes very well for quantum spaces also. We shall model in this way quantum spaces for orthogonal and symplectic quantum groups.

1. Yang-Baxter equation and ordering

A quantum space, being defined by only a part of projectors of an $\hat{R}$-matrix, does not carry the whole information about the $\hat{R}$-matrix itself. It is not difficult to find a quantum space which can be defined by several different $\hat{R}$-matrices (*Exercise*: give an example).

However, there is a convenient way to encode an $\hat{R}$-matrix in a framework of quantum spaces. It requires several copies of a quantum space. Let x^i be coordinates of some quantum space. We shall not be interested in commutation relations between the elements x^i; rather, we introduce copies, say $x(1)^i$, $x(2)^i$ *etc*, and define commutation relations between different copies to be

$$x(M)^i x(N)^j = \hat{R}^{ij}_{kl}\, x(N)^k x(M)^l \tag{4.3.1}$$

for $M < N$. The relations (4.3.1) allow to reorder any multilinear combination $x(M_1)^{i_1} x(M_2)^{i_2} \dots x(M_p)^{i_p}$ with pairwise distinct labels $M_1, M_2, \dots, M_p$ in the descending (with respect to the labels $M_1, M_2, \dots, M_p$) order.

There are two ways to reorder a monomial $x(M_1)^{i_1} x(M_2)^{i_2} x(M_3)^{i_3}$, with $M_1 < M_2 < M_3$, in a descending way, starting from $x(M_1)^{i_1} x(M_2)^{i_2}$ or $x(M_2)^{i_2} x(M_3)^{i_3}$. The equality of two resulting ordered expressions is a compatibility condition. Assume that monomials $x(M_1)^{i_1} x(M_2)^{i_2} \dots x(M_p)^{i_p}$, where $M_1 > M_2 > \dots > M_p$, are linearly independent. Then the compatibility condition is precisely equivalent to the Yang-Baxter equation for the matrix $\hat{R}$. This interpretation of the Yang-Baxter equation is very useful especially in cases when the index i of coordinates x^i is composite (like, for instance, a pair of indices $\{\alpha, \beta\}$ if one wants to view the elements T^α_β of the quantum matrix as coordinates of a quantum space).

In the sequel, to avoid the cumbersome notation $x(M)^i$, we shall write $x^i y^j = \hat{R}^{ij}_{kl}\, y^k x^l$ instead of (4.3.1).

2. Assumptions

Our starting point is a solution $\hat{R}$ of the Yang-Baxter equation. We impose several conditions:

A1. $\hat{R}$ is invertible.
A2. $\hat{R}$ is skew-invertible with a skew inverse $\hat{\Psi}$.
A3. an operator Q defined by (4.1.38) is invertible (thus, $\tilde{Q}$ defined by (4.1.39) is invertible as well).

As any solution of the Yang-Baxter equation, $\hat{R}$ defines a quantum group; in this subsection it will be enough to understand it as an algebra generated by T^i_j

and $(T^{-1})^i_j$ with relations

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12} \tag{4.3.2}$$

and

$$TT^{-1} = T^{-1}T = I\ , \tag{4.3.3}$$

or $T^i_j(T^{-1})^j_k = (T^{-1})^i_jT^j_k = \delta^i_k$.

We need one more assumption. The relations $\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}$ imply that $W_{12}T_1T_2 = T_1T_2W_{12}$ for any polynomial W in $\hat{R}$, $W = \sum_\alpha c_\alpha \hat{R}^\alpha$. We shall say that $\hat{R}$ is rigid if every W for which $W_{12}T_1T_2 = T_1T_2W_{12}$ is a polynomial in $\hat{R}$. And this is our assumption:

A4. $\hat{R}$ is rigid.

3. Auxiliary formulas

Here are some immediate consequences from (4.3.2) and (4.3.3). First,

$$T_1^{-1}\hat{R}_{12}T_1 \quad = \quad T_2\hat{R}_{12}T_2^{-1}\ , \tag{4.3.4}$$

$$\hat{R}_{12}T_2^{-1}T_1^{-1} \quad = \quad T_2^{-1}T_1^{-1}\hat{R}_{12}\ . \tag{4.3.5}$$

Multiplying (4.3.4) by $\hat{\Psi}_{a1}$ from the left, by $\hat{\Psi}_{2b}$ from the right and taking traces in the spaces 1 and 2, we obtain (as usual, after relabeling spaces)

$$\mathrm{tr}_0(\hat{\Psi}_{10}T_0^{-1}P_{02}T_0) = \mathrm{tr}_0(T_0P_{10}T_0^{-1}\hat{\Psi}_{02})\ . \tag{4.3.6}$$

Attention: one cannot move T_0 cyclically under the tr_0 in (4.3.6) because the matrix elements of T_0 do not commute with matrix elements of other operators in the expression.

Tracing (4.3.6) in the spaces 1 or 2 gives

$$\mathrm{tr}_0(T_0P_{10}T_0^{-1}Q_0) \quad = \quad Q_1\ , \tag{4.3.7}$$

$$\mathrm{tr}_0(\tilde{Q}_0T_0^{-1}P_{01}T_0) \quad = \quad \tilde{Q}_1\ . \tag{4.3.8}$$

Sometimes it is more transparent to write eq. (4.3.6), as well as eqs. (4.3.7) and (4.3.8), in indices:

$$\hat{\Psi}^{ua}_{vb}(T^{-1})^b_iT^j_a = T^s_v(T^{-1})^u_t\hat{\Psi}^{tj}_{si} \tag{4.3.9}$$

and

$$T^a_i(T^{-1}Q)^b_a = Q^j_i \qquad \text{and} \qquad (\tilde{Q}T^{-1})^a_iT^j_a = \tilde{Q}^j_i\ . \tag{4.3.10}$$

Operators Q and $\tilde{Q}$ are invertible, so we can rewrite (4.3.10) in terms of an associative operation $(X\circ Y)^i_j := X^a_jY^i_a$ (or $X\circ Y = (X^tY^t)^t$, where t means the transposition) as

$$T\circ(Q^{-1}T^{-1}Q) = I \qquad \text{and} \qquad (\tilde{Q}T^{-1}\tilde{Q}^{-1})\circ T = I\ , \tag{4.3.11}$$

where I is the identity (with respect to the usual multiplication as well as to the multiplication $\circ$). Left and right inverses coincide, so

$$TQ\tilde{Q} = Q\tilde{Q}T\ . \tag{4.3.12}$$

It follows from (4.3.11) that T^{-1} also has an inverse with respect to $\circ$:

$$(QTQ^{-1}) \circ T^{-1} = I \ . \tag{4.3.13}$$

Note that (4.3.6) can be rewritten as $(\hat{\Psi}_{12}T_2^{-1}) \circ T_2 = T_1 \circ (T_1^{-1}\hat{\Psi}_{12})$ or

$$T_2^{-1} \circ \hat{\Psi}_{12} \circ T_2 = T_1 \circ \hat{\Psi}_{12} \circ T_1^{-1} \ , \tag{4.3.14}$$

since, if the matrix elements of an operator X commute with the matrix elements of an operator Y then $X \circ Y = YX$.

4. Covariance

As explained in the part 1 of this subsection, the operator $\hat{R}$ provides a consistent set of relations

$$x^i y^j = \hat{R}^{ij}_{kl} y^k x^l \quad . \tag{4.3.15}$$

The relations (4.3.15) are covariant under the following (co)action of the quantum group (generators T^i_j commute with x^i):

$$x^i \to T^i_j x^j \ , \tag{4.3.16}$$

and the same for y^i.

We are going to build a quantum analog of the direct sum $V \oplus V^*$, so we need, in addition to x^i, another set of generators, x_i. To mimic that the generators x_i describe a dual space, we require their transformation law to be

$$x_i \to x_j (T^{-1})^j_i \tag{4.3.17}$$

(the same for y_i).

A little later we will restrict ourselves to the case when $\hat{R}$ has only two eigenvalues. But already now we can partly analyze possible ordering relations. We have two "multiplets", $x^\bullet = \{x^i\}$ and $x_\bullet = \{x_i\}$. For the moment, let $S \in End(V \otimes V)$ be an arbitrary operator. Let us say that a matrix element $S^{\alpha\beta}_{\gamma\delta}$ is "ice" if either $\alpha = \gamma$ and $\beta = \delta$ or $\alpha = \delta$ and $\beta = \gamma$. If all non-vanishing matrix elements of S are ice then we have an ice matrix in the sense of the subsection 4.2. We shall apply the same terminology to the whole multiplets $x^\bullet$ and $x_\bullet$: if x belongs to a multiplet A (A can be $^\bullet$ or $_\bullet$) and y belongs to a multiplet B then the parts in the ordered expression for xy, which contain the same multiplets will be called "ice". We shall see that the ice part of ordering relations is strongly governed by the covariance.

We fix the ordering relations for $x^i y^j$ to be as in (4.3.15).

In the ordered expression for $x_i y_j$ the ice terms are $y_k x_l$,

$$x_i y_j = E^{kl}_{ij} y_k x_l + \dots \ , \tag{4.3.18}$$

where dots stand for terms with other structures of indices. Then the covariance under the transformations (4.3.17) requires $T_1^{-1}T_2^{-1}E_{21} = E_{21}T_1^{-1}T_2^{-1}$ so, by rigidity, E_{12} is a polynomial in $\hat{R}_{21}$, $E_{12} = e(\hat{R}_{21})$.

In the ordered expression for $x^i y_j$ we may have terms like $y_k x^l$ and $y^k x_l$,

$$x^i y_j = A^{ik}_{jl} y_k x^l + B^{il}_{jk} y^k x_l + \dots \ , \tag{4.3.19}$$

dots stand for terms with other structures of indices. Then the covariance under the transformations (4.3.16) and (4.3.17) requires

$$T_1 A_{12} T_1^{-1} = T_2^{-1} A_{12} T_2 \ , \tag{4.3.20}$$

$$T_1 B_{12} T_1^{-1} = \mathrm{tr}_0 (B_{10} T_0 P_{02} T_0^{-1}) \ . \tag{4.3.21}$$

Similarly, if, in the ordered expression for $x_i y^j$, we have terms like $y^k x_l$ and $y_k x^l$,

$$x_i y^j = C_{ik}^{jl} y^k x_l + D_{il}^{jk} y_k x^l + \dots \ , \tag{4.3.22}$$

where dots stand for terms with other structures of indices, then the covariance under the transformations (4.3.16) and (4.3.17) requires

$$\mathrm{tr}_0 (T_0^{-1} P_{01} T_0 C_{02}) = \mathrm{tr}_0 (C_{10} T_0 P_{02} T_0^{-1}) \ , \tag{4.3.23}$$

$$\mathrm{tr}_0 (T_0^{-1} P_{01} T_0 D_{02}) = T_2^{-1} D_{12} T_2 \ . \tag{4.3.24}$$

Due to rigidity of $\hat{R}$, it follows from eq. (4.3.20) that A_{12} is a polynomial in $\hat{R}_{21}$, $A_{12} = a(\hat{R}_{21})$.

Multiply (4.3.21) by $\hat{R}_{a2}$ from the right and take tr_2. The l.h.s. becomes $T_1 \tilde{B}_{1a} T_1^{-1}$, where $\tilde{B}_{1a} = \mathrm{tr}_2 (B_{12} \hat{R}_{a2})$. The r.h.s. becomes

$$\begin{aligned} &\mathrm{tr}_{02} (B_{10} T_0 P_{02} T_0^{-1} \hat{R}_{a2}) = \mathrm{tr}_{02} (B_{10} T_0 P_{02} \hat{R}_{a2} T_0^{-1}) \\ &= \mathrm{tr}_{02} (P_{02} B_{12} T_2 \hat{R}_{a2} T_0^{-1}) \stackrel{(c)}{=} \mathrm{tr}_{02} (B_{12} T_2 \hat{R}_{a2} T_0^{-1} P_{02}) \\ &= \mathrm{tr}_{02} (B_{12} T_2 \hat{R}_{a2} P_{02} T_2^{-1}) \stackrel{\mathrm{tr}_0}{=} \mathrm{tr}_2 (B_{12} T_2 \hat{R}_{a2} T_2^{-1}) \\ &\stackrel{(4.3.4)}{=} \mathrm{tr}_2 (B_{12} T_a^{-1} \hat{R}_{a2} T_a) = T_a^{-1} \tilde{B}_{1a} T_a \ . \end{aligned} \tag{4.3.25}$$

We used: the cyclic property of the trace to move P_{02}, it is indicated by (c) over "="; we took tr_0 (it is indicated over "="); and we used eq. (4.3.4).

Therefore, $\tilde{B}_{1a}$ is, by rigidity of $\hat{R}$, a polynomial in $\hat{R}_{a1}$, $\mathrm{tr}_2 (B_{12} \hat{R}_{a2}) = b(\hat{R}_{a1})$ with some polynomial b. Multiplying by $\hat{\Psi}_{ba}$ from the left and taking tr_a, we find

$$B_{12} = \mathrm{tr}_0 (\hat{\Psi}_{20} b(\hat{R}_{01})) \ . \tag{4.3.26}$$

Similarly, multiplying (4.3.24) by $\hat{R}_{1a}$ from the left and taking tr_1, we find

$$T_2^{-1} \tilde{D}_{a2} T_2 = T_a \tilde{D}_{a2} T_a^{-1} \ , \tag{4.3.27}$$

where $\tilde{D}_{a2} = \mathrm{tr}_1 (\hat{R}_{1a} D_{12})$. Therefore, $\tilde{D}_{a2}$ is a polynomial in $\hat{R}_{2a}$. Thus,

$$D_{12} = \mathrm{tr}_0 (d(\hat{R}_{20}) \hat{\Psi}_{01}) \tag{4.3.28}$$

for some polynomial d.

Finally, multiply eq. (4.3.23) from the left by $\hat{R}_{1a}$, from the right by $\hat{R}_{b2}$ and take tr_{12} to obtain

$$T_a \tilde{C}_{ab} T_a^{-1} = T_b^{-1} \tilde{C}_{ab} T_b \ , \tag{4.3.29}$$

where $\tilde{C}_{ab} = \mathrm{tr}_{12} (\hat{R}_{1a} C_{12} \hat{R}_{b2})$. Therefore, $\tilde{C}_{ab}$ is a polynomial in $\hat{R}_{ba}$. Thus,

$$C_{12} = \mathrm{tr}_{03} (\hat{\Psi}_{23} c(\hat{R}_{30}) \hat{\Psi}_{01}) \tag{4.3.30}$$

for some polynomial c.

5. Ansatz

We keep in mind that the multiplets $x^\bullet$ and $x_\bullet$ are associated to the group GL_N. For general N, the only invariant tensors with four indices are the permutation and the identity in $V\otimes V$. This motivates the following Ansatz:

$$x^iy^j = \hat{R}^{ij}_{kl}y^kx^l \ , \tag{4.3.31}$$

$$x^iy_j = A^{ik}_{jl}y_kx^l + B^{il}_{jk}y^kx_l \ , \tag{4.3.32}$$

$$x_iy^j = C^{jl}_{ik}y^kx_l + D^{jk}_{il}y_kx^l \ , \tag{4.3.33}$$

$$x_iy_j = E^{lk}_{ij}y_lx_k \ . \tag{4.3.34}$$

Here

$$A_{12} = a(\hat{R}_{21}) \ , \ C_{12} = \ \mathrm{tr}_{03}(\hat{\Psi}_{23}c(\hat{R}_{30})\hat{\Psi}_{01}) \ , \ B_{12} = \ \mathrm{tr}_0(\hat{\Psi}_{20}b(\hat{R}_{01})) \ ,$$

$$D_{12} = \ \mathrm{tr}_0(d(\hat{R}_{20})\hat{\Psi}_{01}) \ , \ E_{12} = e(\hat{R}_{21}) \tag{4.3.35}$$

with some polynomials a, b, c, d and e.

The original matrix $\hat{R}$ is a matrix of the size $N^2\times N^2$, where N is the dimension of the space V, the range of indices of multiplets x^i and x_i. A solution $\hat{\mathbf{R}}^{IJ}_{KL}$ of the consistency conditions for the ordering relations (4.3.31)-(4.3.34) is a matrix of a bigger size $(2N^2)\times(2N)^2$, each of four indices of $\hat{\mathbf{R}}$ runs from 1 to $2N$. The new index is the union of upper and lower indices of the original multiplets. To remember it, we shall write, for the new index I, $I=(\underline{k})$ for a value of the original index from the multiplet x^k or $I=(\overline{k})$ for a value of the original index from the multiplet x_k. In this notation, the nonzero matrix elements of $\hat{\mathbf{R}}$ are

$$\begin{array}{c} \hat{\mathbf{R}}^{(\underline{i})\,(\underline{j})}_{(\underline{k})\,(\underline{l})} = \hat{R}^{ij}_{kl} \ , \ \hat{\mathbf{R}}^{(\underline{i})\,(\overline{j})}_{(\overline{k})\,(\underline{l})} = A^{ik}_{jl} \ , \ \hat{\mathbf{R}}^{(\underline{i})\,(\overline{j})}_{(\underline{k})\,(\overline{l})} = B^{il}_{jk} \ , \\ \hat{\mathbf{R}}^{(\overline{i})\,(\underline{j})}_{(\underline{k})\,(\overline{l})} = C^{jl}_{ik} \ , \ \hat{\mathbf{R}}^{(\overline{i})\,(\underline{j})}_{(\overline{k})\,(\underline{l})} = D^{jk}_{il} \ , \ \hat{\mathbf{R}}^{(\overline{i})\,(\overline{j})}_{(\overline{l})\,(\overline{k})} = E^{lk}_{ij} \ . \end{array} \tag{4.3.36}$$

We are looking for a skew-invertible $\hat{\mathbf{R}}$. In the notation, as in (4.3.36), it is easy to see that if A is zero then the matrix $\hat{\mathbf{R}}^{IJ}_{KL}$ has a zero eigenvector with respect to the skew multiplication, that is, a quantity v^I_K which satisfies $v^I_K\hat{\mathbf{R}}^{IJ}_{KL}=0$: one may take any v whose non-zero elements are only $v^{(\overline{k})}_{(\underline{i})}$; so, such $\hat{\mathbf{R}}$ cannot be skew invertible. In fact, this argument shows that the skew invertibility of $\hat{\mathbf{R}}$ requires that the operator A is invertible (with respect to the usual multiplication). Similarly, C must be invertible. The conditions

$$A \ \text{ and } C \ \text{ are invertible} \tag{4.3.37}$$

we will use in the process of solving the Yang-Baxter equation for $\hat{\mathbf{R}}$.

6. Yang-Baxter equation for $\hat{\mathbf{R}}$

As explained in the beginning of this subsection, the Yang-Baxter equation for $\hat{\mathbf{R}}$ we obtain by ordering in two different ways expressions $x^A y^B z^C$, where the indices A, B and C can belong now to any of multiplets, $^\bullet$ or $_\bullet$.

Ordering $x^\bullet y^\bullet z_\bullet$:

$$\hat{R}_{23}A_{21}A_{32} = A_{21}A_{32}\hat{R}_{12} \ , \tag{4.3.38}$$

$$\hat{R}_{21}B_{23}A_{12} = P_{23}\ \mathrm{tr}_0(A_{02}B_{10}\hat{R}_{30}) + P_{23}\ \mathrm{tr}_0(A_{10}D_{02}P_{03}B_{03}) \ , \tag{4.3.39}$$

$$\hat{R}_{12}B_{23}\hat{R}_{13} = \ \mathrm{tr}_0(B_{20}\hat{R}_{10}B_{03}) + \ \mathrm{tr}_0(P_{01}B_{01}A_{20}C_{03}) \ . \tag{4.3.40}$$

Ordering $x^\bullet y_\bullet z^\bullet$:

$$\begin{aligned} &P_{12}\ \mathrm{tr}_0(A_{10}\hat{R}_{02}C_{03}) + P_{12}\ \mathrm{tr}_0(B_{10}B_{03}P_{02}D_{02}) \\ &\qquad = \ \mathrm{tr}_0(C_{10}\hat{R}_{20}A_{03})P_{23} + \ \mathrm{tr}_0(D_{10}D_{03}P_{02}B_{02})P_{23} \ , \end{aligned} \tag{4.3.41}$$

$$A_{23}D_{12}\hat{R}_{32} = P_{12}\ \mathrm{tr}_0(\hat{R}_{02}D_{03}A_{10}) + P_{12}\ \mathrm{tr}_0(B_{10}A_{03}P_{02}D_{02}), \tag{4.3.42}$$

$$C_{23}B_{12}\hat{R}_{23} = P_{12}\ \mathrm{tr}_0(\hat{R}_{20}B_{03}C_{10}) + P_{12}\ \mathrm{tr}_0(D_{10}C_{03}P_{02}B_{02}) \ . \tag{4.3.43}$$

Ordering $x_\bullet y^\bullet z^\bullet$:

$$C_{23}C_{12}\hat{R}_{23} = \hat{R}_{12}C_{23}C_{12} \ , \tag{4.3.44}$$

$$\hat{R}_{12}D_{23}C_{12} = P_{23}\ \mathrm{tr}_0(C_{02}D_{10}\hat{R}_{03}) + P_{23}\ \mathrm{tr}_0(C_{10}B_{02}P_{03}D_{03}) \ , \tag{4.3.45}$$

$$\hat{R}_{13}D_{12}\hat{R}_{23} = \ \mathrm{tr}_0(D_{10}\hat{R}_{03}D_{02}) + \ \mathrm{tr}_0(C_{10}A_{02}P_{03}D_{03}) \ . \tag{4.3.46}$$

Exercise. Verify eqs. (4.3.38)-(4.3.46).

Equations, arising from ordering $x^\bullet y_\bullet z_\bullet$, $x_\bullet y^\bullet z_\bullet$ and $x_\bullet y_\bullet z^\bullet$, can be quickly obtained by noticing that the system (4.3.31)-(4.3.34) is invariant under a substitution $x^\bullet \leftrightarrow x_\bullet$, $y^\bullet \leftrightarrow y_\bullet$, $\hat{R} \leftrightarrow E^t$, $A \leftrightarrow C^t$ and $B \leftrightarrow D^t$, where t stands for the transposition. We have:
for $x^\bullet y_\bullet z_\bullet$:

$$A_{12}A_{23}E_{12} = E_{23}A_{12}A_{23} \ , \tag{4.3.47}$$

$$A_{12}B_{23}E_{12} = \ \mathrm{tr}_0(E_{03}B_{10}A_{02})P_{23} + \ \mathrm{tr}_0(P_{03}B_{30}D_{02}A_{10})P_{23} \ , \tag{4.3.48}$$

$$E_{23}B_{12}E_{13} = \ \mathrm{tr}_0(B_{02}E_{03}B_{10}) + \ \mathrm{tr}_0(P_{03}B_{30}C_{02}A_{10}) \ ; \tag{4.3.49}$$

for $x_\bullet y^\bullet z_\bullet$:

$$\begin{aligned} &\mathrm{tr}_0(A_{03}E_{02}C_{10})P_{12} + \ \mathrm{tr}_0(P_{02}B_{20}D_{03}D_{10})P_{12} \\ &\qquad = P_{23}\ \mathrm{tr}_0(C_{03}E_{20}A_{10}) + P_{23}\ \mathrm{tr}_0(P_{02}D_{20}B_{03}B_{10}) \ , \end{aligned} \tag{4.3.50}$$

$$E_{23}D_{12}A_{23} = \ \mathrm{tr}_0(A_{10}D_{03}E_{20})P_{12} + \ \mathrm{tr}_0(P_{02}D_{20}A_{03}B_{10})P_{12} \ , \tag{4.3.51}$$

$$E_{32}B_{12}C_{23} = \ \mathrm{tr}_0(C_{10}B_{03}E_{02})P_{12} + \ \mathrm{tr}_0(P_{02}B_{20}C_{03}D_{10})P_{23} \ ; \tag{4.3.52}$$

for $x_\bullet y_\bullet z^\bullet$:

$$E_{12}C_{32}C_{21} = C_{32}C_{21}E_{23} \ , \tag{4.3.53}$$

$$C_{12}D_{23}E_{21} = \ \mathrm{tr}_0(E_{30}D_{10}C_{02})P_{23} + \ \mathrm{tr}_0(P_{03}D_{30}B_{02}C_{10})P_{23} \ , \tag{4.3.54}$$

$$E_{12}D_{32}E_{13} = \ \mathrm{tr}_0(D_{02}E_{10}D_{30}) + \ \mathrm{tr}_0(P_{10}D_{10}A_{02}C_{30}) \ . \tag{4.3.55}$$

Finally, ordering $x_\bullet y_\bullet z_\bullet$ implies the Yang-Baxter equation for E:

$$E_{12}E_{23}E_{12} = E_{23}E_{12}E_{23} \ . \tag{4.3.56}$$

7. Specifying to the Hecke case

We shall solve the system (4.3.38)-4.3.56) in the Hecke case - when the matrix $\hat{R}$ satisfies a quadratic equation $\hat{R}^2 = \lambda\hat{R} + 1$. Note that $\hat{R}$ cannot be proportional to a constant, it would contradict the skew invertibility.

As we have seen in the subsection 4.1.2 (see eq. (4.1.76)), in the Hecke case the product $Q\tilde{Q}$ is proportional to a unity, $Q\tilde{Q} = r^2 I$ (r corresponds to the ribbon element in the quasi-triangular case). Due to the assumption **A3**, $r \neq 0$. Therefore, by (4.1.68) (and (4.1.49))

$$1 - \lambda\ \mathrm{tr}(Q) = 1 - \lambda\ \mathrm{tr}(\tilde{Q}) \neq 0 \ . \tag{4.3.57}$$

Because of Hecke condition, the polynomials in (4.3.35) contain only constant and linear terms.

8. Block triangularity

The standard $\hat{R}$-matrix (4.0.1) has a following property:

$$\hat{R}^{ij}_{kl} = 0 \qquad \text{if} \qquad ji < kl \ , \tag{4.3.58}$$

where $<$ is the lexicographic ordering (*i.e.*, $ji < kl$ when $j < k$ or $j = k$ and $i < l$). This means that the matrix $P\hat{R}$ is lower triangular.

The standard $\hat{R}$-matrix (4.0.1) has also another triangularity property: $\hat{R}^{ij}_{kl} = 0$ if $ij > lk$; this means that the matrix $\hat{R}P$ is upper triangular.

For the ordering relations $x^i y^j = \hat{R}^{ij}_{kl}$, the property (4.3.58) says that the ordered expression for $x^i y^j$ can contain only monomials which are lexicographically not bigger than $y^j x^i$.

As a first step towards a solution of the system (4.3.38)-4.3.56) in the Hecke case, we shall prove that the relations (4.3.31)-(4.3.34) are "block triangular": say, block upper triangularity means that we define an order on the set $\mathcal{S} = \{^\bullet, _\bullet\}$ of multiplets $x^\bullet$ and $x_\bullet$, $x^\bullet > x_\bullet$ and then the ordered expression for $x^{\mathcal{I}} y^{\mathcal{J}}$, $\mathcal{I}, \mathcal{J} \in \mathcal{S}$, contains only monomials which are not bigger than $y^{\mathcal{J}} x^{\mathcal{I}}$.

In the simple situation of eqs. (4.3.31)-(4.3.34), the block triangularity means that either $B = 0$ or $D = 0$.

To prove the block triangularity, it is enough to consider two equations, (4.3.38) and (4.3.39). Eqn. (4.3.38) implies that A_{12} is proportional to either $\hat{R}_{21}$ or $\hat{R}^{-1}_{21}$. We shall write it as $A_{12} \propto \hat{R}_{12} - \epsilon\lambda I_1 I_2$, where $\epsilon = 0$ or 1. The coefficient of proportionality is different from 0 due to (4.3.37).

The expressions (4.3.35) for B and D reduce in the Hecke case to

$$B_{12} = \mu I_1 Q_2 + \nu P_{12} \ , \tag{4.3.59}$$

$$D_{12} = \alpha \tilde{Q}_1 I_2 + \beta P_{12} \tag{4.3.60}$$

with some constants μ, ν, α and β.

Substituting the expressions for A, B and D into (4.3.39), we obtain, after using identities from the subsection 4.1.2, an equality

$$\begin{aligned} &(\mu\lambda(1-\epsilon) + \alpha\nu(1-\epsilon\lambda\ \mathrm{tr}(\tilde{Q})) - \epsilon\lambda\beta\nu - \epsilon\lambda\alpha\mu)P_{23}I_1 \\ &\quad +(\beta\mu - \mu(1-\epsilon)\lambda)\hat{R}_{21}Q_3 + \alpha\mu r^2 P_{23}\hat{R}_{31} \\ &\quad +\beta\nu P_{23}\hat{R}_{21} - \epsilon\lambda\beta\mu I_1 I_2 Q_3 = 0 \ . \end{aligned} \tag{4.3.61}$$

The tensors $P_{23}I_1$, $\hat{R}_{21}Q_3$, $P_{23}\hat{R}_{31}$, $P_{23}\hat{R}_{21}$ and $I_1I_2Q_3$, entering eqn. (4.3.61) are linearly independent: to see it, multiply them from the right by $\hat{\Psi}_{14}$ and take tr_1; the tensors become $P_{23}\tilde{Q}_4$, $P_{24}Q_3$, $P_{23}P_{34}$, $P_{23}P_{24}$ and $I_2Q_3\tilde{Q}_4$, which are obviously independent. Thus, the coefficients must vanish,

$$\begin{aligned} &\mu\lambda(1-\epsilon) + \alpha\nu(1-\epsilon\lambda\ \mathrm{tr}(\tilde{Q})) - \epsilon\lambda\beta\nu - \epsilon\lambda\alpha\mu = 0 \ , \\ &\beta\mu - \mu(1-\epsilon)\lambda = 0 \ , \ \alpha\mu = 0 \ , \ \beta\nu = 0 \ , \ \epsilon\lambda\beta\mu = 0 \ . \end{aligned} \tag{4.3.62}$$

For $\epsilon = 0$, it follows from eqs. (4.3.62) that $\alpha\mu = 0$, $\beta\nu = 0$ and $\beta\mu + \alpha\nu = 0$, which implies that either B or D is zero.

For $\epsilon = 1$, it follows from eqs. (4.3.62) that $\beta\mu = 0$, $\alpha\nu(1-\lambda\ \mathrm{tr}(\tilde{Q}) = 0$, $\alpha\mu = 0$ and $\beta\nu = 0$; in view of (4.3.57) we conclude again that either B or D is zero.

It is enough to consider the case $B = 0$; another case can be reduced to it by considering the opposite ordering (if we read (4.3.31)-(4.3.34) from the right to the left, as instructions to order yx to the form xy).

9. Solution

With $B = 0$ the system (4.3.38)-4.3.56) simplifies drastically, can be fully analyzed and one can write down all solutions. It is lengthy and we shall not do it here.

It turns out that solutions which give rise to the orthogonal and symplectic quantum groups are those for which the coefficient α in (4.3.60) is different from 0.

Proposition. Let $\hat{R}$ be a solution of the Yang-Baxter equation with $\hat{R}^2 = \lambda\hat{R} + 1$. If $\hat{R}$ satisfies assumptions **A1-A4** then the ordering relations

$$x^i y^j = \hat{R}^{ij}_{kl} y^k x^l \ , \tag{4.3.63}$$

$$x_i y_j = \hat{R}^{kl}_{ji} y_l x_k \ , \tag{4.3.64}$$

$$x^i y_j = \kappa (\hat{R}^{-1})^{ki}_{lj} y_k x^l \ , \tag{4.3.65}$$

$$x_i y^j = \kappa^{-1} \hat{\Psi}^{uj}_{vi} y^v x_u + \lambda y_i x^j + \nu\lambda \tilde{Q}^j_i y_k x^k \ , \tag{4.3.66}$$

where κ is an arbitrary non-zero number, provide an invertible and skew invertible solution $\hat{\mathbf{R}}$ of the Yang-Baxter equation when $\nu^2 + \lambda\nu - 1 = 0$.

If $\hat{R}$ is of GL_N-type then $\hat{\mathbf{R}}$ is of SO_{2N} type for $\nu = -q$ and of Sp_{2N} type for $\nu = q^{-1}$.

10. $SO(2N+1)$

Without going into details we shall describe the situation with the odd-dimensional orthogonal groups.

One has to add a new generator x^0 to the multiplets $x^\bullet$ and $x_\bullet$. The matrix $\hat{\mathbf{R}}$ again turns out to be block-triangular; we will write the answer for the order $x^\bullet > x^0 > x_\bullet$. Relations (4.3.63), (4.3.64) and (4.3.65) are the same. Relation (4.3.66) has to be replaced by

$$x_i y^j = \hat{\Psi}^{uj}_{vi} y^v x_u + \lambda y_i x^j - \lambda \tilde{Q}^j_i y_k x^k - q^{-1/2}\lambda \tilde{Q}^j_i y^0 x^0 \ . \tag{4.3.67}$$

Finally, when one of generators has an index 0, the ordering relations are

$$\begin{aligned} x^i y^0 &= y^0 x^i \ , \\ x^0 y^0 &= y^0 x^0 - q^{1/2}\lambda y_l x^l \ , \\ x_i y^0 &= y^0 x_i + \lambda y_i x^0 \ , \\ x^0 y^i &= y^i x^0 + \lambda y^0 x^i \ , \\ x^0 y_i &= y_i x^0 \ . \end{aligned} \tag{4.3.68}$$

Proposition. Under the same conditions as in the Proposition above, the ordering relations (4.3.63)-(4.3.65), (4.3.67) and (4.3.68) provide an invertible and skew invertible solution $\hat{\mathbf{R}}$ of the Yang-Baxter equation.

If $\hat{R}$ is of GL_N-type then $\hat{\mathbf{R}}$ is of SO_{2N+1} type.

Remarks. 1. For a standard $\hat{R}$ for GL, it was noted in [**32**], that the commutation relations between coordinates and derivatives (even or odd) can be given by projectors of $\hat{R}$ for Sp and SO. Our propositions in this subsection generalize it to the construction of the whole $\hat{R}$-matrix for SO and Sp from the $\hat{R}$-matrix for GL, which works in all cases, not only for the standard deformation.

2. If one starts with $\hat{R}$ corresponding to a supergroup $GL(M|N)$, the constructions of the propositions from this subsection produce Yang-Baxter matrices for the quantum supergroups of OSp type.

5. Real forms

In this subsection we explain how to classify real forms of RTT-algebras using quantum spaces [**33, 34**][6].

[6]The description of real forms of the dual algebra for a generic q is given in [**35**]. Our description is more precise, it requires only that $q^4 \neq 1$ in SL case and $q^8 \neq 1$ in the SO and Sp cases

5.1. General linear quantum groups. We shall start with a standard Drinfeld-Jimbo $\hat{R}$ matrix (4.0.1) for the quantum group $GL_q(N)$. We shall assume that $q^4 \neq 1$.

Exercise. Show that the $\hat{R}$-matrix (4.0.1) satisfies the Yang-Baxter equation. Show that the spectral decomposition of $\hat{R}$ is $\hat{R} = qS - q^{-1}A$ (S and A are projectors, $S^2 = S$, $A^2 = A$) with $\operatorname{rk} S = \frac{N(N+1)}{2}$ and $\operatorname{rk} A = \frac{N(N-1)}{2}$.

Let $*$ be an involution on the RTT-algebra, that is, an antilinear operation, satisfying $*(ab) = *(b) * (a)$ and $(* \otimes *) \circ \Delta = \Delta \circ *$. Then $*x^i$ form a comodule for the $SL_q(N)$. There are two comodules of dimension N: one is generated by x^i, another one is generated by x_i. So we may have two different types of conjugations, $*$ can map A^l_+ to itself or to A^r_+.

We shall consider in some details the first possibility. So, we assume

$$*x^i = J^i_j x^j \ . \tag{5.1.1}$$

Since the matrix T coacts on the vector x, we have $*(Tx) = JTx$; on the other hand, $*(Tx)^i = *(T^i_j x^j) = *x^j * T^i_j = *T^i_j * x^j = *T^i_j J^j_k x^k$ (we used that T^i_j commutes with x^k). It follows then that

$$*T = JTJ^{-1} \ . \tag{5.1.2}$$

Conjugate now the relation $\hat{R}T_1T_2 = T_1T_2\hat{R}$:

$$\overline{\hat{R}} * T_2 * T_1 = *T_2 * T_1 \overline{\hat{R}}$$

(here is the complex conjugate), or

$$\overline{\hat{R}}_{21} * T_1 * T_2 = *T_1 * T_2 \overline{\hat{R}}_{21} \ .$$

Substituting $*T$ from (5.1.2) we find

$$\Phi T_1 T_2 = T_1 T_2 \Phi \ , \tag{5.1.3}$$

where $\Phi = J_1^{-1} J_2^{-1} \overline{\hat{R}}_{21} J_1 J_2$.

Proposition. Let $\hat{R}$ be the standard $SL_q(N)$ $\hat{R}$-matrix (4.0.1). If an operator $\Phi = \Phi_{12}$ satisfies an equality

$$[\Phi, T_1T_2] = 0 \ , \tag{5.1.4}$$

then Φ is a polynomial in $\hat{R}$.

Sketch of the proof. Take a 1-dimensional representation for T, $T^i_j \mapsto \mu_j \delta^i_j$ with some commuting variables μ_i. Then it follows from (5.1.4) that Φ is of "ice" type, that is, Φ^{ij}_{kl} can be different from zero only if $i = k$, $j = l$ or $i = l$, $j = k$.

Take now another representation, $(T^i_j)^a_b = \hat{R}^{ai}_{jb}$. Writing (5.1.4) in this representation with an ice Φ, one arrives at the statement of the proposition. $\square$

$\hat{R}$ satisfies the Yang-Baxter equation (YBe) $\Rightarrow$ $\hat{R}_{21}$ satisfies YBe $\Rightarrow$ $\overline{\hat{R}}_{21}$ satisfies YBe $\Rightarrow$ $\Phi = J_1^{-1} J_2^{-1} \overline{\hat{R}}_{21} J_1 J_2$ satisfies YBe.

The following proposition is easy:

Proposition. A non-constant polynomial in $\hat{R}$ which satisfies YBe is either $\alpha\hat{R}$ or $\alpha\hat{R}^{-1}$ for some constant α.

The operator $\hat{R}_{21} = P\hat{R}P$ (P is the permutation) has the same spectrum as $\hat{R}$. Therefore, the spectrum of $\Phi = J_1^{-1}J_2^{-1}\overline{\hat{R}}_{21}J_1J_2$ contains an eigenvalue $\overline{q}$ with the multiplicity $\frac{N(N+1)}{2}$ and the eigenvalue $(-\overline{q}^{-1})$ with the multiplicity $\frac{N(N-1)}{2}$.

According to the Proposition, we have to consider two possibilities, $\Phi = \alpha\hat{R}$ or $\Phi = \alpha\hat{R}^{-1}$.

Comparing spectra, we find that if $\Phi = \alpha\hat{R}$ then $\overline{q} = \alpha q$ and $-\overline{q}^{-1} = -\alpha q^{-1}$. Therefore, $\alpha^2 = 1$ and $\overline{q} = \pm q$.

Similarly, if $\Phi = \alpha\hat{R}^{-1}$ then $\overline{q} = \alpha q^{-1}$ and $-\overline{q}^{-1} = -\alpha q$. Therefore, $\alpha^2 = 1$ and $\overline{q} = \pm q^{-1}$.

We have four cases. Let us see which equations we have to solve. For example, for $\overline{q} = q$ we have $\Phi = \hat{R}$; for q real, $\overline{\hat{R}} = \hat{R}$ and we have therefore equations $\hat{R}_{21}J_1J_2 = J_1J_2\hat{R}$. This is a system of quadratic equations and it turns out that for the $\hat{R}$-matrix (4.0.1) one can completely solve the system. One can solve the corresponding system in the other three cases as well.

For the other type of conjugation (when $*$ of a quantum vector is a quantum covector), the operator J has two lower indices, $*x_i = J_{ij}x^j$. Again at the end one arrives at a system of quadratic equations for j which admits a complete solution.

The last step is to impose the condition that the square of the conjugation is the identity, $** =$ Id; this produces a further restriction on the operator J.

The final result is presented below. We use a notation $\Omega(c_1, \ldots, c_N)$ for an antidiagonal matrix $\begin{pmatrix} & & & & c_1 \\ & & & c_2 & \\ & & \cdots & & \\ & c_{N-1} & & & \\ c_N & & & & \end{pmatrix}$.

In the formulation of the theorem below, a letter "a" appears sometimes in the name of a real form. The letter "a" stands for "alternative"; it signifies that there are several real forms having the same classical limit.

Theorem. (i) There are no real forms in the nonquasiclassical cases $\overline{q} = -q^{\pm 1}$; all real forms admit the classical limit.

(ii) For $\overline{q} = q^{-1}$ the real forms are:

$SL_q(N, \mathbb{R})$; here $J = 1$.

$SU_q^a(N - [N/2], [N/2])$; $J_{ij} = \delta_{i+j}^{N+1}$.

(iii) For $\overline{q} = q$ the real forms are:

$SL_q^a(N, \mathbb{R})$; $J_j^i = \delta_{i+j}^{N+1}$.

$SU_q^*(2n)$, $N = 2n$; $J =$ antidiag$(\underbrace{1, \ldots, 1}_{n \text{ times}}, \underbrace{-1, \ldots, -1}_{n \text{ times}})$.

$SU_q(\mu_1, \ldots, \mu_N)$; $J_{ij} = \mu_i\delta_i^j$, $\mu_i = \pm 1$.

In the last case the sequences $\{\mu_i\}$ and $\{-\mu_i\}$ produce equivalent real forms. What is more interesting is that the sequences $\{\mu_i\}$ and $\{\mu_i'\}$ where $\mu_i' = \mu_{i'}$, where $i' = N + 1 - i$ produce equivalent real forms as well. An explanation: classically, there is an outer automorphism $T \mapsto (T^{-1})^t$ of the algebra, corresponding to the symmetry $T_{\leftrightarrow}$ of the Dynkin diagram A_l. For the quantum T, we have $(T^{-1})^i_k T^k_j = \delta^i_j = T^i_k (T^{-1})^k_j$ but $(T^{-1})^k_i T^j_k \neq \delta^j_i$. The correct version is[7] $(T^{-1})^k_i (QTQ^{-1})^j_k = \delta^j_i$ where the numerical matrix Q is defined by (4.1.38) (we remind that the standard $\hat{R}$-matrix (4.0.1) is skew-invertible, see (4.1.41)). It is, up to a factor, the same Q which cyclically rotated the E-tensor.

Set $\tau(T^i_j) = (T^{-1})^{j'}_{i'}$.

Proposition. The map τ preserves the RTT-relations.

The proof follows from the fact that $\hat{R}Q_1Q_2 = Q_1Q_2\hat{R}$ and $\hat{R}^{i'j'}_{k'l'} = \hat{R}^{ji}_{lk}$ for the standard $\hat{R}$. Moreover, $\tau((T^{-1})^j_k) = (Q^{-1})^{k'}_v T^v_u Q^u_{j'}$. The effect of τ on the sequence $\{\mu_i\}$ is exactly $\{\mu_i\} \mapsto \{\mu'_i\}$.

5.2. Orthogonal and symplectic quantum groups. I shall very shortly list the real forms for orthogonal and symplectic quantum groups.

The answer below is written in the basis, in which the ordering relations for the quantum planes have the form as in (4.3.63)-(4.3.66), with $\kappa = 1$, for $SO_q(2N)$ and $Sp_q(2N)$, or (4.3.63)-(4.3.65) and (4.3.67)-(4.3.68) for $SO_q(2N+1)$.

Let $B = \begin{pmatrix} 1 &&&&&&& \\ & \cdots &&&&&& \\ && 1 &&&&& \\ &&& . & 1 &&& \\ &&& 1 & . &&& \\ &&&&& 1 && \\ &&&&&& \cdots & \\ &&&&&&& 1 \end{pmatrix}$ (the nondiagonal 2 by 2 block is in the middle); in the formulation of the theorem below, a letter "b" in the name of a real form signifies that the matrix J involves the matrix B.

Theorem. (i) Again, all real forms admit the classical limit.

(ii) For $\overline{q} = q^{-1}$ the real forms are:
$SO_q([N/2], N - [N/2])$; $J = 1$;
$SO^b_q(n+1, n-1)$, $N = 2n$; $J = B$.
$Sp_q(N, \mathbb{R})$; $J = 1$.

(iii) For $\overline{q} = q$ the real forms are ($\mu_i = \pm 1$):
$SO_q(\mu_1, \dots, \mu_N)$; $J = \Omega(\mu_1, \dots, \mu_N)$ with $J^t = J$.
$SO^b_q(\mu_1, \dots, \mu_N)$; $J = B\Omega(\mu_1, \dots, \mu_N)$ with $\mu_{i'} = \mu_i$.
$SO^*_q(\mu_1, \dots, \mu_N)$; $J = \Omega(\mu_1, \dots, \mu_N)$ with $J^t = -J$.
$USp_q(\mu_1, \dots, \mu_N)$; $J = \Omega(\mu_1, \dots, \mu_N)$ with $J^t = J$.
$Sp_q(\mu_1, \dots, \mu_N; \mathbb{R})$; $J = \Omega(\mu_1, \dots, \mu_N)$ with $J^t = -J$.

I shall end the lectures by a comparison with the classical (Cartan) way of classifying the real forms (see, eg. [**36**]).

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[7]see eqs. (4.3.7), (4.3.8) and (4.3.12).

1. One proves that there exists a unique compact real form u; denote the corresponding $*$ by τ.

2. For an arbitrary real form σ one proves that there exists an equivalent to it real form $\tilde{\sigma}$ such that the automorphism $\theta = \tilde{\sigma}\tau$ is involutive, $\theta^2 = 1$. For a description of involutive automorphisms one should analyze each Cartan data concretely.

3. The automorphism θ acts on u; under this action, u decomposes according to the eigenvalues of θ, $u = u_1 \oplus u_{-1}$. The real form corresponding to $\tilde{\sigma}$ is $u_1 \oplus \sqrt{-1} u_{-1}$.

In the classification of real forms of quantum groups given above, these steps become hidden because quantum spaces are more "rigid" (they admit less automorphisms).

Acknowledgements. I am deeply grateful to my friends and colleagues R. Coquereaux and A. Isaev for inspiration and numerous valuable comments.

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