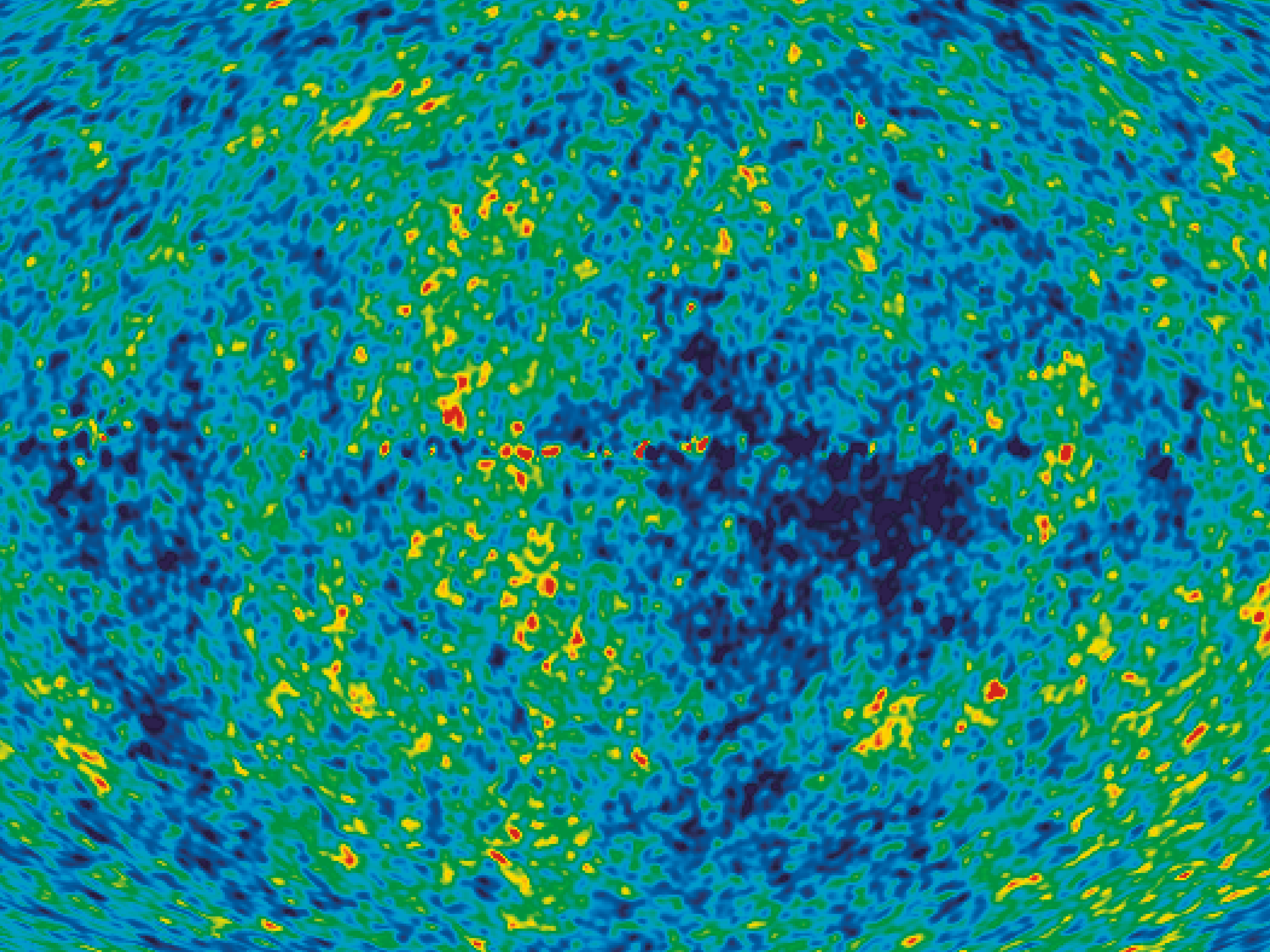
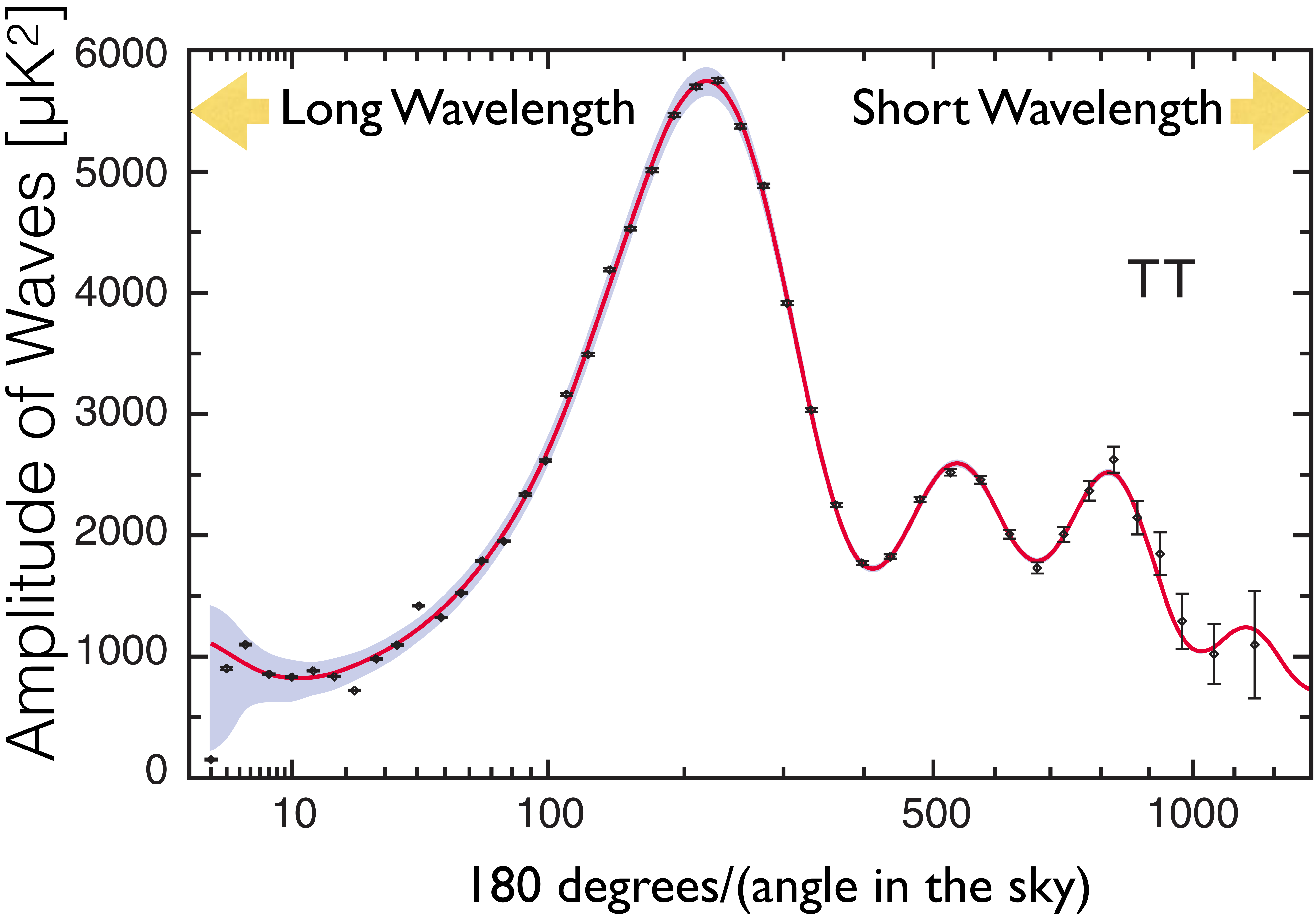
Lecture 2

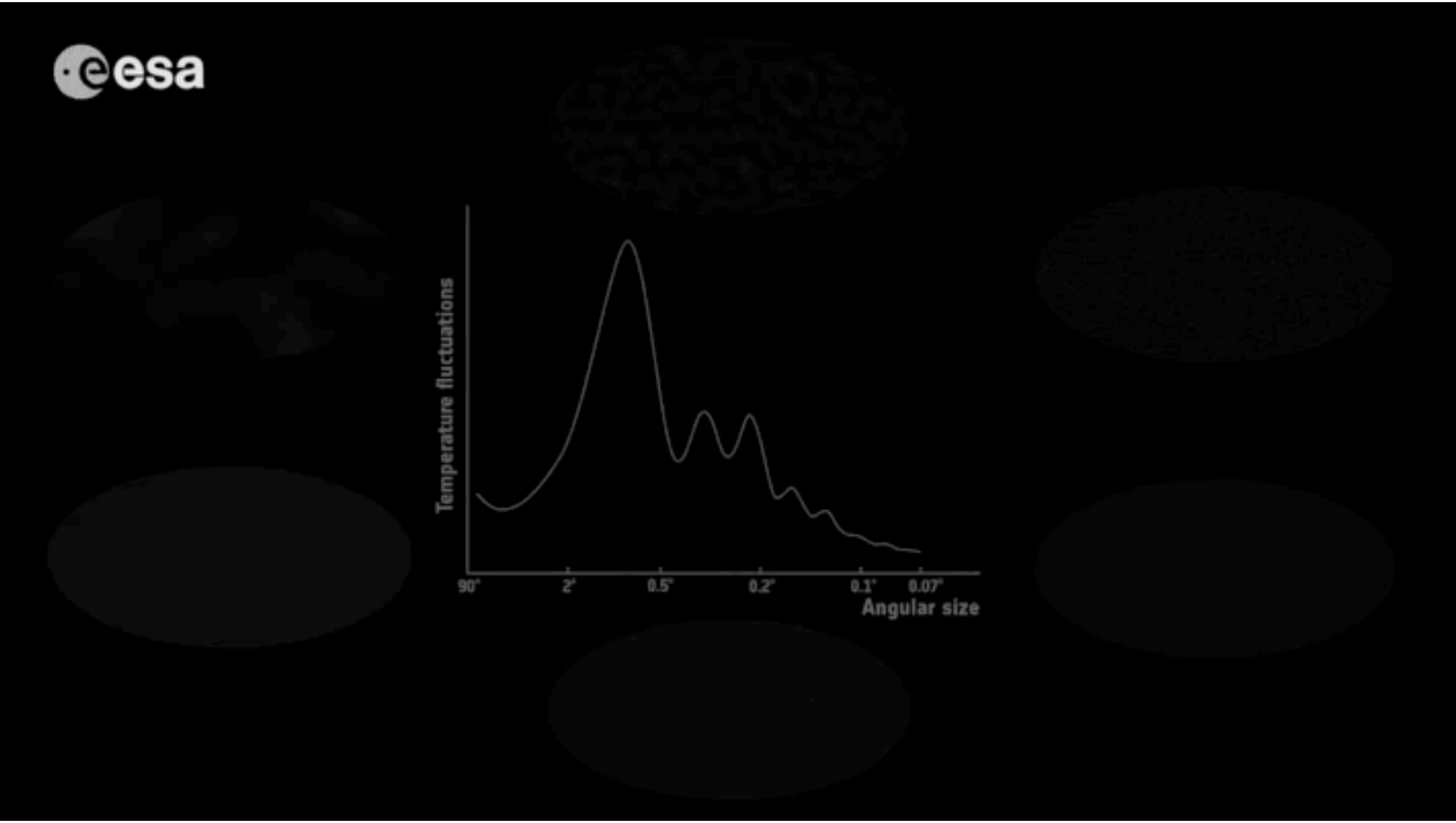
- Power spectrum
- Temperature anisotropy from sound waves

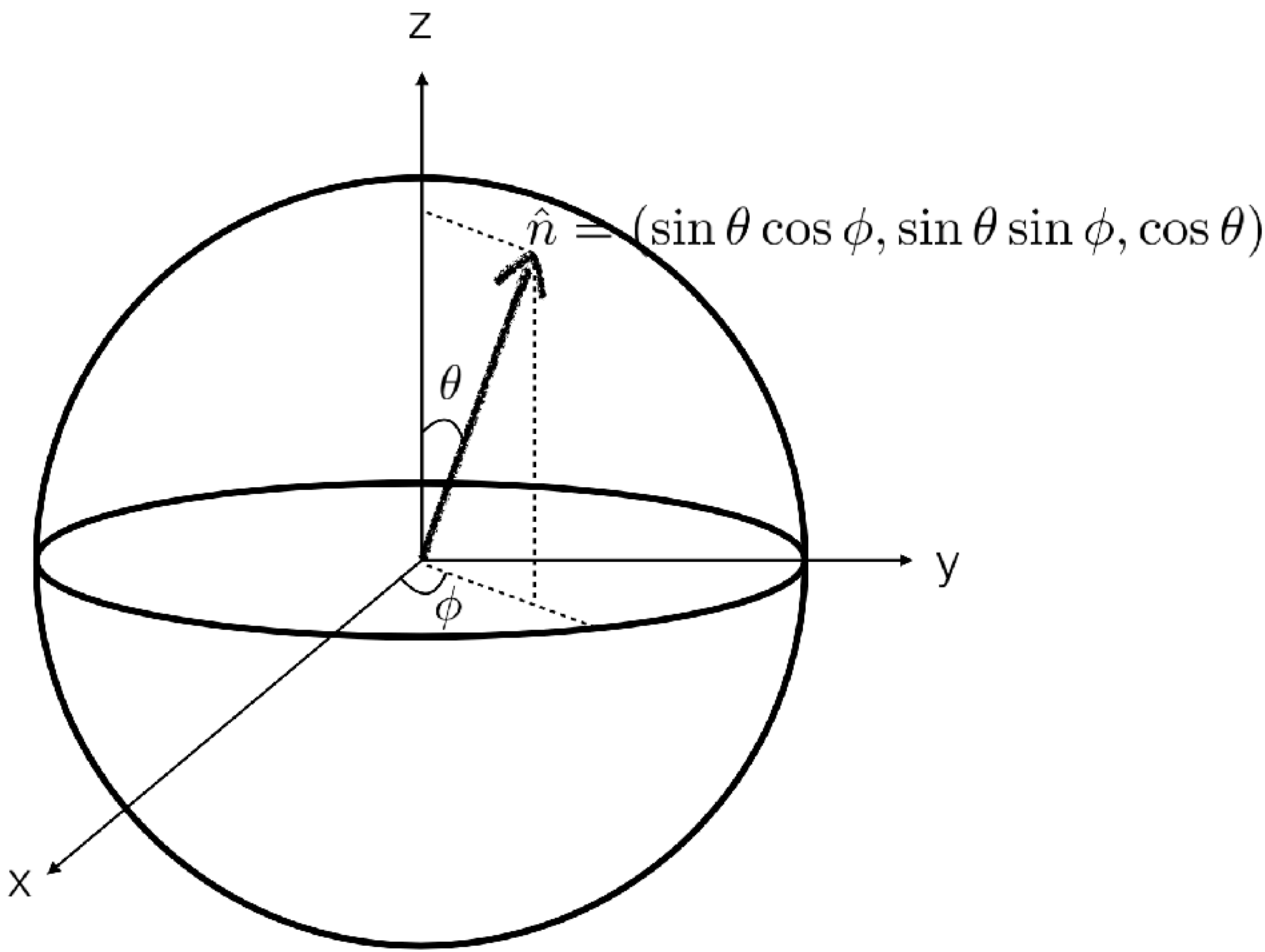


Data Analysis

- Decompose temperature fluctuations in the sky into a set of waves with various wavelengths
- Make a diagram showing the strength of each wavelength


Amplitude of Waves [μK²]
6000
5000
4000
3000
2000
1000
0
Long Wavelength
Short Wavelength
TT
10
100
500
1000
180 degrees/(angle in the sky)


Temperature fluctuations
90°
2°
0.5°
0.2°
0.1°
0.07°
Angular size

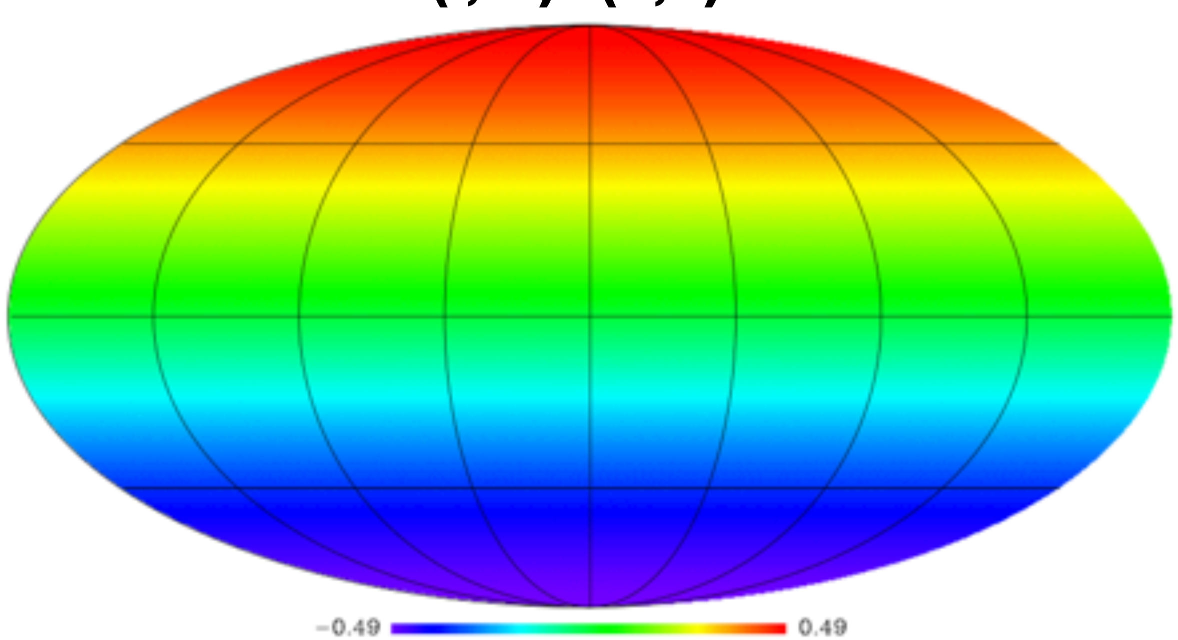

z
$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
θ
y
ϕ
x

Spherical Harmonic Transform

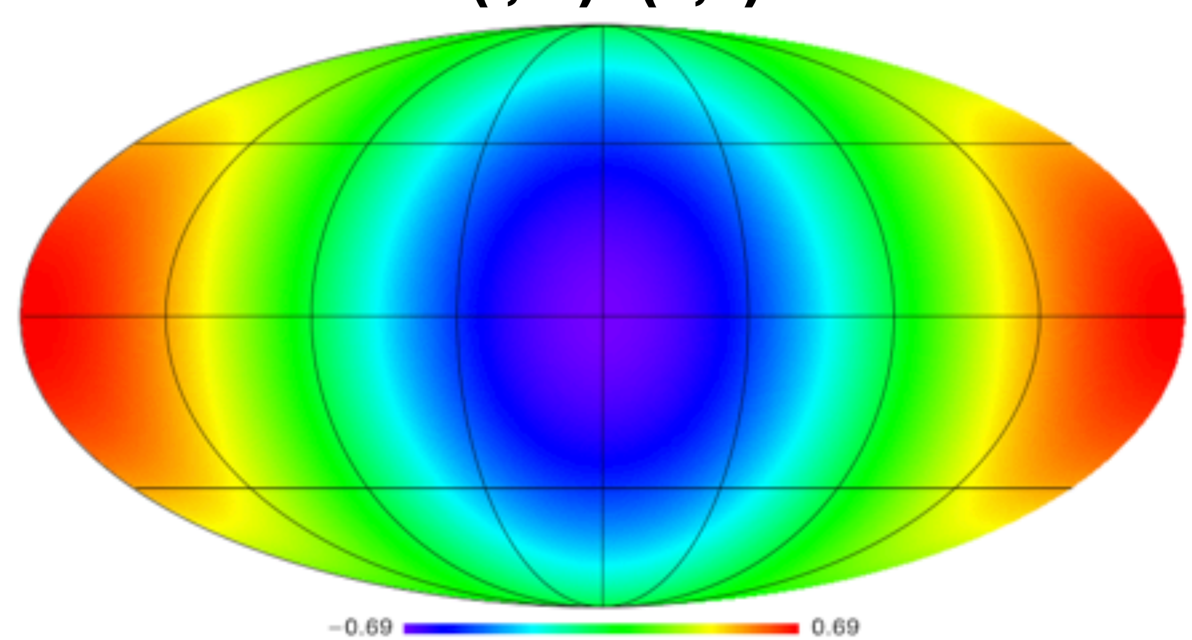
$$\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n})$$

- Values of a_{lm} depend on coordinates, but the squared amplitude, $\sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$, does not depend on coordinates

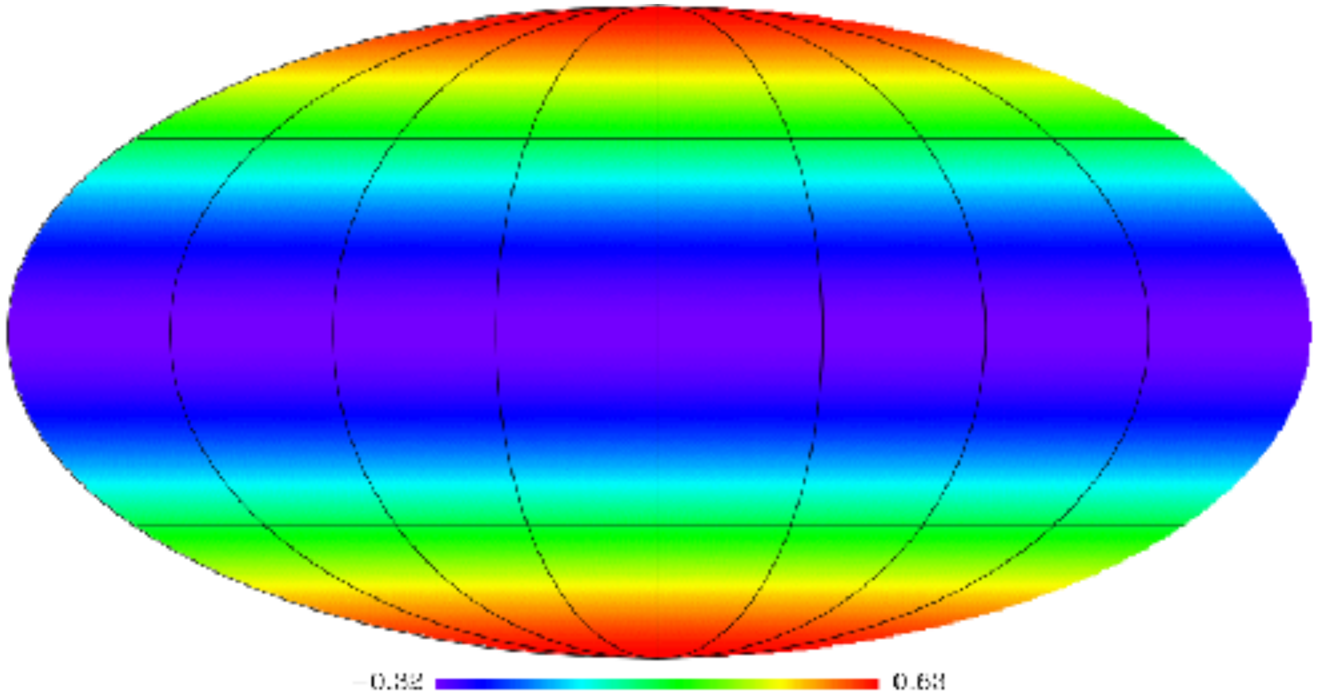
(l,m)=(1,0)



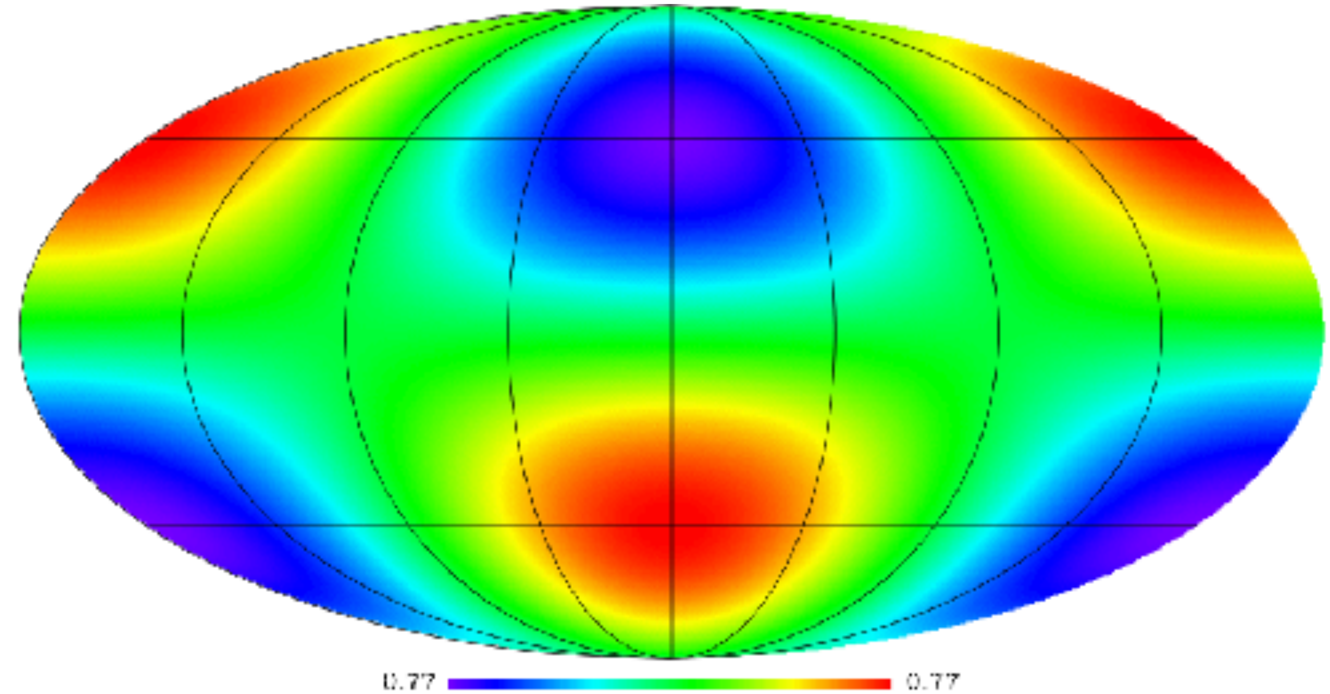
(l,m)=(1,1)



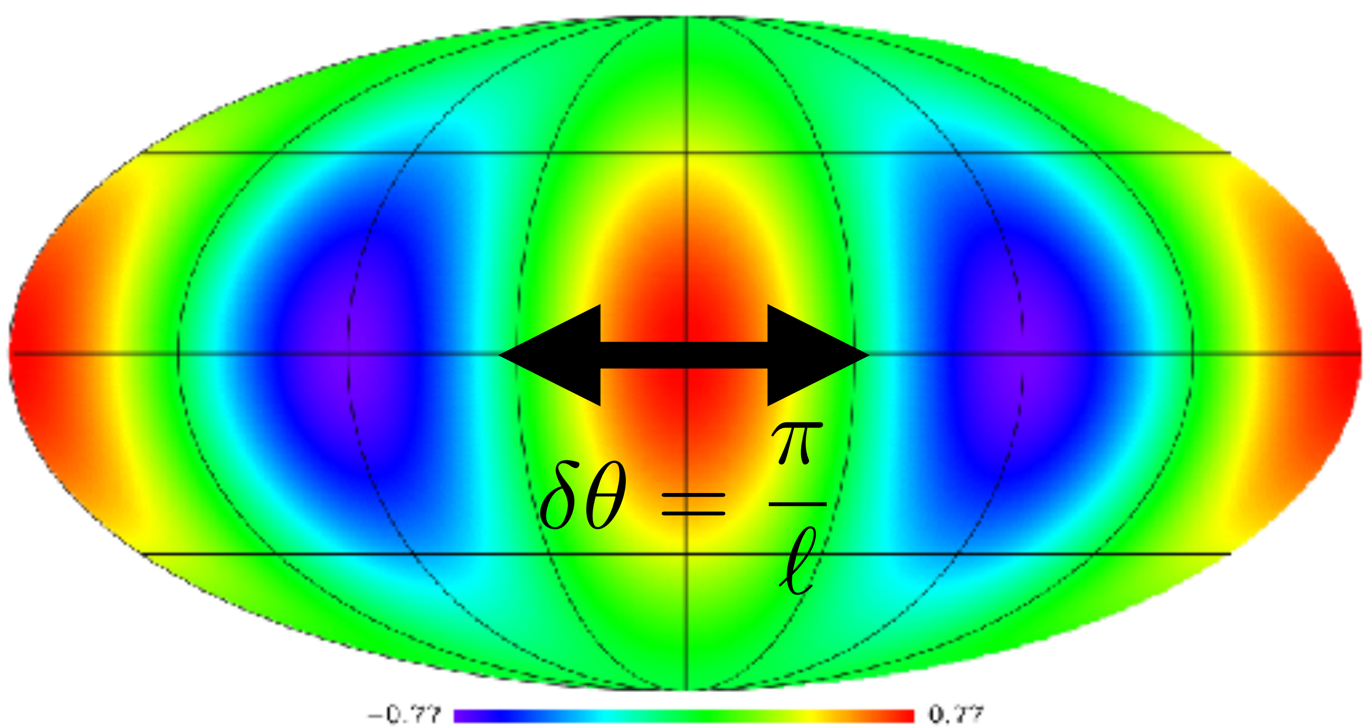
(l,m)=(2,0)



(l,m)=(2,1)

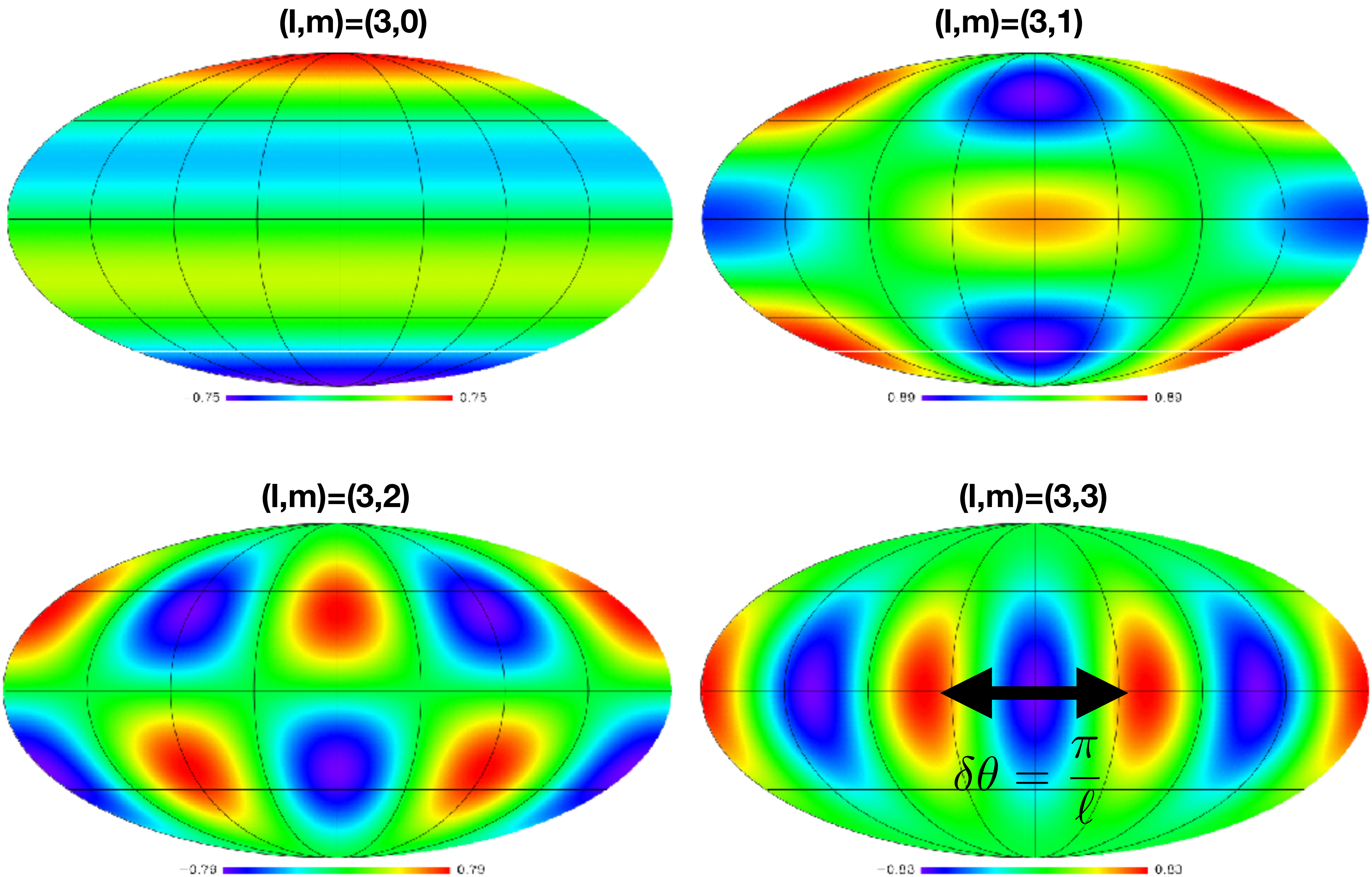


(l,m)=(2,2)



For l=m, a half-wavelength, $\lambda_\theta/2$, corresponds to π/l.

Therefore, $\lambda_\theta = 2\pi/l$


(l,m)=(3,0)
−0.75
0.75
(l,m)=(3,1)
0.89
0.89
(l,m)=(3,2)
−0.79
0.79
(l,m)=(3,3)
−0.83
0.83
$\delta\theta = \frac{\pi}{\ell}$

a_{lm} of the SW effect

- Using the inverse transform $a_{\ell m} = \int d\Omega \Delta T(\hat{n}) Y_\ell^{m*}(\hat{n})$

  on the Sachs-Wolfe (SW) formula $\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{r}_L)$

  and Fourier-transforming the potential, we obtain:

$$a_{\ell m}^{\rm SW} = \frac{T_0}{3} \int d\Omega \; Y_\ell^{m*}(\hat{n}) \int \frac{d^3 q}{(2\pi)^3} \; \Phi_{\boldsymbol{q}} \exp(i\boldsymbol{q} \cdot \hat{n} r_L)$$

***q is the 3d Fourier wavenumber**

The left hand side is the coefficients of <u>2d spherical waves</u>, whereas the right hand side is the coefficients of <u>3d plane waves</u>. How can we make the connection?

Spherical wave decomposition of a plane wave

$$\exp(i\boldsymbol{q}\cdot\hat{n}r_L) = 4\pi \sum_{\ell=0}^{\infty} i^{\ell} j_{\ell}(qr_L) \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(\hat{n}) Y_{\ell}^{m*}(\hat{q})$$

- This "partial-wave decomposition formula" (or Rayleigh's formula) then gives

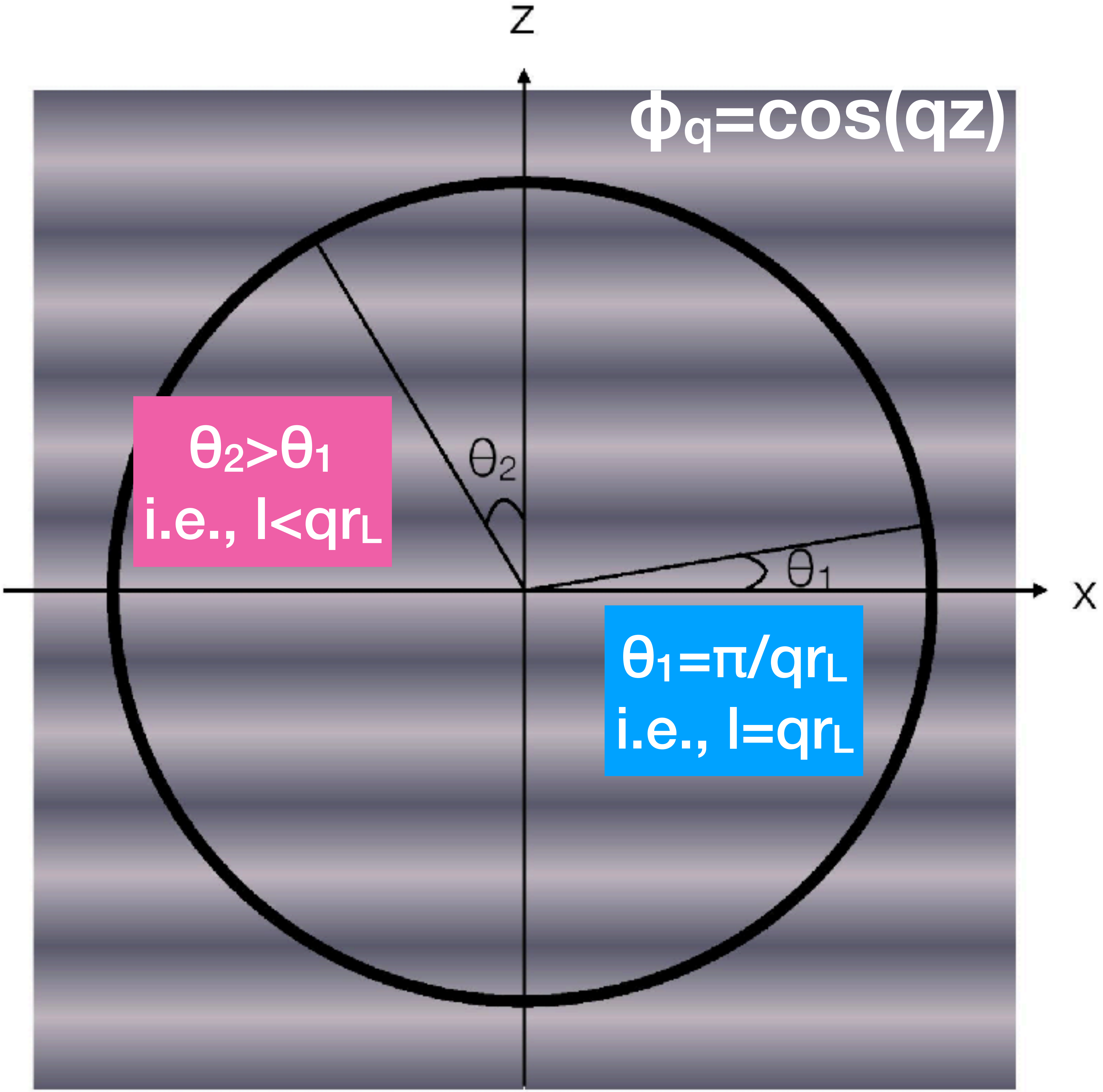
$$a_{\ell m}^{\mathrm{SW}} = \frac{4\pi T_0 i^{\ell}}{3} \int \frac{d^3q}{(2\pi)^3}\, \Phi_{\boldsymbol{q}} j_{\ell}(qr_L) Y_{\ell}^{m*}(\hat{q})$$

- This is the exact formula relating 3d potential at the last scattering surface onto a_{lm}. **How do we understand this?**

q -> l projection

$$a_{\ell m}^{\rm SW} = \frac{4\pi T_0 i^\ell}{3} \int \frac{d^3q}{(2\pi)^3}\, \Phi_q \boxed{j_\ell(qr_L)} Y_\ell^{m*}(\hat{q})$$

- A half wavelength, $\lambda/2$, at the last scattering surface subtends an angle of $\lambda/2r_L$. Since $q=2\pi/\lambda$, the angle is given by $\delta\theta=\pi/qr_L$. Comparing this with the relation $\delta\theta=\pi/l$ (for l=m), we obtain **$l=qr_L$**. How can we see this?
- For l>>1, the spherical Bessel function, **$j_l(qr_L)$, peaks at $l=qr_L$** and falls gradually toward $qr_L>l$. Thus, a given q mode contributes to large angular scales too.



z
x
ϕq=cos(qz)
θ2
θ1
θ2>θ1
i.e., l<qrL
θ1=π/qrL
i.e., l=qrL

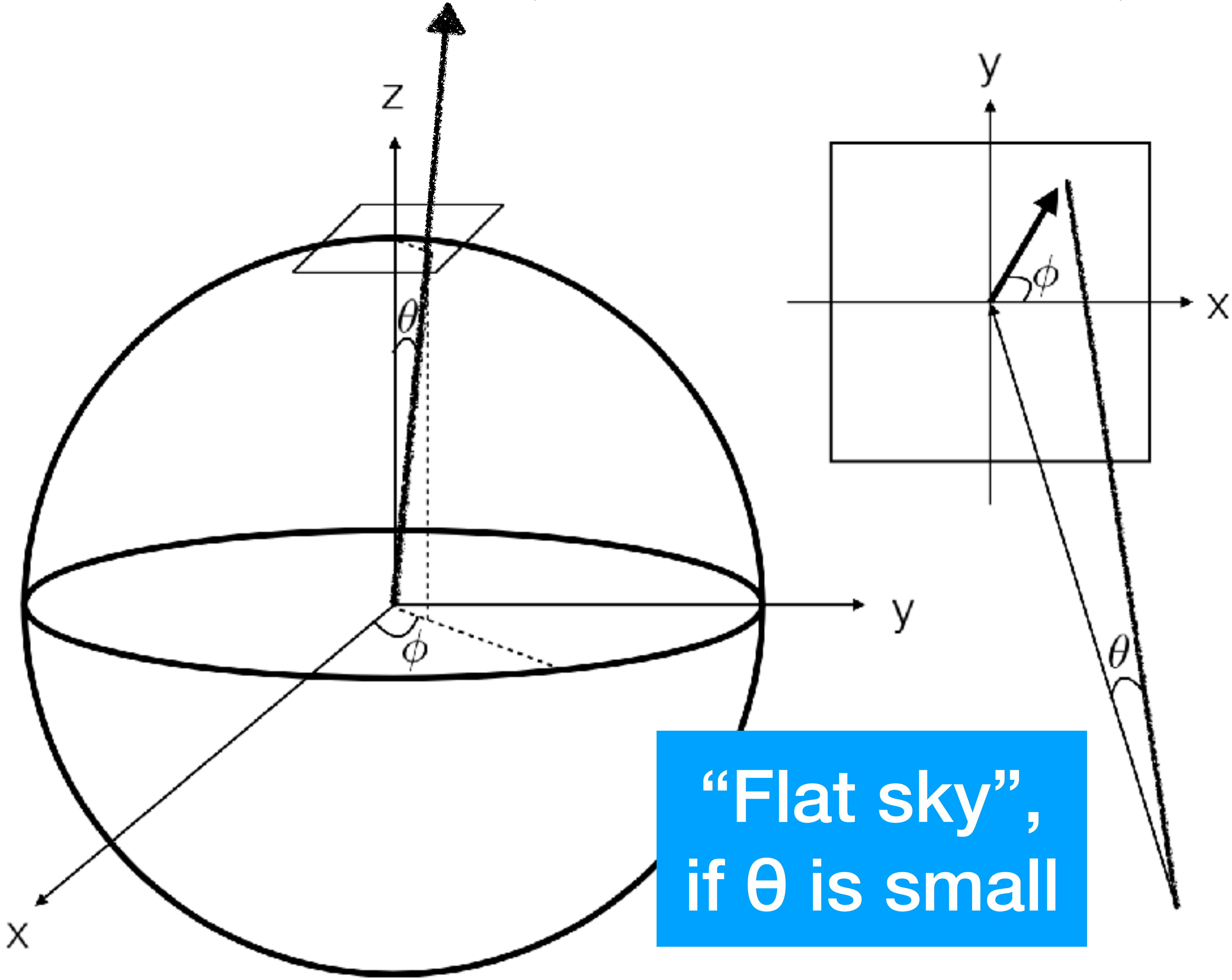
More intuitive approach: Flay-sky Approximation

- Not all of us are familiar with spherical bessel functions…
  - The fundamental complication here is that we are trying to relate a 3d plane wave with a spherical wave.
  - More intuitive approach would be to relate a 3d plane wave with **a 2d plane wave**

Decomposition

- **Full sky**
  - Decompose temperature fluctuations using **spherical harmonics**
- **Flat sky**
  - Decompose temperature fluctuations using **Fourier transform**
- The former approaches the latter in the small-angle limit

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



“Flat sky”,
if θ is small

2d Fourier Transform

$$\Delta T(\hat{n}) = \int \frac{d^2\ell}{(2\pi)^2}\, a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

$$= \int_0^\infty \frac{\ell d\ell}{2\pi} \int_0^{2\pi} \frac{d\phi_\ell}{2\pi}\, a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

C.f.,

$$\left(\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n}) \right)$$

a(l) of the SW effect

- Using the inverse 2d Fourier transform on the Sachs-Wolfe (SW) formula

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{r}_L)$$

and Fourier-transforming the potential, we obtain:

$$a_{\boldsymbol{\ell}}^{\mathrm{SW}} = \frac{T_0}{3}\int d^2\theta\ \exp(-i\boldsymbol{\ell}\cdot\boldsymbol{\theta}) \times \int \frac{d^3q}{(2\pi)^3}\,\Phi_{\boldsymbol{q}}\ \exp(i\boldsymbol{q}_\perp r_L\cdot\boldsymbol{\theta} + iq_{\|} r_L \boxed{\cos\theta})$$

$\cos\theta \to 1$ **flat-sky approx.**

Flat-sky Result

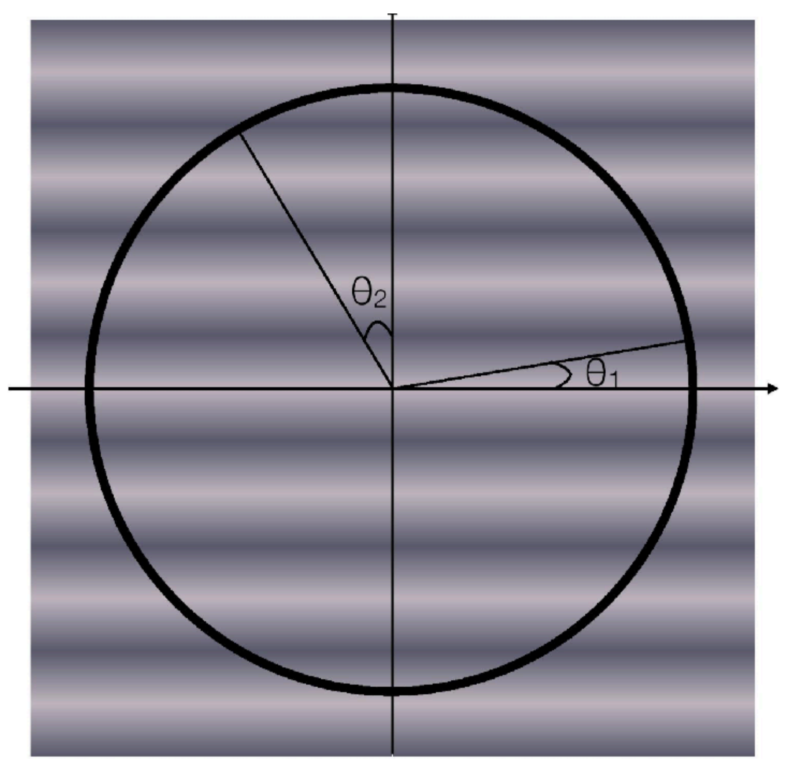
$$a_{\boldsymbol{\ell}}^{\mathrm{SW}} = \frac{T_0}{3r_L^2}\int_{-\infty}^{\infty}\frac{dq_{\|}}{2\pi}\,\Phi_{\boldsymbol{q}}\left(\boldsymbol{q}_{\perp} = \frac{\boldsymbol{\ell}}{r_L}, q_{\|}\right)\exp(iq_{\|}r_L)$$

$q = \sqrt{\ell^2/r_L^2 + q_{\|}^2}$ **i.e.,** $q \geq \ell/r_L$

C.f.,

$$\left(a_{\ell m}^{\mathrm{SW}} = \frac{4\pi T_0 i^{\ell}}{3}\int\frac{d^3q}{(2\pi)^3}\,\Phi_{\boldsymbol{q}}j_{\ell}(qr_L)Y_{\ell}^{m*}(\hat{q}) \right)$$

- It is **now manifest** that only the perpendicular wavenumber contributes to l, i.e., **$l=q_{perp}r_L$**, giving $l<qr_L$



Angular Power Spectrum

- The angular power spectrum, C_l, quantifies how much correlation power we have at a given angular separation.

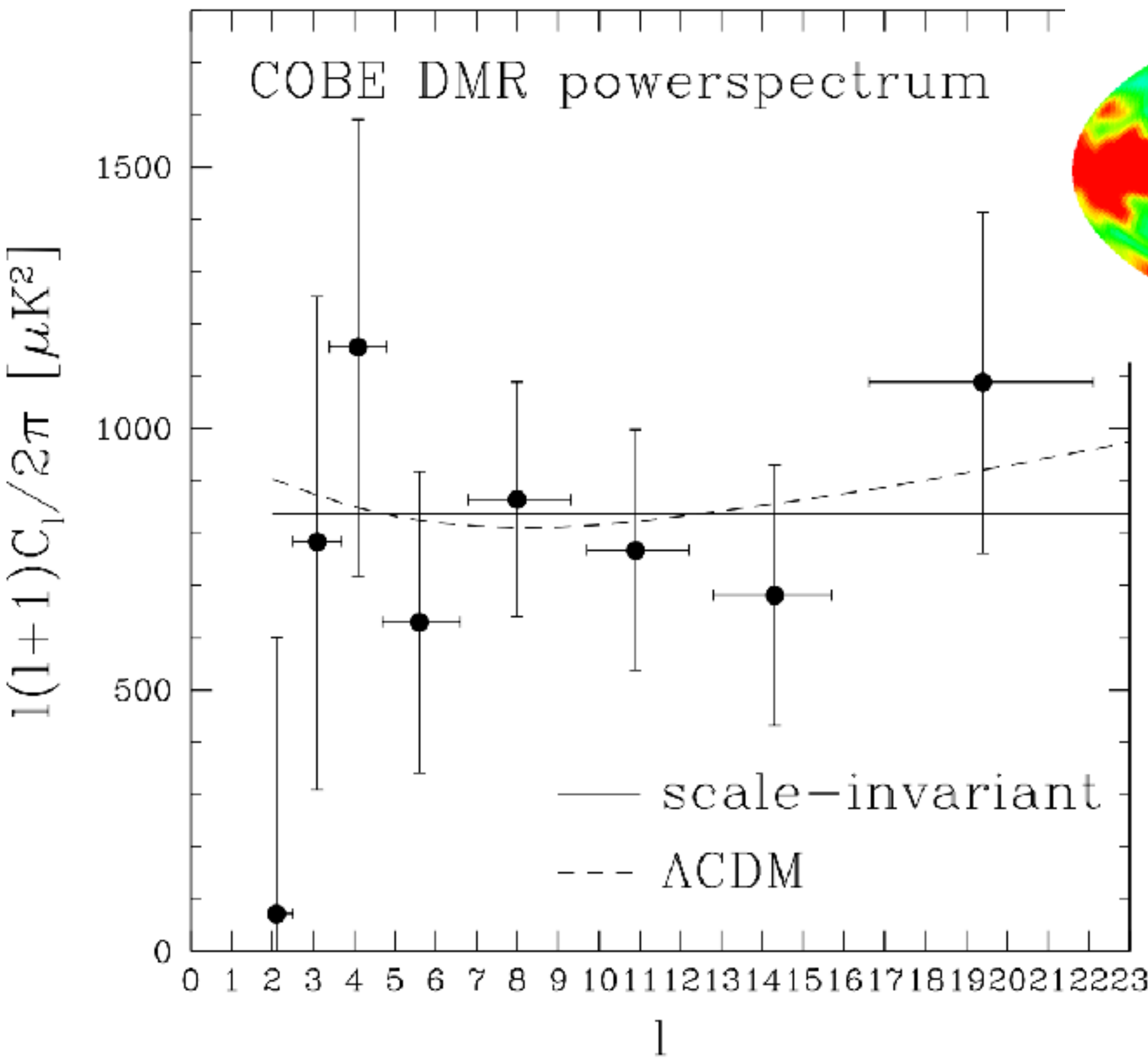
$$C_\ell \equiv \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} a_{\ell m}a^*_{\ell m}$$

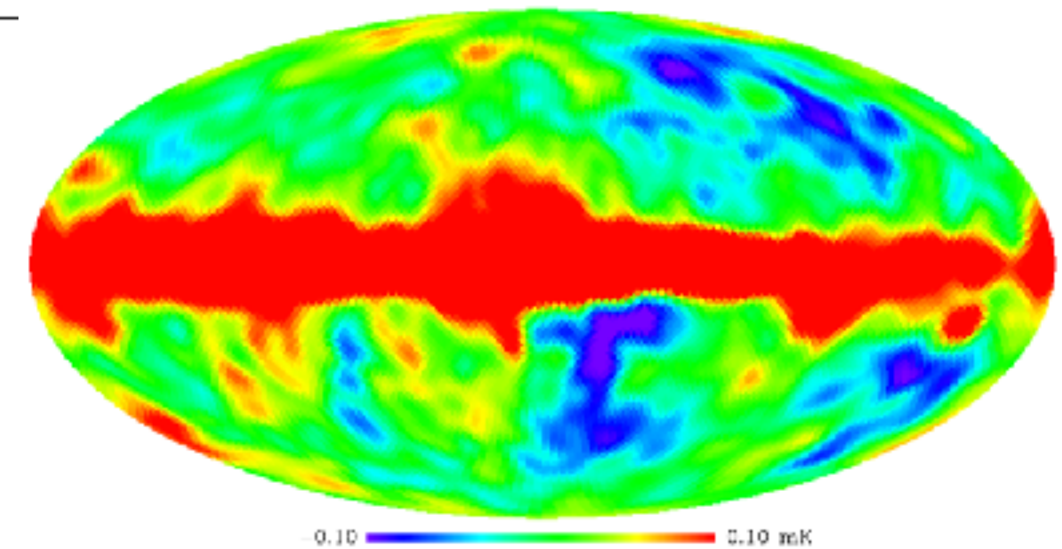
- More precisely: it is **$l(2l+1)C_l/4\pi$** that gives the fluctuation power at a given angular separation, ~π/l. We can see this by computing **variance**:

$$\int \frac{d\Omega}{4\pi}\Delta T^2(\hat{n}) = \frac{1}{4\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}a^*_{\ell m} = \sum_{\ell=2}^{\infty}\frac{2\ell+1}{4\pi}C_\ell$$

Bennett et al. (1996)

COBE 4-year Power Spectrum





The SW formula allows us to determine the **3d power spectrum of ϕ** at the last scattering surface from C_l.

But how?

SW Power Spectrum

$$a_{\ell m}^{\rm SW} = \frac{4\pi T_0 i^\ell}{3} \int \frac{d^3q}{(2\pi)^3}\, \Phi_{\boldsymbol{q}} j_\ell(q r_L) Y_\ell^{m*}(\hat{q})$$

$$C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$$

gives…

$$C_{\ell,\rm SW} = \frac{4\pi T_0^2}{9} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3}\, \Phi_{\boldsymbol{q}} \Phi_{\boldsymbol{q}'}^* j_\ell(q r_L) j_\ell(q' r_L) P_\ell(\hat{q}\cdot\hat{q}')$$

- But this is not exactly what we want. We want the **statistical average** of this quantity.

Power Spectrum of ϕ

- Statistical average of the right hand side contains

$$\langle \Phi_{\boldsymbol{q}} \Phi^*_{\boldsymbol{q}'} \rangle = \int d^3x \int d^3r \, \langle \Phi(\boldsymbol{x}) \Phi(\boldsymbol{x}+\boldsymbol{r}) \rangle \exp\left[i(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{x} - i\boldsymbol{q}'\cdot\boldsymbol{r} \right]$$

two-point correlation function

If $\langle \Phi(\boldsymbol{x}) \Phi(\boldsymbol{x}+\boldsymbol{r}) \rangle$ **does not depend on locations (x) but only on separations between two points (r), then**

$$\langle \Phi_{\boldsymbol{q}} \Phi^*_{\boldsymbol{q}'} \rangle = (2\pi)^3 \delta_D^{(3)}(\boldsymbol{q}-\boldsymbol{q}') \int d^3r \, \xi_\phi(\boldsymbol{r}) \exp(-i\boldsymbol{q}\cdot\boldsymbol{r})$$

consequence of "statistical homogeneity"

where we defined $\xi_\phi(\boldsymbol{r}) \equiv \langle \Phi(\boldsymbol{x}) \Phi(\boldsymbol{x}+\boldsymbol{r}) \rangle$

and used $\int d^3x \; \exp(i\boldsymbol{q}\cdot\boldsymbol{x}) = (2\pi)^3 \delta_D^{(3)}(\boldsymbol{q})$

Power Spectrum of ϕ

- In addition, if $\xi_\phi(\boldsymbol{r}) \equiv \langle \Phi(\boldsymbol{x})\Phi(\boldsymbol{x}+\boldsymbol{r})\rangle$ depends only on the magnitude of the separation r and not on the directions, then

$$\langle \Phi_{\boldsymbol{q}}\Phi^*_{\boldsymbol{q}'}\rangle = (2\pi)^3\delta_D^{(3)}(\boldsymbol{q}-\boldsymbol{q}')\int 4\pi r^2 dr\ \xi_\phi(r)\frac{\sin(qr)}{qr}$$

$$= (2\pi)^3\delta_D^{(3)}(\boldsymbol{q}-\boldsymbol{q}')\boxed{P_\phi(q)}$$

Power spectrum!

Generic definition of the power spectrum for statistically homogeneous and isotropic fluctuations

SW Power Spectrum

- Thus, the power spectrum of the CMB in the SW limit is

$$\langle C_{\ell,\mathrm{sw}} \rangle = \frac{16\pi^2 T_0^2}{9} \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \, P_\phi(q) j_\ell^2(q r_L)$$

- In the flat-sky approximation,

$$\langle C_{\ell,\mathrm{sw}} \rangle = \frac{T_0^2}{9 r_L^2} \int_{-\infty}^\infty \frac{dq_\parallel}{2\pi} \, P_\phi \left(\sqrt{\frac{\ell^2}{r_L^2} + q_\parallel^2} \right)$$

SW Power Spectrum

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For a power-law form, $P_\phi(q) = (2\pi)^3 N_\phi^2 q^{n-4}$**, we get**

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{8\pi^2 N_\phi^2 T_0^2}{9\ell^2} \left(\frac{\ell}{r_L}\right)^{n-1} \frac{\sqrt{\pi}}{2} \frac{\Gamma[(3-n)/2]}{\Gamma[(4-n)/2]}$$

SW Power Spectrum

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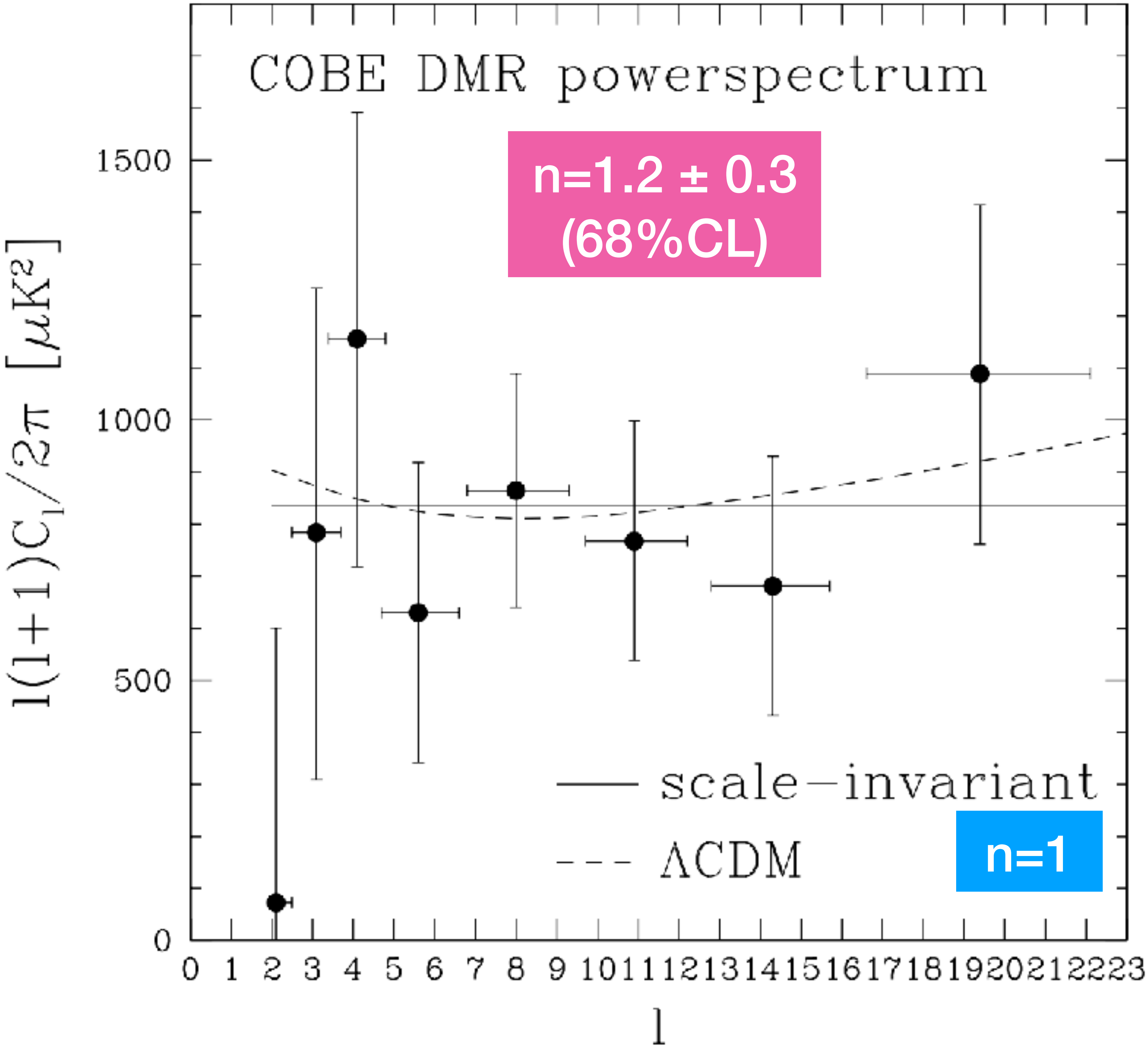
n=1 → $$\frac{8\pi^2 N_\phi^2 T_0^2}{9\ell(\ell+1)}$$

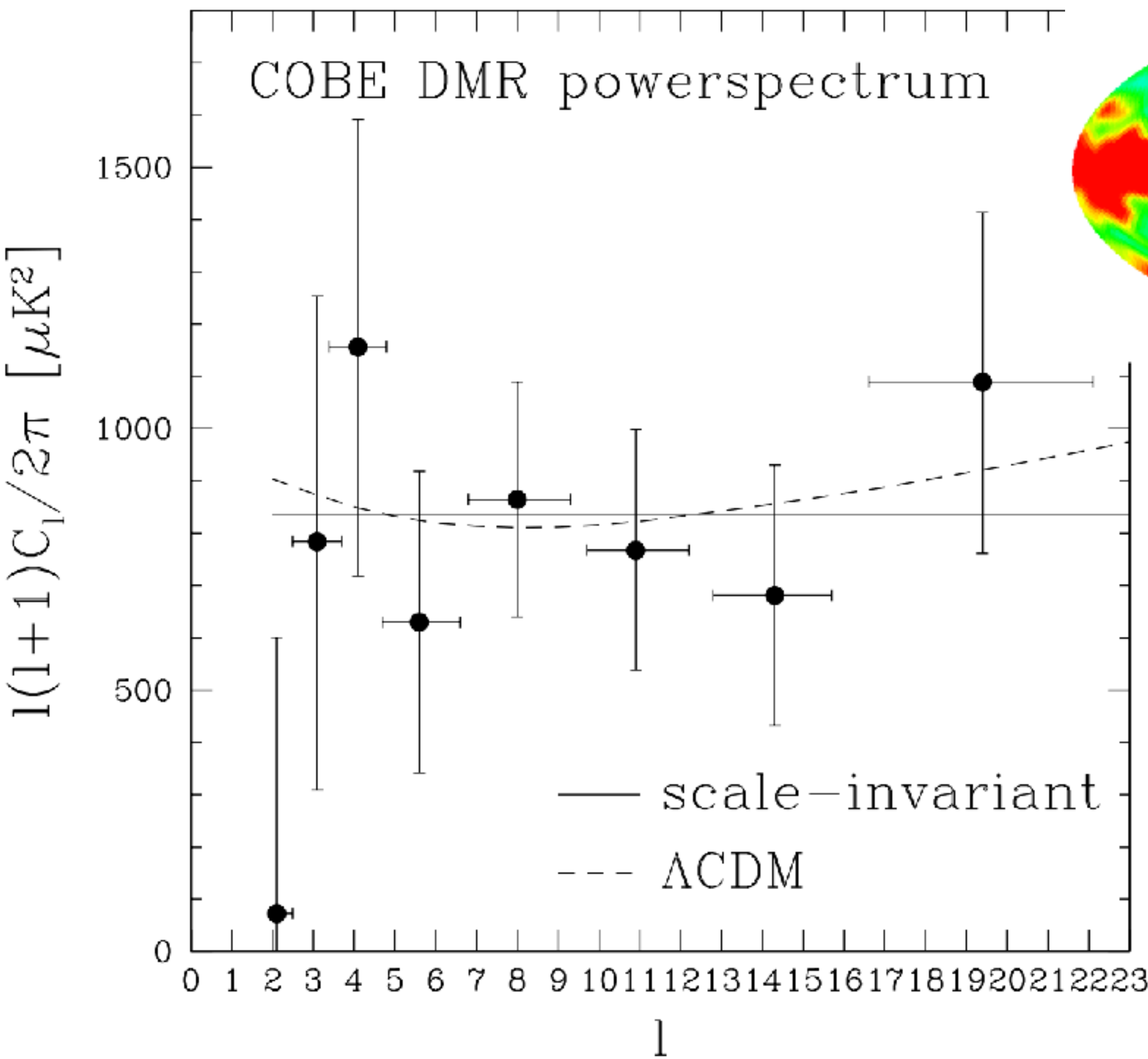
full-sky correction

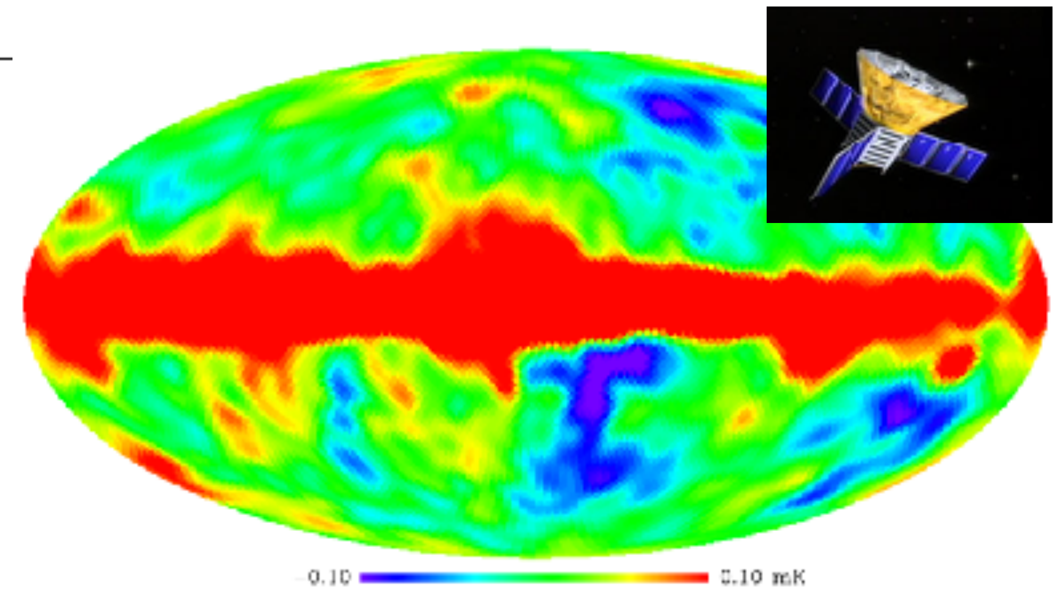
Bennett et al. (1996)



Bennett et al. (1996)

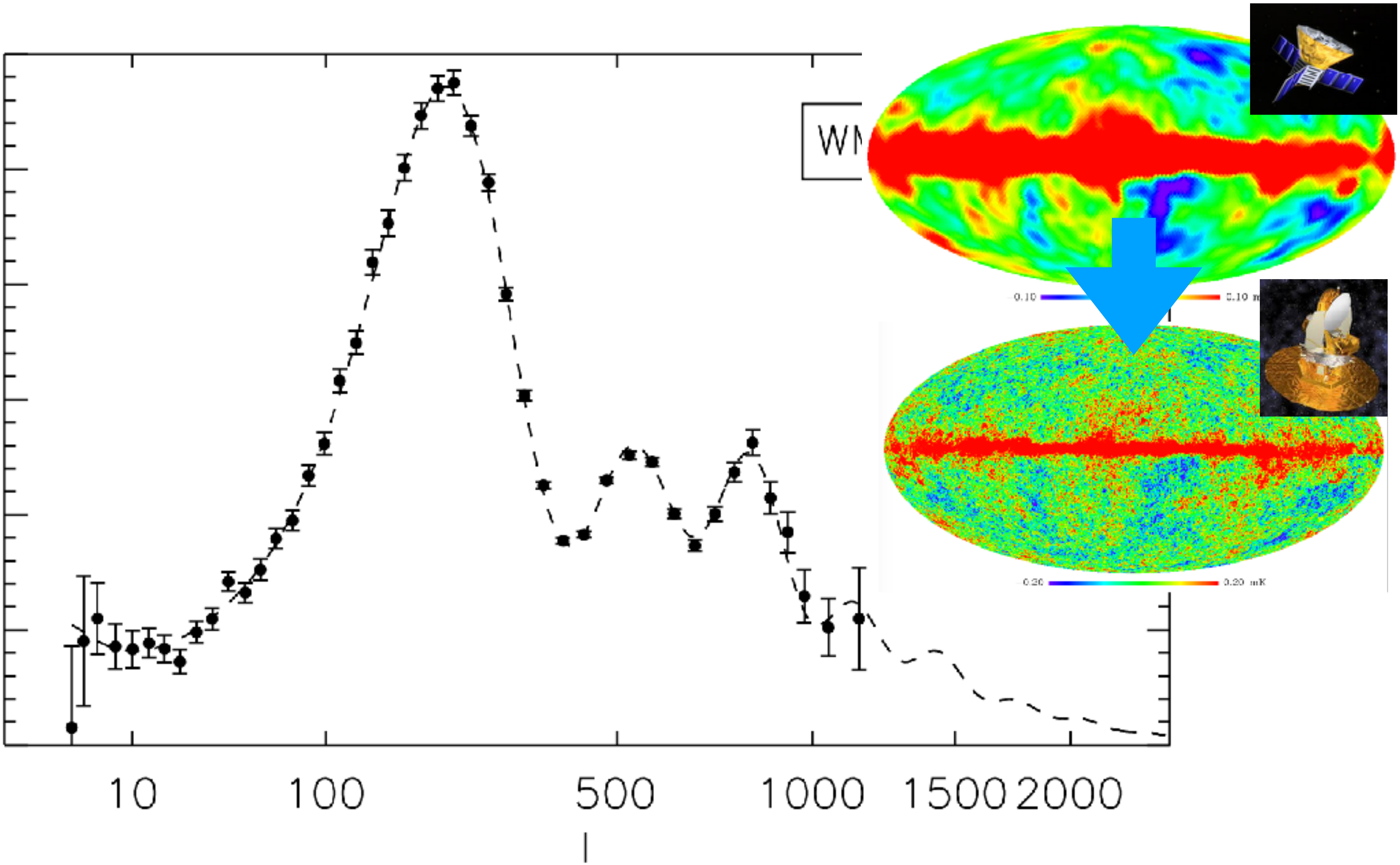
COBE 4-year Power Spectrum





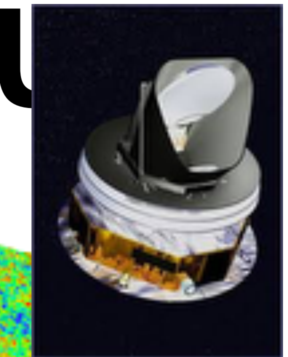
Bennett et al. (2013)

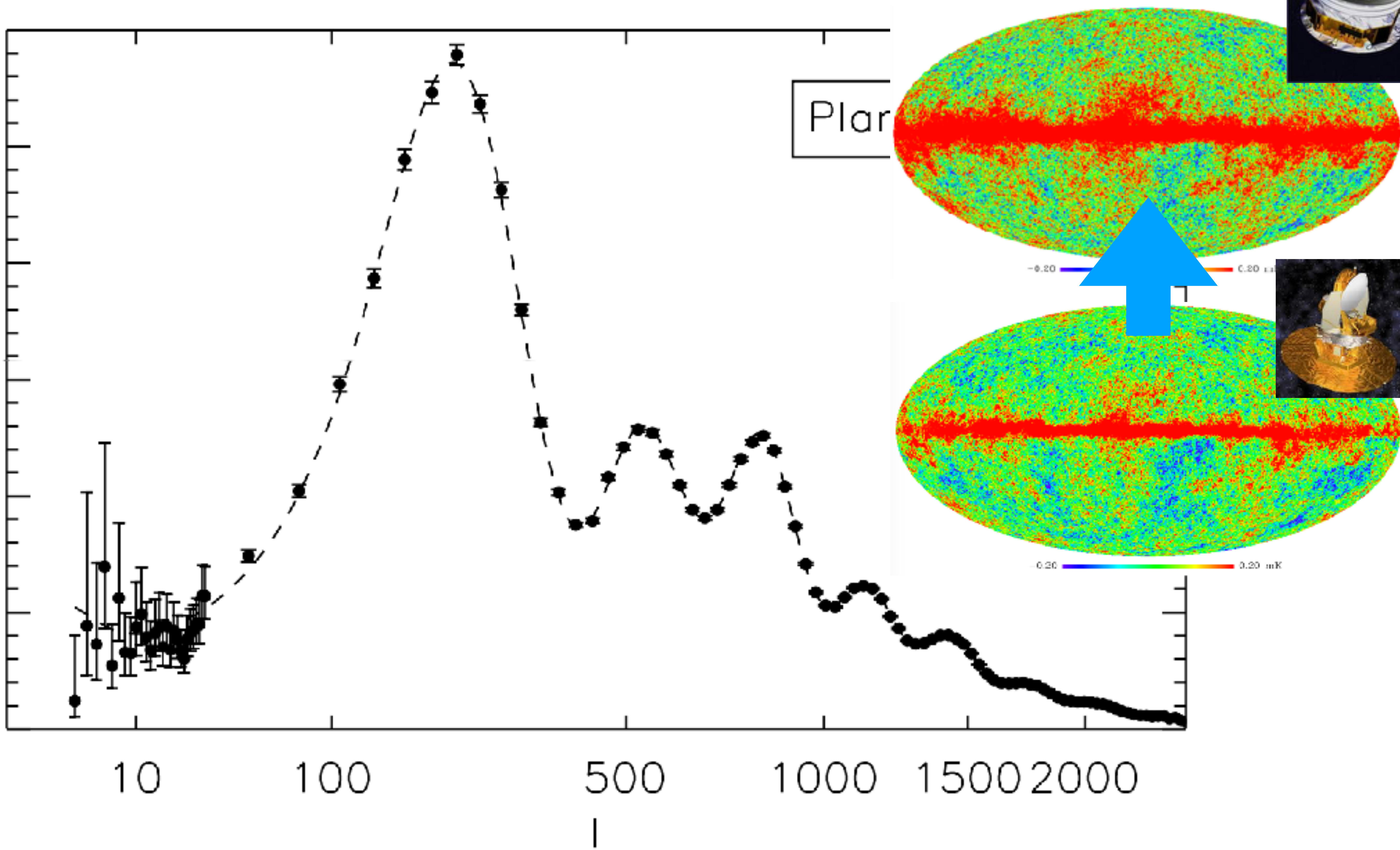
WMAP 9-year Power Spectrum



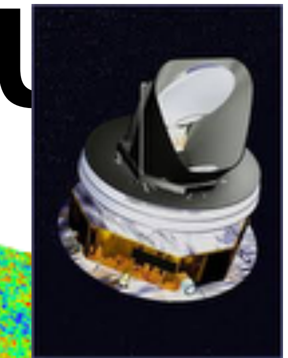
Planck Collaboration (2016)

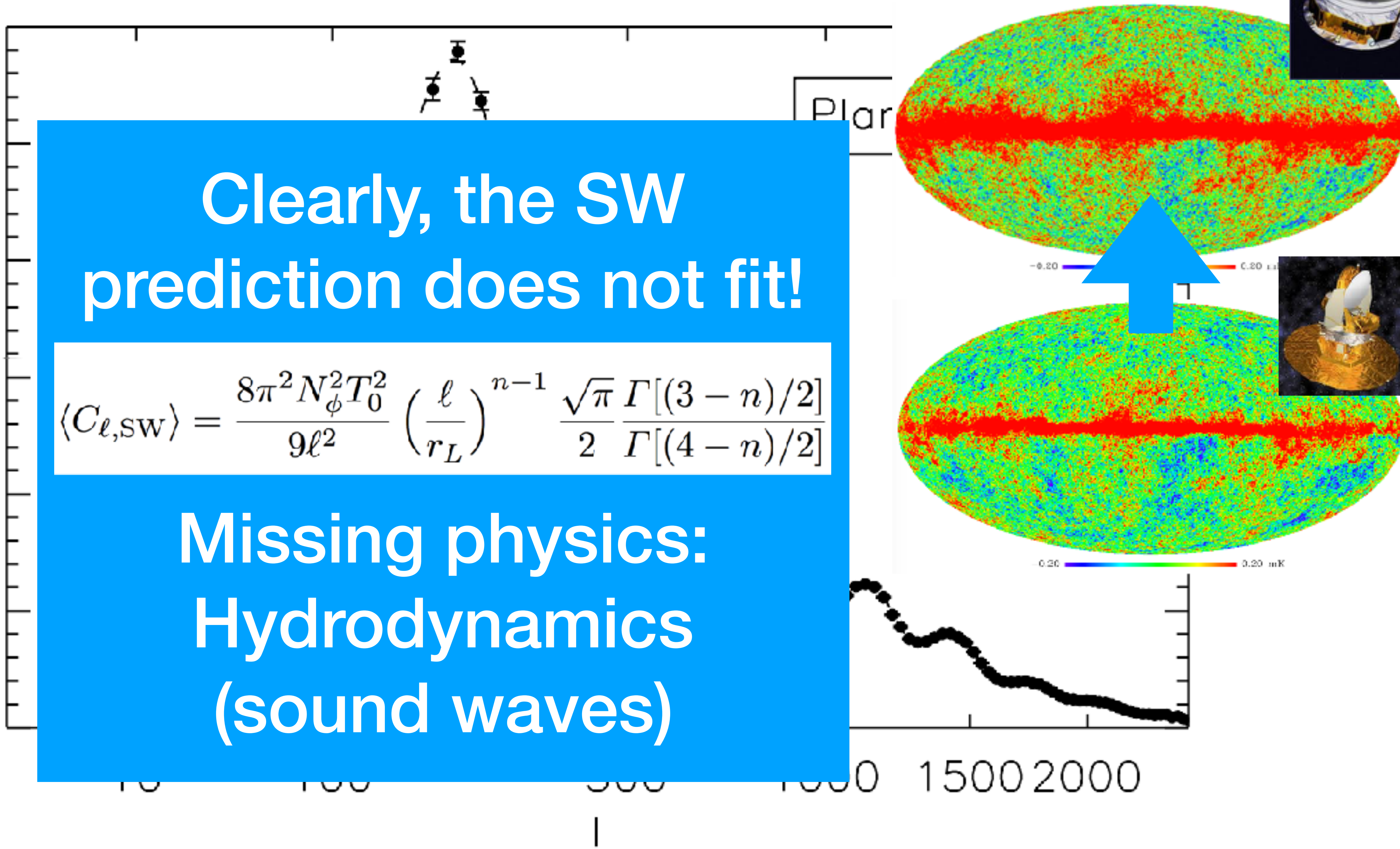
Planck 29-mo Power Spectru





Planck 29-mo Power Spectrum





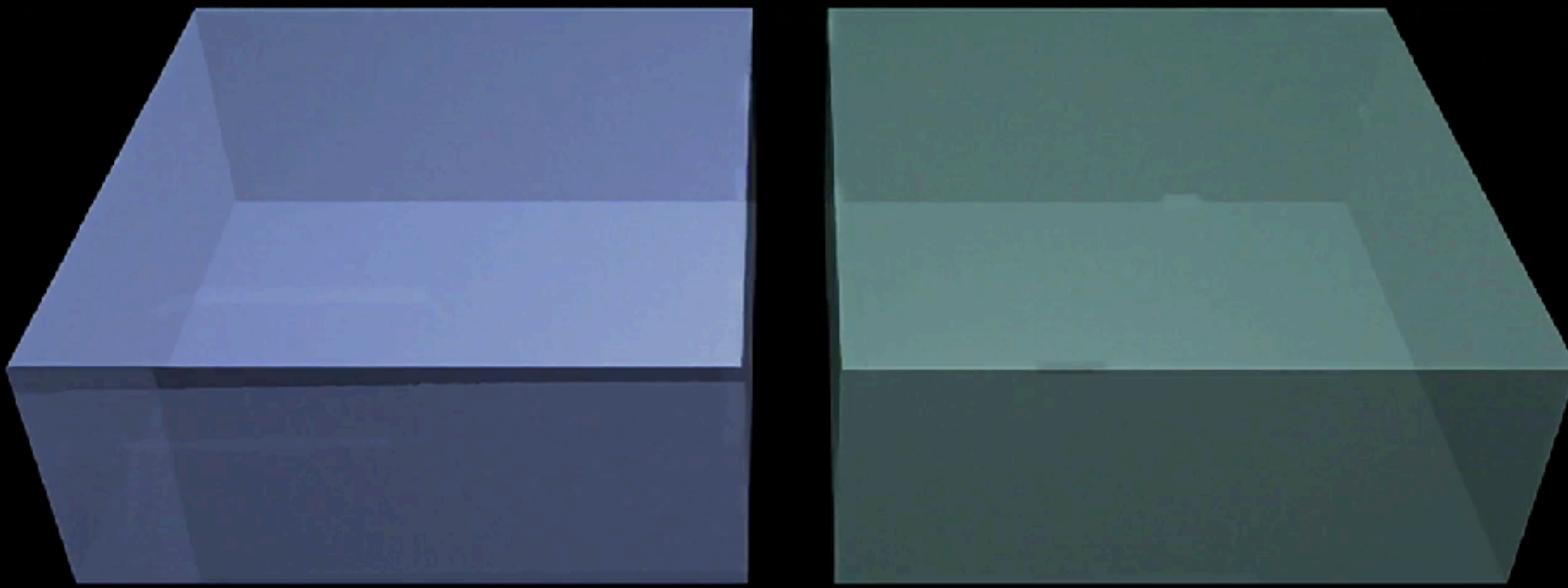
Clearly, the SW prediction does not fit!

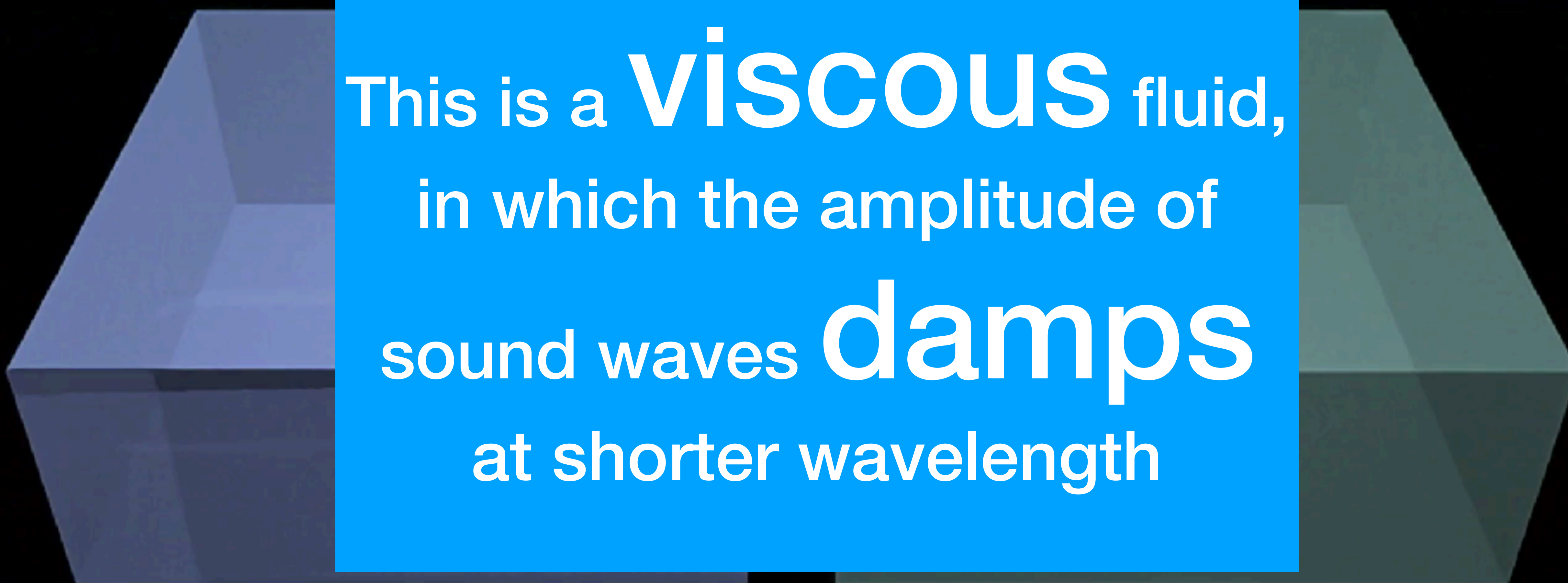
$$\langle C_{\ell,\mathrm{SW}}\rangle = \frac{8\pi^2 N_\phi^2 T_0^2}{9\ell^2}\left(\frac{\ell}{r_L}\right)^{n-1}\frac{\sqrt{\pi}}{2}\frac{\Gamma[(3-n)/2]}{\Gamma[(4-n)/2]}$$

Missing physics: Hydrodynamics (sound waves)

Cosmic Miso Soup

- When matter and radiation were hotter than 3000 K, matter was completely ionised. The Universe was filled with plasma, which behaves just like a soup
- Think about a Miso soup (if you know what it is). Imagine throwing Tofus into a Miso soup, while changing the density of Miso
- And imagine watching how ripples are created and propagate throughout the soup



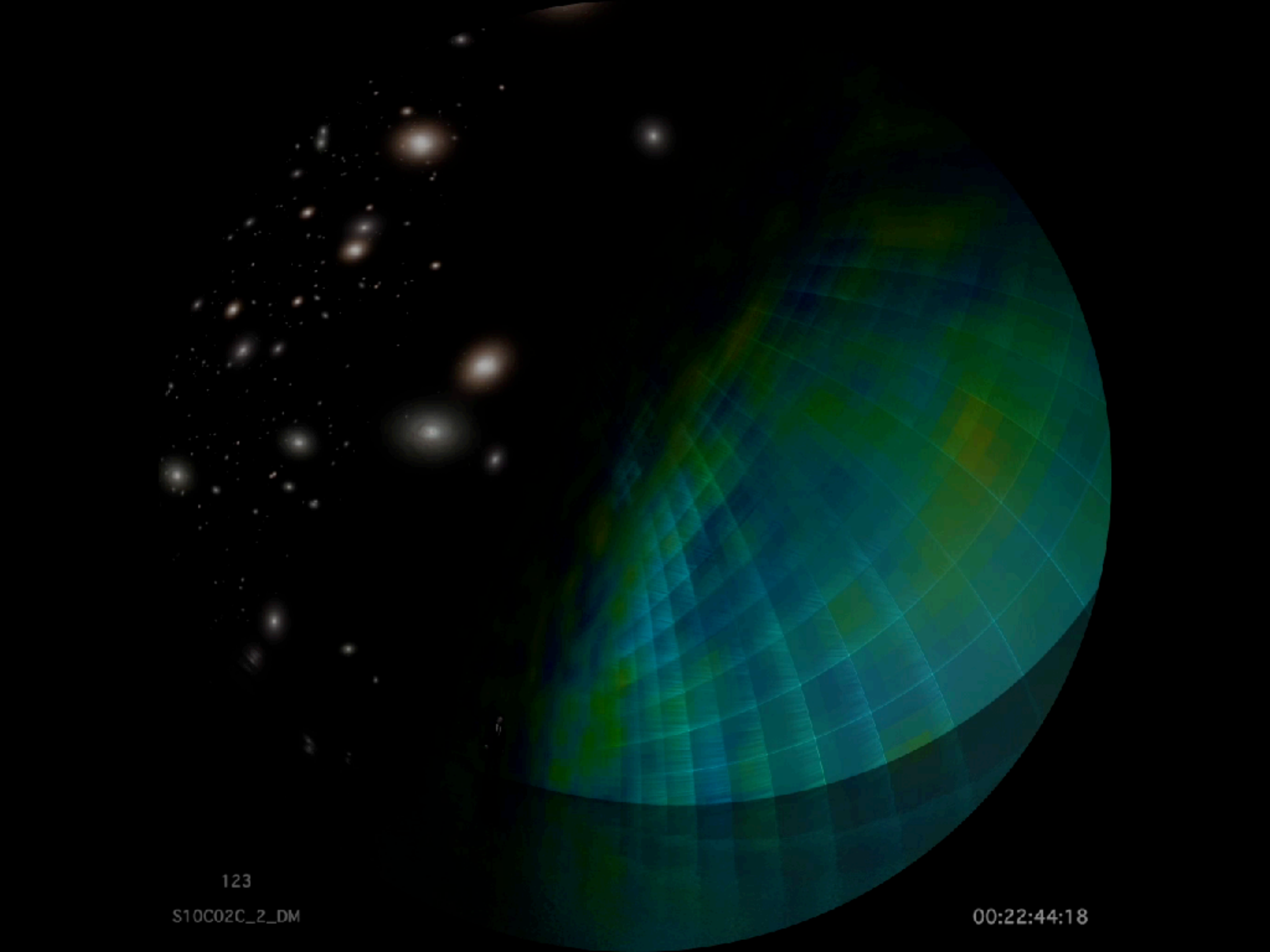


This is a **viscous** fluid,

in which the amplitude of

sound waves **damps**

at shorter wavelength


123
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When do sound waves become important?

- In other words, when would the Sachs-Wolfe approximation (purely gravitational effects) become invalid?
- The key to the answer: **Sound-crossing Time**
- Sound waves cannot alter temperature anisotropy at a given angular scale if there was not enough time for sound waves to propagate to the corresponding distance at the last-scattering surface
    - The distance traveled by sound waves within a given time = **The Sound Horizon**

Comoving Photon Horizon

- First, the comoving distance traveled by photons is given by setting the space-time distance to be null:

$$ds^2 = -c^2 dt^2 + a^2(t) dr^2 = 0$$

$$\Rightarrow \quad r_{\text{photon}} = c \int_0^t \frac{dt'}{a(t')}$$

Comoving **Sound** Horizon

- Then, we replace the speed of light with a time-dependent speed of sound:

$$r_s = \int_0^t \frac{dt'}{a(t')} c_s(t')$$

- We cannot ignore the effects of sound waves if **$qr_s > 1$**

Sound Speed

- Sound speed of an adiabatic fluid is given by

$$c_s^2 = \delta P / \delta \rho$$

- **δP: pressure perturbation**
- **$\delta \rho$: density perturbation**

- For a baryon-photon system:

$$c_s^2 = \delta P_\gamma / (\delta \rho_\gamma + \delta \rho_B)$$

We can ignore the baryon pressure because it is much smaller than the photon pressure

Sound Speed

- Using the adiabatic relationship between photons and baryons:

$$\delta\rho_B/\bar{\rho}_B = \delta\rho_\gamma/(\bar{\rho}_\gamma + \bar{P}_\gamma) = 3\delta\rho_\gamma/4\bar{\rho}_\gamma$$

[i.e., the ratio of the number densities of baryons and photons is equal everywhere]

- and pressure-density relation of a relativistic fluid, **$\delta P_\gamma = \delta\rho_\gamma/3$**, We obtain

$$c_s^2 = \delta P_\gamma/(\delta\rho_\gamma + \delta\rho_B) = 1/3(1 + 3\bar{\rho}_B/4\bar{\rho}_\gamma)$$

- Or equivalently

$$c_s = \frac{1}{\sqrt{3(1+R)}}$$

sound speed is reduced!

where

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Value of R?

- The baryon mass density goes like a^{-3}, whereas the photon energy density goes like a^{-4}. Thus, the ratio of the two, R, goes like a.

- The proportionality constant is:

$$R = \frac{3\Omega_B}{4\Omega_\gamma}\frac{a}{a_0} = 0.6120\left(\frac{\Omega_B h^2}{0.022}\right)\frac{1091}{1+z}$$

where we used

$$\Omega_\gamma \equiv \frac{8\pi G\rho_{\gamma 0}}{3H_0^2} = 2.471 \times 10^{-5}\ h^{-2} \quad \textbf{for} \quad T_0 = 2.725\ \mathrm{K}$$

For the last-scattering redshift of z_L=1090
(or last-scattering temperature of T_L=2974 K),

$$r_s = 145.3 \text{ Mpc}$$

We cannot ignore the effects of sound waves if $qr_s>1$. Since $l \sim qr_L$, this means

$$l > r_L/r_s = 96$$

where we used r_L=13.95 Gpc

Creation of Sound Waves: Basic Equations

1. Conservation equations (energy and momentum)

2. Equation of state, relating pressure to energy density

$$P = P(\rho)$$

3. General relativistic version of the "Poisson equation", relating gravitational potential to energy density

$$\nabla^2 \Phi(t, \boldsymbol{x}) = 4\pi G a^2(t) \delta\rho_M(t, \boldsymbol{x})$$

4. Evolution of the "anisotropic stress" (viscosity)

Energy Conservation

- **Total energy conservation:**

$$\sum_\alpha \left\{ \delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \nabla^2\pi_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\nabla^2 \delta u_\alpha \right\} = 0\,,$$

anisotropic stress:
[or, viscosity]

$$\Delta T_{ij} = a^2 \partial_i \partial_j \pi$$

velocity *potential*

$$\mathbf{v}_\alpha = \frac{1}{a}\nabla \delta u_\alpha$$

(

- C.f., Total energy conservation [unperturbed]

$$\sum_\alpha \left[\dot{\bar{\rho}}_\alpha + \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha) \right] = 0$$

)

Energy Conservation

- **Total energy conservation:**

$$\sum_\alpha \left\{ \delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \nabla^2\pi_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\nabla^2\delta u_\alpha \right\} = 0\,,$$

- **Again, this is the effect of locally-defined inhomogeneous scale factor**, i.e.,
  - The spatial metric is given by $ds^2 = a^2(t)\exp(-2\Psi)d\mathbf{x}^2$
  - Thus, locally we can define a new scale factor:

$$\tilde{a}(t,\mathbf{x}) = a(t)\exp(-\Psi)$$

Energy Conservation

- **Total energy conservation:**

$$\sum_\alpha \left\{ \delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \nabla^2\pi_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \boxed{\frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\nabla^2\delta u_\alpha} \right\} = 0\,,$$

- **Momentum flux going outward (inward) -> reduction (increase) in the energy density**

(**C.f., for a non-expanding medium:** $\dot{\rho} + \nabla\cdot(\rho\mathbf{v}) = 0$)

Momentum Conservation

- **Total momentum conservation**

$$\sum_\alpha \left\{ \frac{\partial}{\partial t}[(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha] + \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha \right.$$
$$\left. + (\bar{\rho}_\alpha + \bar{P}_\alpha)\Phi + \delta P_\alpha + \nabla^2 \pi_\alpha \right\} = 0\,,$$

- **Cosmological redshift of the momentum**
- **Gravitational force given by potential gradient**
- **Force given by pressure gradient**
- **Force given by gradient of anisotropic stress**

Equation of State

- Pressure of non-relativistic species (i.e., baryons and cold dark matter) can be ignored relative to the energy density. Thus, we set them to zero: **$P_B=0=P_D$** and **$\delta P_B=0=\delta P_D$**

- Unperturbed pressure of relativistic species (i.e., photons and relativistic neutrinos) is given by the third of the energy density, i.e., **$P_\gamma=\rho_\gamma/3$** and **$P_\nu=\rho_\nu/3$**

- Perturbed pressure involves contributions from the bulk viscosity:

$$\delta P_\gamma = (\delta\rho_\gamma - \nabla^2\pi_\gamma)/3$$

$$\delta P_\nu = (\delta\rho_\nu - \nabla^2\pi_\nu)/3$$

The reason for this is that **trace** of the stress-energy of relativistic species **vanishes**: $\sum_{\mu=0,1,2,3} T_\mu{}^\mu = 0$

$$T_0^0 + \sum_{i=1}^{3} T_i^i = -\rho + 3P + \nabla^2\pi = 0$$

- Pres … d cold dark … ensity. Thu … D
- Unp … otons and … ene …
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$$\delta P_\nu = (\delta\rho_\nu - \nabla^2\pi_\nu)/3$$

Two Remarks

- In the standard scenario:
  - Energy densities are conserved separately; thus we do not need to sum over all species
  - Momentum densities of photons and baryons are NOT conserved separately but they are coupled via Thomson scattering. This must be taken into account when writing down separate conservation equations

Conservation Equations for Photons and Baryons

- Fourier transformation replaces $\nabla^2 \to -q^2$

$$X(t, \boldsymbol{x}) = (2\pi)^{-3} \int d^3q \; X_{\boldsymbol{q}}(t) \exp(i\boldsymbol{q}\cdot\boldsymbol{x})$$

$$\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma) - \frac{4q^2}{3a^2}\delta u_\gamma = 4\dot{\Psi}$$

$$\frac{\partial}{\partial t}(\delta\rho_B/\bar{\rho}_B) - \frac{q^2}{a^2}\delta u_B = 3\dot{\Psi}$$

momentum transfer via scattering

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

$$\delta\dot{u}_B + \Phi = -\frac{\sigma_{\mathcal{T}}\bar{n}_e}{R}(\delta u_B - \delta u_\gamma)$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

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what about photon's viscosity?

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_T\bar{n}_e(\delta u_B - \delta u_\gamma)$$

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$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Peebles & Yu (1970); Sunyaev & Zeldovich (1970)

Formation of a Photon-baryon Fluid

- Photons are not a fluid. Photons free-stream at the speed of light
  - The conservation equations are not enough because we need to specify the evolution of viscosity
  - Solving for viscosity requires information of the phase-space distribution function of photons: **Boltzmann equation**
- However, frequent scattering of photons with baryons* can make photons behave as a fluid: **Photon-baryon fluid**

Photons scatter with electrons via Thomson scattering. Protons scatter with electrons via Coulomb scattering. Thus we can say, effectively, photons scatter with baryons

Let's solve them!

- Fourier transformation replaces $\nabla^2 \to -q^2$

$$X(t, \boldsymbol{x}) = (2\pi)^{-3} \int d^3q \; X_{\boldsymbol{q}}(t) \exp(i\boldsymbol{q} \cdot \boldsymbol{x})$$

$$\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma) - \frac{4q^2}{3a^2}\delta u_\gamma = 4\dot{\Psi}$$

$$\frac{\partial}{\partial t}(\delta\rho_B/\bar{\rho}_B) - \frac{q^2}{a^2}\delta u_B = 3\dot{\Psi}$$

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

$$\delta\dot{u}_B + \Phi = -\frac{\sigma_{\mathcal{T}}\bar{n}_e}{R}(\delta u_B - \delta u_\gamma)$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Tight-coupling Approximation

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d/\sigma_\mathcal{T}\bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_\mathcal{T}\bar{n}_e \to \infty$ *. We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} = d\,, \qquad \delta\dot{u}_\gamma + \Phi = -\frac{d}{R}$$

In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.

Tight-coupling Approximation

- Eliminating *d* and using the fact that R is proportional to the scale factor, we obtain

$$a\frac{\partial}{\partial t}\left[(1+R)\delta u_\gamma/a\right]+(1+R)\Phi+\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}=0$$

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain

$$\frac{1}{a(1+R)}\frac{\partial}{\partial t}\left[a(1+R)\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma-4\Psi)\right]+\frac{4q^2}{3a^2}\Phi+\frac{q^2}{a^2}\frac{\delta\rho_\gamma/\bar{\rho}_\gamma}{\boxed{3(1+R)}}=0$$

Wave Equation, with the speed of sound of $c_s^2 = 1/3(1+R)$!

Sound Wave!

- To simplify the equation, let's first look at the high-frequency solution
  - Specifically, we take q >> aH (the wavelength of fluctuations is much shorter than the Hubble length). Then we can ignore time derivatives of R and Ψ because they evolve in the Hubble time scale:

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

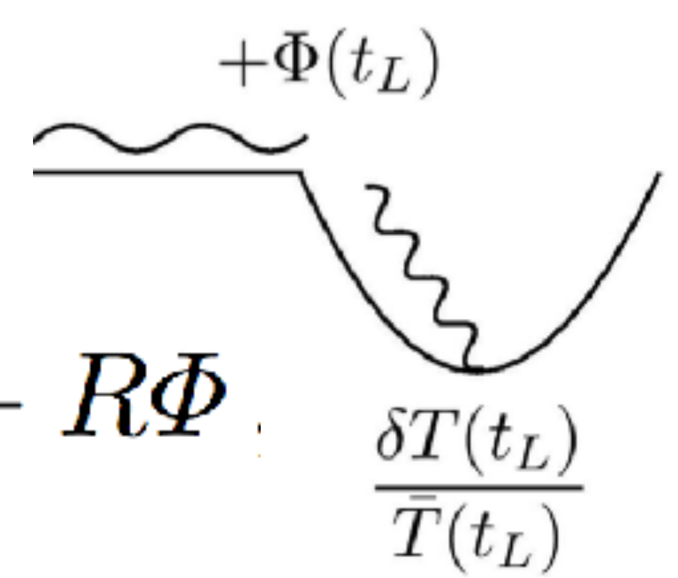
Solution: SOUND WAVE!

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=A\cos(qr_s)+B\sin(qr_s)-R\Phi$$

Recap

- Photons are not a fluid; but Thomson scattering couples photons to baryons, forming a **photon-baryon fluid**
- The reduced sound speed, $c_s^2=1/3(1+R)$, emerges automatically
- $\delta\rho_\gamma/4\rho_\gamma$ is the temperature anisotropy at the bottom of the potential well. Adding gravitational redshift, the observed temperature anisotropy is $\delta\rho_\gamma/4\rho_\gamma + \Phi$, which is given by

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$



Effect of Baryon-Density