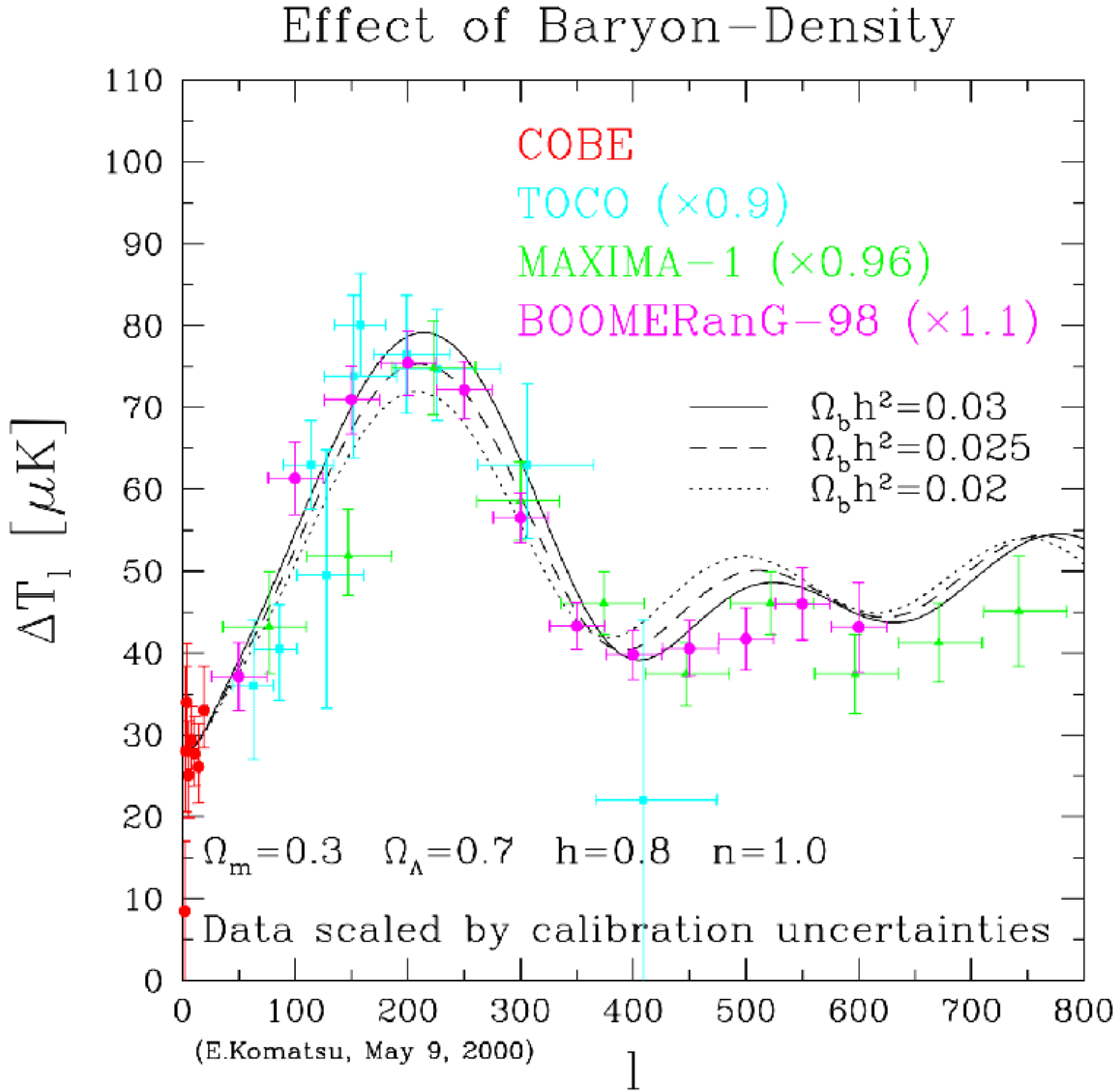
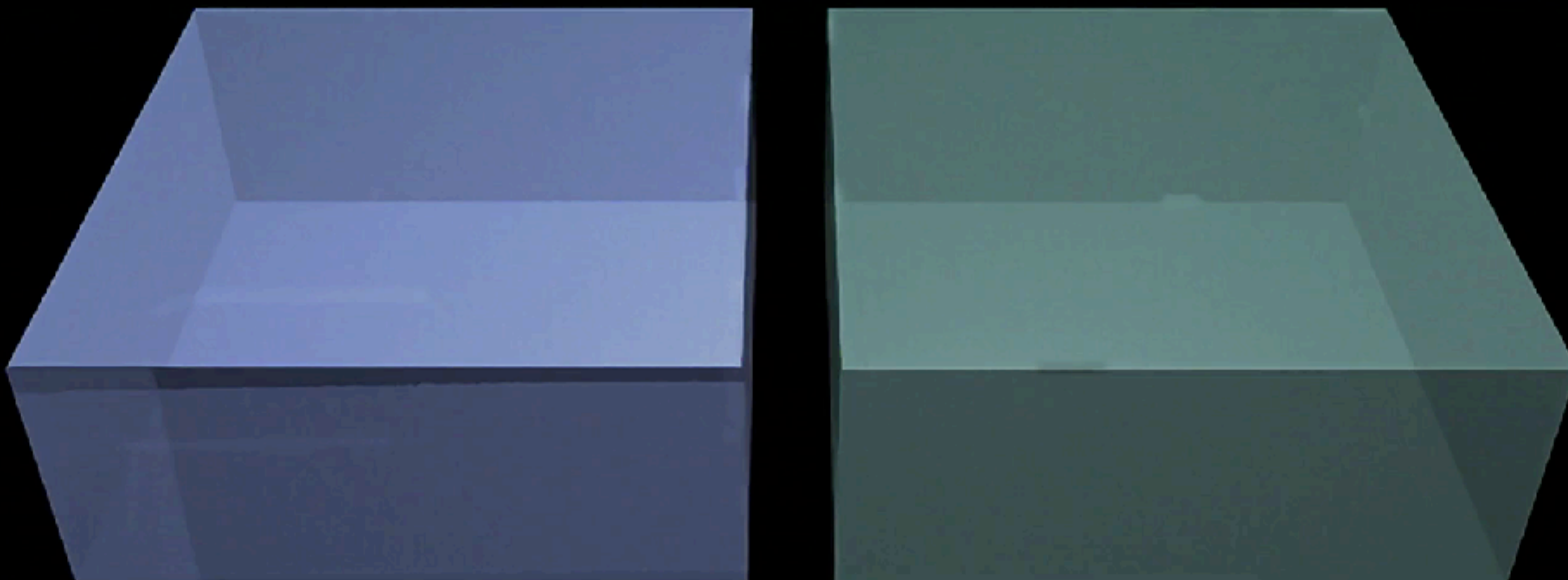
Lecture 3

- Temperature anisotropy from sound waves (continued)

- Cosmological parameter dependence of the temperature power spectrum

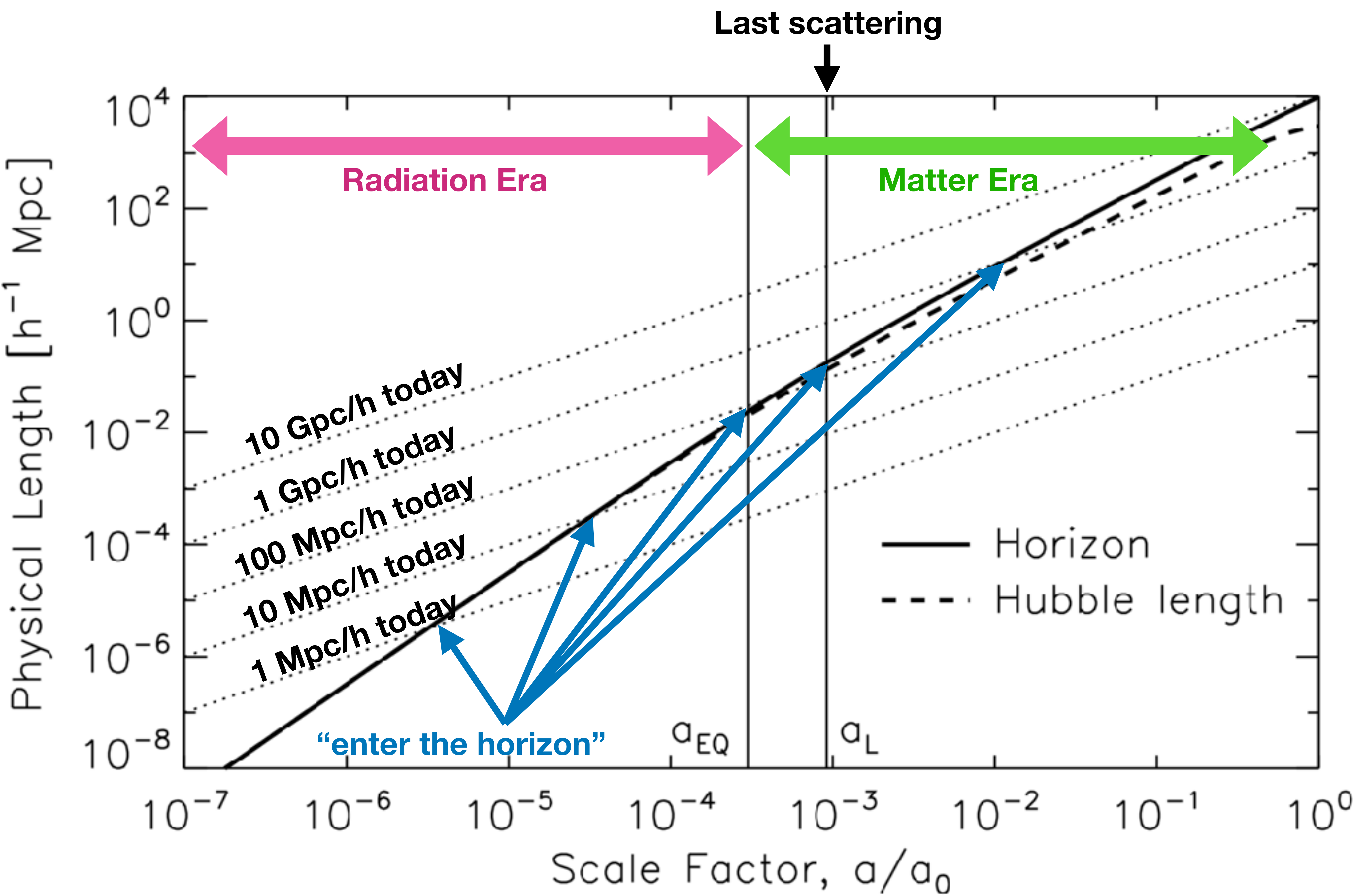


Effect of Baryon−Density
COBE
TOCO (×0.9)
MAXIMA−1 (×0.96)
BOOMERanG−98 (×1.1)
$\Omega_b h^2=0.03$
$\Omega_b h^2=0.025$
$\Omega_b h^2=0.02$
$\Omega_m=0.3$ $\Omega_\Lambda=0.7$ h=0.8 n=1.0
Data scaled by calibration uncertainties
ΔT_l [μK]
l
(E.Komatsu, May 9, 2000)
0
10
20
30
40
50
60
70
80
90
100
110
0
100
200
300
400
500
600
700
800



Stone: Fluctuations "entering the horizon"

- This is a tricky concept, but it is important
- Suppose that there are fluctuations at all wavelengths, including the ones that exceed the Hubble length (which we loosely call our "horizon")
  - Let's not ask the origin of these "super-horizon fluctuations", but just assume their existence
- As the Universe expands, our horizon grows and we can see longer and longer wavelengths
  - **Fluctuations "entering the horizon"**



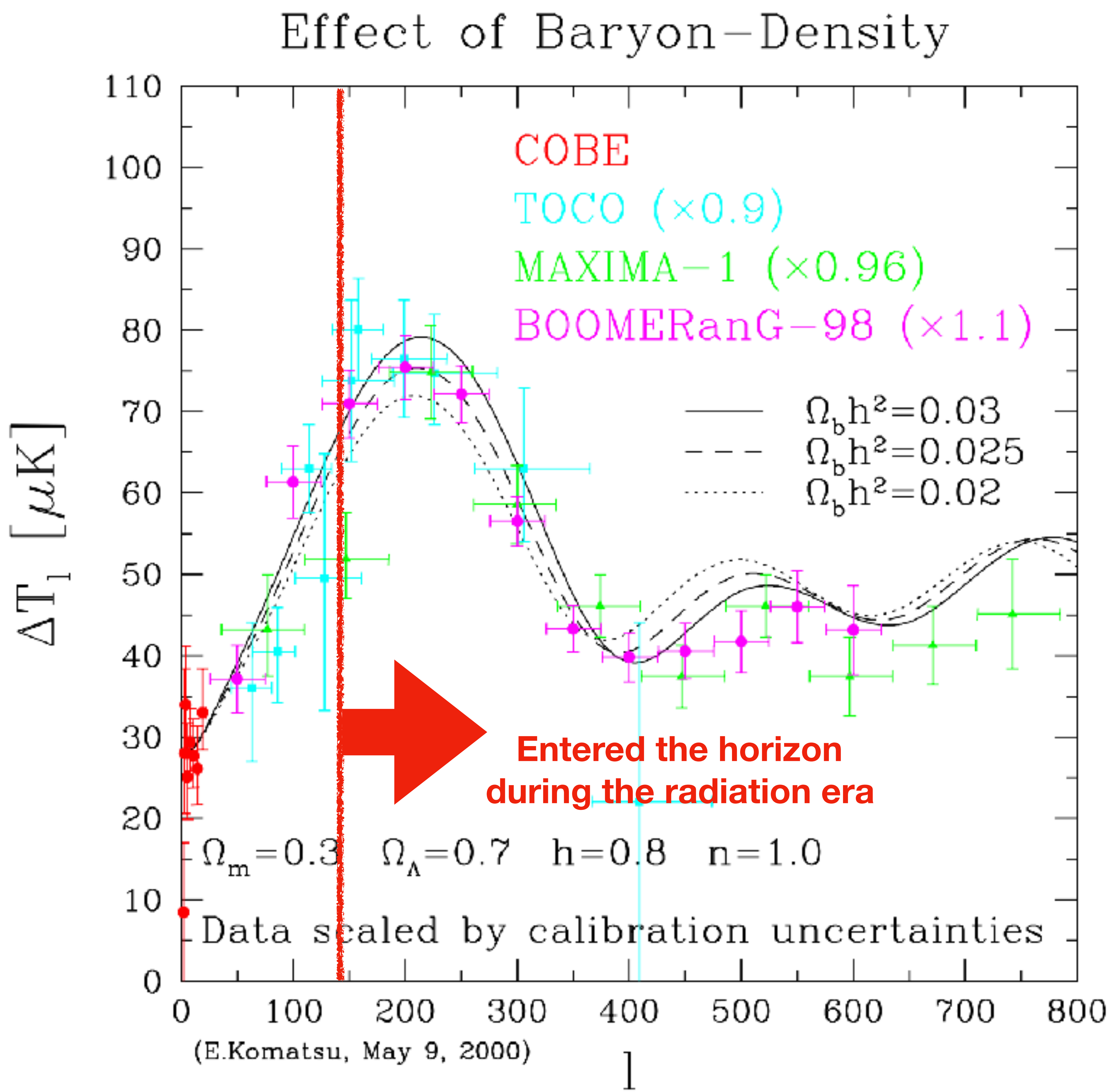
Last scattering
Radiation Era
Matter Era
10 Gpc/h today
1 Gpc/h today
100 Mpc/h today
10 Mpc/h today
1 Mpc/h today
"enter the horizon"
a_{EQ}
a_L
Horizon
Hubble length
Physical Length [h^{-1} Mpc]
Scale Factor, a/a_0
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

Three Regimes

- **Super-horizon scales [q < aH]**
    - Only gravity is important
    - Evolution differs from Newtonian
- **Sub-horizon but super-sound-horizon [aH < q < aH/c_s]**
    - Only gravity is important
    - Evolution similar to Newtonian
- **Sub-sound-horizon scales [q > aH/c_s]**
    - Hydrodynamics important -> Sound waves

q_{EQ}

- Which fluctuation entered the horizon before the matter-radiation equality?
- $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ (\Omega_M h^2/0.14)\ \mathrm{Mpc}^{-1}$
- At the last scattering surface, this subtends the multipole of $l_{EQ} = q_{EQ}r_L \sim \mathbf{140}$



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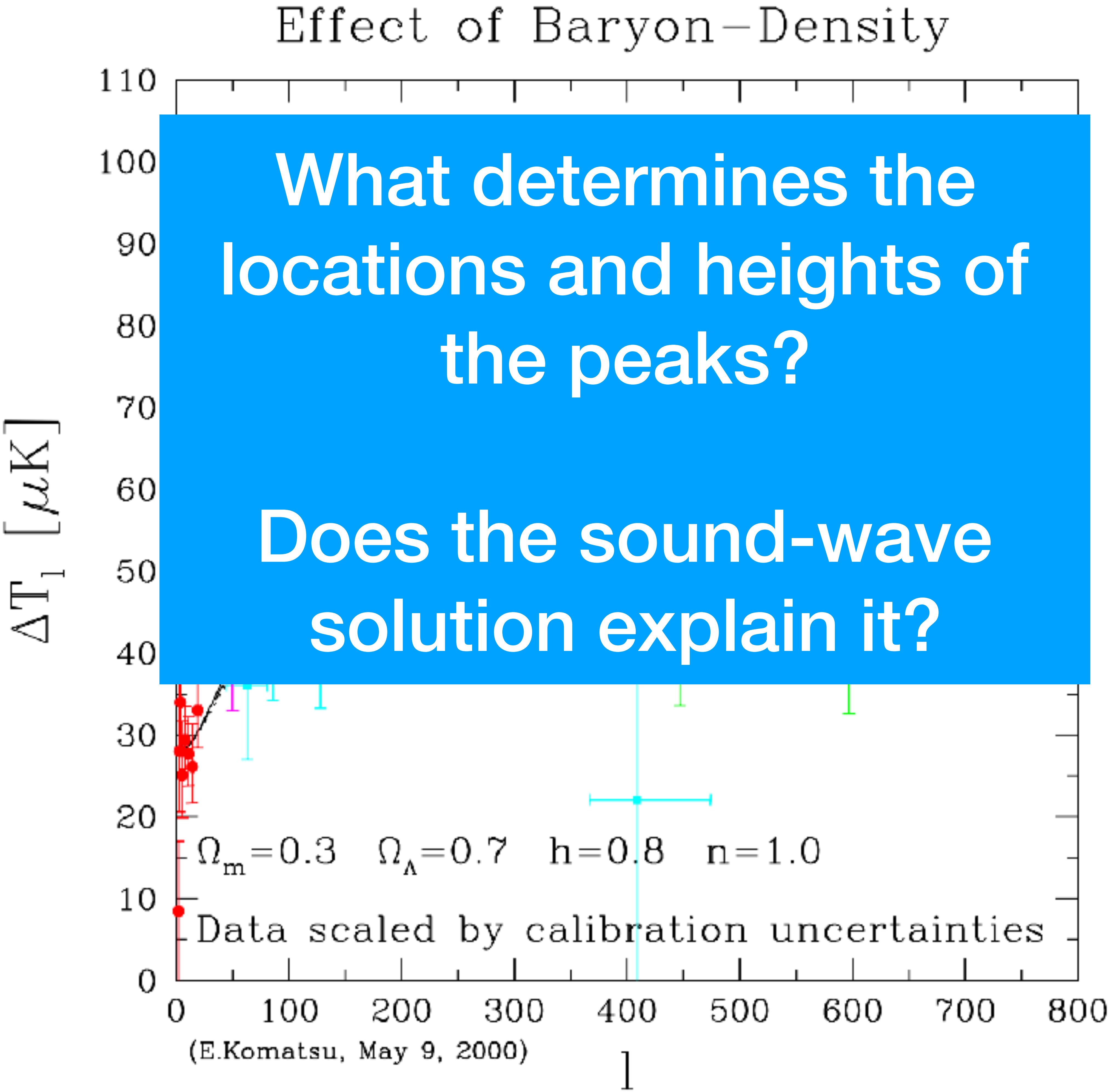
Entered the horizon
during the radiation era
$\Omega_m=0.3$ $\Omega_\Lambda=0.7$ h=0.8 n=1.0
Data scaled by calibration uncertainties
ΔT_l [μK]
l
(E.Komatsu, May 9, 2000)



What determines the locations and heights of the peaks?

Does the sound-wave solution explain it?

Peak Locations?

High-frequency solution, for q >> aH

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$

- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi\right]^2$ with $q \to l/r_L$
    - Question: What are the integration constants, **A** and **B**?
    - Answer: They depend on the initial conditions; namely, adiabatic or not?
    - For adiabatic initial condition, **A >> B *when q is large***

[We will show it later.]

Peak Locations?

<u>High-frequency solution, for q >> aH</u>

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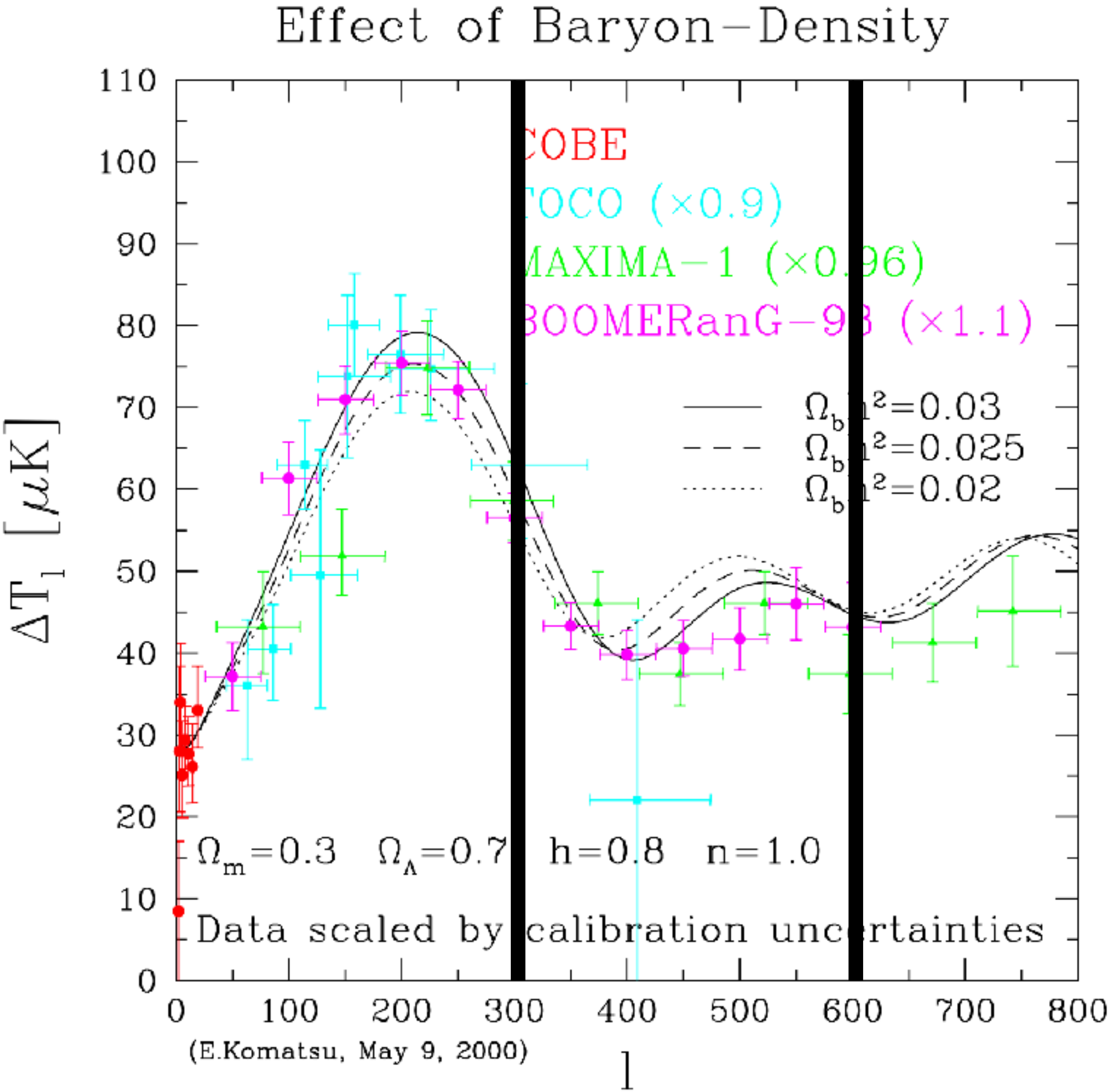
- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi\right]^2$ with q -> l/r_L

- If A>>B, the locations of peaks are

$$\ell = (1,\ 2,\ \cdots)\pi r_L/r_s(t_L) = (1,\ 2,\ \cdots) \times 302$$



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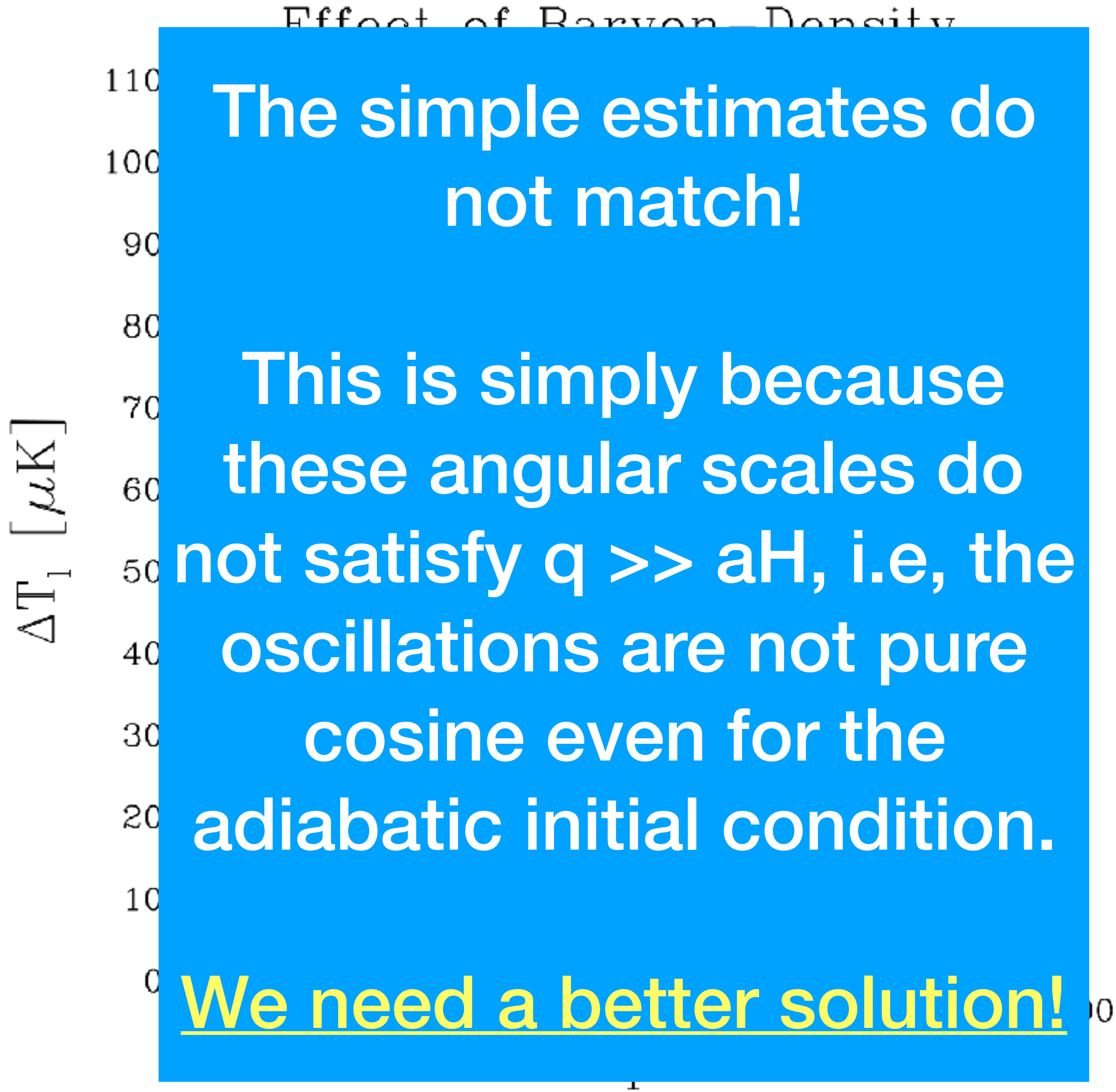

The simple estimates do not match!

This is simply because these angular scales do not satisfy q >> aH, i.e, the oscillations are not pure cosine even for the adiabatic initial condition.

<u>We need a better solution!</u>

Better Solution in Radiation-dominated Era

Going back to the original tight-coupling equation..

$$\frac{1}{a(1+R)}\frac{\partial}{\partial t}\left[a(1+R)\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma - 4\Psi)\right] + \frac{4q^2}{3a^2}\Phi + \frac{q^2}{a^2}\frac{\delta\rho_\gamma/\bar{\rho}_\gamma}{3(1+R)} = 0$$

- In the radiation-dominated era, R << 1
- Change the independent variable from the time (t) to

$$\varphi \equiv qr_s = 2qt/\sqrt{3}a$$

Better Solution in Radiation-dominated Era

Then the equation simplifies to

$$\partial^2 X/\partial\varphi^2 + X + \Phi + \Psi = 0$$

where $X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$

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The solution is

$$X = \tilde{A}\cos\varphi + \tilde{B}\sin\varphi - \int_0^\varphi d\varphi' \sin(\varphi - \varphi')(\Phi + \Psi)(\varphi')$$

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$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi'),$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

Einstein's Equations

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.

- Einstein's equations - let's look up any text books:

$$\nabla^2\Psi = 4\pi G a^2 \sum_\alpha \left[\delta\rho_\alpha - \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha\right]$$

$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -4\pi G \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha$$

$$\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \sum_\alpha \pi_\alpha$$

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$$\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \pi_\nu$$

**Will come back to this later.
For now, let's ignore any viscosity.**

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$$\Phi = \Psi$$

Will come back to this later.
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Einstein's Equations in Radiation-dominated Era

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.

- Einstein's equations - let's look up any text books:

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{4}{\varphi}\frac{\partial \Phi}{\partial \varphi} + \Phi = \frac{3}{2\varphi^2}\frac{\delta \mathcal{P}}{\bar{\rho}_R}$$

$$\sum_\alpha \delta P_\alpha(t, \boldsymbol{x}) = \frac{\sum_\alpha \dot{\bar{P}}_\alpha(t)}{\sum_\alpha \dot{\bar{\rho}}_\alpha(t)} \sum_\alpha \delta\rho_\alpha(t, \boldsymbol{x}) + \delta \mathcal{P}(t, \boldsymbol{x})$$

"non-adiabatic" pressure

Einstein's Equations in Radiation-dominated Era

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We shall ignore this

$$\sum_\alpha \delta P_\alpha(t, \boldsymbol{x}) = \frac{\sum_\alpha \dot{\bar{P}}_\alpha(t)}{\sum_\alpha \dot{\bar{\rho}}_\alpha(t)} \sum_\alpha \delta \rho_\alpha(t, \boldsymbol{x}) + \delta \mathcal{P}(t, \boldsymbol{x})$$

"non-adiabatic" pressure

Solution (Adiabatic) in Radiation-dominated Era

$$\Phi_{\mathrm{ADI}} = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

where

$$\varphi \equiv qr_s = 2qt/\sqrt{3}a$$

- Low-frequency limit (*super-sound-horizon scales, $qr_s \ll 1$*)
  - $\Phi_{\mathrm{ADI}} \to -2\zeta/3$ = **constant**
- High-frequency limit (*sub-sound-horizon scales, $qr_s \gg 1$*)
  - $\Phi_{\mathrm{ADI}} \to 2\zeta\cos\varphi/\varphi^2 \propto a^{-2}$ **damp**

Solution (Adiabatic) in Radiation-dominated Era

$$\Phi_{\mathbf{ADI}} = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

wh

Poisson Equation

$$-q^2\Phi = 4\pi G a^2 \delta\rho$$

& oscillation solution for radiation

$$\delta\rho_R/\bar{\rho}_R \propto \cos\varphi$$

- Low-frequency ... *cales, $qr_s << 1$*)
  - Φ_{ADI} -> $-2\zeta/3$
- High-frequency li... *(sub-sound-horizon scales, $qr_s >> 1$*)
  - Φ_{ADI} -> $2\zeta\cos\varphi/\varphi^2 \propto a^{-2}$ **damp**

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  - Φ_{ADI} -> $-2\zeta/3$ = **constant**
- High-frequency limit (*sub-sound-horizon scales, qr_s >> 1*)
  - Φ_{ADI} -> $2\zeta\cos\varphi/\varphi^2 \propto a^{-2}$ **damp**

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

ζ: Conserved on large scales

- For the adiabatic initial condition, there exists a useful quantity, **ζ**, which remains constant on large scales (*super-horizon scales, $q \ll aH$*) regardless of the contents of the Universe
  - ζ is conserved regardless of whether the Universe is radiation-dominated, matter-dominated, or whatever
- Energy conservation for $q \ll aH$:

$$\delta\dot{\rho}_\alpha + \frac{3\dot{a}}{a}(\delta\rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} = 0$$

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

ζ: Conserved on large scales

- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$, then

$$\frac{1}{3}\frac{\delta\rho_\alpha(t,\boldsymbol{x})}{\bar\rho_\alpha(t)+\bar P_\alpha(t)} - \Psi(t,\boldsymbol{x}) = \zeta_\alpha(\boldsymbol{x})$$

integration constant

Integrate

$$\delta\dot\rho_\alpha + \frac{3\dot a}{a}(\delta\rho_\alpha + \delta P_\alpha) - 3(\bar\rho_\alpha + \bar P_\alpha)\dot\Psi = 0$$

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

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- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$, then

$$\frac{1}{3}\frac{\delta\rho_\alpha(t,\boldsymbol{x})}{\bar{\rho}_\alpha(t)+\bar{P}_\alpha(t)} - \Psi(t,\boldsymbol{x}) = \zeta_\alpha(\boldsymbol{x})$$

integration constant

For the adiabatic initial condition, all species share the same value of ζ_α, i.e., $\zeta_\alpha=\zeta$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2$$

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Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi \xrightarrow{\varphi \ll 1} \zeta$$

i.e., $\tilde{A}_{\text{ADI}} = \zeta, \quad \tilde{B}_{\text{ADI}} = 0$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

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$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2$$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

The adiabatic solution is

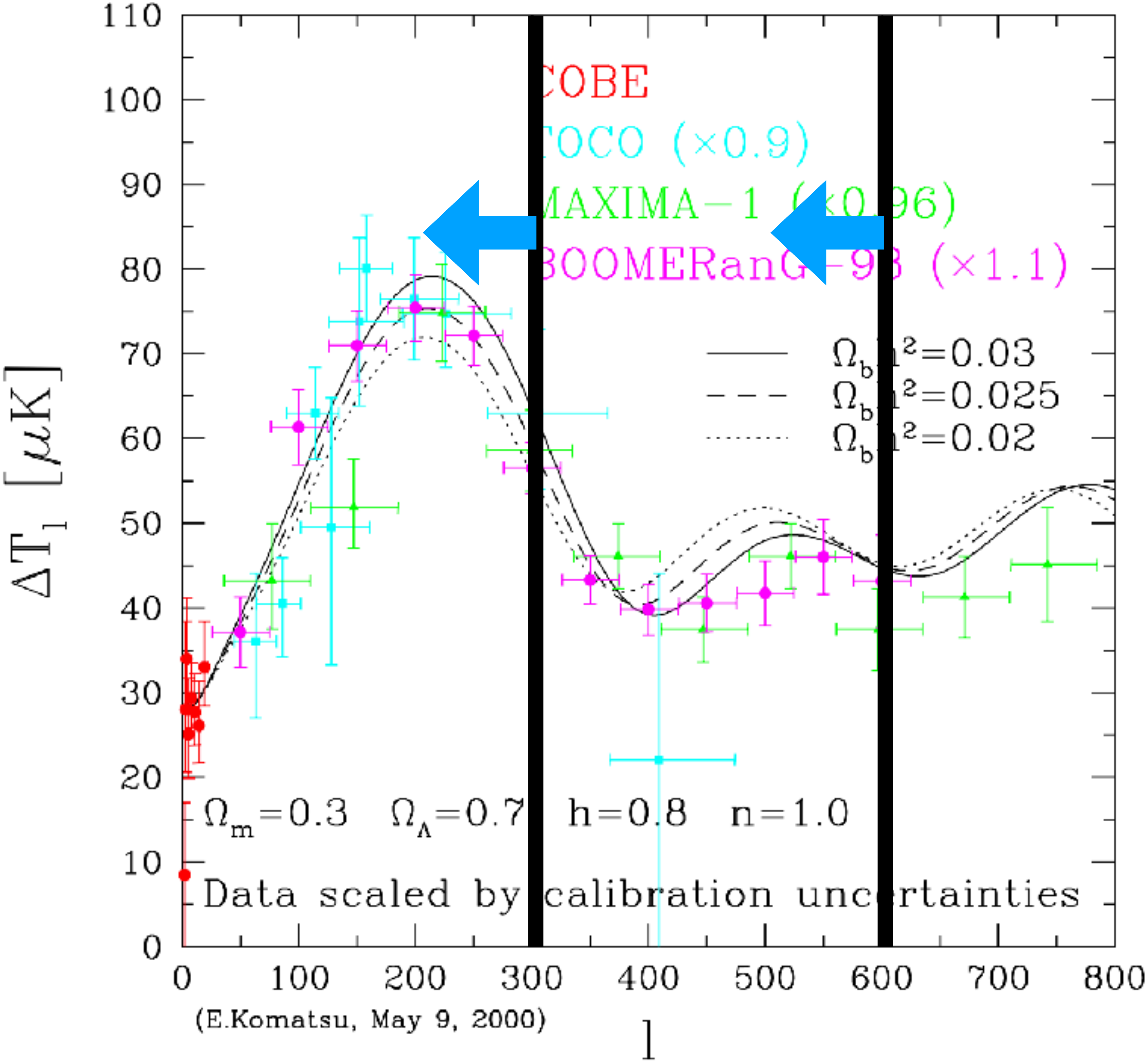
$$X = \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

with

$$\Phi = \Psi = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

Therefore, the solution is a **pure cosine** only in the **high-frequency** limit!

Effect of Baryon-Density



Roles of viscosity

- **Neutrino viscosity**
  - Modify potentials: $\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \pi_\nu$
- **Photon viscosity**
  - Viscous photon-baryon fluid: **damping of sound waves**

Silk (1968) "Silk damping"

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

High-frequency solution without neutrino viscosity

The solution is

$$X = (\zeta + \Delta A)\cos\varphi + (\qquad \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2) \longrightarrow -2\zeta$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2 \longrightarrow 0$$

Chluba & Grin (2013)

High-frequency solution with neutrino viscosity

The solution is

$$X = (-\zeta + \Delta A_\nu) \cos\varphi + \Delta B_\nu \sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A_\nu \longrightarrow 0.338 R_\nu \zeta$$

$$\Delta B_\nu \longrightarrow 0.418 R_\nu \zeta$$

non-zero value!

$$R_\nu \equiv \bar{\rho}_\nu / (\bar{\rho}_\gamma + \bar{\rho}_\nu) \approx 0.409$$

High-frequency solution **with** neutrino viscosity

The solution is

$$X = -C\cos(\varphi + \theta)$$

where

$$C \equiv \sqrt{(-\zeta + \Delta A_\nu)^2 + \Delta B_\nu^2}$$

$$\approx \zeta(1 + 4R_\nu/15)^{-1}$$

Hu & Sugiyama (1996)

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

Bashinsky & Seljak (2004)

High-frequency solution

with neutrino viscosity

The s

X =

where

O

Thus, the neutrino viscosity will:

(1) Reduce the amplitude

of sound waves at large multipoles

(2) Shift the peak positions

of the temperature power spectrum

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

Bashinsky & Seljak (2004)

Photon Viscosity

- In the tight-coupling approximation, the photon viscosity damps exponentially
- To take into account a non-zero photon viscosity, we go to a higher order in the tight-coupling approximation

Tight-coupling Approximation (1st-order)

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d/\sigma_\mathcal{T}\bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_\mathcal{T}\bar{n}_e \to \infty$ *. We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} = d\,, \qquad \delta\dot{u}_\gamma + \Phi = -\frac{d}{R}$$

In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.

Tight-coupling Approximation (2nd-order)

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d_1/\sigma_T \bar{n}_e + q d_2/(\sigma_T \bar{n}_e)^2$$

where

$$d_1 = -R(\delta\dot{u}_\gamma + \Phi)$$

[d_2 is an arbitrary dimensionless variables]

- And take $\sigma_T \bar{n}_e \to \infty$. We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = -R(\delta\dot{u}_\gamma + \Phi) + \frac{q}{\sigma_T \bar{n}_e} d_2$$

$$\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma + \Phi)}{\sigma_T \bar{n}_e}\right] = \frac{q}{R\sigma_T \bar{n}_e} d_2$$

Tight-coupling Approximation (2nd-order)

- Eliminating d_2 and using the fact that R is proportional to the scale factor, we obtain

$$a\frac{\partial}{\partial t}\left[(1+R)\delta u_\gamma/a\right]+(1+R)\Phi+\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}-\frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma}+R\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma+\Phi)}{\sigma_{\mathcal{T}}\bar{n}_e}\right]=0$$

- Getting π_γ requires an approximate solution of the Boltzmann equation in the 2nd-order tight coupling. We do not derive it here. The answer is

$$\pi_\gamma = -\frac{32}{45}\frac{\bar{\rho}_\gamma}{\sigma_{\mathcal{T}}\bar{n}_e}\frac{\delta u_\gamma}{a^2}$$

Kaiser (1983)

Tight-coupling Approximation (2nd-order)

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given by the velocity potential - a well-known result in fluid dynamics

$$\pi_\gamma=-\frac{32}{45}\frac{\bar{\rho}_\gamma}{\sigma_T\bar{n}_e}\frac{\delta u_\gamma}{a^2}$$

Kaiser (1983)

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for $q >> aH$,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

where

$$\Gamma(q,t)\equiv\frac{q^2}{6a^2\sigma_{\mathcal{T}}\bar{n}_e}\left[\frac{16}{15(1+R)}+\frac{R^2}{(1+R)^2}\right]$$

Damped Oscillator

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$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

where

Important for high frequencies (large multipoles)

$$\Gamma(q,t)\equiv\frac{q^2}{6a^2\sigma_T\bar{n}_e}\left[\frac{16}{15(1+R)}+\frac{R^2}{(1+R)^2}\right]$$

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for q >> aH,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

SOLUTION:

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=\left[A\cos(qr_s)+B\sin(qr_s)\right]\exp\left[-\int_0^t dt'\ \Gamma(q,t')\right]-R\Phi$$

Exponential dampling!

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SOLUTION:

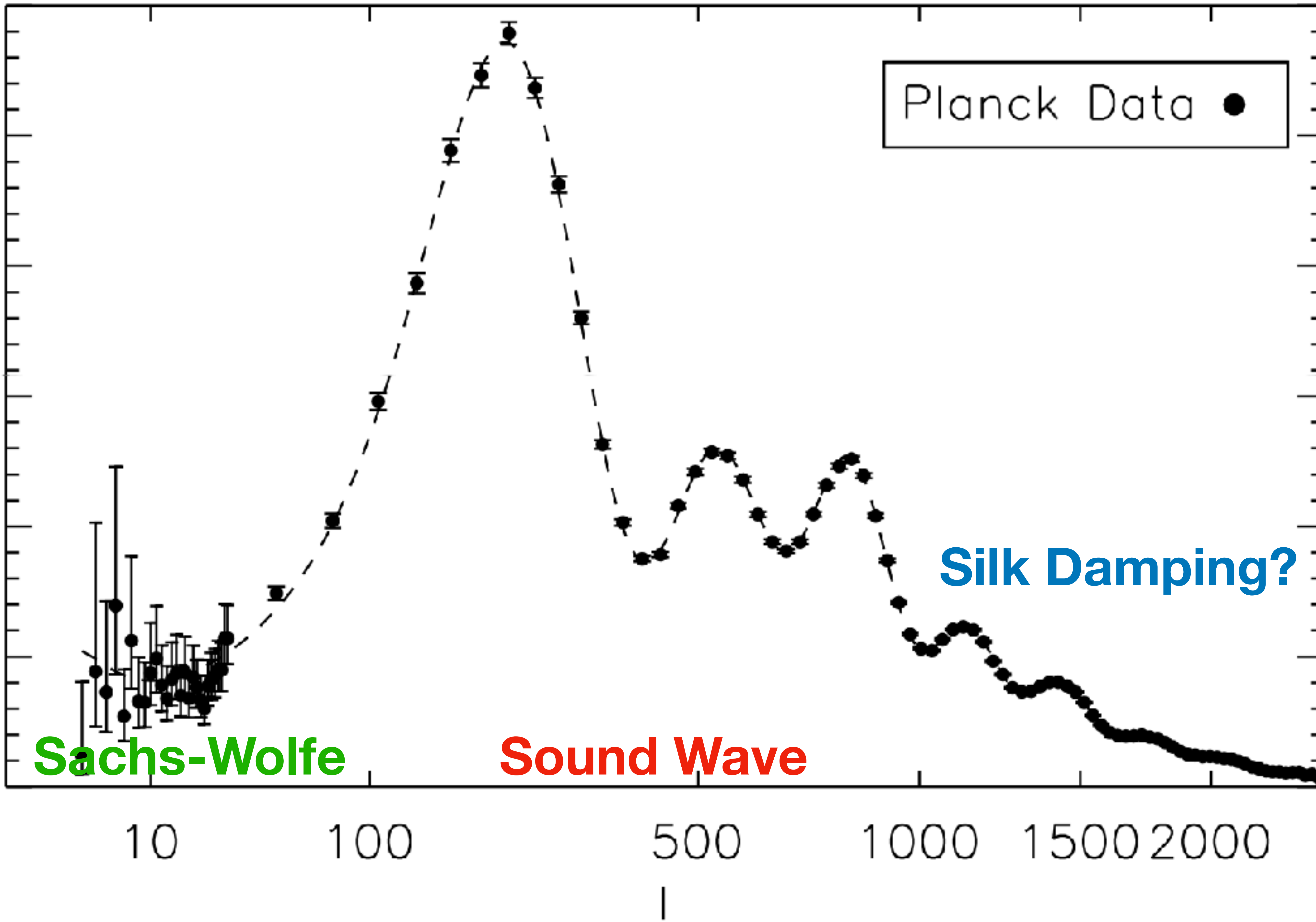
$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=[A\cos(qr_s)+B\sin(qr_s)]\exp(-q^2/q_{\text{Silk}}^2)-R\Phi$$

Exponential dampling!

$$a/q_{\text{Silk}}\approx(\sigma_T\bar{n}_eH)^{-1/2}$$

"diffusion length"
= length traveled by photon's random walks

Planck Collaboration (2016)



Additional Damping

- The power spectrum is $[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi]^2$ with q -> l/r_L. The damping factor is thus $\exp(-\mathbf{2}q^2/q_{silk}^2)$
- $q_{silk}(t_L)$ = 0.139 Mpc^{-1}. This corresponds to a multipole of **$l_{silk} \sim q_{silk} r_L/\sqrt{2}$ = 1370.** Seems too large, compared to the exact calculation
- There is an additional damping due to a finite width of the last scattering surface, **σ~250 K**
  - "Fuzziness damping" – Bond (1996)
  - "Landau damping" - Weinberg (2001)

$$q_{\mathbf{Landau}}^{-2} = \frac{3\sigma^2 t_L^2}{8a_0^2 T_0^2 (1+R_L)} \approx \left(0.20\ \mathrm{Mpc}^{-1}\right)^{-2}$$

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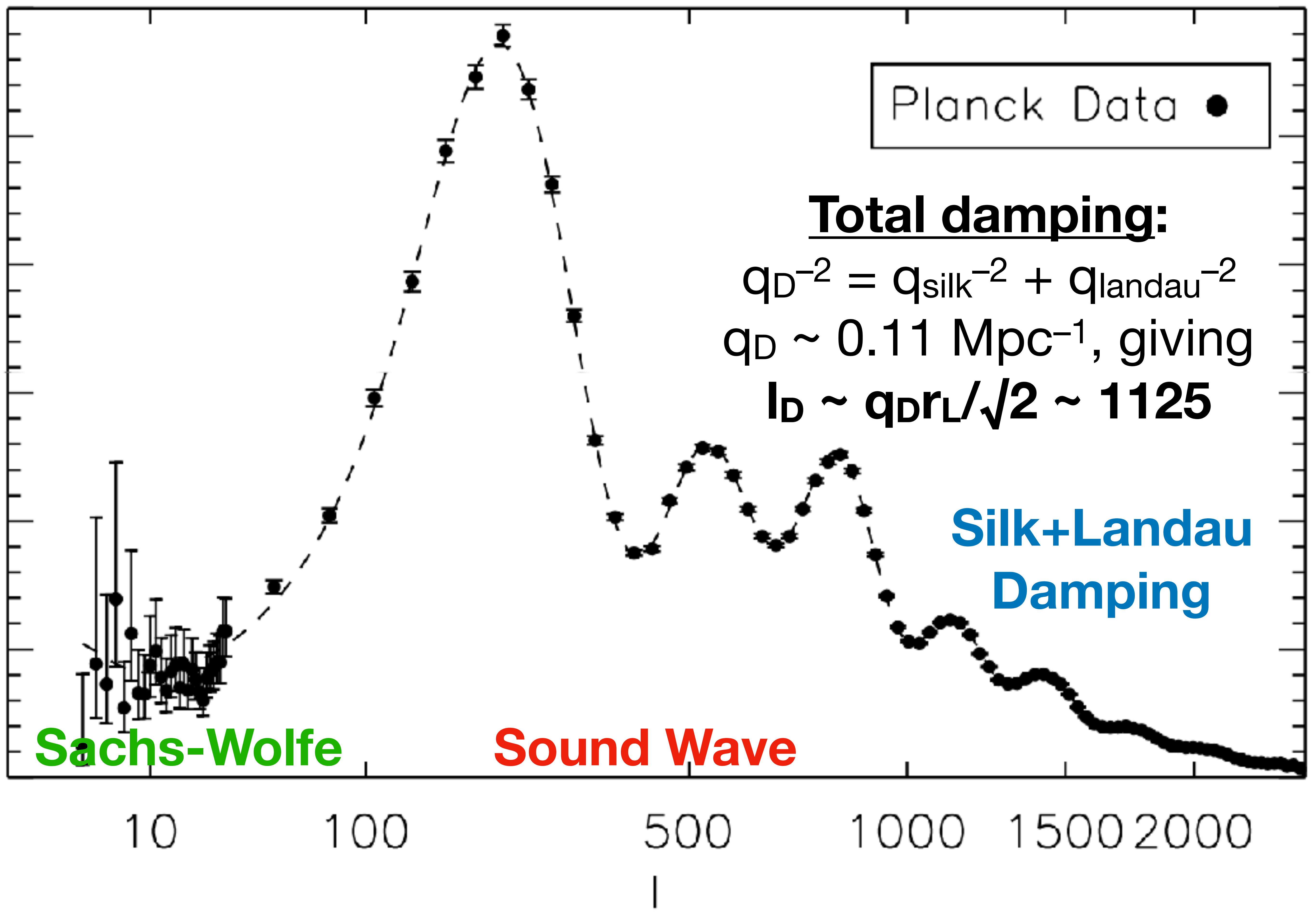
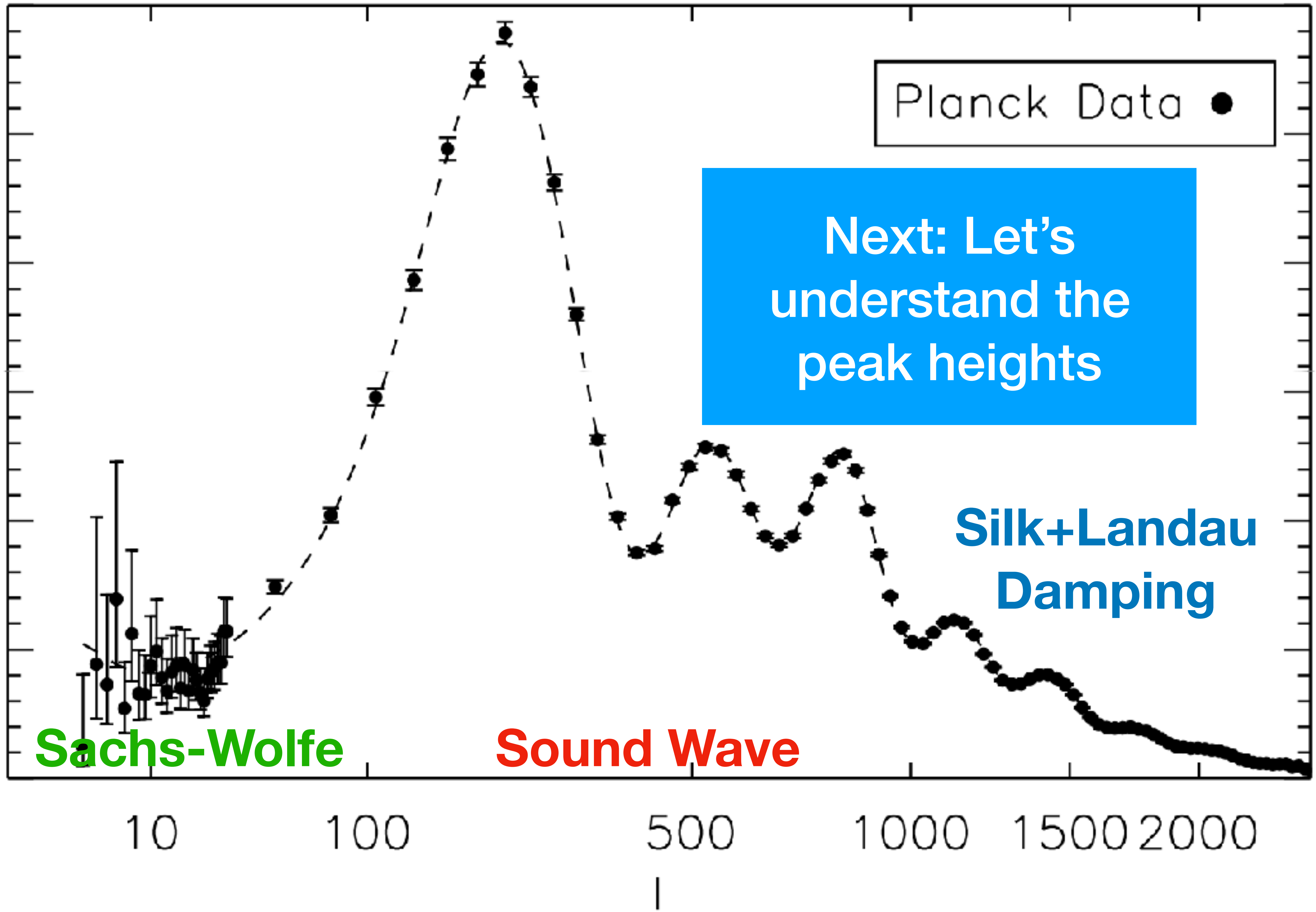


Planck Collaboration (2016)



Matching Solutions

- We have a very good analytical solution valid at low and high frequencies during the radiation era:

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

- Now, match this to a high-frequency solution valid at the last-scattering surface (when R is no longer small)

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$

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Slightly improved solution, with a weak time dependence of R using the WKB method [*Peebles & Yu (1970)*]

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = (1+R)^{-1/4}[A\cos(qr_s) + B\sin(qr_s)] - R\Phi$$

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

where *T*(q), *S*(q), *θ*(q) are "transfer functions" that smoothly interpolate two limits as

q << q_{EQ}: $\mathcal{S} \to 1,\ \mathcal{T} \to 1,\ \theta \to 0$

q >> q_{EQ}: $\mathcal{S} \to 5,\ \mathcal{T} \propto \ln q/q^2,\ \theta \to 0.062\pi$

"EQ" for "matter-radiation Equality epoch"

with $q_{EQ} = a_{EQ}H_{EQ}$ ~ 0.01 Mpc^{-1}, giving $l_{EQ}=q_{EQ}r_L$ ~ 140

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales

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“EQ” for “matter radiation Equality epoch”

with q_{EQ} … qr_L ~ 140

Due to the decay of gravitational potential during the radiation dominated era

- (*) To a … solution is given b… not importa… at large scales

Weinberg "Cosmology", Eq. (6.5.7)

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

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Due to the neutrino anisotropic stress

Weinberg "Cosmology", Eq. (6.5.7)

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

q -> 0(*)

$$\longrightarrow -\frac{\zeta}{5}$$

This should agree with the Sachs-Wolfe result: Φ/3; thus,

$$\Phi = -3\zeta/5$$ **in the matter-dominated era**

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales

Weinberg “Cosmology”, Eq. (6.5.7)

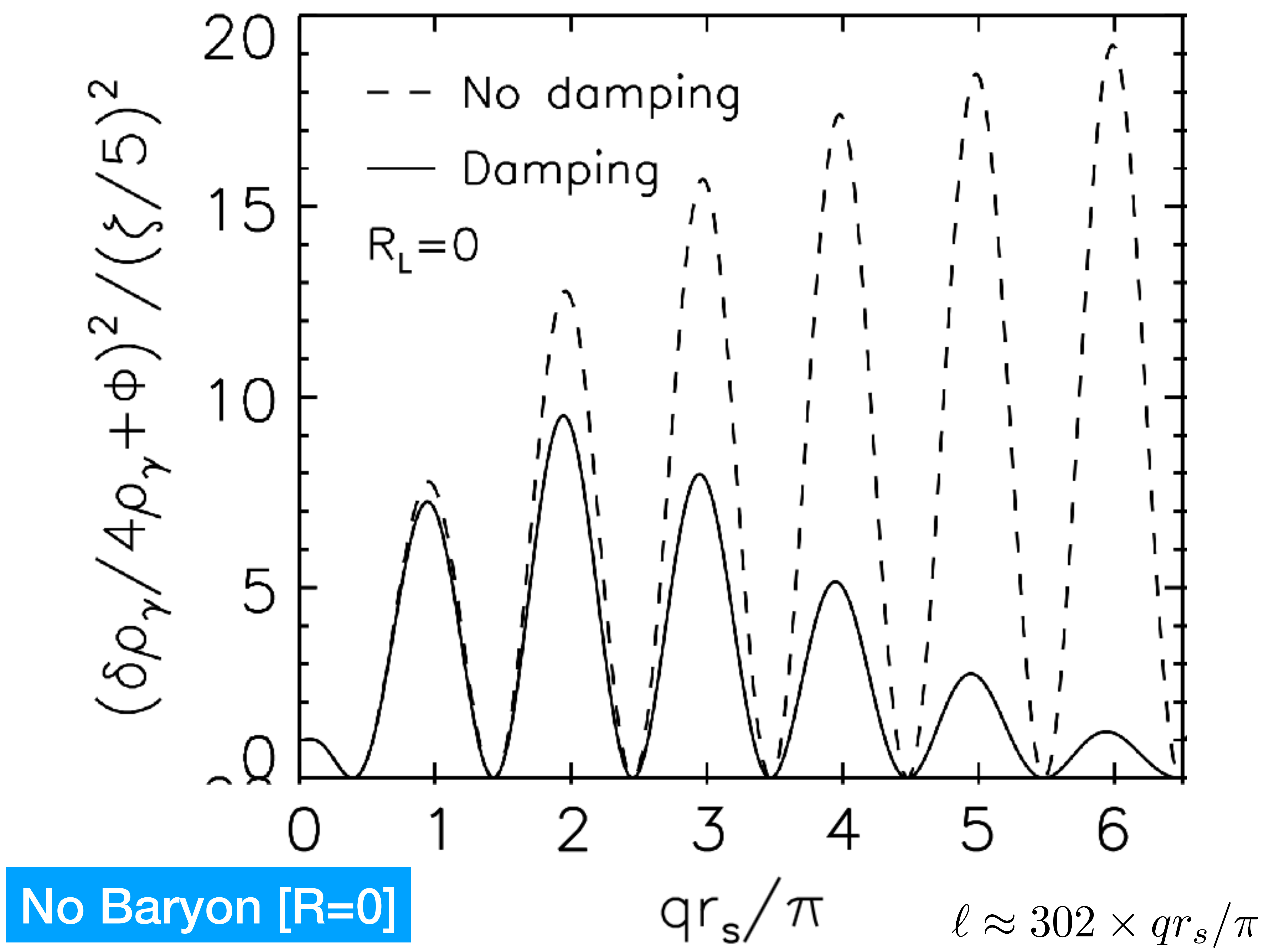
Effect of Baryons

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{ 3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)] \right\}$$

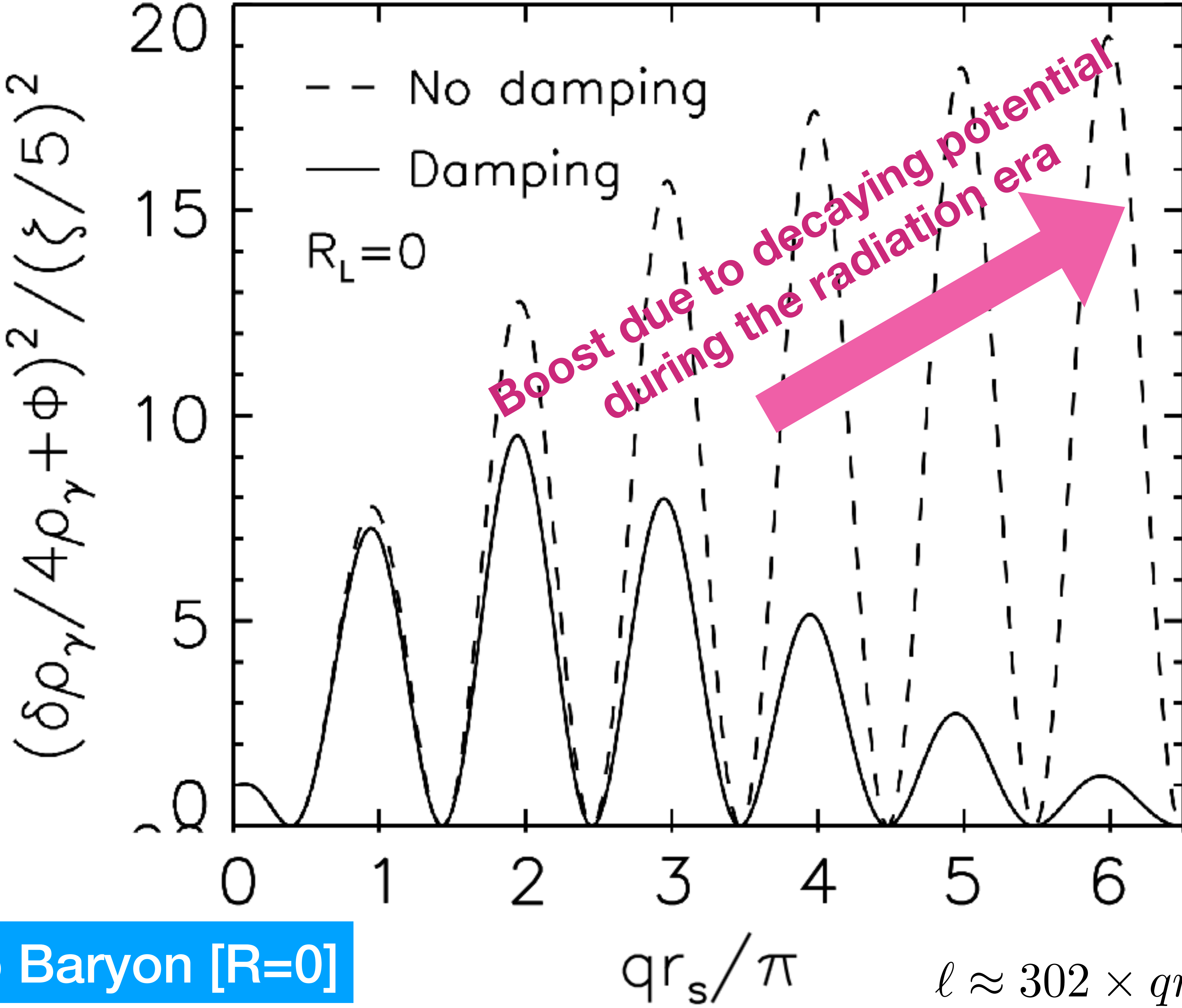
Shift the zero-point of oscillations

Reduce the amplitude of oscillations

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales



No damping
Damping
$R_L=0$
20
15
10
5
0
0
1
2
3
4
5
6
$(\delta\rho_\gamma/4\rho_\gamma+\phi)^2/(\zeta/5)^2$
qr_s/π
No Baryon [R=0]
$\ell \approx 302 \times qr_s/\pi$



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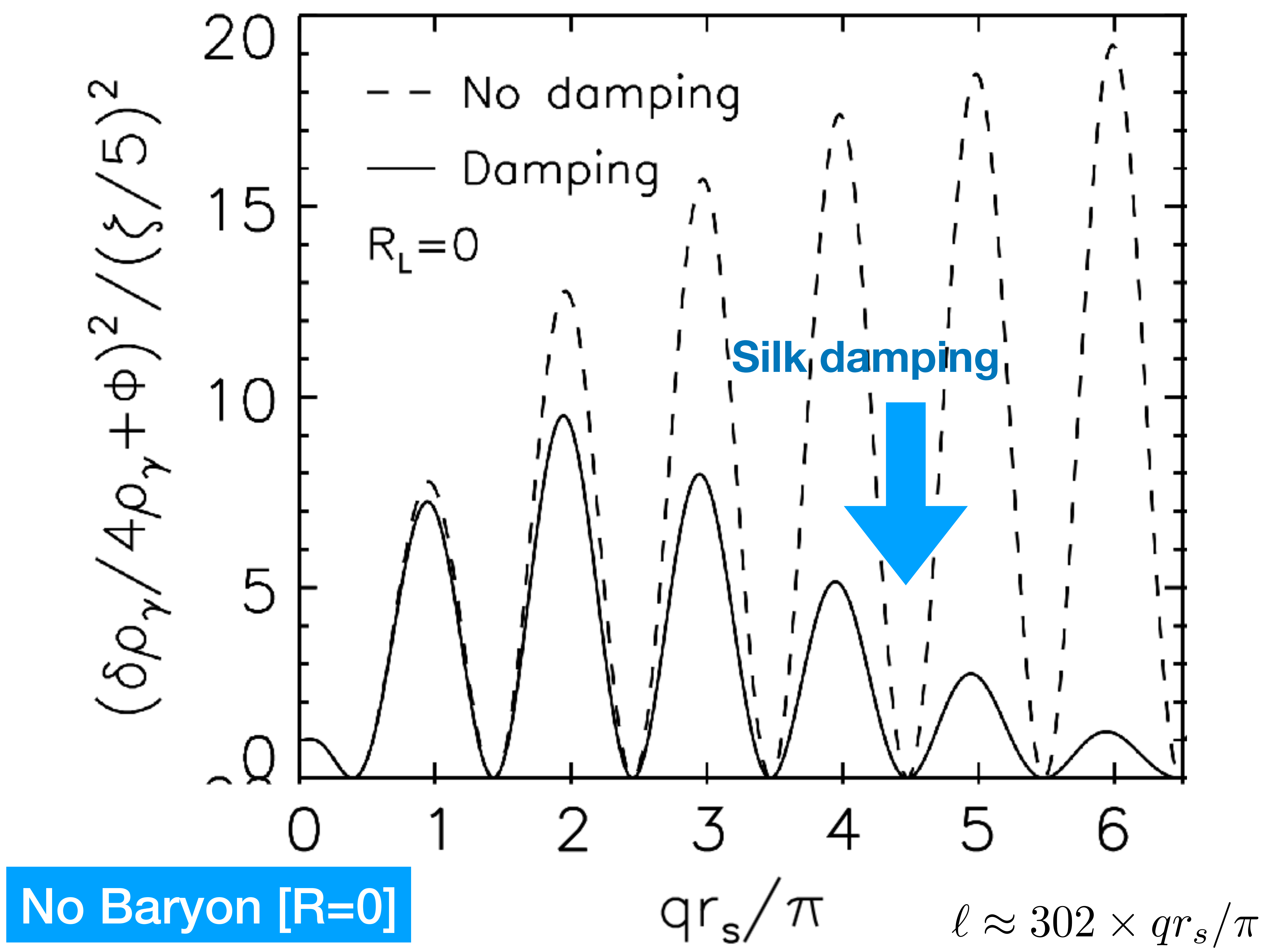

No damping
Damping
$R_L=0$
Silk damping
$(\delta\rho_\gamma/4\rho_\gamma+\Phi)^2/(\zeta/5)^2$
qr_s/π
0
5
10
15
20
0
1
2
3
4
5
6
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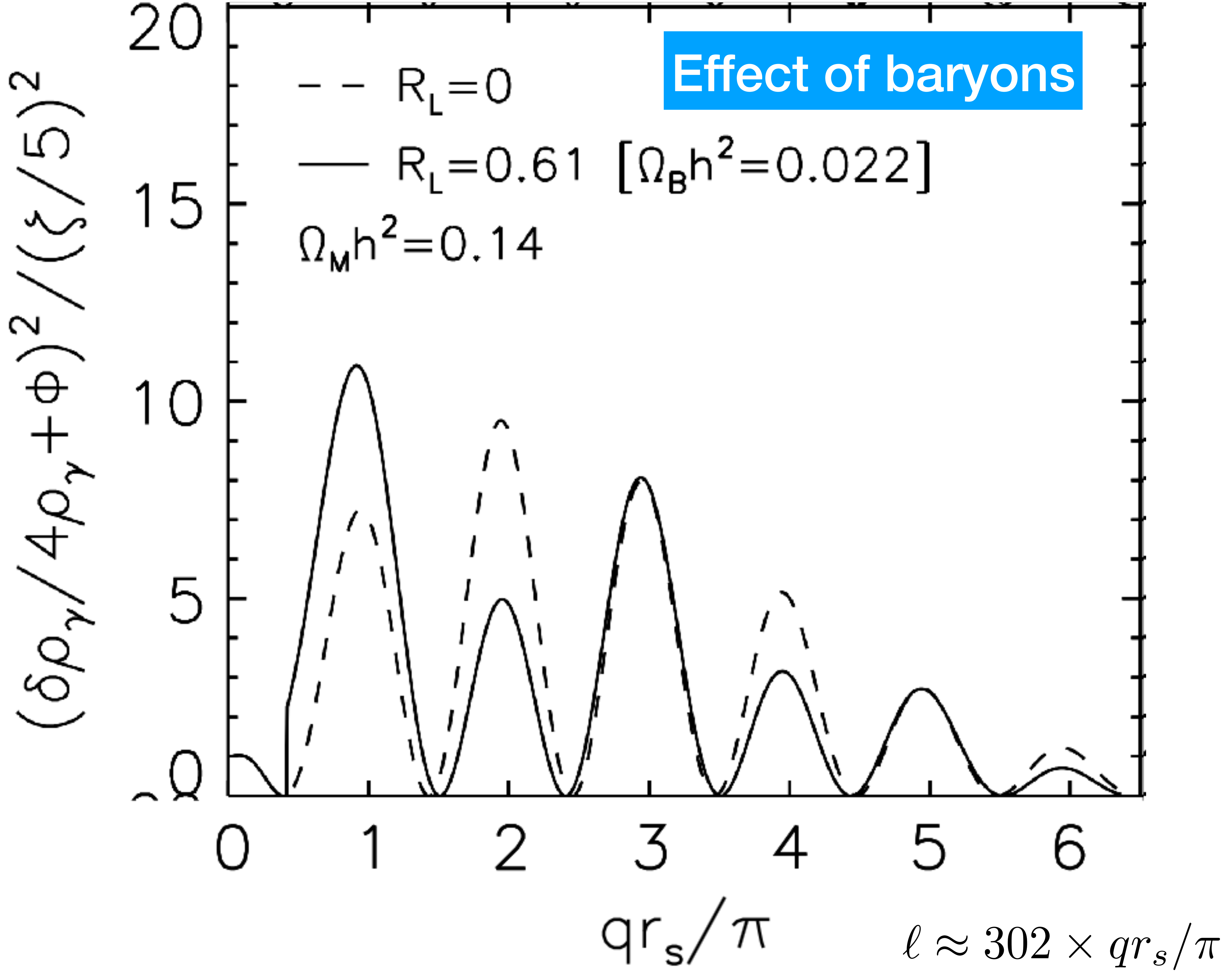
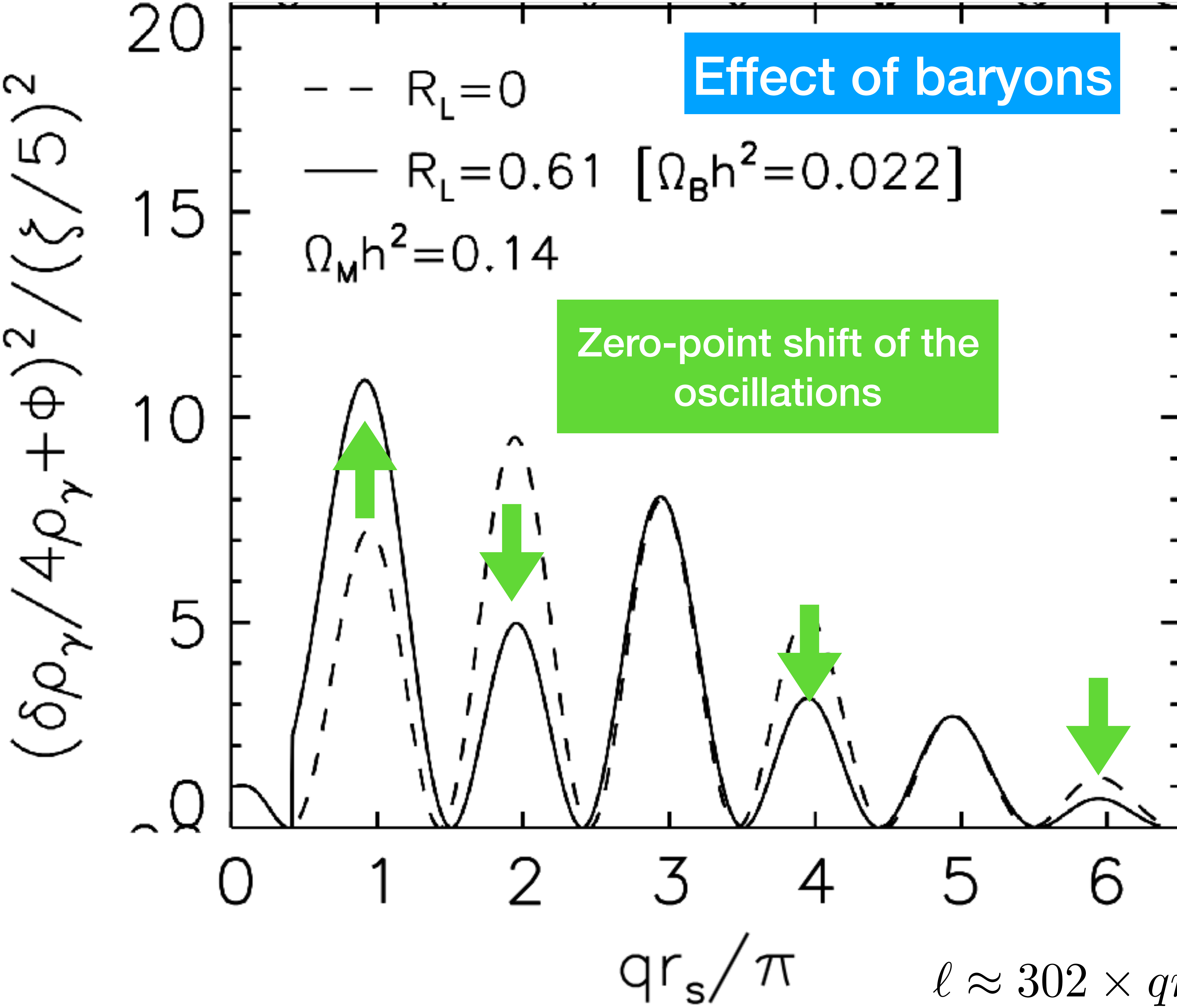


Effect of baryons
$R_L=0$
$R_L=0.61$ $[\Omega_B h^2=0.022]$
$\Omega_M h^2=0.14$
$(\delta\rho_\gamma/4\rho_\gamma+\Phi)^2/(\zeta/5)^2$
0
5
10
15
20
0
1
2
3
4
5
6
qr_s/π
$\ell \approx 302 \times qr_s/\pi$



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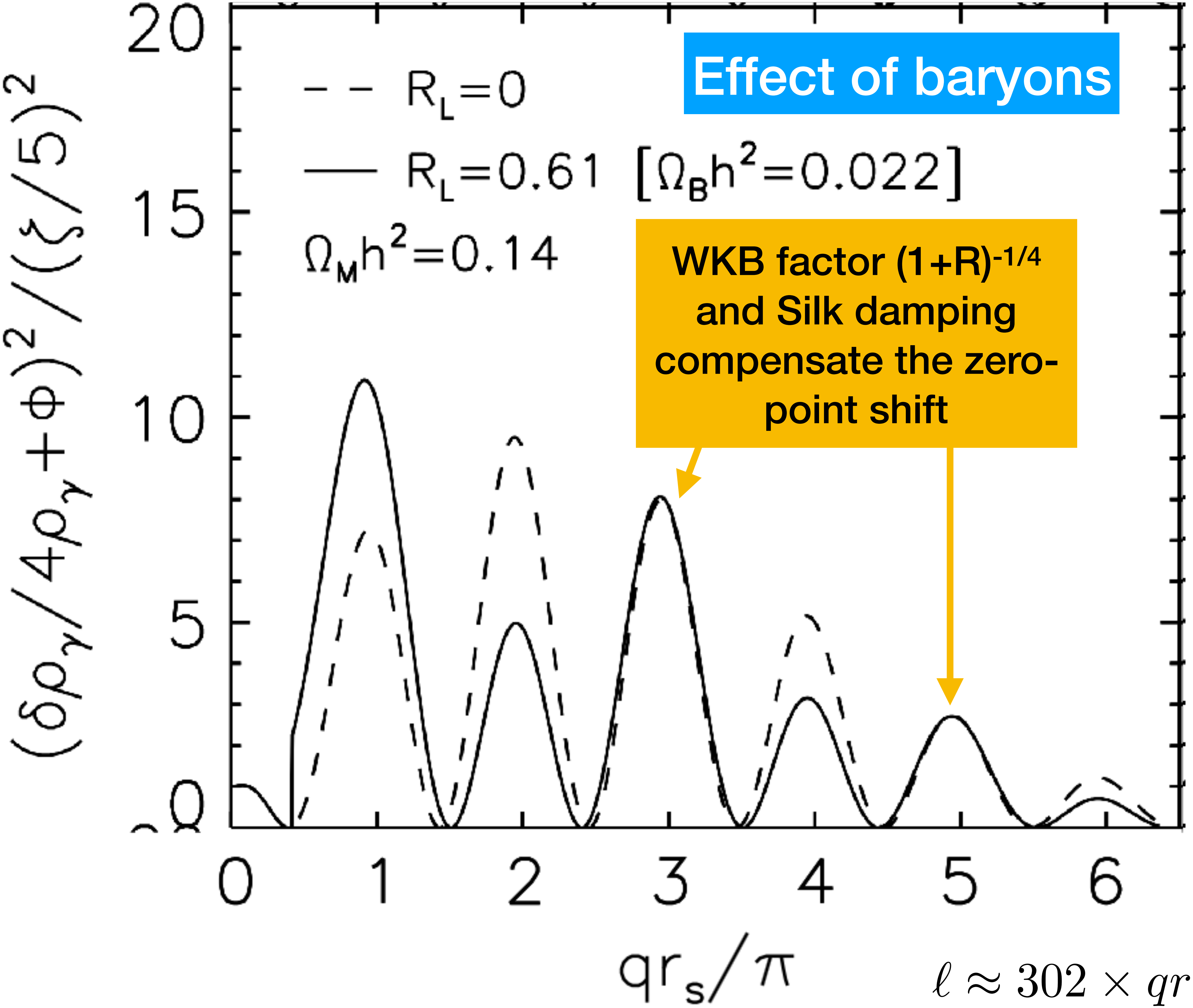

$\ell \approx 302 \times qr_s/\pi$

Effect of Total Matter

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

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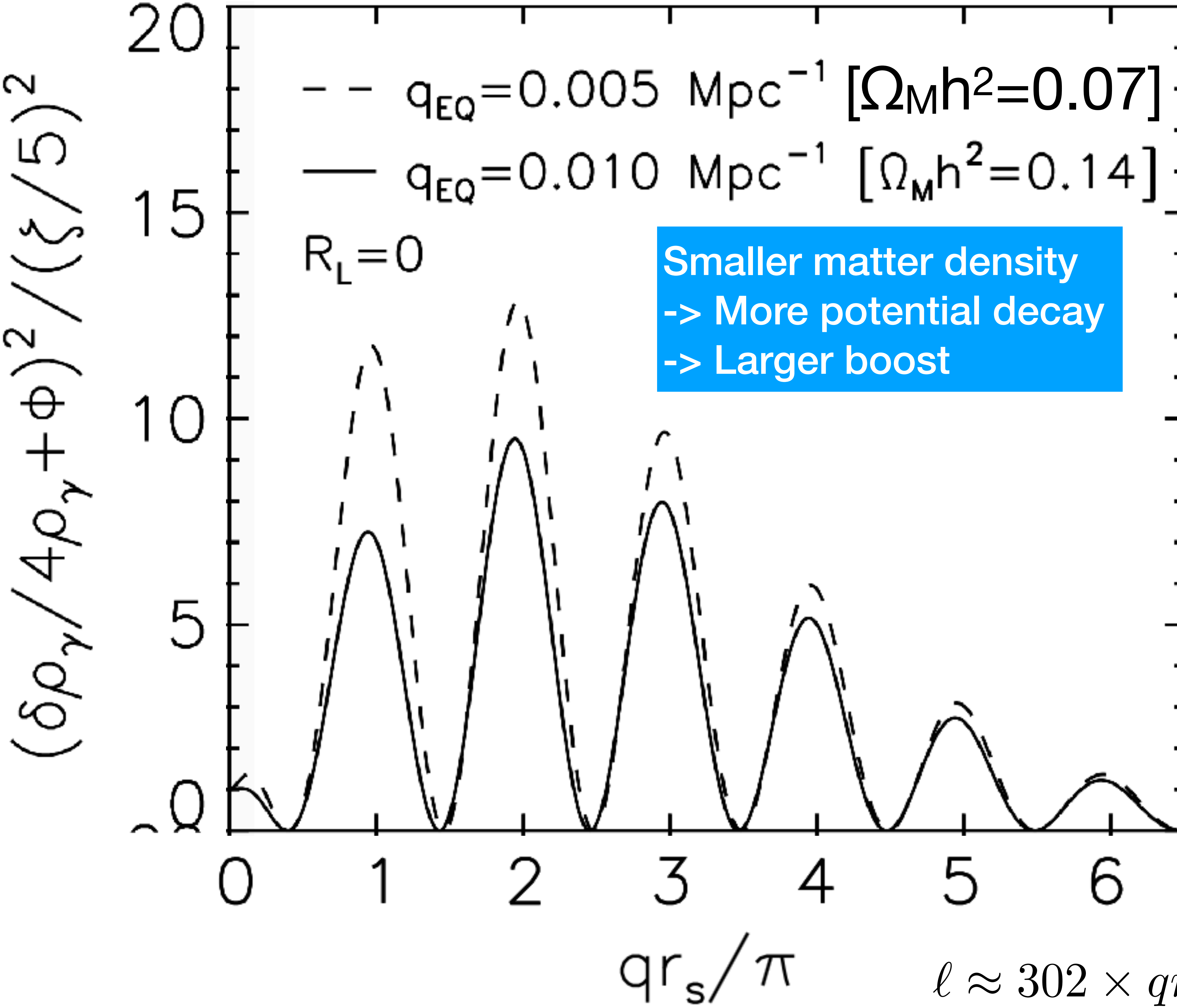

$\ell \approx 302 \times qr_s/\pi$

Recap

- The basic structure of the temperature power spectrum is
  - The Sachs-Wolfe “plateau” at low multipoles
  - Sound waves at intermediate multipoles
    - 1st-order tight-coupling
  - Silk damping and Landau damping at high multipoles
    - 2nd-order tight-coupling

In more details…

- Decay of gravitational potentials boosts the temperature power at high multipoles by **a factor of 5 compared to the Sachs-Wolfe plateau**
    - Where this boost starts depends on the total matter density
- Baryon density shifts the zero-point of the oscillation, boosting the odd peaks relative to the even peaks
    - However, the WKB factor $(1+R)^{-1/4}$ and damping make the **boosting of the 3rd and 5th peaks not so obvious**

Not quite there yet…

- **The first peak is too low**
    - We need to include the “integrated Sachs-Wolfe effect”
- **How to fill zeros between the peaks?**
    - We need to include the Doppler shift of light