An introduction to gravitational waves

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Outline of lectures (1/2)

- The world's shortest introduction to General Relativity
- The linearized Einstein equations and the degrees of freedom of General Relativity
- Gravitational waves in linearized gravity and the quadrupole formula
- Gravitational waves in the geometric optics regime and their stress energy tensor
- A detector's response to gravitational waves: geodesic deviation and Weyl scalars

Outline of lectures (2/2)

- GW detectors and their sources
- Source modelling:

  - Numerical relativity in a nutshell: 3+1 form of the Einstein equations

  - Analytic approximations: The Post-Newtonian expansion, the self-force formalism, the effective one-body model
- Fundamental physics, astrophysics and cosmology with gravitational-wave detectors: a few examples

References

- Einstein equations: any GR textbook (Misner, Thorne & Wheeler, Wald, Carroll, …)
- Basics of gravitational waves:

  - Flanagan, E. E. & Hughes, S. A. 2005, New Journal of Physics, 7, 204 (arXiv:gr-qc/0501041)

  - Rezzolla, L. 2003, ICTP Lecture Series, Vol. 3 (arXiv:gr-qc/0302025)

  - Thorne, K., ”Gravitational Waves and Experimental Tests of General Relativity” www.pma.caltech.edu/Courses/ph136/yr2004/0426.1.K.pdf
- 3+1 formulation of Einstein equations and numerical relativity: Gourgoulhon, E., gr-qc/0703035
- LISA: Pau Amaro-Seoane et al, arXiv:1201.3621
- More specialized references for some slides

General Relativity: a description of gravity

- Newtonian mechanics (v<< c and weak gravitational fields M/r << c^2): gravity is a force

  - Gravitational potentials satisfies Poisson's equation (aka Newton's law of gravitation): $\nabla^2 \varphi = 4\pi\rho$

  - Motion described by 3 laws of Newtonian mechanics and namely $\vec{F} = m\vec{a}$

- Special relativity generalizes Newtonian mechanics (but not Newton's law of gravitation) to v ~ c by requiring that speed of light be the same and finite in all inertial reference systems (cf Michelson-Morley experiment!) $\Rightarrow$

  Minkoswki metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

- General relativity generalizes Newton's law of gravitation to v ~ c and strong gravitational fields, but gravity is not a force any more!

General Relativity in a nutshell (1/4)

- Gravity is not a force, but geometrical effect encoded in 4D metric
  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
- Metric measures "distance" between events $x_1^\mu = (ct, x, y, z)$ and $x_2^\mu = (ct, x, y, z)$, is symmetric, has signature Lorentz signature (-,+,+,+)
- Particles move along lines that minimize distance (geodesics)

$$u^\mu = \frac{dx^\mu}{d\lambda} \qquad a^\mu = u^\nu \nabla_\nu u^\mu = 0 \qquad \nabla_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\alpha u^\nu = 0$$

$$g_{\mu\nu} u^\mu u^\nu = -1 \quad \text{(particles with mass)}$$

$$\Gamma^\mu_{\nu\alpha} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\alpha\sigma} + \partial_\alpha g_{\nu\sigma} - \partial_\sigma g_{\alpha\nu}) \qquad g_{\mu\nu} u^\mu u^\nu = 0 \quad \text{(light rays)}$$

- General covariance: equations of motion take same form in any coordinate system (because defined in terms of spacetime geometry)

  In locally flat coordinates near moving particle (ie free-falling frame), $g_{\mu\nu} = \eta_{\mu\nu} + O(x)^2$ $\Longrightarrow$ non-gravitational law of physics reduce to special relativity, and gravitational forces disappear (cf free-falling spacecraft in Newtonian gravity)

General Relativity in a nutshell (2/4)

- Geodesic motion generalizes Newtonian/special relativistic mechanics, but how do we choose the metric, ie how do we generalize Poisson's equation?
- Requirements for generalization

1) Must reduce to Poisson equation for v<<c and weak fields

2) General covariance: equation for the gravitational field must be the same in all coordinate systems (must be defined in terms of 4D tensors)

3) Gravity described by metric alone (eg no gravitational scalars)

4) Possion equation is linear and second order in the derivatives of φ : look for simplest equation that is linear in 2^{nd} derivatives of metric and satisfies first 3 conditions

Einstein equations

General Relativity in a nutshell (3/4)

The Einstein equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}$

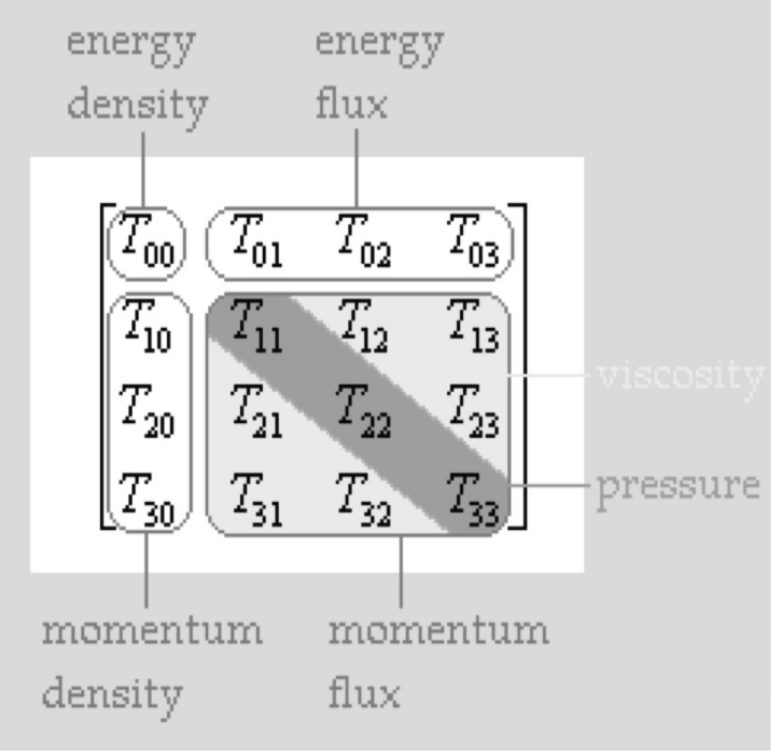
$$R^{\alpha}_{\beta\gamma\delta} \equiv \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\nu}_{\beta\delta}\Gamma^{\alpha}_{\nu\gamma} - \Gamma^{\nu}_{\beta\gamma}\Gamma^{\alpha}_{\nu\delta}$$ (Riemann tensor)

$$R_{\alpha\beta} \equiv R^{\gamma}_{\alpha\gamma\beta}$$ (Ricci tensor)

$$R = g^{\alpha\beta} R_{\alpha\beta}$$ (Ricci scalar)

- Stress-energy tensor $T^{\mu\nu}$ describes matter content of spacetime,

  eg for perfect fluid $T^{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} + p g^{\mu\nu}$



General Relativity in a nutshell (4/4)

Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$ **+** $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}$

$\Rightarrow$ $\nabla_\nu T^{\mu\nu} = 0$

- 4 independent components: conservation of energy and linear momentum
- For a perfect fluid, energy conservation and Euler equation

$$u^\mu \partial_\mu \rho = -(p+\rho) \nabla_\mu u^\mu \qquad a^\mu = -\frac{(g^{\mu\nu} + u^\mu u^\nu) \partial_\nu p}{p+\rho}$$

- For dust (p=0) we get the geodesic equation. Same if we use stress energy tensor for a single particle

Equations of motion of matter follow from Einstein equations

The degrees of freedom of GR

- 4D metric has 10 independent components vs 1 potential of Newtonian theory. What are the other degrees of freedom?
- Let's consider linear perturbations over Minkoskwi background metric, ie $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, with $|h_{\mu\nu}|\ll 1$ and $|T_{\mu\nu}|\ll 1$ (from now on, *G=c=1*)
- If $T_{\mu\nu}, h_{\mu\nu}\to 0$ as $r\to\infty$, most general decomposition is

$$
\begin{aligned}
h_{tt} &= 2\phi \,, \\
h_{ti} &= \beta_i + \partial_i\gamma \,, \\
h_{ij} &= h_{ij}^{\mathrm{TT}} + \frac{1}{3}H\delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda \,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_i\beta_i &= 0 \\
\partial_i\varepsilon_i &= 0 \\
\partial_i h_{ij}^{\mathrm{TT}} &= 0 \\
\delta^{ij} h_{ij}^{\mathrm{TT}} &= 0
\end{aligned}
$$

$$
\begin{aligned}
T_{tt} &= \rho \,, \\
T_{ti} &= S_i + \partial_i S \,, \\
T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma,
\end{aligned}
$$

$$
\begin{aligned}
\partial_i S_i &= 0 \,, \\
\partial_i\sigma_i &= 0 \,, \\
\partial_i\sigma_{ij} &= 0 \,, \\
\delta^{ij}\sigma_{ij} &= 0,
\end{aligned}
$$

Gauge transformations

- Physics does not depend on choice of coordinates, ie we are free to use any coordinate system
- Metric and stress energy transform as

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\alpha\beta}(x(\tilde{x}))\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}}(\tilde{x})\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}}(\tilde{x}) \qquad \tilde{T}_{\mu\nu}(\tilde{x}) = T_{\alpha\beta}(x(\tilde{x}))\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}}(\tilde{x})\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}}(\tilde{x})$$

- For a "small" coordinate change $\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}, \quad |\xi^{\mu}| \ll 1$

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} \qquad \tilde{T}_{\mu\nu} = T_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$$

- Decomposing $(\xi_t, \xi_i) \equiv (A, B_i + \partial_i C)$

$$\begin{array}{lcl} \phi & \to & \phi - \dot{A}\,, \\ \beta_i & \to & \beta_i - \dot{B}_i\,, \\ \gamma & \to & \gamma - A - \dot{C}\,, \\ H & \to & H - 2\nabla^2 C\,, \end{array} \qquad \begin{array}{lcl} \lambda & \to & \lambda - 2C\,, \\ \varepsilon_i & \to & \varepsilon_i - 2B_i\,, \\ h_{ij}^{\mathrm{TT}} & \to & h_{ij}^{\mathrm{TT}}\,. \end{array}$$

Similar expressions for perturbations of stress-energy tensor

The Poisson gauge

- Defined $\partial_i h^{ti} = \partial_i h^{ij} = 0 \Rightarrow \gamma = \lambda = \epsilon_i = 0$

$$
\begin{aligned}
h_{tt} &= 2\phi \,, \\
h_{ti} &= \beta_i + \cancel{\partial_i \gamma} \,, \\
h_{ij} &= h_{ij}^{\mathrm{TT}} + \frac{1}{3} H \delta_{ij} + \cancel{\partial_{(i}\varepsilon_{j)}} + \cancel{\left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \lambda} \,,
\end{aligned}
$$

- Equivalent to using gauge invariant combinations

$$
\begin{aligned}
\Phi &\equiv -\phi + \dot{\gamma} - \frac{1}{2} \ddot{\lambda} \,, \\
\Theta &\equiv \frac{1}{3} \left(H - \nabla^2 \lambda \right) \,, \\
\Xi_i &\equiv \beta_i - \frac{1}{2} \dot{\varepsilon}_i \,;
\end{aligned}
$$

and h_{ij}^{TT}

(already gauge-invariant)

The linearized Einstein equations

$$
\begin{aligned}
G_{tt} &= -\nabla^2\Theta\,,\\
G_{ti} &= -\frac{1}{2}\nabla^2\Xi_i - \partial_i\dot{\Theta}\,,\\
G_{ij} &= -\frac{1}{2}\Box h_{ij}^{\mathrm{TT}} - \partial_{(i}\dot{\Xi}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi+\Theta\right)\\
&\qquad + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi+\Theta\right) - \ddot{\Theta}\right]\,.
\end{aligned}
$$

$$
\begin{aligned}
T_{tt} &= \rho\,,\\
T_{ti} &= S_i + \partial_i S\,,\\
T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma,
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$$
\begin{aligned}
\nabla^2 S &= \dot{\rho}\,,\\
\nabla^2\sigma &= -\frac{3}{2}P + \frac{3}{2}\dot{S}\,,\\
\nabla^2\sigma_i &= 2\dot{S}_i\,.
\end{aligned}
$$

$\left(\text{from } \partial_\mu T^{\mu\nu} = 0\right)$

$$
\begin{aligned}
\nabla^2\Theta &= -8\pi\rho\,,\\
\nabla^2\Phi &= 4\pi\left(\rho + 3P - 3\dot{S}\right)\,,\\
\nabla^2\Xi_i &= -16\pi S_i\,,\\
\Box h_{ij}^{\mathrm{TT}} &= -16\pi\sigma_{ij}\,.
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$\longrightarrow$ h_{tt}, generalizes Newtonian potential

The linearized Einstein equations

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$\nabla^2\Theta$ → h_i^i, appears at 1PN order, ie suppressed by $(v/c)^2$

$\nabla^2\Phi$ → h_{tt}, generalizes Newtonian potential

The linearized Einstein equations

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(from $\partial_\mu T^{\mu\nu}=0$)

$$\nabla^2\Theta = -8\pi\rho\ ,$$

h^i_i, appears at 1PN order, ie suppressed by $(v/c)^2$

$$\nabla^2\Phi = 4\pi\left(\rho + 3P - 3\dot{S}\right)\ ,$$

h_{tt}, generalizes Newtonian potential

$$\nabla^2\Xi_i = -16\pi S_i\ ,$$

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$$\Box h^{\rm TT}_{ij} = -16\pi\sigma_{ij}\ .$$

The linearized Einstein equations

$$G_{tt} = -\nabla^2\Theta\ ,$$
$$G_{ti} = -\frac{1}{2}\nabla^2\Xi_i - \partial_i\dot{\Theta}\ ,$$
$$G_{ij} = -\frac{1}{2}\Box h^{\mathrm{TT}}_{ij} - \partial_{(i}\dot{\Xi}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi+\Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi+\Theta\right) - \ddot{\Theta}\right]\ .$$

$$T_{tt} = \rho\ ,$$
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(from $\partial_\mu T^{\mu\nu} = 0$)

$\nabla^2\Theta = -8\pi\rho\ ,$ → h^i_i, appears at 1PN order, ie suppressed by $(v/c)^2$

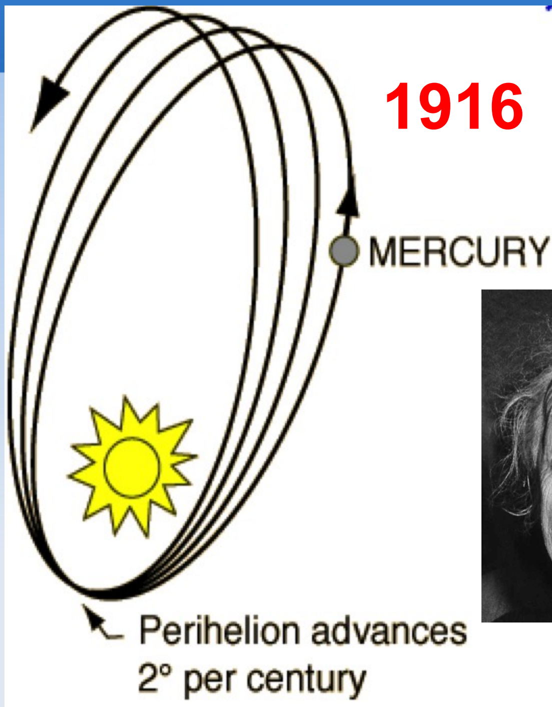
$\nabla^2\Phi = 4\pi\left(\rho + 3P - 3\dot{S}\right)\ ,$ → h_{tt}, generalizes Newtonian potential

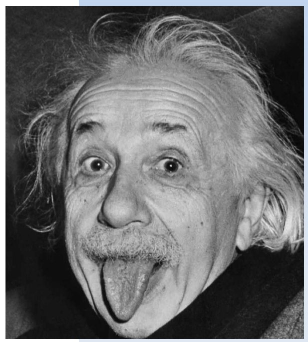
$\nabla^2\Xi_i = -16\pi S_i\ ,$ → h_{ti}, appears at 1PN order, ie suppressed by $(v/c)^2$

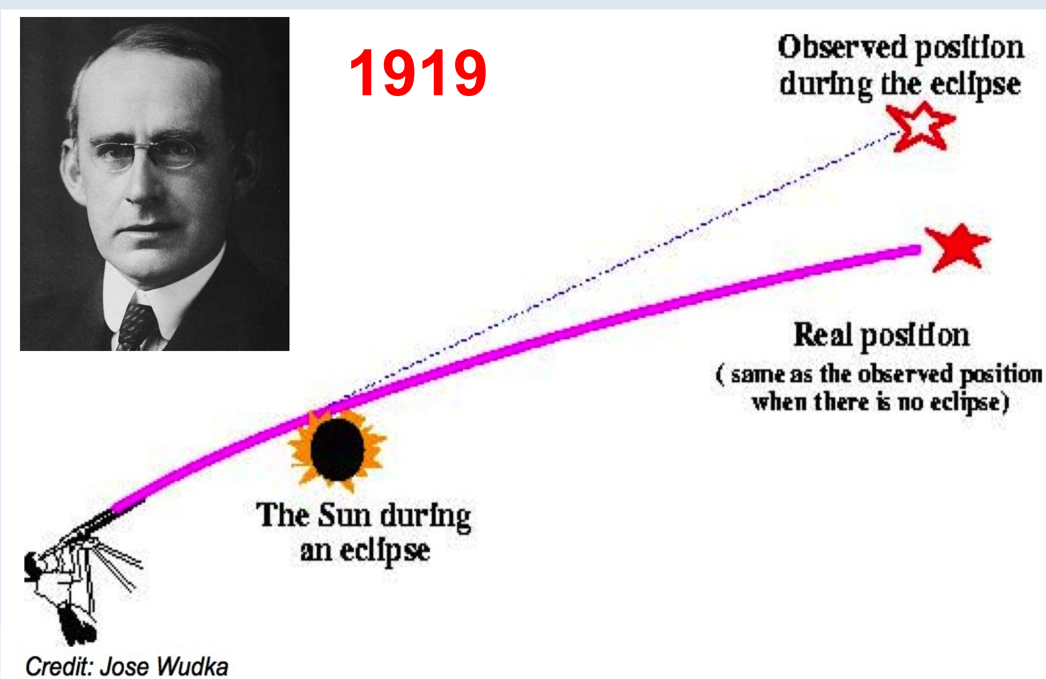
$\Box h^{\mathrm{TT}}_{ij} = -16\pi\sigma_{ij}\ .$ → TT part of h_{ij},

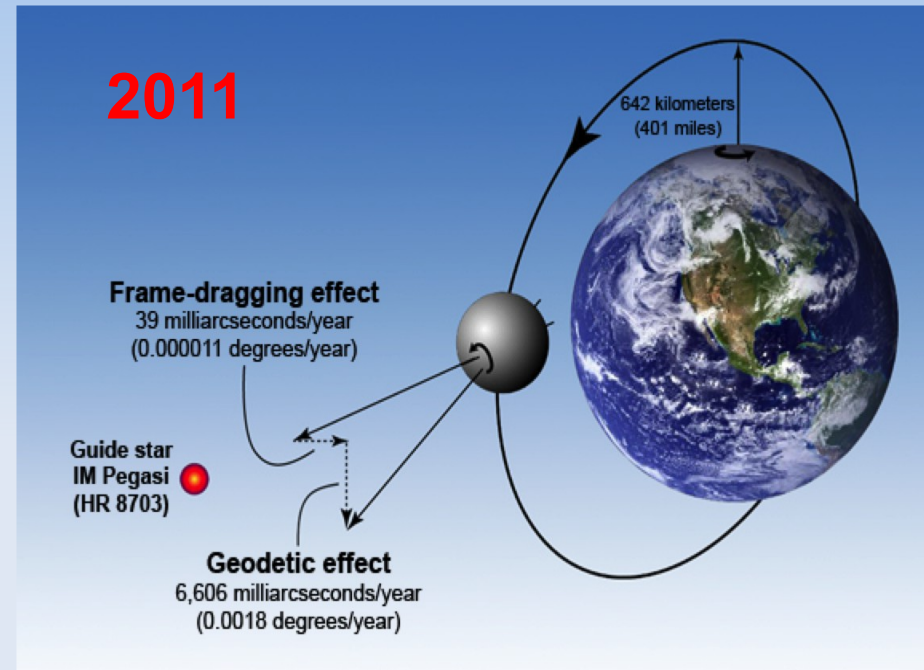
appears at 2PN (conservative part) and 2.5PN order (dissipative part)

1PN effects observed for a century!





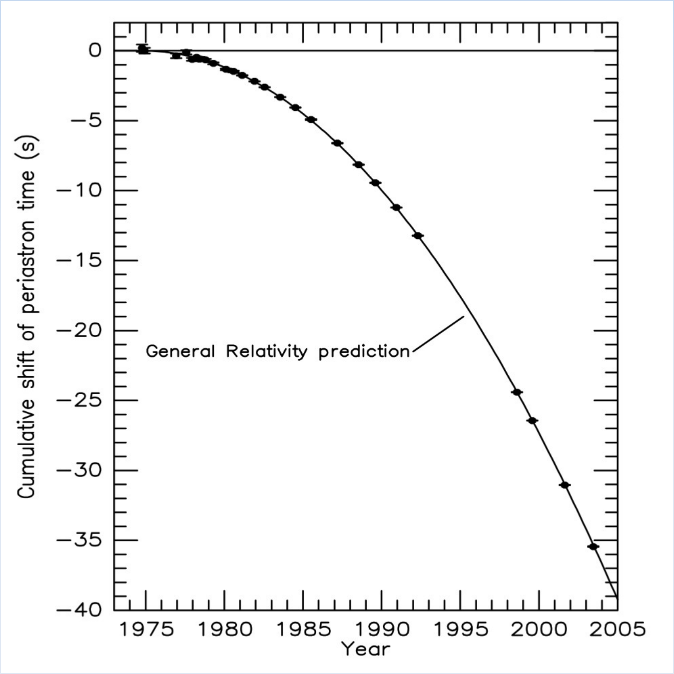


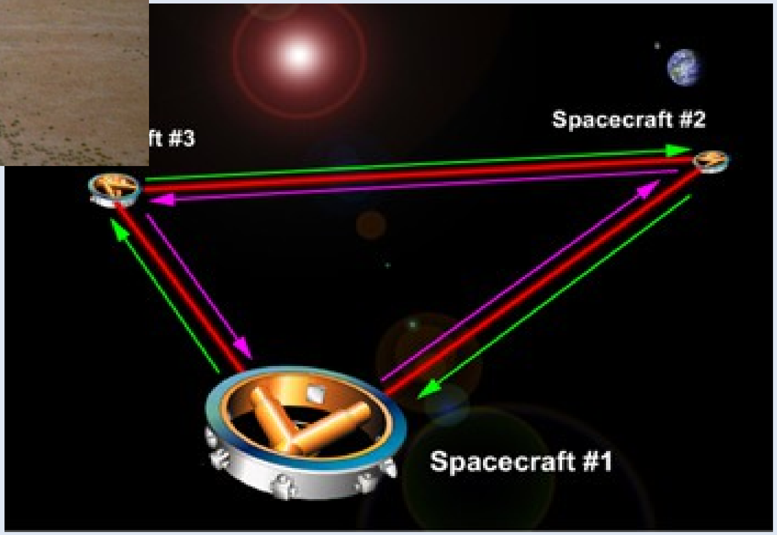


	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

How about h_{ij}^{TT} ?

Gravitational waves!





Indirect detection: GWs carry energy away from binary, which shrinks (ie period decreases)

Direct detection: 2015-19?

The generation of GWs

$$\Box h_{ij}^{\mathrm{TT}} = -16\pi\sigma_{ij}$$

$$G(t,\boldsymbol{x}) = -\frac{1}{4\pi|\boldsymbol{x}|}\delta(t-|\boldsymbol{x}|),$$

$$\Box G(t,\boldsymbol{x}) = \delta(t)\delta^{(3)}(\boldsymbol{x}),$$

$$\chi_{ij}^{\mathrm{TT}}(t,x^i) = 4\int \frac{\sigma_{ij}(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x'$$

$$\sigma_{ij} = P_i^k P_j^l T_{kl} - P_{ij}P^{kl}T_{kl}/2$$

$$P_{ij} = \delta_{ij} - \nabla^{-2}\partial_i\partial_j$$

The generation of GWs

$$\chi_{ij}^{\mathrm{TT}} = -16\pi\Box^{-1}\sigma_{ij} = -16\pi\Box^{-1}\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)T_{kl}$$

$$= -16\pi\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\Box^{-1}T_{kl}$$

$$= 4\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\int\frac{T_{ij}\left(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x'$$

$$\approx \frac{4}{r}\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\int T_{ij}(t-r,\boldsymbol{x}')\mathrm{d}^3x'$$

far from the source, **slow motion**

The generation of GWs

From stress-energy tensor conservation:

$$\partial_t^2\left(T^{tt}x^i x^j\right) = \partial_k\partial_l\left(T^{kl}x^i x^j\right) - 2\partial_k\left(T^{ik}x^j + T^{kj}x^i\right) + 2T^{ij}$$

$$\frac{4}{r}\int d^3x'\,T_{ij} = \frac{4}{r}\int d^3x'\left[\frac{1}{2}\partial_t^2\left(T^{tt}x'^i x'^j\right) + \partial_k\left(T^{ik}x'^j + T^{kj}x'^i\right) - \frac{1}{2}\partial_k\partial_l\left(T^{kl}x'^i x'^j\right)\right]$$

$$= \frac{2}{r}\int d^3x'\,\partial_t^2\left(T^{tt}x'^i x'^j\right) = \frac{2}{r}\frac{\partial^2}{\partial t^2}\int d^3x'\,\rho\,x'^i x'^j = \frac{2}{r}\frac{d^2 I_{ij}(t-r)}{dt^2}$$

$$I_{ij}(t) = \int d^3x'\,\rho(t,\mathbf{x}')\,x'^i x'^j$$

The quadrupole formula, finally!

$$\chi_{ij}^{\mathrm{TT}} \approx \frac{4}{r}\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\int T_{ij}(t-r,\boldsymbol{x}')\mathrm{d}^3x'$$

$$= \frac{2}{r}\frac{d^2\mathcal{I}_{kl}(t-r)}{dt^2}\left[P_{ik}(\mathbf{n})P_{jl}(\mathbf{n}) - \frac{1}{2}P_{kl}(\mathbf{n})P_{ij}(\mathbf{n})\right]\frac{G}{c^4}$$

$$P_{ij} = \delta_{ij} - n_i n_j$$

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I,$$

Quadrupole tensor

small number!

$$I_{ij}(t) = \int d^3x'\,\rho(t,\mathbf{x}')\,x'^i x'^j$$

$$I = I_{ii}\ .$$

An example: a binary system

- Binary with total mass *M*, reduced mass *μ*, separation *R*, orbital frequency *Ω*; orbit lies in xy plane
- Consider GWs along z axis at distance r

$$h_{ij}^{\mathrm{TT}} = h \times \begin{bmatrix} \cos 2\Omega t & \sin 2\Omega t & 0 \\ -\sin 2\Omega t & -\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h = \frac{4\mu\Omega^2 R^2}{r} = \frac{4\mu M^{2/3}\Omega^{2/3}}{r}$$

$$h \simeq 10^{-21}\left(\frac{M}{2\,M_\odot}\right)^{5/3}\left(\frac{1\,\mathrm{hour}}{P}\right)^{2/3}\left(\frac{1\,\mathrm{kiloparsec}}{r}\right)$$

$$\simeq 10^{-22}\left(\frac{M}{2.8\,M_\odot}\right)^{5/3}\left(\frac{0.01\,\mathrm{sec}}{P}\right)^{2/3}\left(\frac{100\,\mathrm{Megaparsecs}}{r}\right)$$

vs $h_{\mathrm{Sun}} \sim M_{\mathrm{sun}}/R_{\mathrm{sun}} \sim 2 \times 10^{-6}$

Generalizing the quadrupole formula

- Why? Approximate because based on slow-motion, weak gravity approximations
- Drop slow-motion approximation = include mass octupole, current quadrupole and higher order terms

$$\bar{h}^{jk} = \frac{2}{r}\left[\ddot{I}^{jk} - 2n_i\ddot{S}^{ijk} + n_i\dddot{M}^{ijk}\right]_{t'=t-r},$$

$$I^{jk}(t') = \int x'^j x'^k T^{00}(t', \mathbf{x}')d^3x'$$

mass quadrupole

current quadrupole

$$S^{ijk}(t') = \int x'^j x'^k T^{0i}(t', \mathbf{x}')d^3x',$$

$$M^{ijk}(t') = \int x'^i x'^j x'^k T^{00}(t', \mathbf{x}')d^3x'.$$

mass octupole

$$\bar{h}^{jk}(t,\mathbf{x}) = \frac{2}{r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int\left[\left(T^{00} - 2T^{0l}n_l + T^{lm}n_l n_m\right)x'^j x'^k\right]_{t'=t-|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x',$$

all multipole moments (Press 1977)

Generalizing the quadrupole formula

- Drop weak-gravity assumption = assume geodesic motion in strongly curved spacetime when calculating source (ie quadrupole, octupole etc)

$$\bar{H}^{\mu\nu} \equiv \eta^{\mu\nu} - (-g)^{1/2} g^{\mu\nu}$$

$$\partial_\beta \bar{H}^{\alpha\beta} = 0$$

harmonic gauge

$$\Longrightarrow \quad \Box_{\text{flat}} \bar{H}^{\alpha\beta} = -16\pi\tau^{\alpha\beta}$$

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + (16\pi)^{-1}\Lambda^{\alpha\beta}$$

Full Einstein equations!

$$\Lambda^{\alpha\beta} = 16\pi(-g)t^{\alpha\beta}_{\text{LL}} + (\bar{H}^{\alpha\mu}{}_{,\nu}\,\bar{H}^{\beta\nu}{}_{,\mu} - \bar{H}^{\alpha\beta}{}_{,\mu\nu}\,\bar{H}^{\mu\nu})$$

$$\begin{aligned}16\pi(-g)t^{\alpha\beta}_{\text{LL}} &\equiv g_{\lambda\mu}g^{\nu\rho}\bar{H}^{\alpha\lambda}_{,\nu}\bar{H}^{\beta\mu}_{,\rho}\\ &+\frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\bar{H}^{\lambda\nu}_{,\rho}\bar{H}^{\rho\mu}_{,\nu} - 2g_{\mu\nu}g^{\lambda(\alpha}\bar{H}^{\beta)\nu}_{,\rho}\bar{H}^{\rho\mu}_{,\lambda}\\ &+\frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\bar{H}^{\nu\tau}_{,\lambda}\bar{H}^{\rho\sigma}_{,\mu}\end{aligned}$$

Generalizing the quadrupole formula

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$$\begin{aligned}
16\pi(-g)t^{\alpha\beta}_{\text{LL}} &\equiv g_{\lambda\mu}g^{\nu\rho}\bar{H}^{\alpha\lambda}_{,\nu}\bar{H}^{\beta\mu}_{,\rho} \\
&+\frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\bar{H}^{\lambda\nu}_{,\rho}\bar{H}^{\rho\mu}_{,\nu} - 2g_{\mu\nu}g^{\lambda(\alpha}\bar{H}^{\beta)\nu}_{,\rho}\bar{H}^{\rho\mu}_{,\lambda} \\
&+\frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\bar{H}^{\nu\tau}_{,\lambda}\bar{H}^{\rho\sigma}_{,\mu}
\end{aligned}$$

From gauge condition,

$$\tau^{\alpha\beta}{}_{,\beta} = 0$$

= geodesic motion in curved metric *g*

Generalizing the quadrupole formula

Quadrupole (or quadrupole + octupole + higher moments) formula + geodesic motion is usually decent approximation, eg for particle around Kerr BH (”kludge” waveforms)

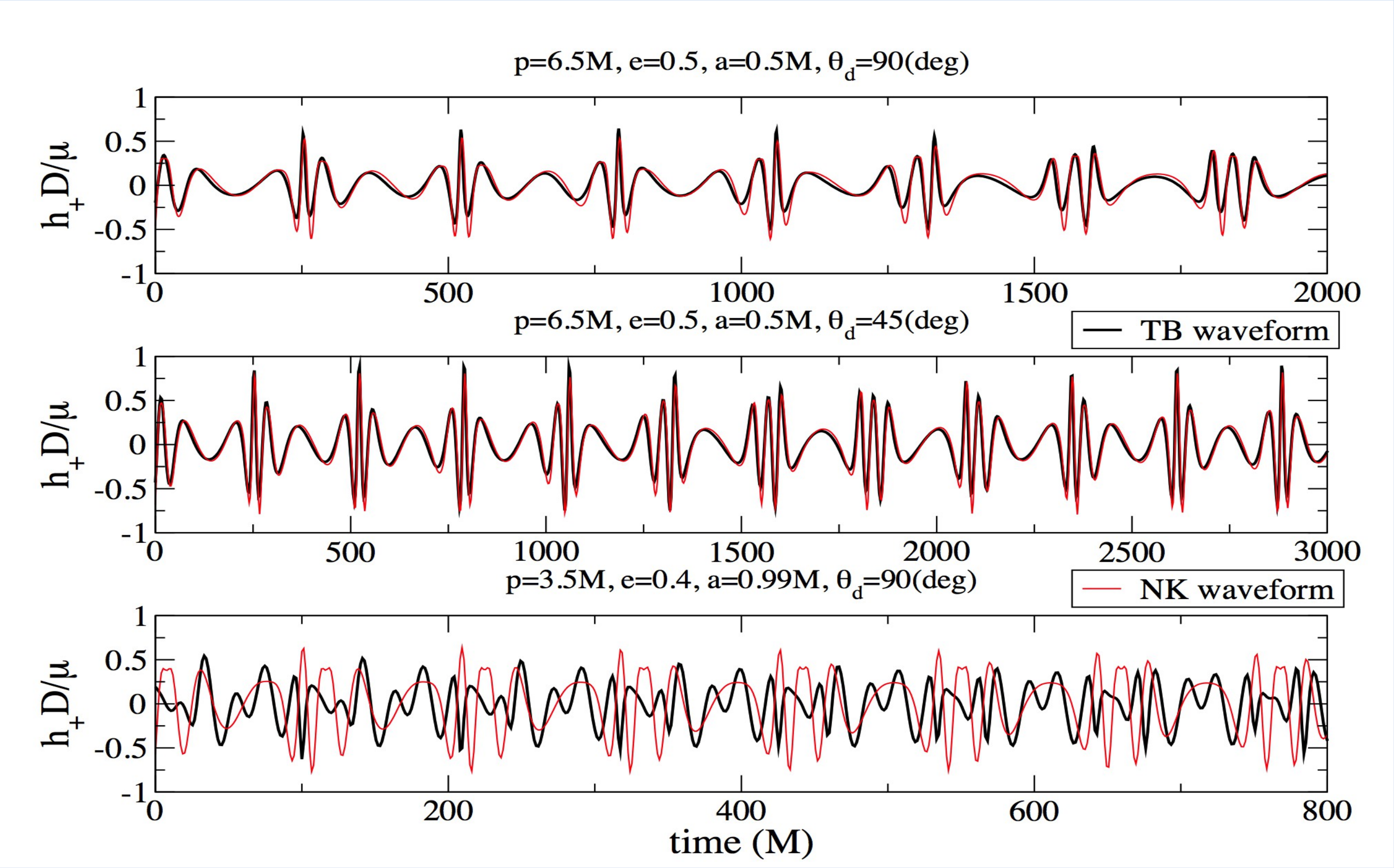
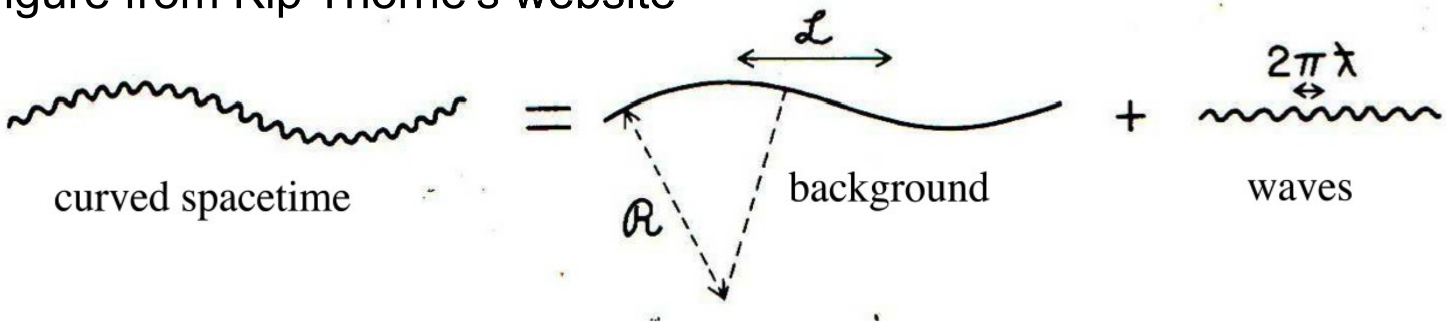


Figure from Babak et al Phys. Rev. D 75, 024005 (2007)

The stress energy tensor of GWs

Figure from Kip Thorne's website



geometric-optics regime

$$g^{\mathrm{B}}_{\alpha\beta} \quad \equiv \quad \langle g_{\alpha\beta} \rangle$$

$$g_{\alpha\beta} = g^{\mathrm{B}}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} + O(\varepsilon^3)$$

$$\begin{aligned} 0 &= G_{\alpha\beta} \\ &= G_{\alpha\beta}[g^{\mathrm{B}}_{cd}] + \varepsilon G^{(1)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] + \varepsilon^2 G^{(1)}_{\alpha\beta}[j_{cd}; g^{\mathrm{B}}_{ef}] + \varepsilon^2 G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] \\ &\quad + O(\varepsilon^3)\,. \end{aligned}$$

$$G_{\alpha\beta}[g^{\mathrm{B}}_{cd}] = 0, \qquad G^{(1)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] = 0\,, \qquad G^{(1)}_{\alpha\beta}[j_{cd}; g^{\mathrm{B}}_{ef}] = -G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}]\,.$$

The stress energy tensor of GWs

Average Einstein equations on scale >> λ and << L

$$\Delta j_{\alpha\beta} = j_{\alpha\beta} - \langle j_{\alpha\beta} \rangle$$

$$G^{(1)}_{\alpha\beta}[\langle j_{cd}\rangle; g^{\mathrm{B}}_{ef}] = -\langle G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}]\rangle$$

$$G^{(1)}_{\alpha\beta}[\Delta j_{cd}] = -G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] + \langle G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}]\rangle$$

$$G_{\alpha\beta}[g^{\mathrm{B}}_{cd} + \varepsilon^2 \langle j_{cd}\rangle] = 8\pi G\, T^{\mathrm{GW,eff}}_{\alpha\beta} + O(\varepsilon^3)$$

$$T^{\mathrm{GW,eff}}_{\alpha\beta} = -\frac{1}{8\pi G}\langle G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}]\rangle$$

Using gauge freedom and intregrating by parts:

$$T^{\mathrm{GW,eff}}_{\alpha\beta} = \frac{1}{32\pi G}\left\langle \nabla^{\mathrm{B}}_{\alpha} h^{\mathrm{TT}}_{\rho\sigma}\, \nabla^{\mathrm{B}}_{\beta} h^{\rho\sigma}_{\mathrm{TT}} \right\rangle$$

The GW luminosity

Quadrupole formula + GW stress energy tensor

$$L_{\text{mass quadrupole}} \equiv \frac{1}{5}\frac{G}{c^5}\langle\, \dddot{I}\, \rangle^2 = \frac{1}{5}\frac{G}{c^5}\langle \dddot{\text{\textit{I}}\!\!\!-}_{jk}\dddot{\text{\textit{I}}\!\!\!-}_{jk}\rangle^2$$

$$\dddot{\text{\textit{I}}\!\!\!-}_{jk} \sim \frac{(\text{mass of the system in motion}) \times (\text{size of the system})^2}{(\text{time scale})^3} \sim \frac{MR^2}{\tau^3} \sim \frac{Mv^2}{\tau}$$

$$L_{\text{mass quadrupole}} \sim \frac{G}{c^5}\frac{Mv^2}{\tau}$$

$$G/c^5 \sim 10^{-59}$$

Conversion of any type of energy into GWs is inefficient, unless large masses and/or v ~ c

Propagation of GWs

GW propagating in z-direction

$$\Box h_{ij}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad h_{ij}^{\mathrm{TT}} = h_{ij}^{\mathrm{TT}}(t-z)$$

$$\partial_z h_{zj}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad h_{zj}^{\mathrm{TT}} = 0$$

$$h_{ii}^{TT} = 0 \quad \Rightarrow \quad \begin{aligned} h_{xx}^{\mathrm{TT}} &= -h_{yy}^{\mathrm{TT}} \equiv h_+(t-z)\,; \\ h_{xy}^{\mathrm{TT}} &= h_{yx}^{\mathrm{TT}} \equiv h_\times(t-z)\,. \end{aligned}$$

Propagation of GWs

$$\boldsymbol{h}^{\mathrm{TT}} = h^{+}(t-z)\boldsymbol{e}^{+} + h^{\times}(t-z)\boldsymbol{e}^{\times}$$

$$\boldsymbol{e}^{+} \equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_x - \boldsymbol{e}_y \otimes \boldsymbol{e}_y \,,$$
$$\boldsymbol{e}^{\times} \equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_y + \boldsymbol{e}_y \otimes \boldsymbol{e}_x \,.$$

Linear polarization

$$h^{+}(t-z) = h(t-z)\cos 2\lambda$$
$$h^{\times}(t-z) = h(t-z)\sin 2\lambda$$

Circular polarization

$$h^{\times}(t-z) = \pm i h(t-z)$$
$$h^{+}(t-z) = h(t-z)$$

Elliptic polarization=
other phase differences

Binary with masses m_1 and m_2, separation R, orbital frequency Ω, distance r;

θ = angle between orbital angular momentum and direction to observer
(θ = 0 or 180 deg: face-on; θ = 90: edge on)

$$h^{+} = \frac{2m_1 m_2}{rR}(1+\cos^2\theta)\cos[2\Omega(t-r)+2\Delta\phi] \,,$$
$$h^{\times} = -\frac{4m_1 m_2}{rR}\cos\theta\sin[2\Omega(t-r)+2\Delta\phi] \,,$$

How to detect GWs

Measure proper (i.e. physical) distance between two free-falling test masses

- Each test mass follows geodesics

Consider flat spacetime + GW (TT perturbation)

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{d x^\alpha}{d\tau} \frac{d x^\beta}{d\tau} = 0$$

$$\frac{d^2 x^i}{dt^2} = -(\Gamma^i_{tt} + 2\Gamma^i_{tj} v^j + \Gamma^i_{jk} v^j v^k) + v^i (\Gamma^t_{tt} + 2\Gamma^t_{tj} v^j + \Gamma^t_{jk} v^j v^k)$$

$$v^i = d x^i d t \ll 1 \quad \Rightarrow \quad \frac{d^2 x^i}{dt^2} + \Gamma^i{}_{tt} = 0 \qquad \Gamma^i{}_{tt} = \frac{1}{2}\left(2\partial_t h^{TT}_{jt} - \partial_j h^{TT}_{tt}\right) = 0$$

- Coordinate positions of test massive unaffected, but how about proper distance?

How to detect GWs

GW in z direction, test masses at x=0, y=0 and x=L_c, y=0

$$L = \int_0^{L_c} dx\,\sqrt{g_{xx}} = \int_0^{L_c} dx\,\sqrt{1 + h_{xx}^{\mathrm{TT}}(t, z = 0)}$$

$$\simeq L_c \left[1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}(t, z = 0)\right]$$

$$\frac{\delta L}{L} \simeq \frac{1}{2} h_{xx}^{\mathrm{TT}}(t, z = 0)$$

Measurable effect!

A better derivation

- Locally flat coordinates $$ds^2 = -dt^2 + d\mathbf{x}^2 + O\left(\frac{\mathbf{x}^2}{\mathcal{R}^2}\right)$$
- Geodesics at $x^i = 0$ and $x^i = L^i(t)$
- Proper distance is $\sqrt{L^i L_i}$ up to errors $\sim h L^2/\lambda^2 \ll 1$ (eg for Earth-based detectors, *L*~ km and *λ* >~ 3000 km, but approximation **not** valid for space detectors)
- Separation vector L^μ between two geodesics obeys geodesic deviation equation (derivation on magic board?)

$$\frac{D^2 L^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\gamma} u^\alpha u^\beta L^\gamma$$

- With $u^\mu = \delta^\mu_t$ and $L^\mu = (0, L^i)$ $$\frac{d^2 L^i(t)}{dt^2} = -R_{itjt}(t, \mathbf{0}) L^j(t)$$

A better derivation

$$R_{itjt} = -\frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij} + \Phi_{,ij} + \dot{\Xi}_{(i,j)} - \frac{1}{2}\ddot{\Theta}\delta_{ij}$$

$$\Theta \sim \Phi \sim \frac{\text{mass}}{r} \qquad \Xi \sim \frac{\text{linear momentum}}{r}$$

$\Rightarrow$ Only TT piece (=GW) contributes far from the source!

More formally, let's show this from field equations on the magic board

$$\begin{aligned} \nabla^2\Theta &= -8\pi\rho\,, \\ \nabla^2\Phi &= 4\pi\left(\rho + 3P - 3\dot{S}\right) \\ \nabla^2\Xi_i &= -16\pi S_i\,, \end{aligned}$$

$$\begin{aligned} \nabla^2 S &= \dot{\rho}\,, \\ \nabla^2\sigma &= -\frac{3}{2}P + \frac{3}{2}\dot{S}\,, \\ \nabla^2\sigma_i &= 2\dot{S}_i\,. \end{aligned}$$

$$\begin{aligned} T_{tt} &= \rho\,, \\ T_{ti} &= S_i + \partial_i S\,, \\ T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \end{aligned}$$

A back-of-the-envelope derivation of the quadrupole formula

Moments of mass and current distributions:

$$M_0 \equiv \int \rho \, d^3x = M$$

$$S_1 \equiv \int \rho v_j \, x_k \, \epsilon_{ijk} \, d^3x = S_i$$

$$M_1 \equiv \int \rho \, x_i \, d^3x = M L_i$$

$$M_2 \equiv \int \rho \, x_i \, x_j \, d^3x = M L_{ij}$$

$$\xcancel{h \sim \frac{G}{c^2}\frac{M_0}{r}}$$

Conservation of mass

$$\xcancel{h \sim \frac{G}{c^3}\frac{d}{dt}\frac{M_1}{r} \sim \frac{G}{c^3}\frac{P}{r}}$$

Conservation of linear momentum

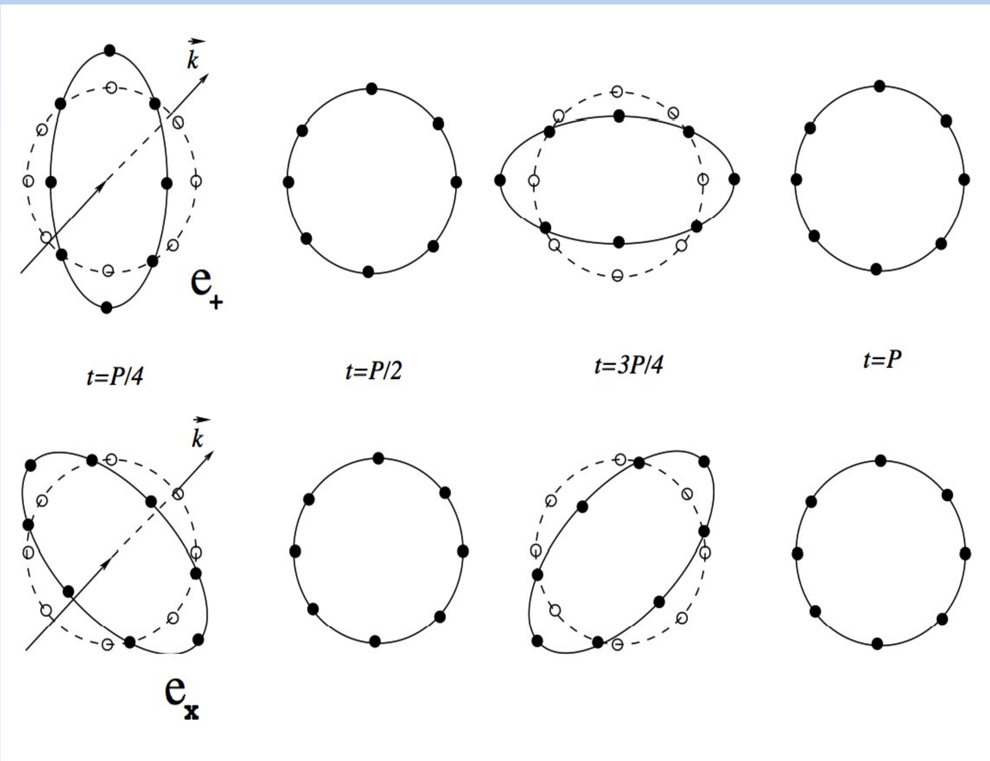
$$\xcancel{h \sim \frac{G}{c^4}\frac{d}{dt}\frac{S_1}{r}}$$

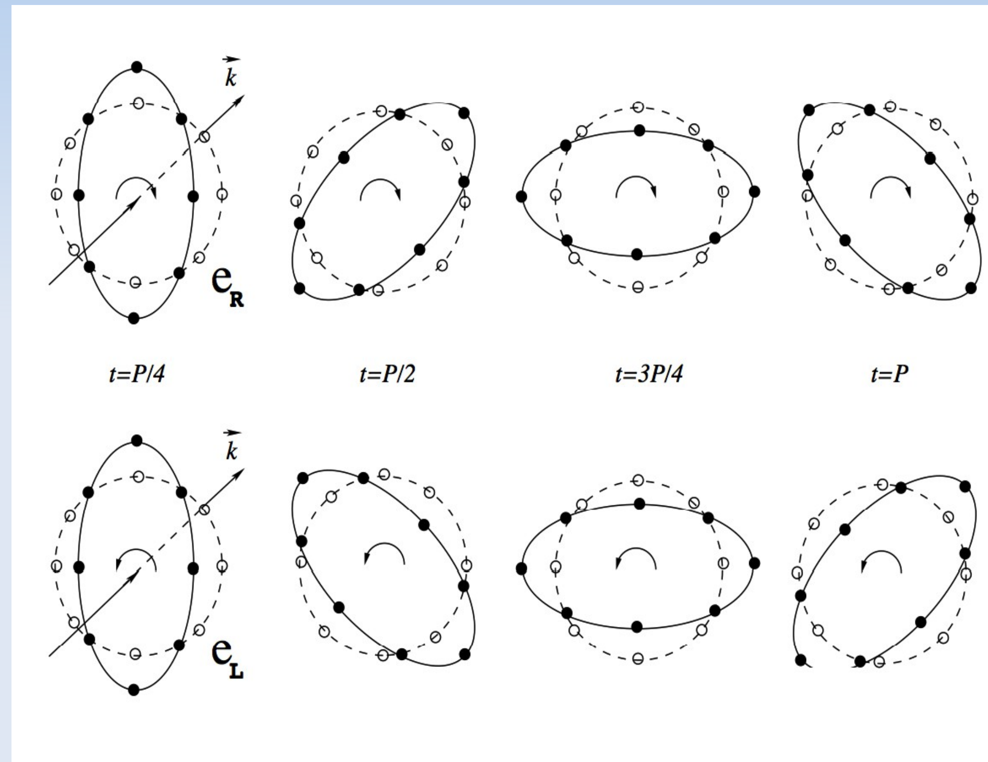
Conservation of angular momentum

$$h \sim \frac{G}{c^4}\frac{d^2}{dt^2}\frac{M_2}{r}$$

Geometrical meaning of h_+ and $h_\times$

Figures from Rezzolla's notes





$$\mathbf{e}_+ \equiv \vec{e}_x \otimes \vec{e}_x - \vec{e}_y \otimes \vec{e}_y$$

$$\mathbf{e}_\times \equiv \vec{e}_x \otimes \vec{e}_x + \vec{e}_y \otimes \vec{e}_y$$

$$\mathbf{e}_{\mathrm{R}} \equiv \frac{\mathbf{e}_+ + i\mathbf{e}_\times}{\sqrt{2}}$$

$$\mathbf{e}_{\mathrm{L}} \equiv \frac{\mathbf{e}_+ - i\mathbf{e}_\times}{\sqrt{2}}$$

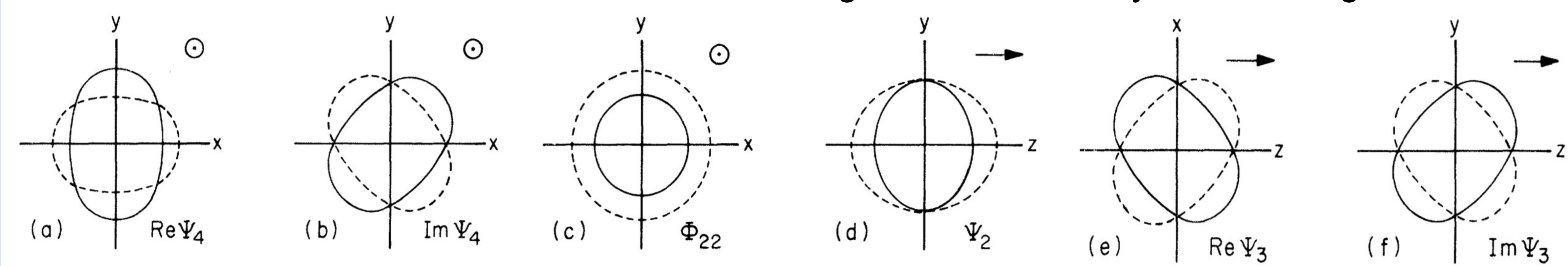
Beyond GR: more polarizations?

Similar decomposition of Riemann tensor in vacuum via Newman-Penrose scalars

$$\Psi_2(u) = -\tfrac{1}{6} R_{z0z0}(u)\,,$$
$$\Psi_3(u) = -\tfrac{1}{2} R_{x0z0} + \tfrac{1}{2} i\, R_{y0z0}\,,$$
$$\Psi_4(u) = -R_{x0x0} + R_{y0y0} + 2i\, R_{x0y0}\,,$$
$$\Phi_{22}(u) = -R_{x0x0} - R_{y0y0}\,.$$

$$\Psi_2(u) \quad (s=0)\,, \qquad \Phi_{22}(u) \quad (s=0)$$
$$\Psi_3(u) \quad (s=\pm 1)\,, \qquad \Psi_4(u) \quad (s=\pm 2)$$

Figures from Eardley, Lee and Lightman 1973



e.g. Dipolar emission if equivalence principle is violated (Brans-Dicke, scalar tensor theories, etc)

$$h \sim \frac{1}{R}\frac{d}{dt}\left(m_{\mathrm{GW},1}\mathbf{x}_1 + m_{\mathrm{GW},2}\mathbf{x}_2\right) \sim \frac{\eta m}{R}\mathbf{v}\left(\frac{m_{\mathrm{GW},1}}{m_{\mathrm{I},1}} - \frac{m_{\mathrm{GW},2}}{m_{\mathrm{I},2}}\right)$$

A real detector: frequencies and not distances

- Detectors are laser interferometers
- Photons acculumulate phase change $\delta\phi = 4\pi\delta L/\lambda$ when proper distance between "mirrors" change

  Analysis valid for $L << \lambda$ (ground-based detectors)
- More in general (space-based detectors), we can integrate photon geodesics between mirrors; photon frequency will change due to GW and produce phase change

$$\frac{\Delta\nu}{\nu} = \frac{1}{2}(1+\cos\theta)\Psi(t) - \cos\theta\,\Psi(t+\tau(1-\cos\theta)/2) - \frac{1}{2}(1-\cos\theta)\Psi(t+\tau)\,,$$

$$\cos\theta \equiv \boldsymbol{\sigma}\cdot\boldsymbol{n}\,,$$

$$\Psi(t) \equiv \frac{h_{ij}^{\rm TT}\sigma^i\sigma^j}{\sin^2\theta}$$

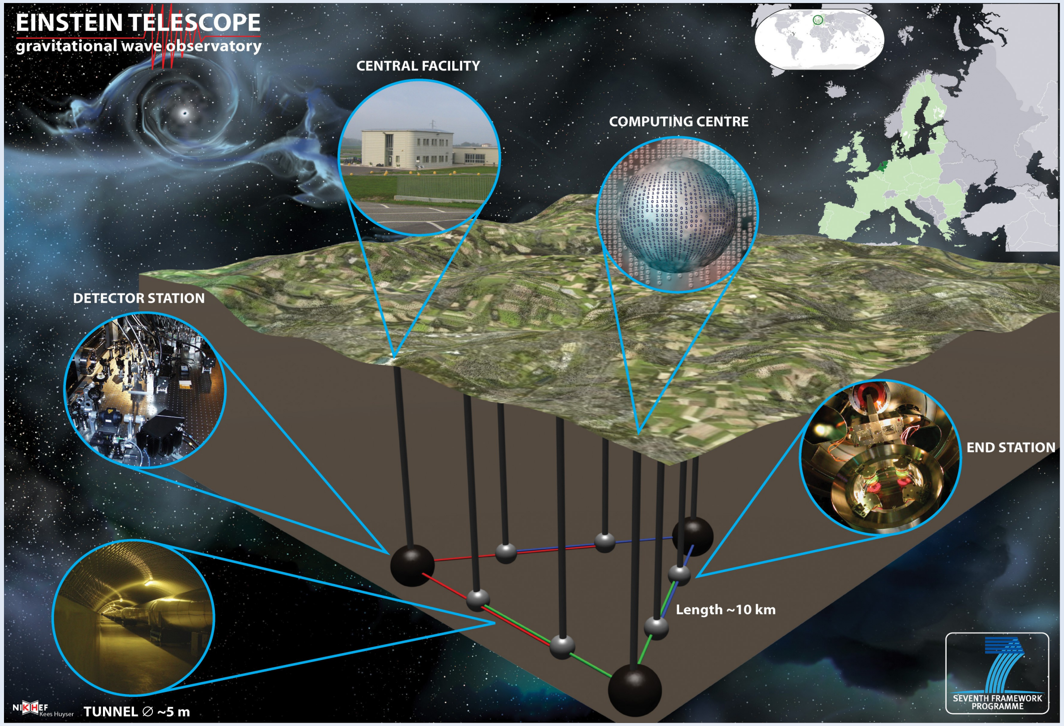
$$\Delta\Phi = \int_0^t \Delta\nu(t')dt'$$

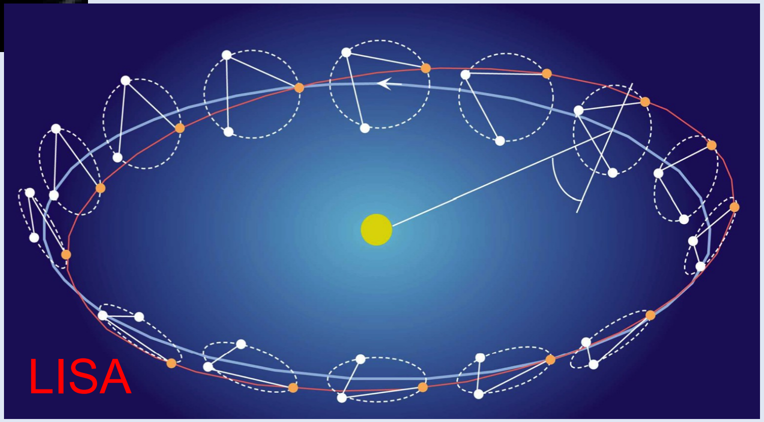
$\tau =$ laser travel time between mirrors

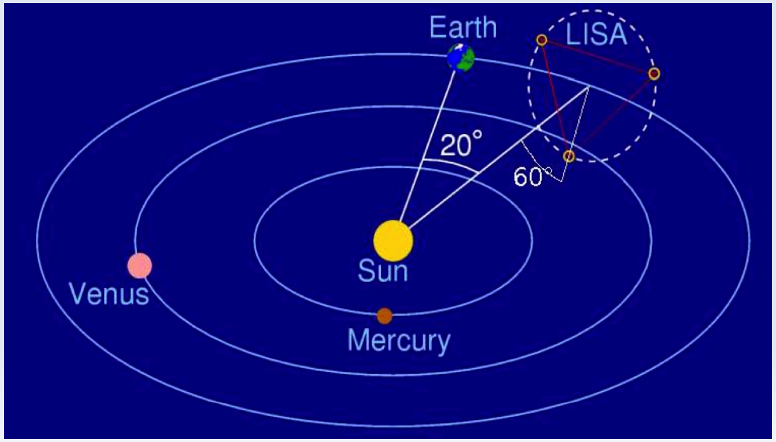
$\boldsymbol{\sigma}$ and $\boldsymbol{n} =$ propagation directions of laser and GW

Existing and future detectors

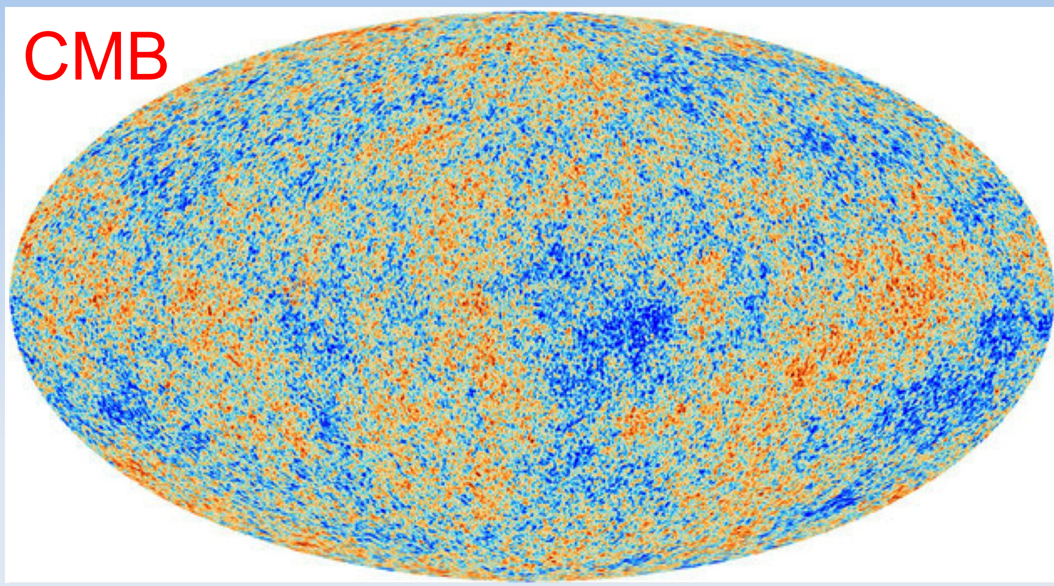
Virgo: 3km
LIGO's: 4 and 2 km
GEO 600: 600m
TAMA 300: 300m
AIGO: 80 m

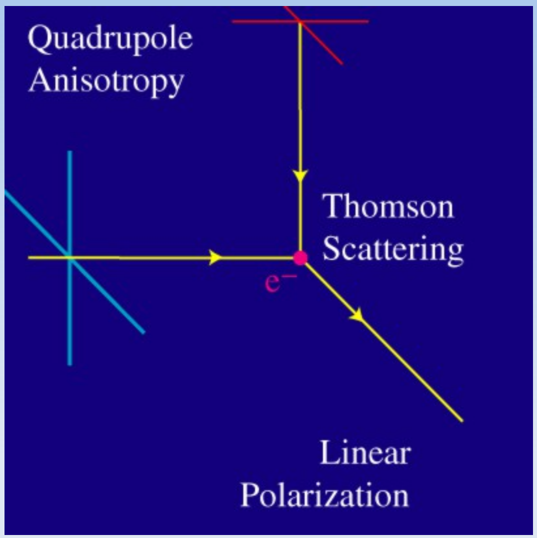




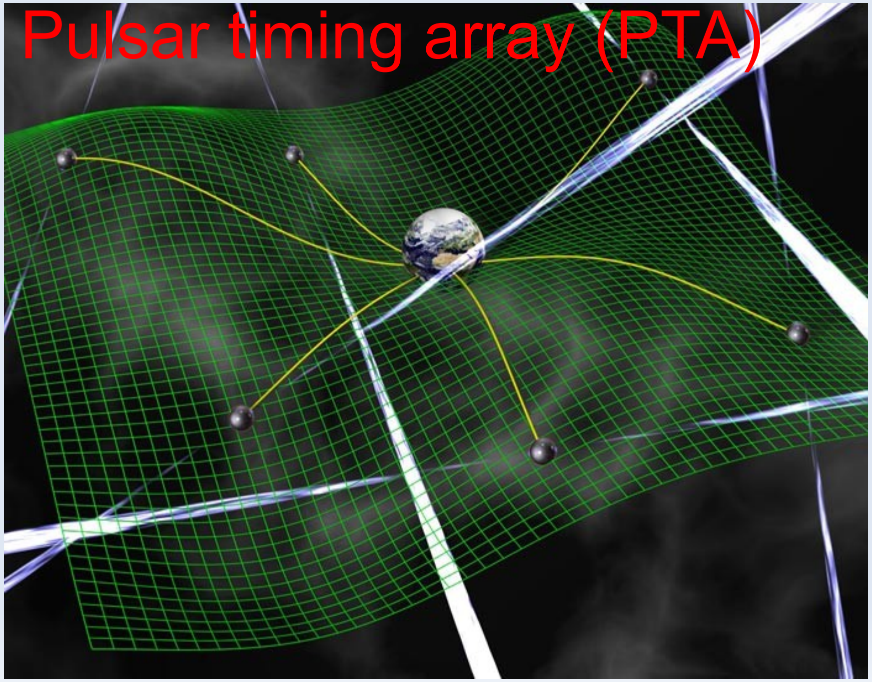


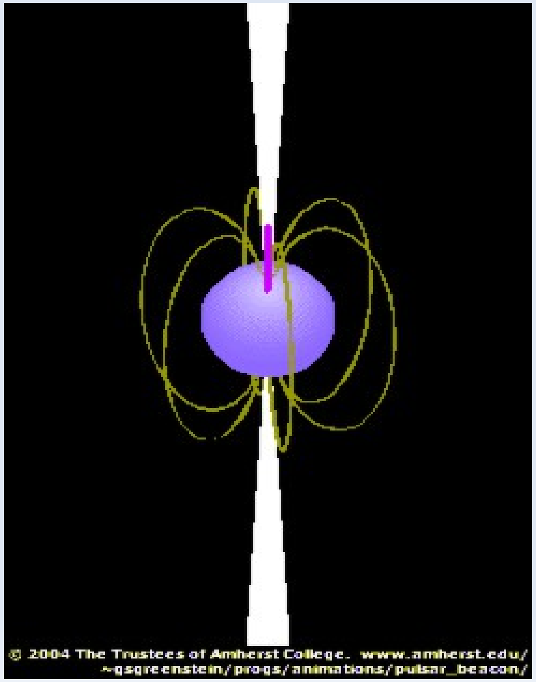
Astrophysical detectors





Animation from Hu 2001





Frequency ranges

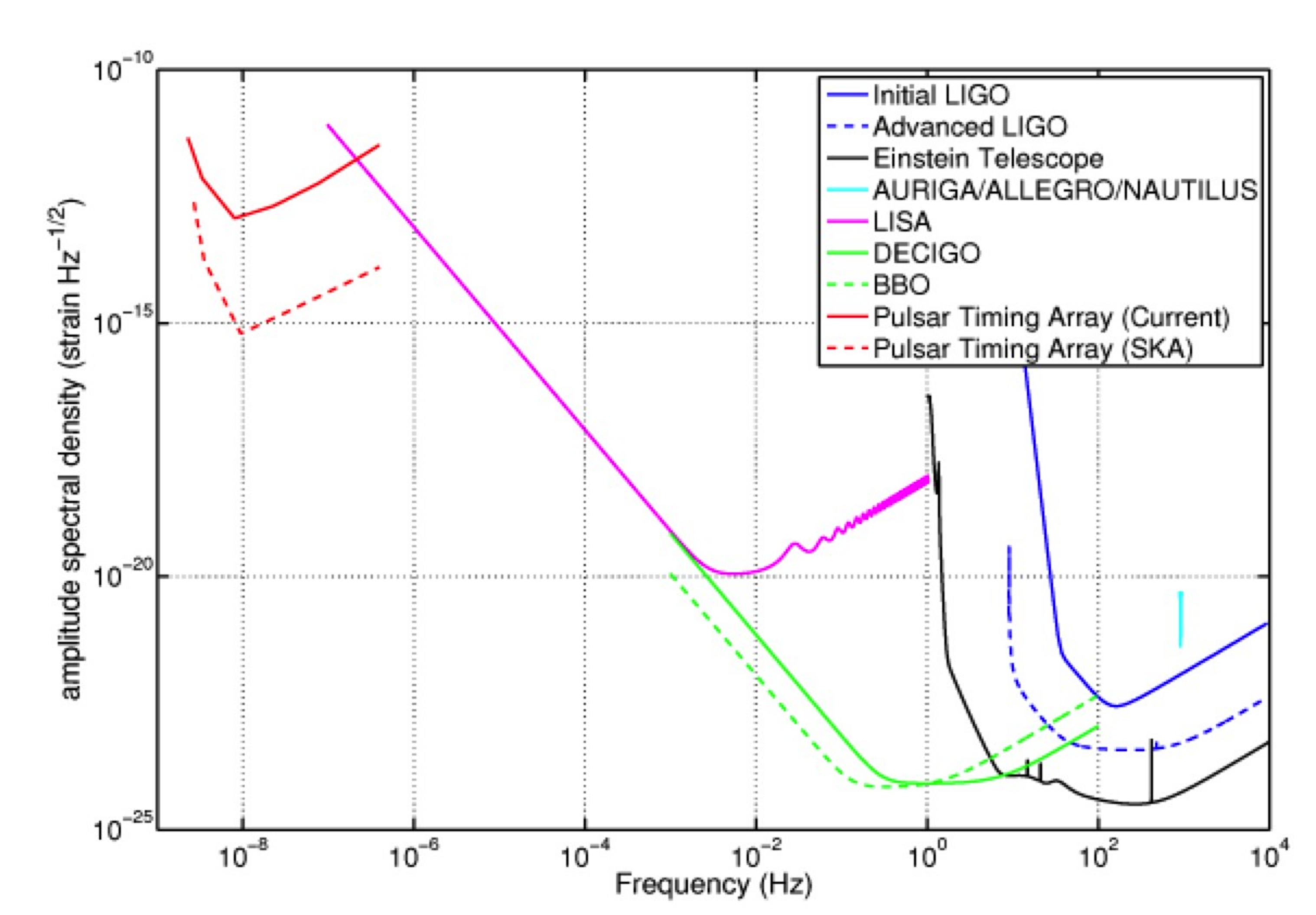


Figure from Pitkin et al Living Rev. Relativity 14, (2011)

GWs from binary systems

From quadrupole formula, GW frequency is twice orbital one

$$f_{\rm GW} = \frac{6 \times 10^4}{\tilde{m}\tilde{R}^{3/2}}\,{\rm Hz}$$

$$\tilde{R} = R/m$$

$$\tilde{m} = m/M_\odot$$

aLIGO/aVirgo:

1) Late inspiral of NS-NS: from few to hundreds of events per year

Binary pulsars observed with masses ~ 1.4 M_{sun}, but isolated NS can have masses   2 M_{sun}

2) BH-NS and BH-BH late inspiral and merger: rates unclear, possibly hundreds per year

BH candidates with mass $\gtrsim$ 10 M_{sun} observed in isolation

3) If intermediate mass BHs exists, IMBH-BH/NS/WD and IMBH-IMBH observable up to total masses ~ 1000 M_{sun}

GWs from binary systems

LISA:

Supermassive BHs observed in center of galaxies with masses ~ 10^5 – 10^9 M_{sun}; believed to merge when galaxies merge (cf double AGNs)

1) Inspiral and merger of SMBH-SMBH (with masses ~ 10^5 – 10^6 M_{sun}): from a few to hundreds per year

2) Inspiral and merger of SMBH – BH/NS/WD (aka Extreme Mass Ratio Inspirals, EMRIs): rates uncertain, from a few to hundreds/thousands per year

3) IMBH-SMBH: rates uncertain

4) WD-WD at separations of a few star radii (~ 10^5 km): thousands of resolved sources, a few guaranteed sources in the Galaxy

Pulsar timing array:

SMBH-SMBH at $0.2 < z < 1.5$, with masses $\gtrsim 5 \times 10^8$ M_{sun} and separations of hundreds gravitational radii

GWs from isolated systems

- Rotating axisymmetric star/spherical collapse do not emit
- Core collapse supernovae (type II) produce burst of GWs if instabilities develop due to high rotational velocities, or if asymmetries are present:

  possible sources for LIGO/Virgo/Einstein telescope
- Rotating pulsar can radiate monochromatically if rotation deviates from axisymmetry: possible sources for LIGO/Virgo/Einstein telescope but no good model for ϵ

$$h \sim \frac{G}{c^4}\frac{I f^2 \epsilon}{r} \qquad \epsilon = (I_{xx} - I_{yy})/I$$

LIGO/Virgo will constrain $\epsilon < 10^{-6}$

Stochastic backgrounds

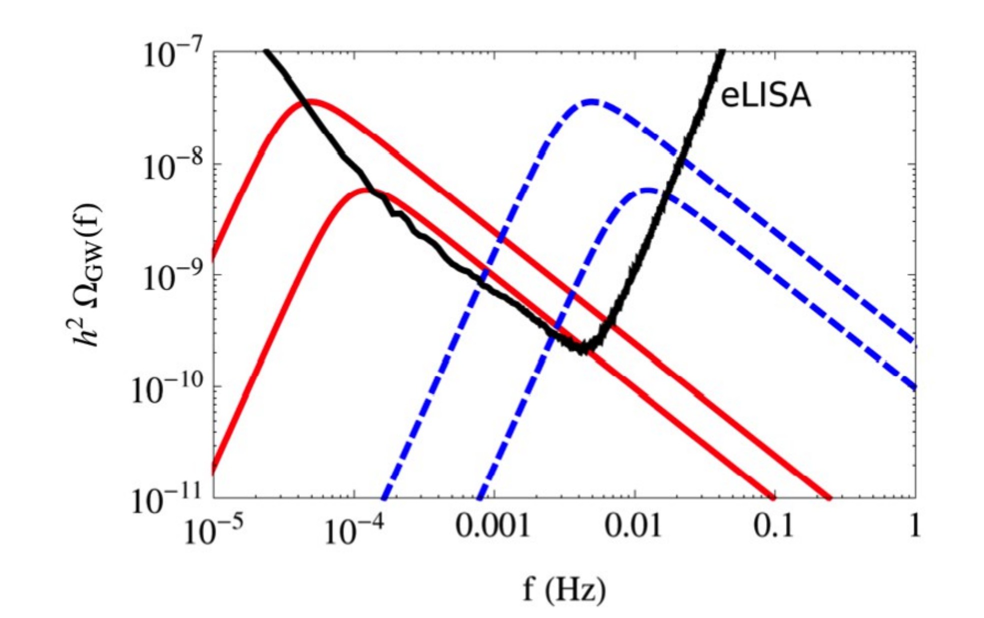
- Isotropic and homogenous (cosmological origin)
- Look like noise by can be detected by cross-correlating detectors
- Inflationary GWs depend on energy scale of inflation

$$\Omega_{gw}(f) \propto (E_{inflation}/M_P)^4 \approx \text{constant} \quad \Rightarrow \quad E_{inflation} < 1.9\times 10^{16}\,\text{GeV}$$

- GWs produced by phase transitions have peaked spectrum

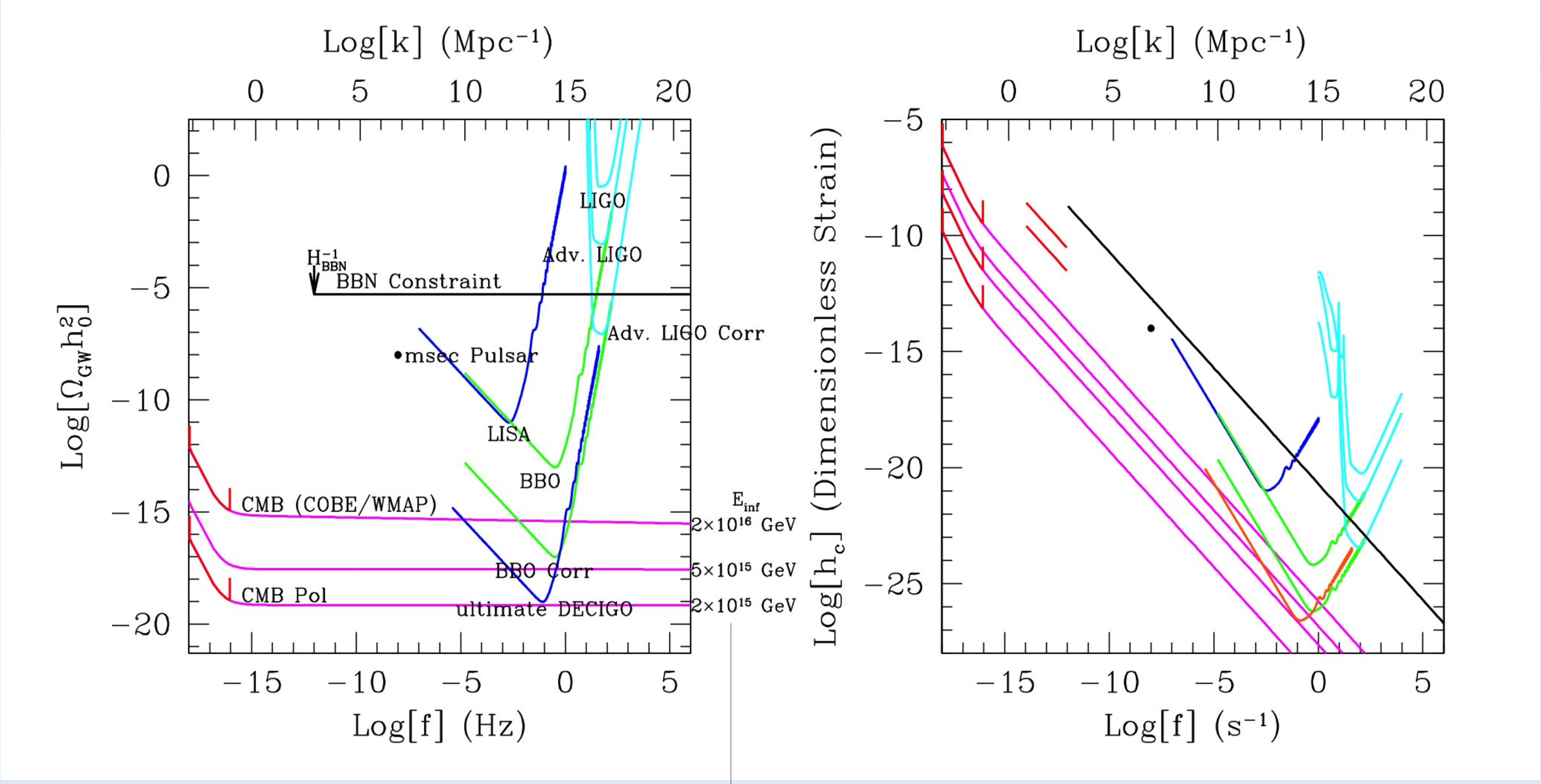
$$f_{\text{peak}} \sim 100\,\text{Hz}\left(\frac{T}{10^5\,\text{TeV}}\right)$$

E.g. some exotic models
(eg extra dimensions, cosmic
strings) could produce phase
transitions observable by LISA
(Dufaux 2012)



Frequency ranges

Figure from A. Cooray, astro-ph/0503118



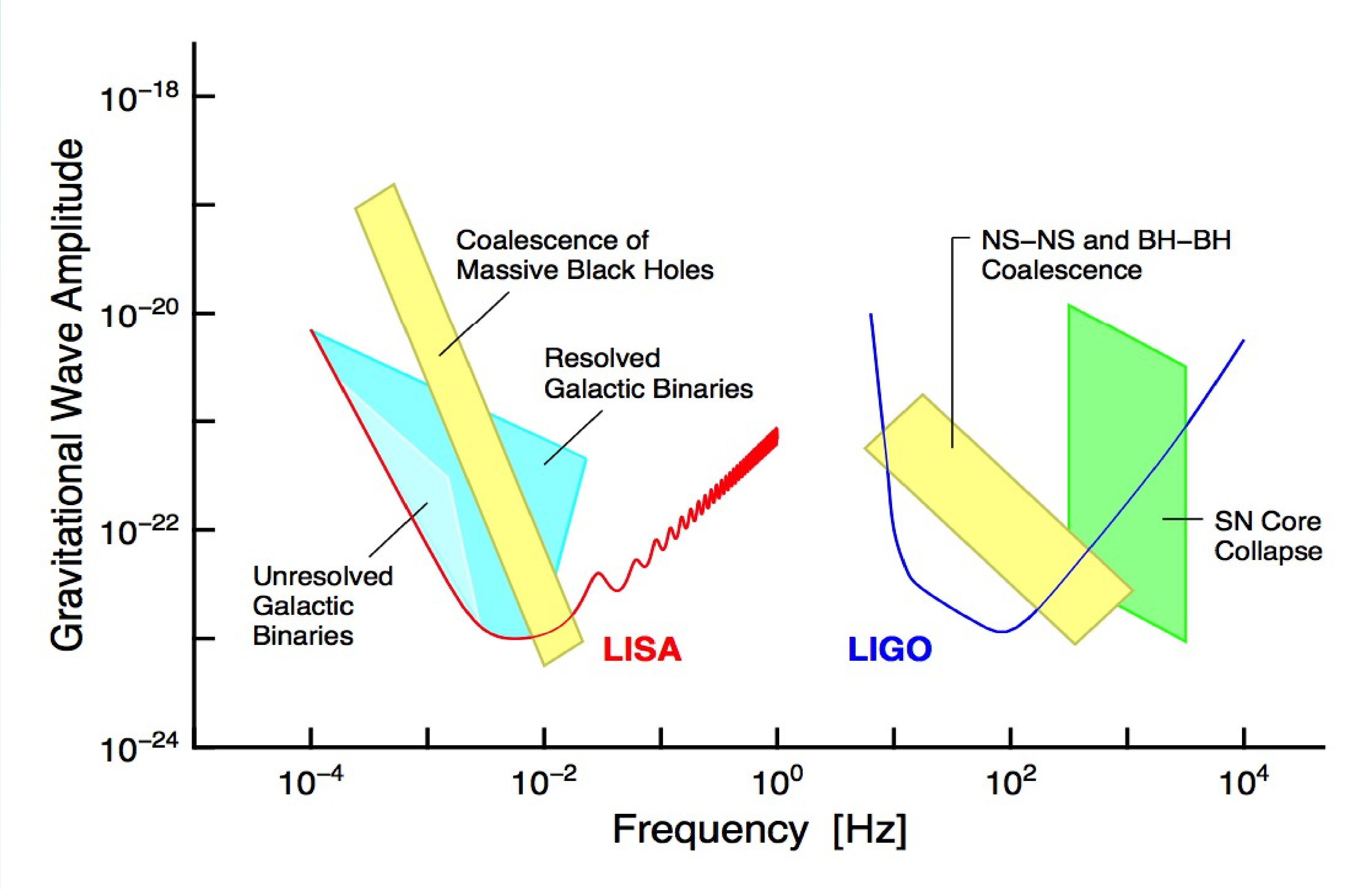
$$\Omega_{\rm gw}(f) = \frac{1}{\rho_{\rm crit}} \frac{d\rho_{\rm gw}}{d\ln f}$$

Energy scale of inflation

$$h \propto f^{-3/2}\sqrt{\Omega_{\rm gw}(f)\Delta f}$$

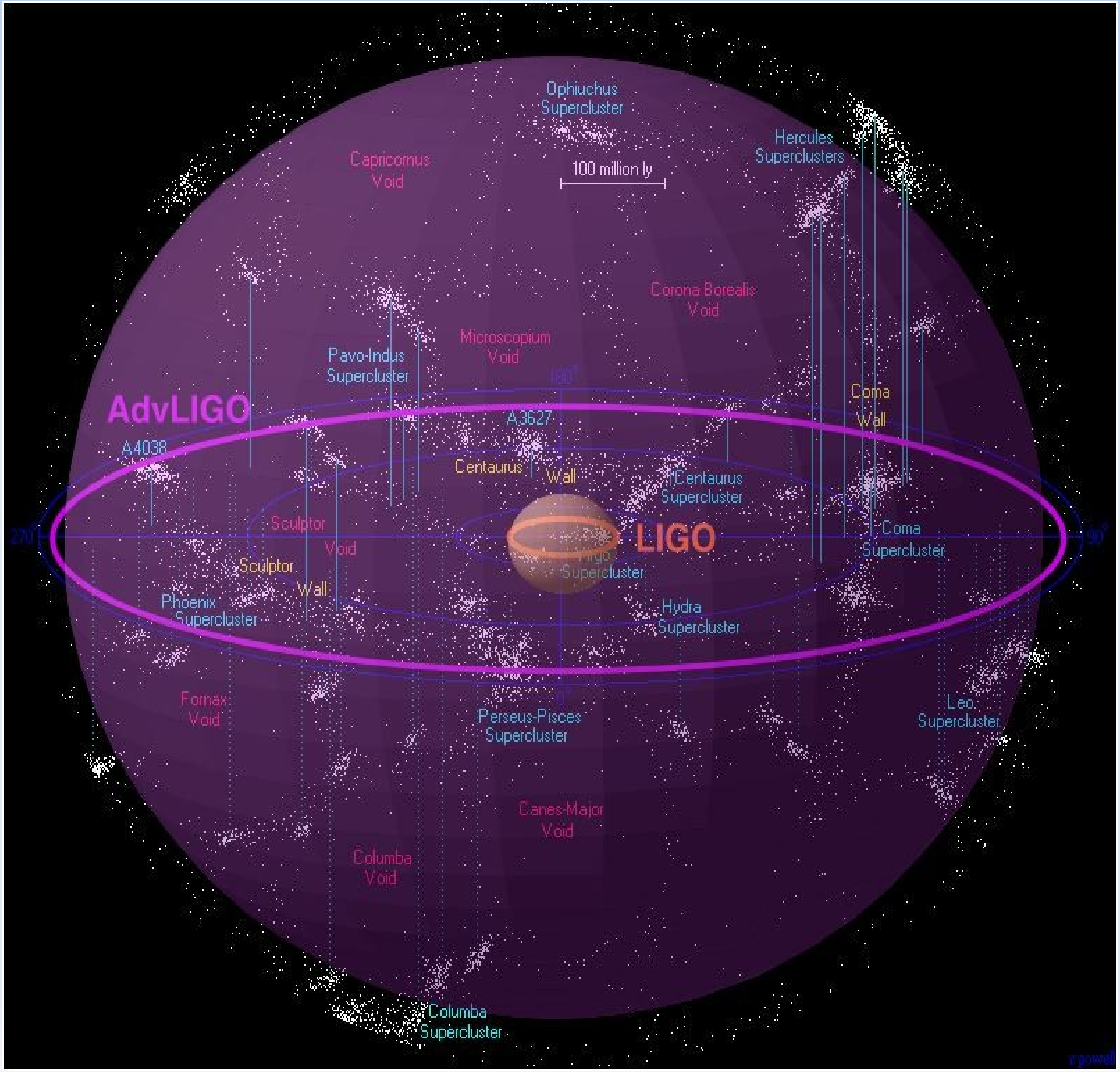
$\Delta f =$ frequency band

LISA vs LIGO/Virgo



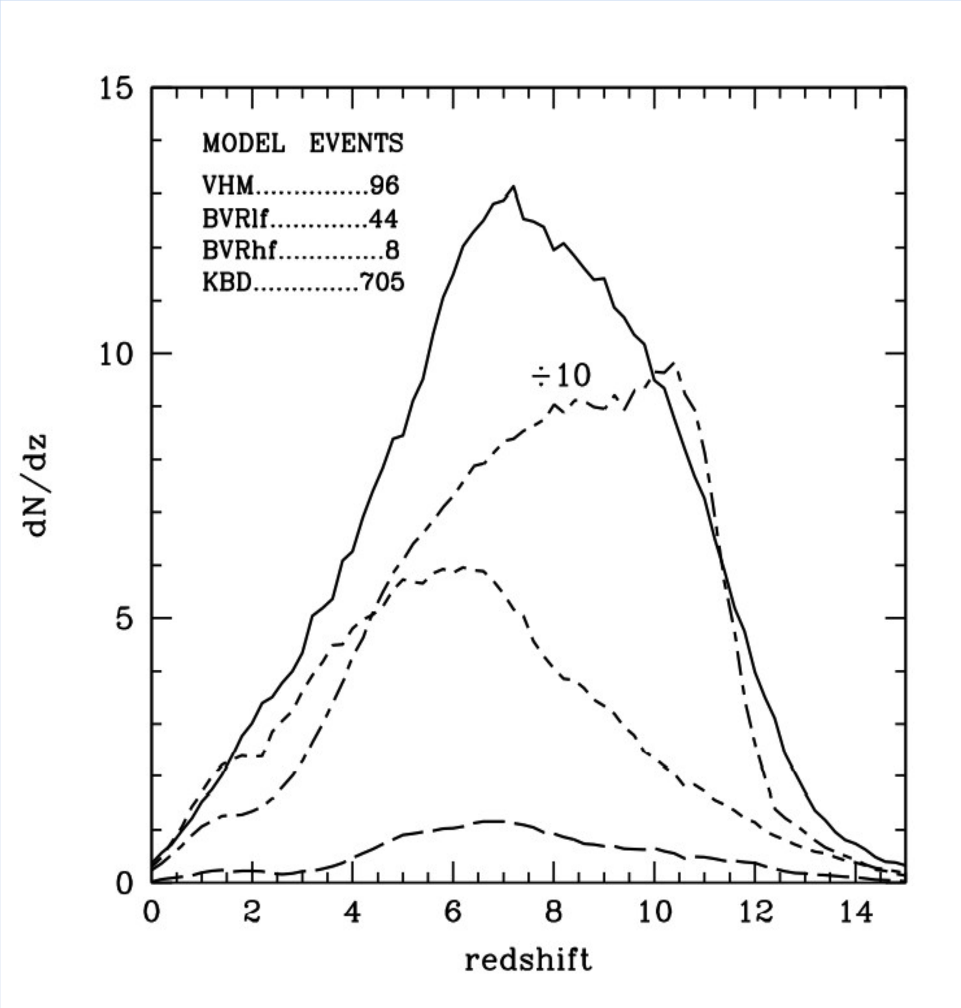
LISA vs LIGO/Virgo

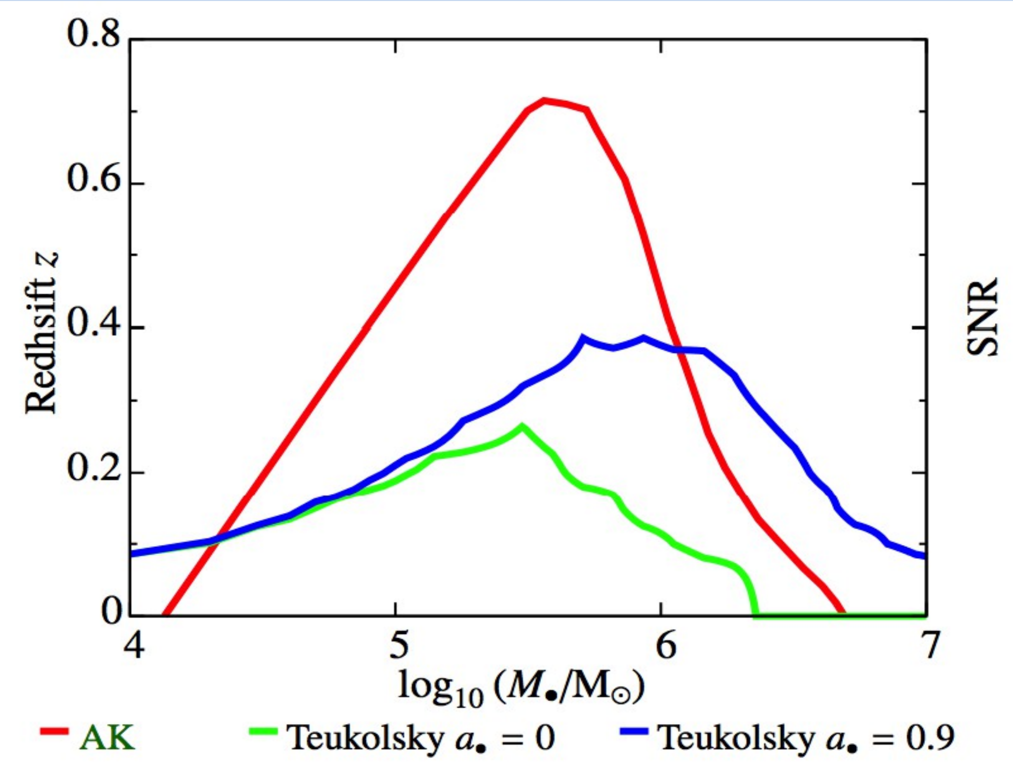
Range depends on sources, but is at most few hundred Mpc for LIGO/Virgo...



LISA vs LIGO/Virgo

... vs z>10 for LISA (for SMBH binaries)





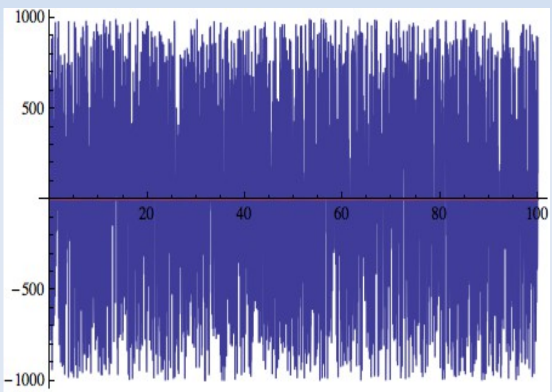
EMRIs (from Pau Amaro-Seoane et al 2012)

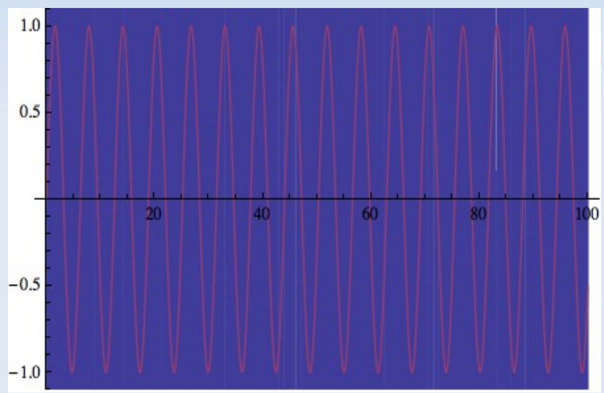
SMBH binaries (from Sesana, Volonteri and Haardt 2007)

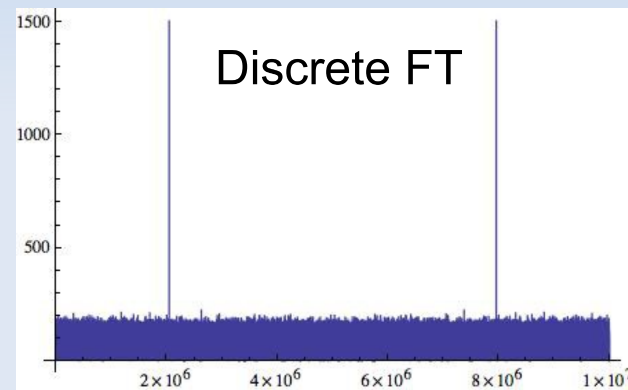
How to dig signal out of noise?

Matched filtering: cross correlate detector's output with bank of templates describing possible sources with all possible parameters

Elementary example







$$(h_1, h_2) \equiv 2\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f) + \tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)}\,df$$

$$= \int\int h_1(t)\,h_2(\tau)\,w(t-\tau)\,dt\,d\tau,$$

$$\text{with} \quad \tilde{w}(f) = 1/S_n(f)$$

Look for template h(t) that maximizes overlap with signal s(t) [ideally, *O(s,h)=1*]

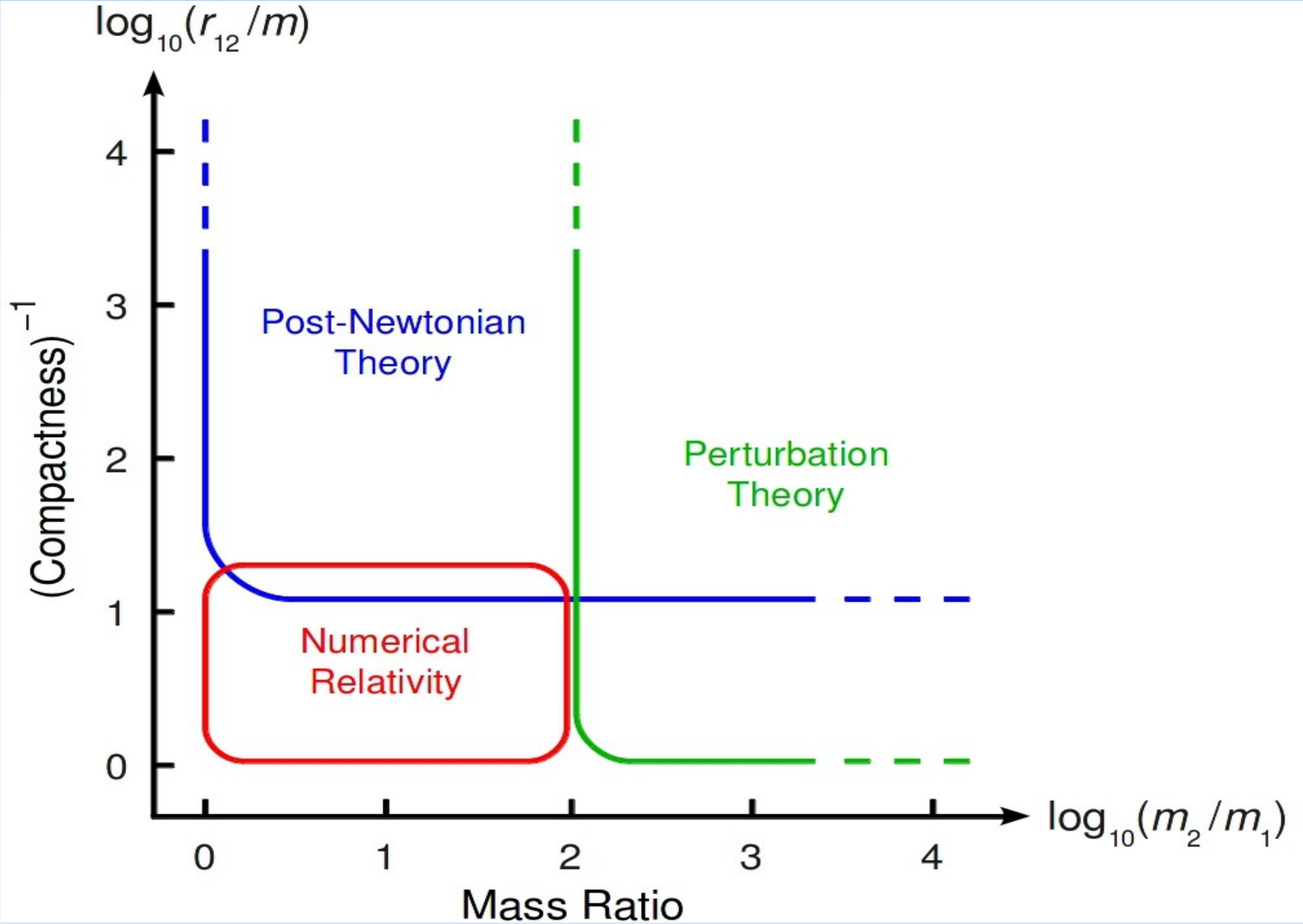
$$O(s,h) = \frac{(s,h)}{(h,h)}$$

Need good templates!

Templates for binary systems

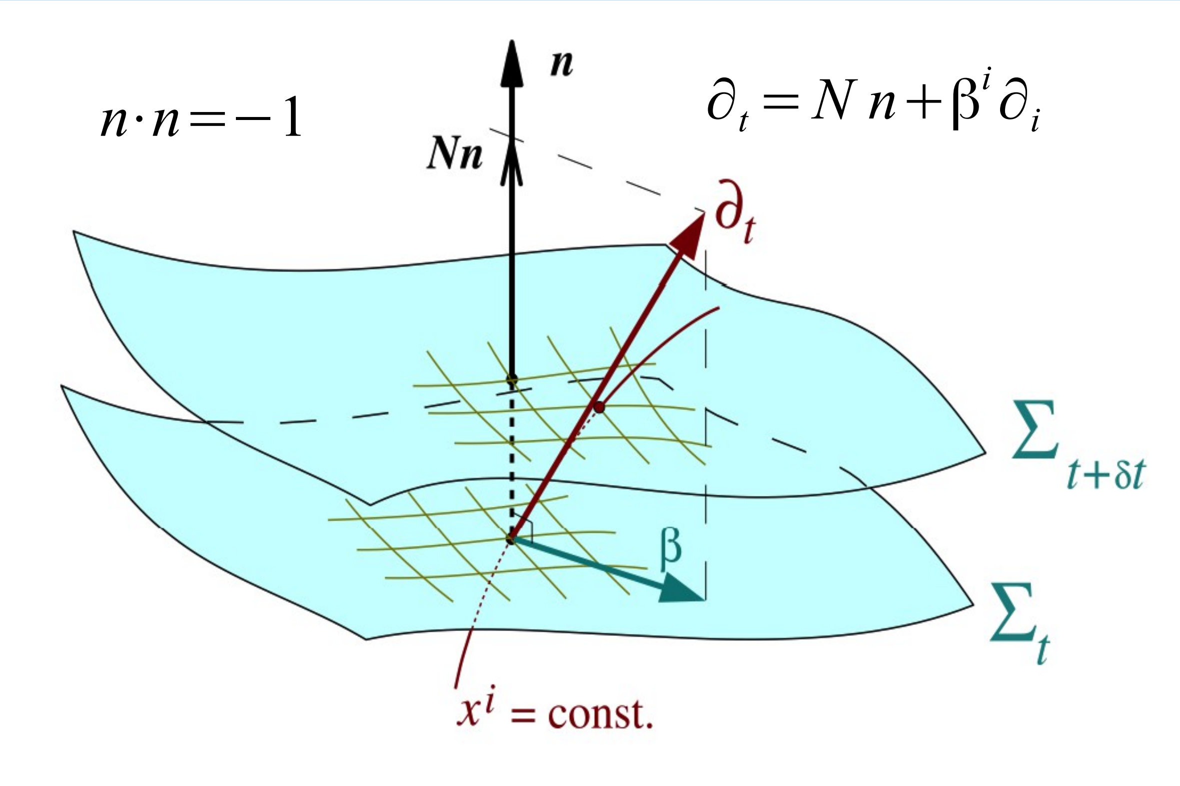
- Two-body problem is solvable analytically in Newtonian theory and for a test-particle particle around Schwarzschild/Kerr black-hole
- Newtonian theory = GR at lowest order in *v/c* $\Rightarrow$ two-body problem solved perturbatively in *v/c* (Post-Newtonian theory)
- Test particle (geodesic motion) = GR at lowest order in mass ratio m_{part}/m_{bh} $\Rightarrow$ two-body problem solved perturbatively in mass ratio (self-force formalism)
- Numerical-relativity simulations can solve for BH binaries (with spins) for mass ratios $\gtrsim 1/100$ and small separations

Templates for binary systems



Numerical relativity, or Einstein equations on a computer

- Introduce 3+1 split of spacetime



$$d\,l^2 = \gamma_{ij}\, dx^i\, dx^j$$

is 3D metric
of spatial slices

N is the "lapse"

β^i is the "shift"

Choice of lapse and
shift is gauge choice!

$$ds^2 = -(N\,dt)^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Figure from Eric Gourgoulhon's lecture notes

Numerical relativity, or Einstein equations on a computer

- Curvature of *t=constant* hypersurface when embedded in 4D spacetime is described by extrinsic curvature

$$K_{\mu\nu} = -h^{\alpha}_{\mu} h^{\beta}_{\nu} \nabla_{\alpha} n_{\beta}$$

$$h^{\mu}_{\nu} = \delta^{\mu}_{\nu} + n^{\mu} n_{\nu}$$

- Matter variable: energy density *E*, momentum density *p* and stress tensor *S*

$$E = T^{\mu\nu} n_{\mu} n_{\nu}$$
$$p_{\alpha} = -T_{\mu\nu} h^{\mu}_{\alpha} n^{\nu}$$
$$S_{\alpha\beta} = T_{\mu\nu} h^{\mu}_{\alpha} h^{\nu}_{\beta}$$

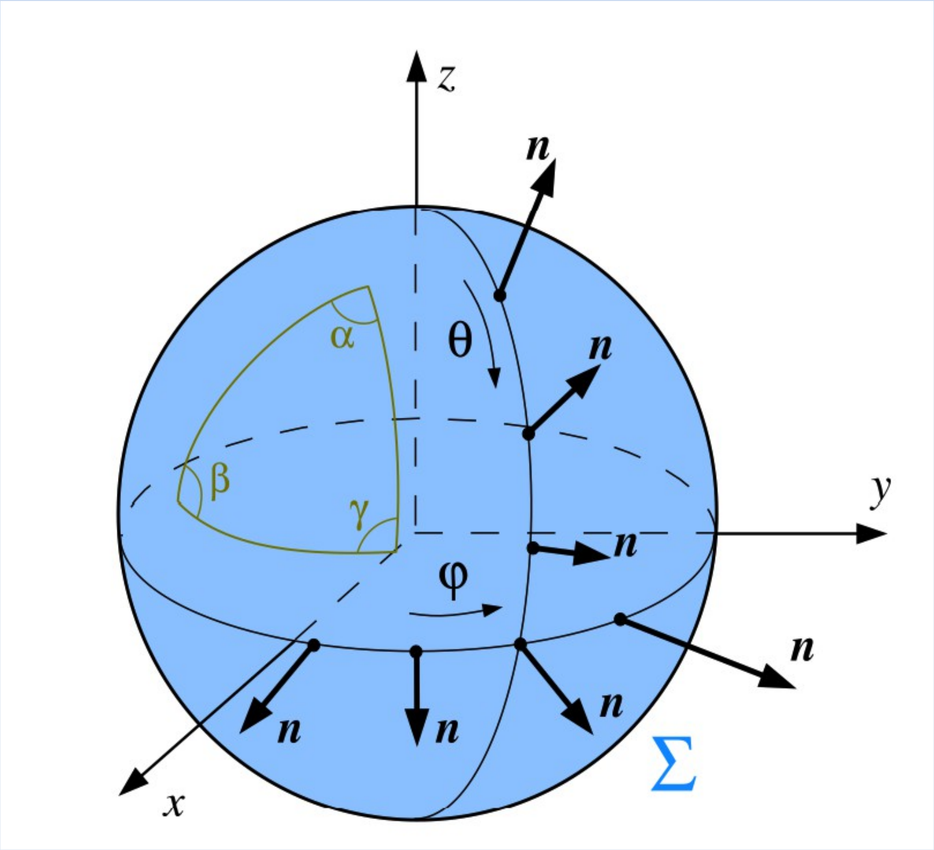


Figure from Eric Gourgoulhon's lecture notes

The Einstein equations in 3+1 form

$$\boxed{\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2NK_{ij}}$$

$$\boxed{\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)K_{ij} = -D_iD_jN + N\left\{R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j + 4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\right\}}$$

Evolution equations, from $G_{ij}=8\pi T_{ij}$

$$\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi E}$$

Energy constraint, from $G_{tt}=8\pi T_{tt}$

$$\boxed{D_jK^j{}_i - D_iK = 8\pi p_i}.$$

Momentum constraint, from $G_{ti}=8\pi T_{ti}$

- Equations do not depend on time derivatives of lapse and shift (ie they are not dynamical variables)

- Energy and momentum constraints are initial value constraints thanks to Bianchi identity

  - Eg in vacuum $\partial_t G^{t\nu} + \partial_i G^{i\nu} + \Gamma G\text{-terms} = 0$ $\Rightarrow$ $G_{t\alpha}=0$ at all times if $G_{t\alpha}=0$ initially and evolution equations satified
  - With matter, need matter's equations of motion $\nabla_\mu T^{\mu\nu}=0$ to evolve system

The Einstein equations in 3+1 form

- Locally we can choose N=1, β^i=0 (normal coordinates, aka synchronous gauge; not possible globally due to caustics)

$$-\frac{\partial^2\gamma_{ij}}{\partial t^2}+\gamma^{kl}\left(\frac{\partial^2\gamma_{ij}}{\partial x^k\partial x^l}+\frac{\partial^2\gamma_{kl}}{\partial x^i\partial x^j}-\frac{\partial^2\gamma_{lj}}{\partial x^i\partial x^k}-\frac{\partial^2\gamma_{il}}{\partial x^j\partial x^k}\right)=8\pi\left[(S-E)\gamma_{ij}-2S_{ij}\right]$$

$$+\mathcal{Q}_{ij}\left(\gamma_{kl},\frac{\partial\gamma_{kl}}{\partial x^m},\frac{\partial\gamma_{kl}}{\partial t}\right)$$

Intial value ("Cauchy") problem

$$\gamma^{ik}\gamma^{jl}\frac{\partial^2\gamma_{ij}}{\partial x^k\partial x^l}-\gamma^{ij}\gamma^{kl}\frac{\partial^2\gamma_{ij}}{\partial x^k\partial x^l}=16\pi E+\mathcal{Q}\left(\gamma_{kl},\frac{\partial\gamma_{kl}}{\partial x^m},\frac{\partial\gamma_{kl}}{\partial t}\right)$$

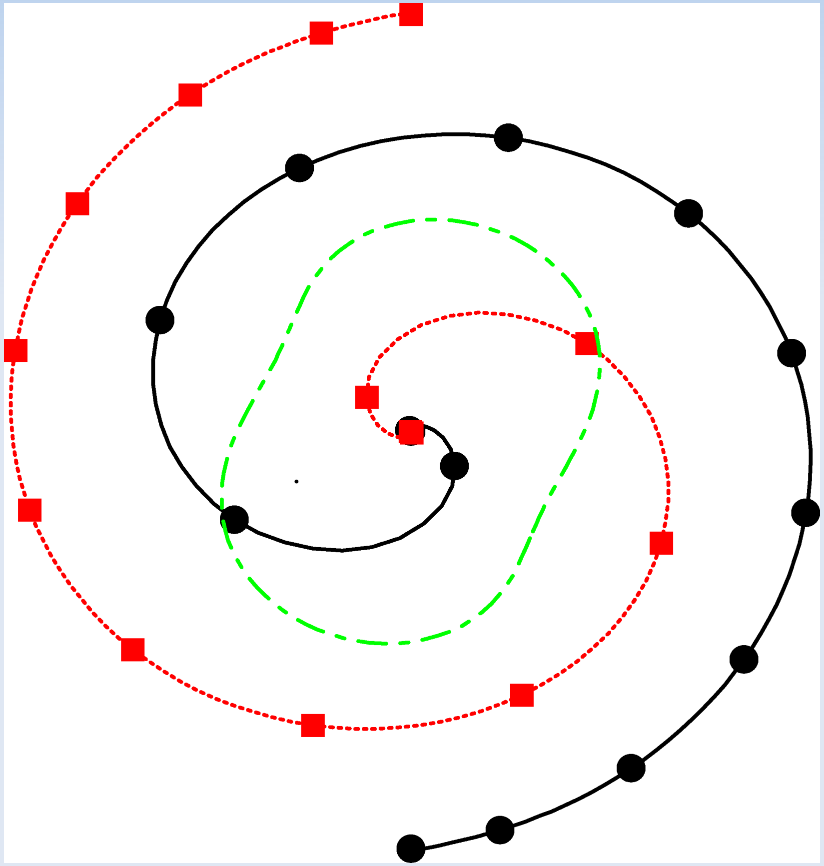
$$\gamma^{jk}\frac{\partial^2\gamma_{ki}}{\partial x^j\partial t}-\gamma^{kl}\frac{\partial^2\gamma_{kl}}{\partial x^i\partial t}=-16\pi p_i+\mathcal{Q}_i\left(\gamma_{kl},\frac{\partial\gamma_{kl}}{\partial x^m},\frac{\partial\gamma_{kl}}{\partial t}\right).$$

Initial value constraints

- In pratice, to evolve system need to choose coordinates (gauge) and evolution scheme (eg BSSN)
- See Gourgoulhon's lecture notes for more details

NR "trajectories" for binary BHs

$a_1 = a_2 = -0.76$



$a_1 = a_2 = 0.76$

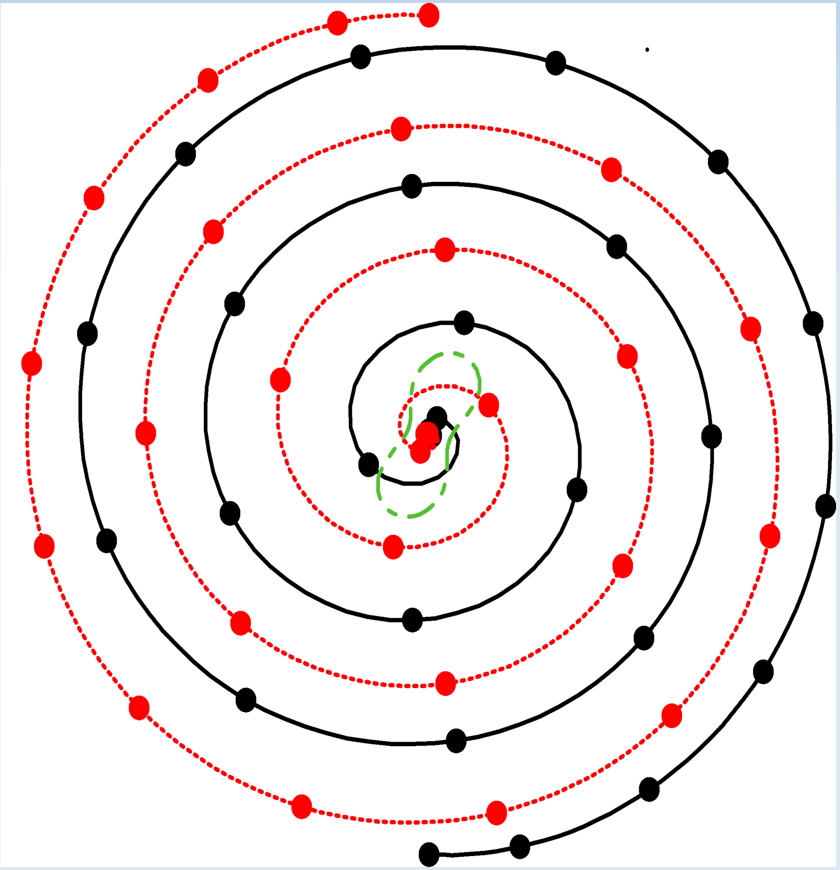
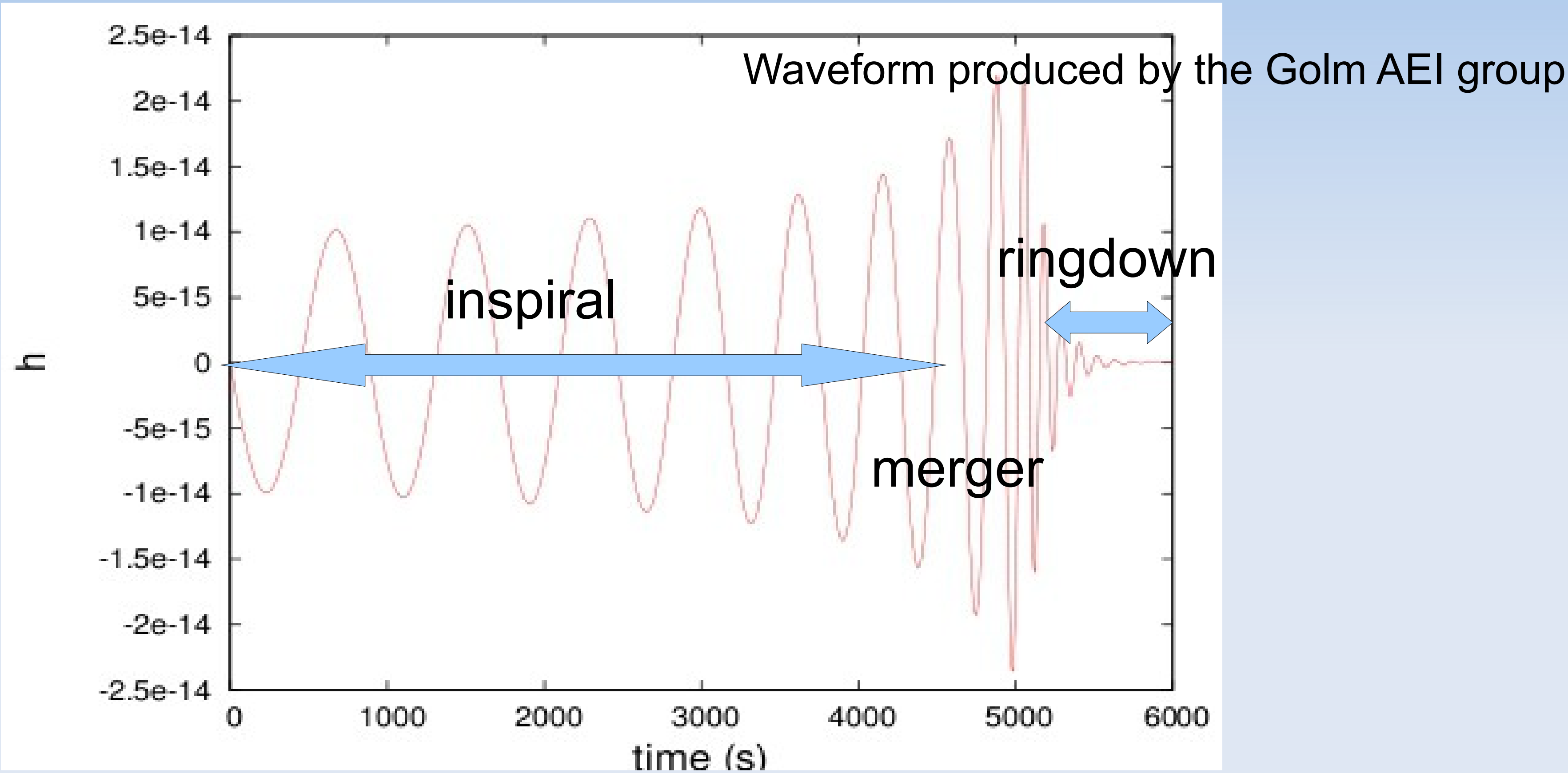


Figure from Campanelli, Lousto & Zlochower 2006

NR waveforms for binary BHs



It takes weeks/months to generate NR waveforms: too slow for data analysis!

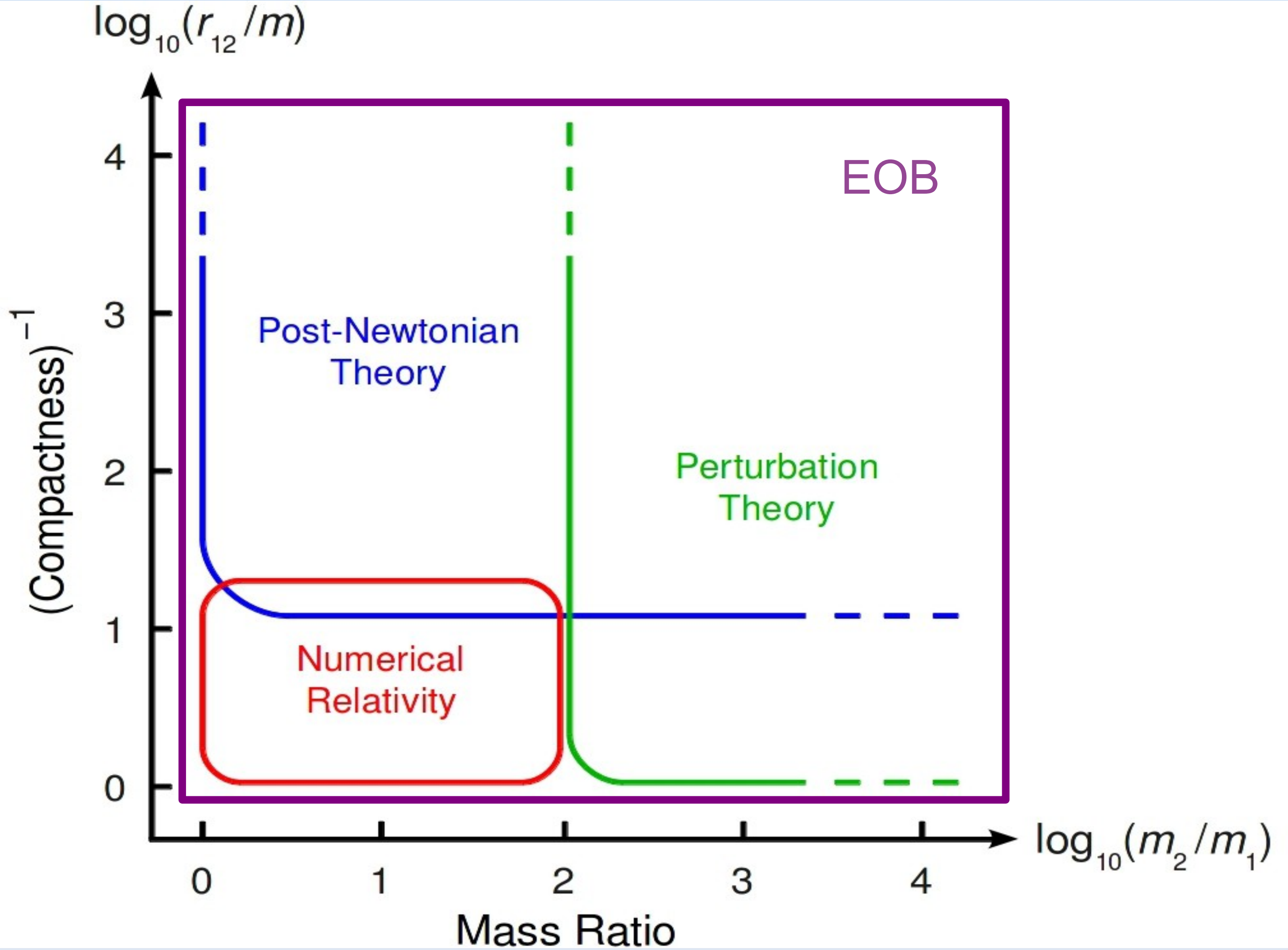
Faster templates

- PN waveforms do not have merger and ringdown (PN theory fails at small separations)
- Perturbative waveforms have merger-ringdown but can't be extrapolated to comparable masses

Possible fixes

- Use PN waveforms as guide for phenomenological fit of NR waveform + add ringdown by attaching quasi-normal modes
- Effective-one-body model ("EOB")

The EOB



EOB = phenomenological way of combining PN, NR and perturbation theory

The EOB

- Main idea: map 2-body problem into test-particle problem

- Newtonian non-spinning binaries can be mapped to non-spinning test-particle with mass = $m_1\ m_2/(m_1\ +m_2)$ around mass $m = m_1+m_2$

- Energy levels of positronium (e^+ - e^-) can be mapped to those of hydrogen through

$$\frac{E_{\mathrm{H}}}{\mu c^2} = \frac{E_{\mathrm{pos}}^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4}$$

$m_1=m_2$ is the electron/antielectron's mass!

The EOB

Same mapping possible in PN theory (Buonanno & Damour 1999)

- PN Hamiltonian $H_{PN,real}$ describes conservative dynamics (no GW emission)
- Particle with mass = $m_1\, m_2/(m_1 + m_2)$ around a $m = m_1 + m_2$ *deformed* Schwarzschild BH ("effective problem") has Hamiltonian

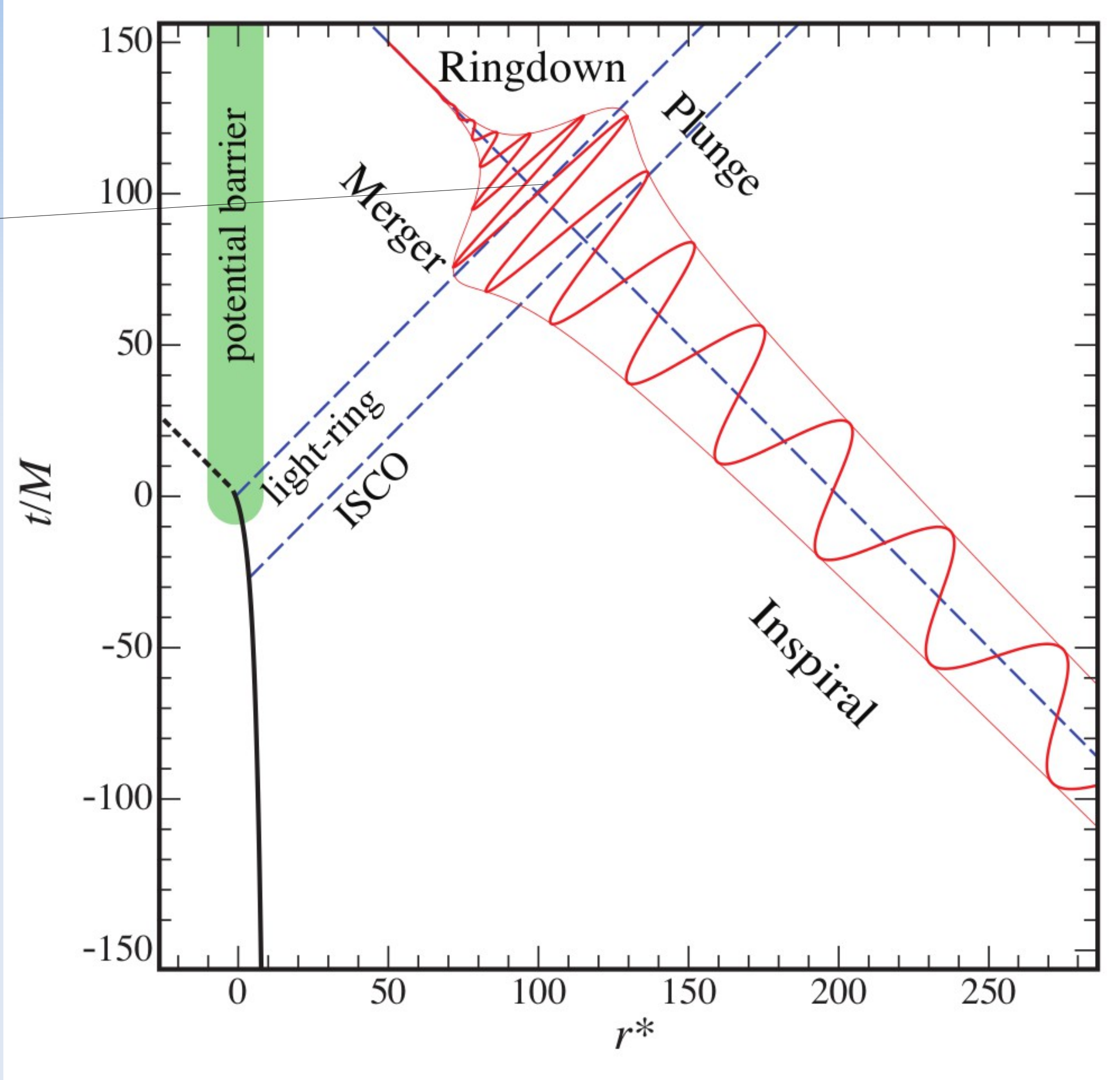
$$\frac{H_{\rm eff}}{\mu c^2} = \frac{H^2_{\rm PN,\,real} - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4} \quad \text{(up to 3 PN)} \ (*)$$

- H_{eff} can be calculated at all PN orders (deformed Schwarzschild metric given at all PN orders) $\Longrightarrow$ invert Eq (*) and get "real" Hamiltonian valid at all PN orders:

$$H_{\rm real} = m\sqrt{1 + 2\frac{\mu}{m}\left(\frac{H_{\rm eff}}{\mu} - 1\right)}\,.$$

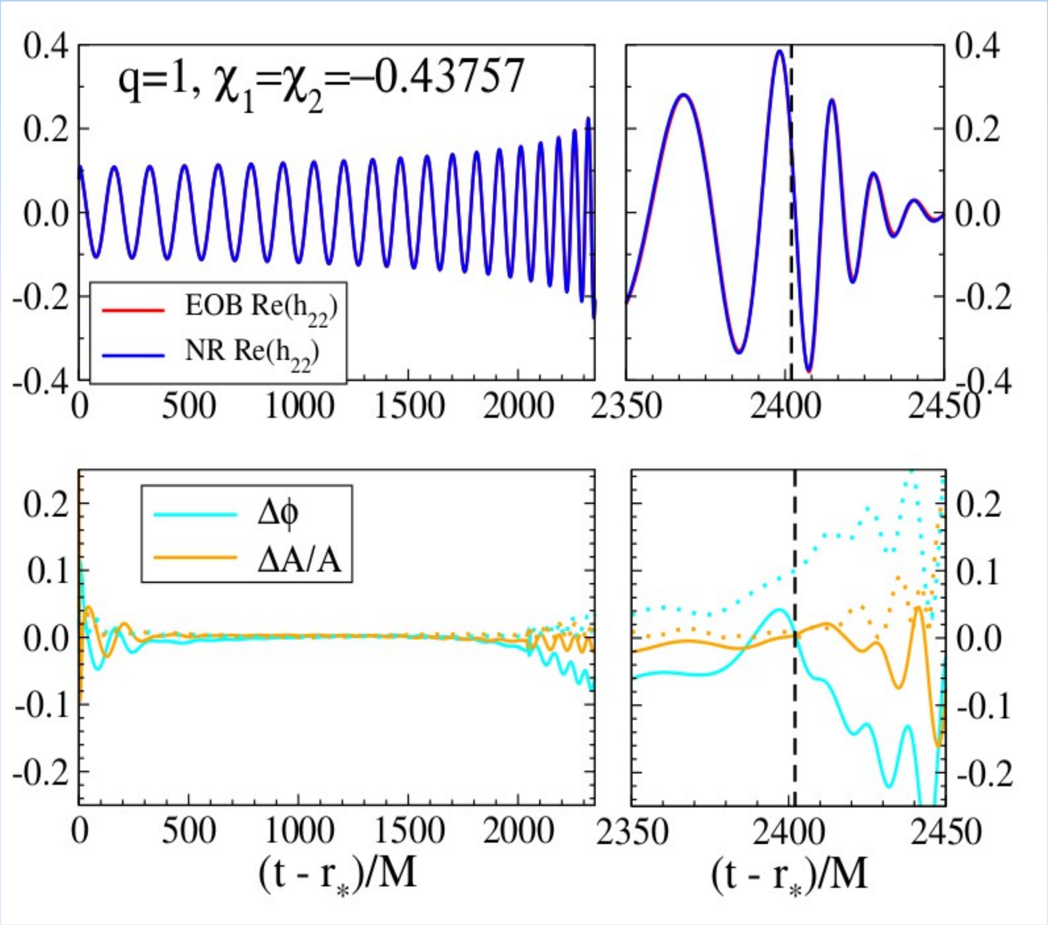
How about dissipative dynamics?

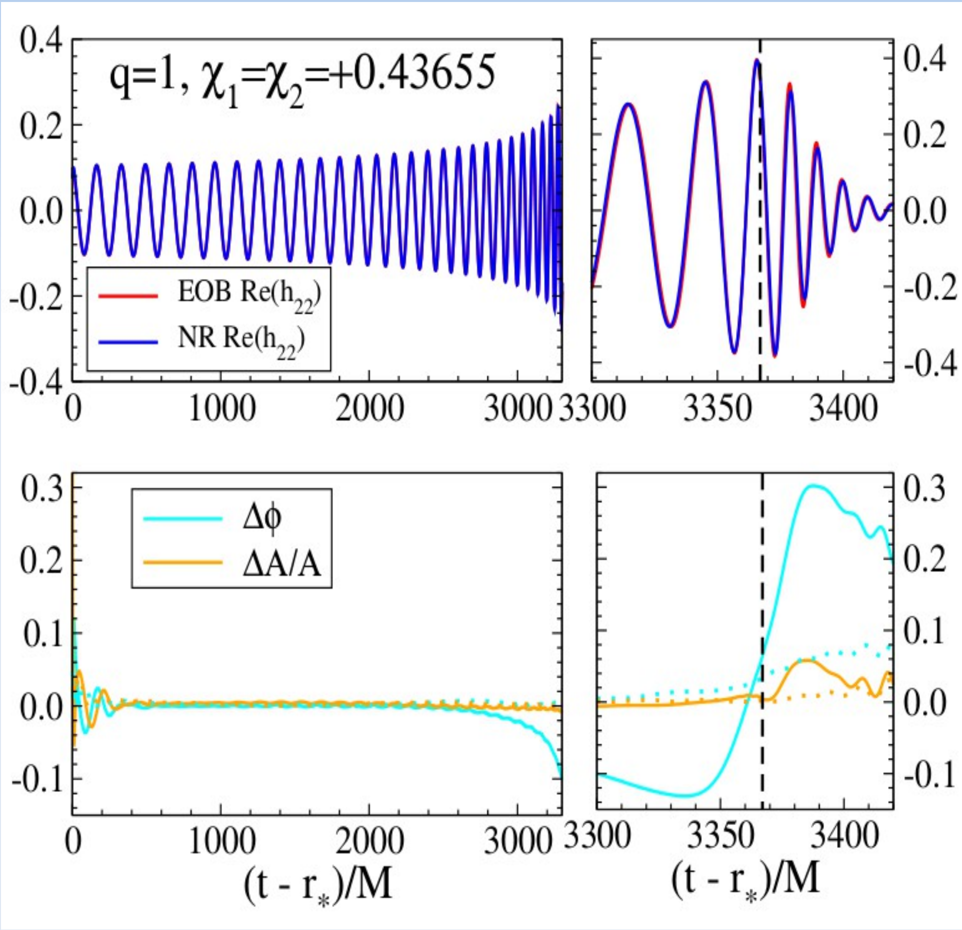
Attach combination of quasinormal modes



EOB waveforms

Figures from Taracchini et al 2012





Model still inaccurate for spins > 0.7
(because PN waveforms are not accurate enough already during inspiral)

GW vs EM astronomy

- GWs interact very weakly with matter, strain h decays as 1/r → GWs visible to very high z, eg SMBHs with LISA, stochastic backgrounds

- Gravitational wavelength >~ source's size (because GWs generated by bulk motion of matter) vs EM wavelengths << source's size (because EM waves generated by moving charges, atomic processes, etc)

  → EM can be used for imaging, GWs do not have angular resolution (akin to sound)

  → EM survey cover small areas, GWs cover whole sky

GW and EM waves are complementary tools for testing fundamental physics, astrophysics and cosmology

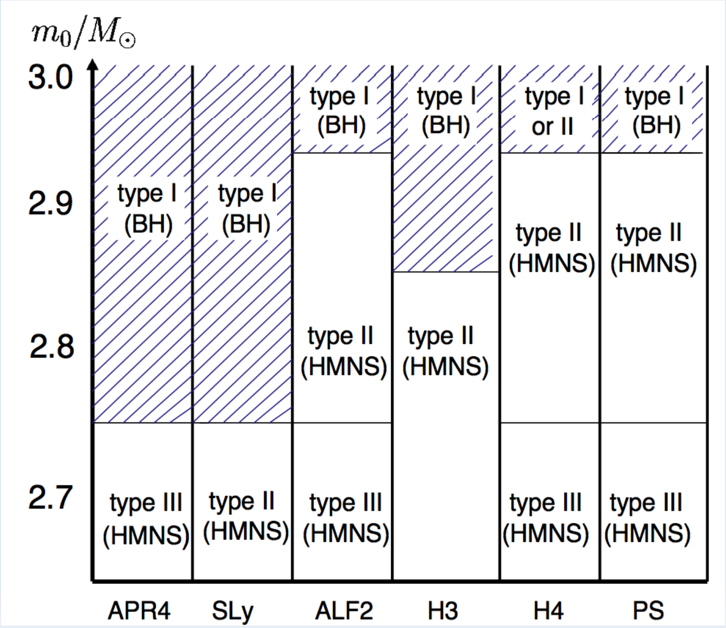
The EOS of nuclear matter

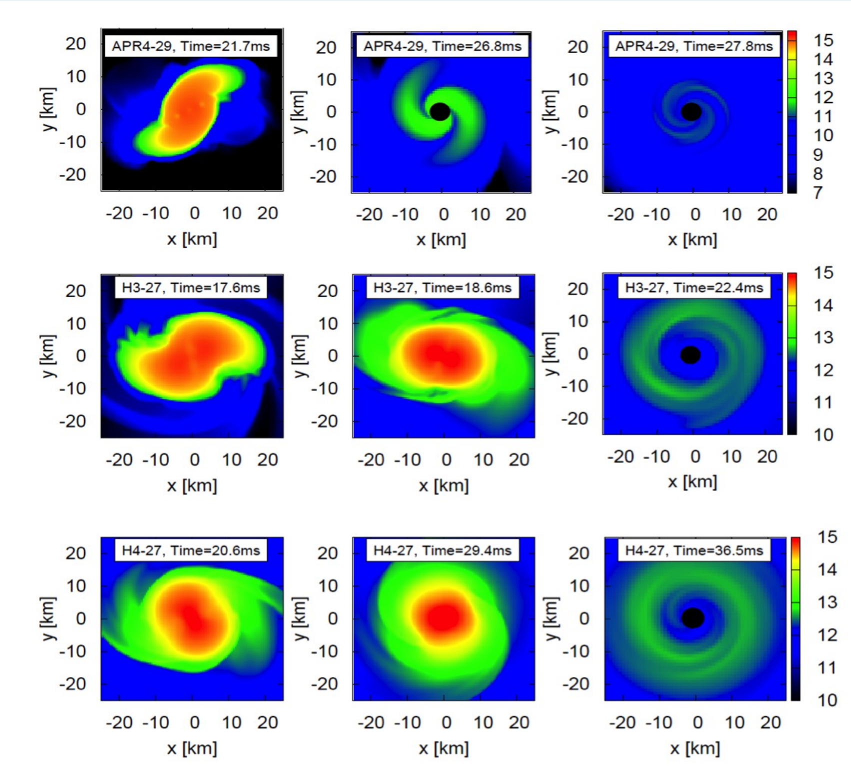
- Equation of state of nuclear matter affects merger and ringdown of binary neutrons stars → effects for Einstein Telescope
- 3 kinds of NS mergers:

  - Type I: BH is promptly formed

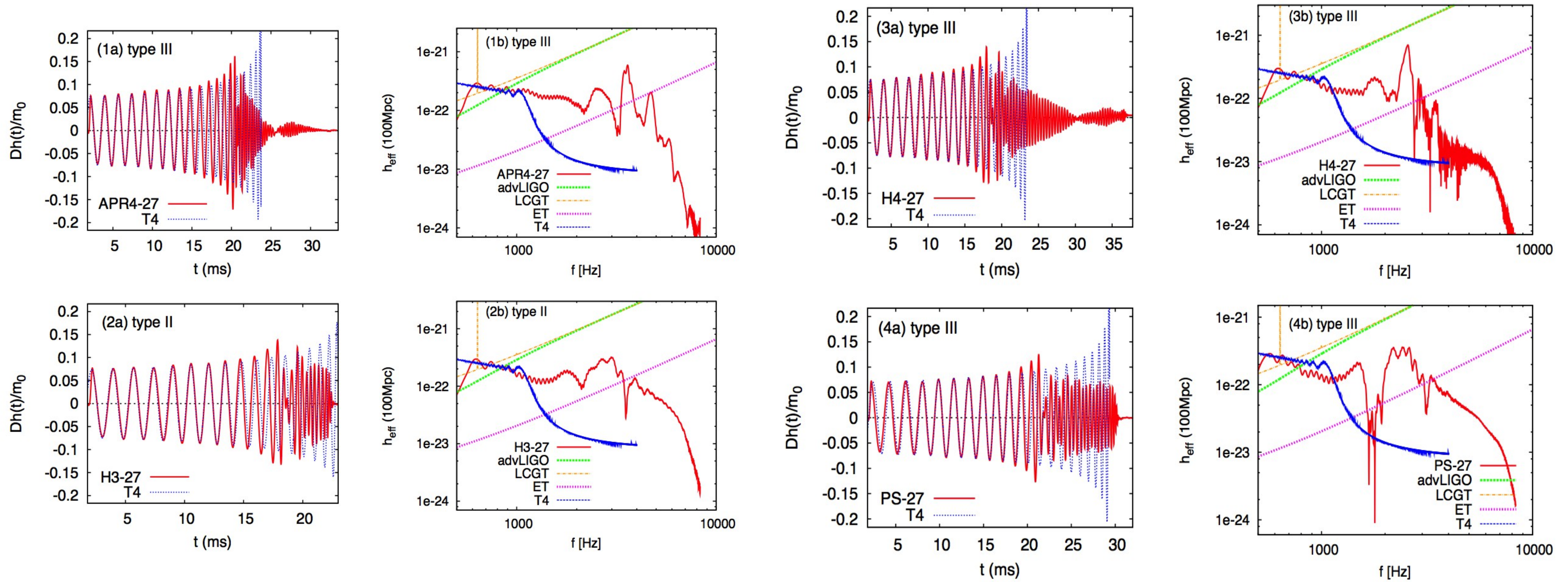
  - Type II: short-lived (< 5 ms) hypermassive NS

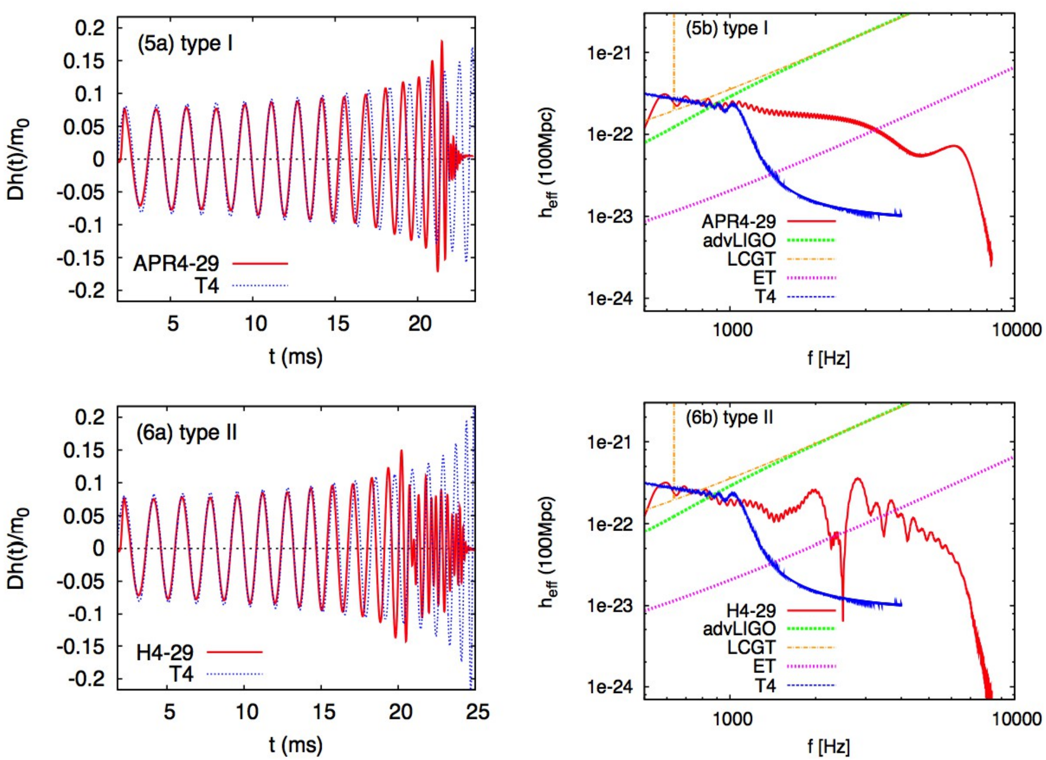
  - Type III: long-lived (> 5 ms) hypermassive NS





Figures from Hotokezaka et al, PRD 83 124008 (2011)

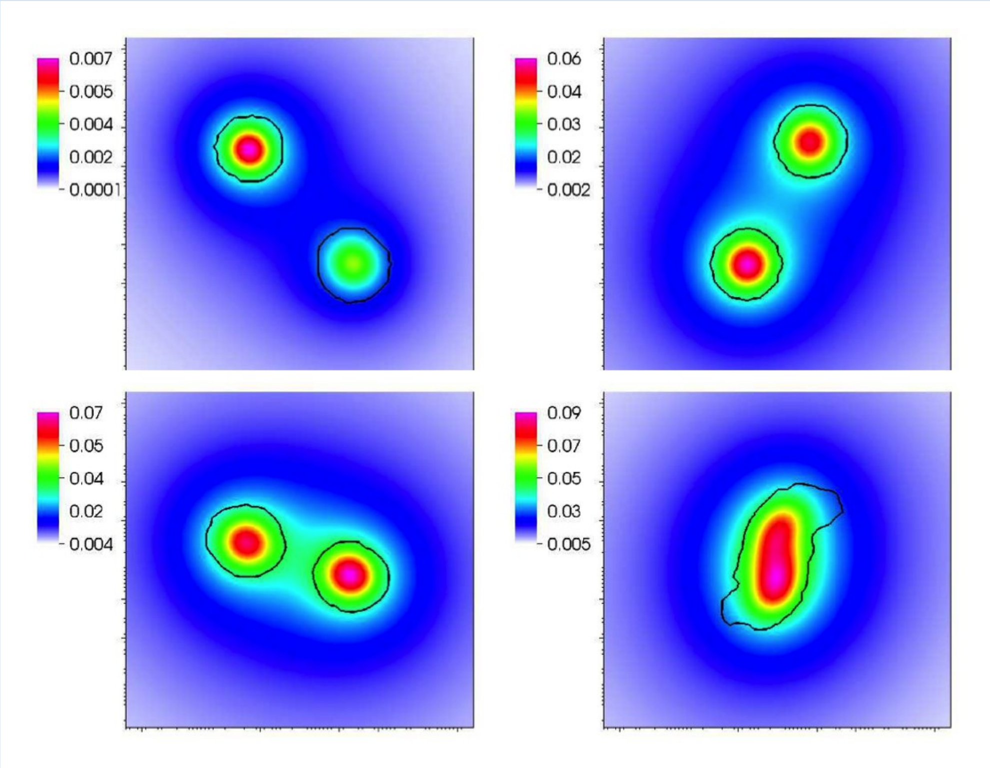


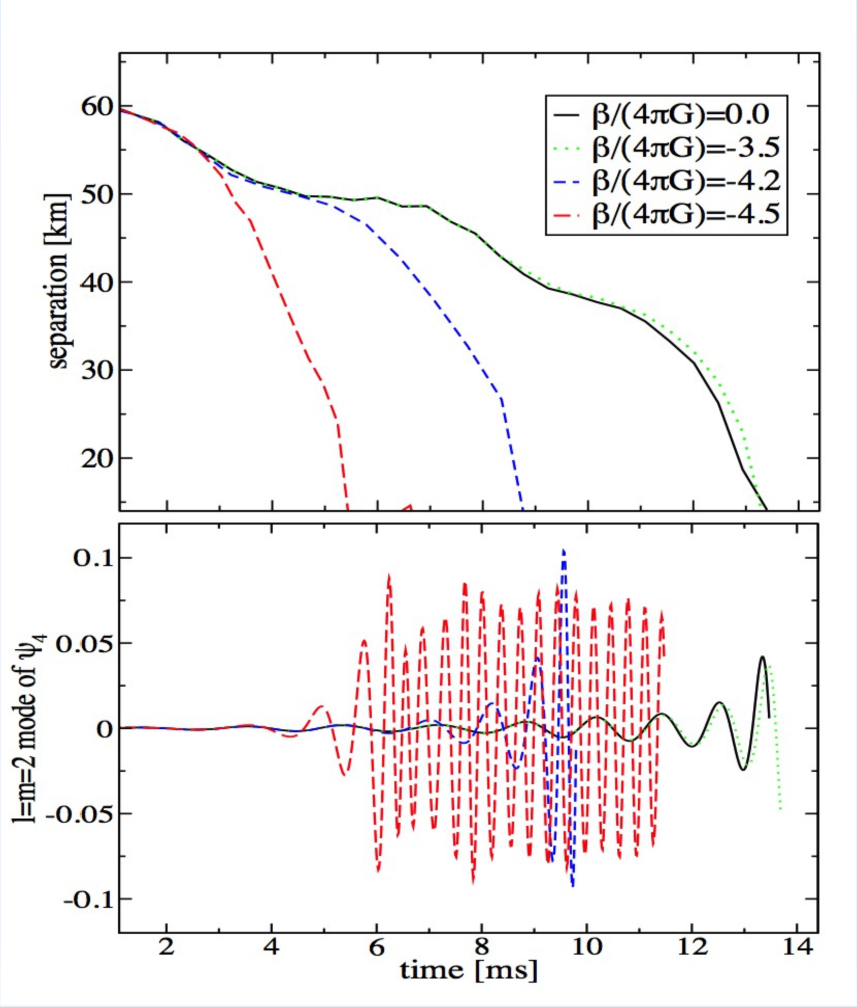


Figures from Hotokezaka et al, PRD 83 124008 (2011)

Test gravity theories with NS mergers

Extra GW polarizations difficult to detect directly, but can backreact on orbital evolution and cause quicker mergers in some (quite special) scalar-tensor theories





From Barausse et al (2013)

Test gravity in the weak field

- Modifications to gravity theory cause different PN inspiral/ringdown than in GR → Parametrized Post-Einsteinian formalism [Yunes and Pretorius PRD 80, 122003 (2009)]

$$\tilde{h}(f) = \begin{cases} \tilde{h}_{\mathrm{I}}^{(\mathrm{GR})}(f) \cdot (1 + \alpha u^a)\, e^{i\beta u^b} & f < f_{\mathrm{IM}}, \\ \gamma u^c e^{i(\delta + \epsilon u)} & f_{\mathrm{IM}} < f < f_{\mathrm{MRD}}, \\ \zeta \frac{\tau}{1 + 4\pi^2 \tau^2 \kappa (f - f_{\mathrm{RD}})^d} & f > f_{\mathrm{MRD}}, \end{cases}$$

GR:

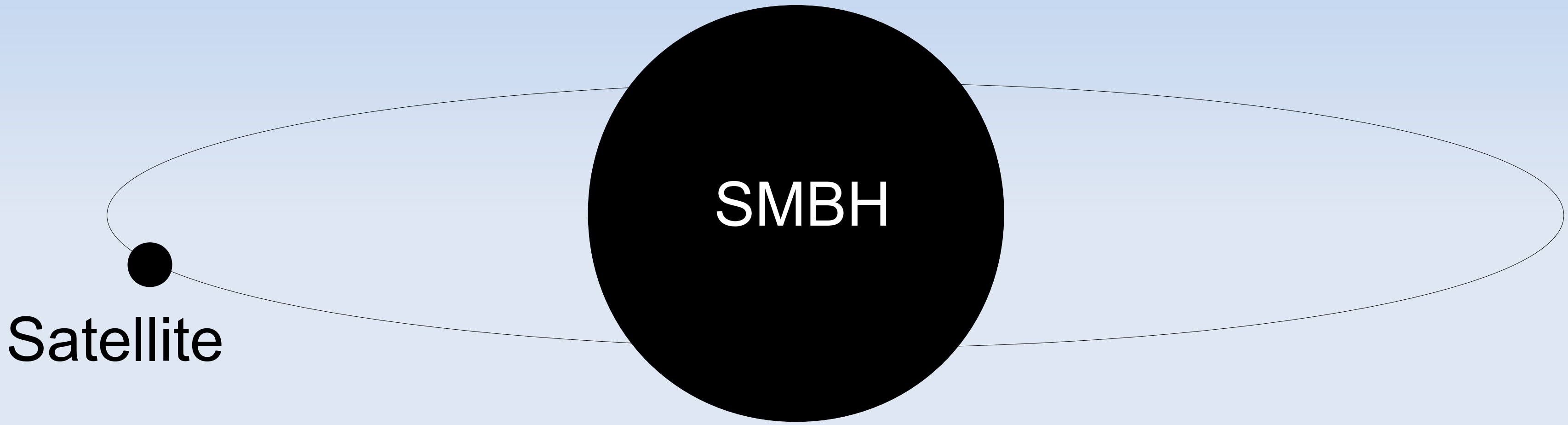
$$(\kappa, d) = (1, 2) \quad (c, \epsilon) = (-2/3, 1)$$

$$(\alpha, a, \beta, b) = (0, a, 0, b)$$

Also: theories with parity or Lorentz violations in the gravity sector

Test the Kerr metric with EMRIs

Stellar-mass BH orbits SMBH ~ 1.e6 times (evolution driven by GW emission, which is weak for small mass ratios)



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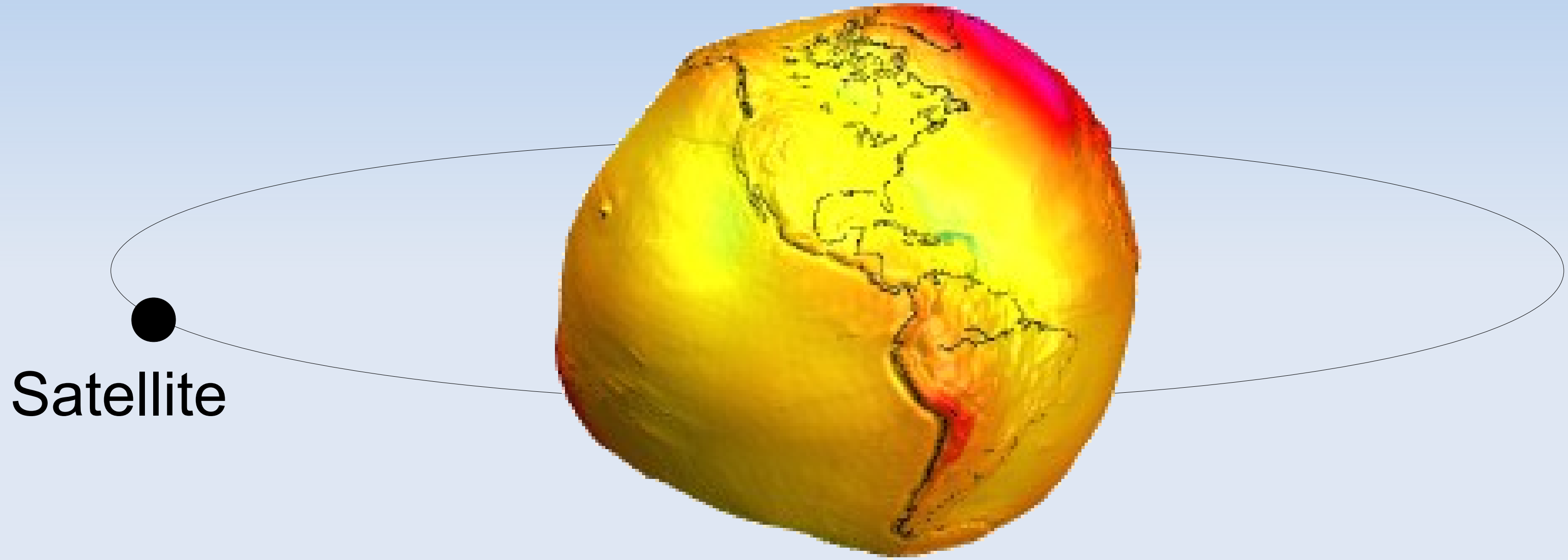


Spacetime mapping (analog of mapping Earth gravitational field with CHAMP and GRACE): is the SMBH really described by Kerr metric? (test of the **no hair theorem**)

Can one see an anomalous quadrupole moment (``bumpy BHs")?

Could it be a BH in a modified gravity theory?

Cosmography with LISA

- LISA's observations of SMBH binaries will measure luminosity distance to within 1-10%, with error mainly coming from pointing uncertainty (GWs are poorly localized in the sky, i.e. to within a few $\deg^2$)
- GWs give no measurement of redshift: masses enter in waveforms through timescale Gm_i/c^3, but timescale redshift binary with (m_1,m_2) is indistinguishable (modulo amplitude) from (m_1,m_2) x $(1+z)$

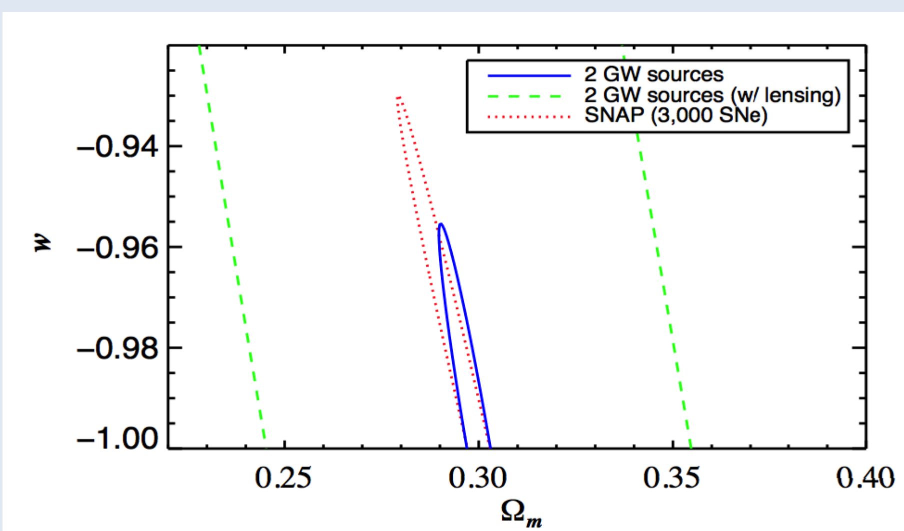
$$h_+ = \frac{2\mathcal{M}_z^{5/3}[\pi f(t)]^{2/3}}{D_L}\left[1+(\hat{L}\cdot\hat{n})^2\right]\cos[\Phi(t)],$$
$$h_\times = \frac{4\mathcal{M}_z^{5/3}[\pi f(t)]^{2/3}(\hat{L}\cdot\hat{n})}{D_L}\sin[\Phi(t)].$$

L = binary's angular momentum
n = observer's direction
Φ = phase (depends strongly on spins and redshifted chirp mass)

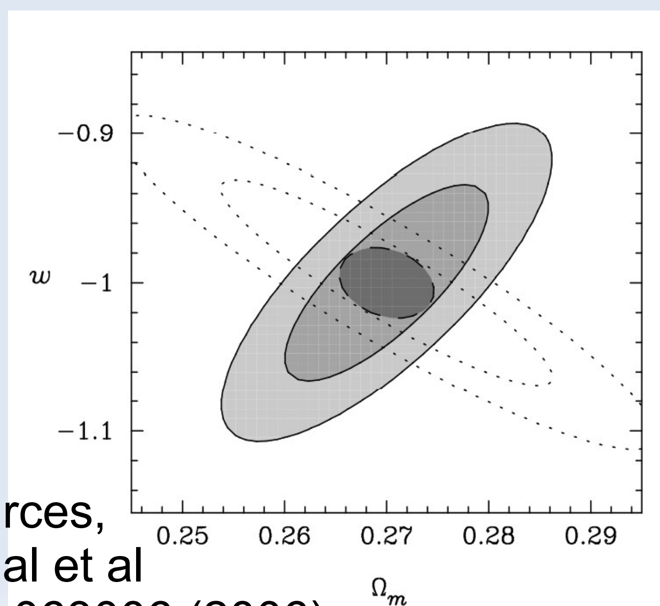
$\mathcal{M}_z = (1+z)(m_1 m_2)^{3/5}/(m_1+m_2)^{1/5}$ is the redshifted chirp mass

Cosmography with LISA

- Luminosity distance degenerate with angular factors depending on observer's direction
- Degeneracy broken thanks to LISA's precession
- Precession of L due to spin helps estimate distance, but measure degraded by weak lensing
- EM counterpart could provide redshift → improved measurement of distance (because of better localization), and test of Dark Energy



Holz & Hughes, ApJ 629, 15 (2005)



1000 sources,
From Dalal et al
PRD, 74, 063006 (2006)

Cosmography with LIGO/Virgo

- NS binaries thought to be progenitors of short gamma-ray bursts (GRBs)

- GRBs will provide EM counterparts and allow localization, improved luminosity-distance estimates, and redshift measurement in the local universe (within a few hundred Mpc)

Measurement of Hubble constant to within ~ 2%

Can GWs be used to learn about galaxy formation?

- GWs produced in highly-dynamical/strong-gravity regimes, i.e. by massive BHs
- Massive BHs are tiny compared to galactic scales

MBH ~ 10^{-6} – 10^{-7} pc MBH accretion disk ~ pc Circumbinary disk ~ 100 pc	MBH scales
Galactic bulge ~ kpc Galactic disk ~ 10 kpc Dark-matter halo ~ Mpc	galactic scales

- There is more to physics than gravity! Highly non-linear, dissipative processed act on small scales

  Star formation, supernova feedback, gas cooling, UV ionizing background, AGN feedback, ...

The basics of galaxy formation

- Galaxy formation is bottom-up: smaller systems form first and merger in larger ones...

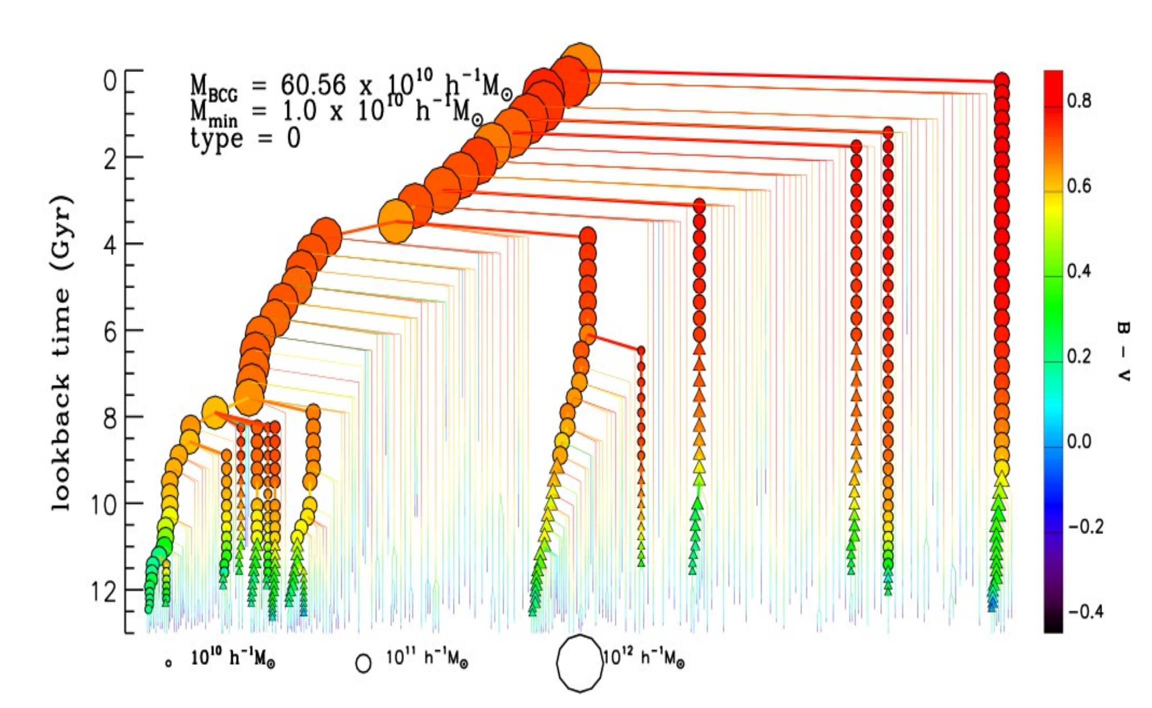
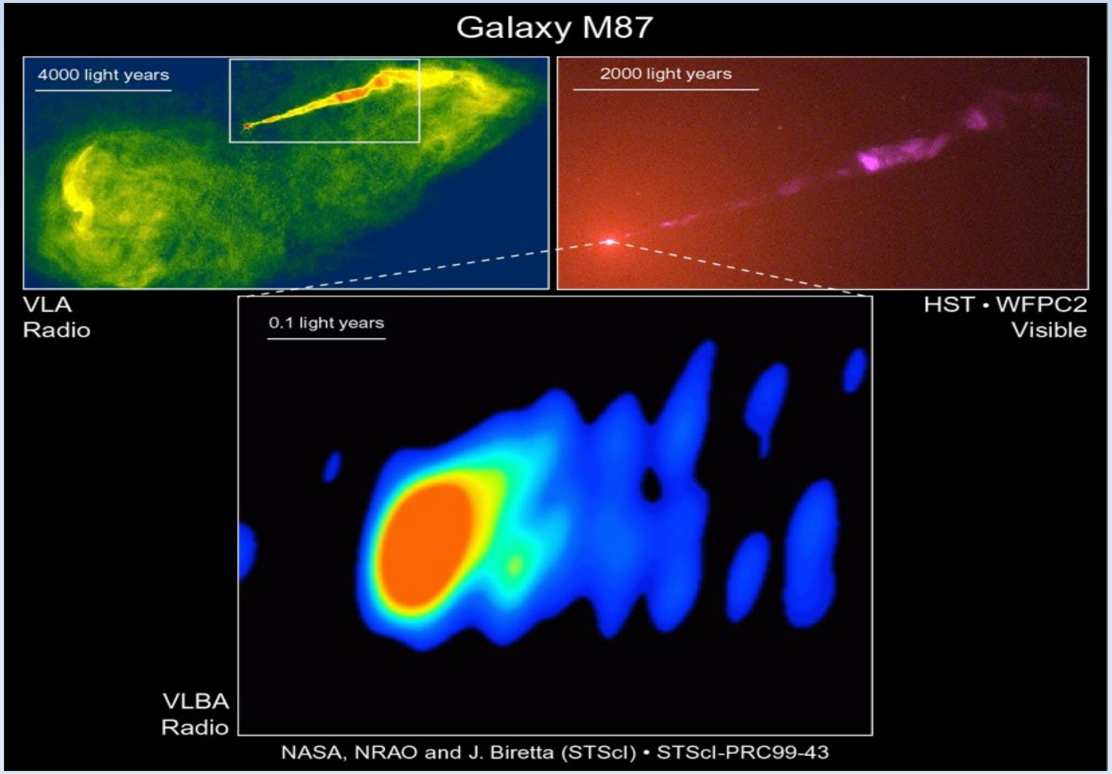


Figure from
De Lucia & Blaizot (2007)

- ...but most massive galaxies have older stars and weaker SF than smaller galaxies (cosmic downsizing)

The basics of galaxy formation

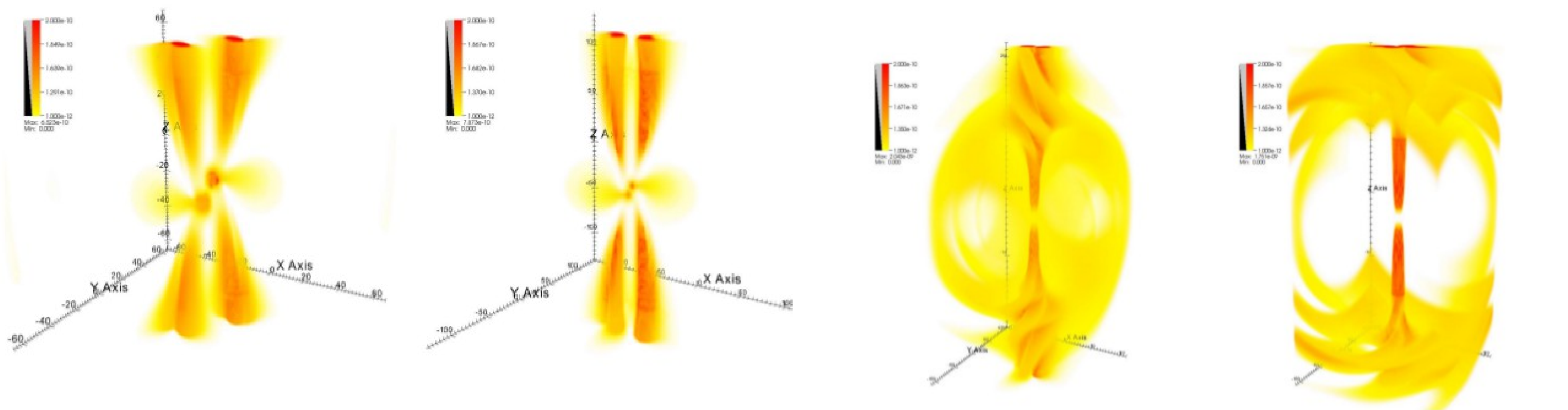
- Solution: AGN feedback (stronger for larger galaxies, which host more massive SMBHs)



- The kinetic energy of the jet is transferred to the galaxy and keeps it “hot”, quenching star formation

How are the jets produced?

- Jets can be produced by isolated spinnning BHs in a magnetic field anchored to accretion disk (Blandford & Znajek 1977)...
- … or by BHs (even non spinning ones) moving a magnetic fields anchored to circumbinary disk (Palenzuela, Lehner and Liebling 2010)

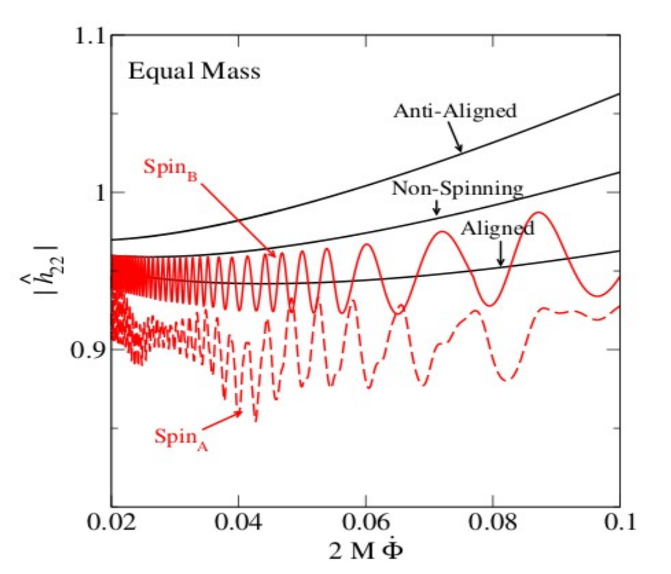


(a) $-11.0\,M_8$ hrs (b) $-3.0\,M_8$ hrs (c) $4.6\,M_8$ hrs (d) $6.8\,M_8$ hrs

GWs and galaxy formation

- AGN feedback (and therefore BH spins and mergers) crucial in modern galaxy formation models
- Galaxy formation regulates gas available to massive BHs for growing
- MBH mergers in gas-rich ("wet") enviroment have aligned spins because they align with circumnuclear disk (Bardeen Petterson effect)
- For BH binaries in gas-poor ("dry") environments, spin-orbit coupling make spins precess around total angular momentum $J=L+S_1+S_2$

→ modulations in gravitational waveforms visible with GWs!

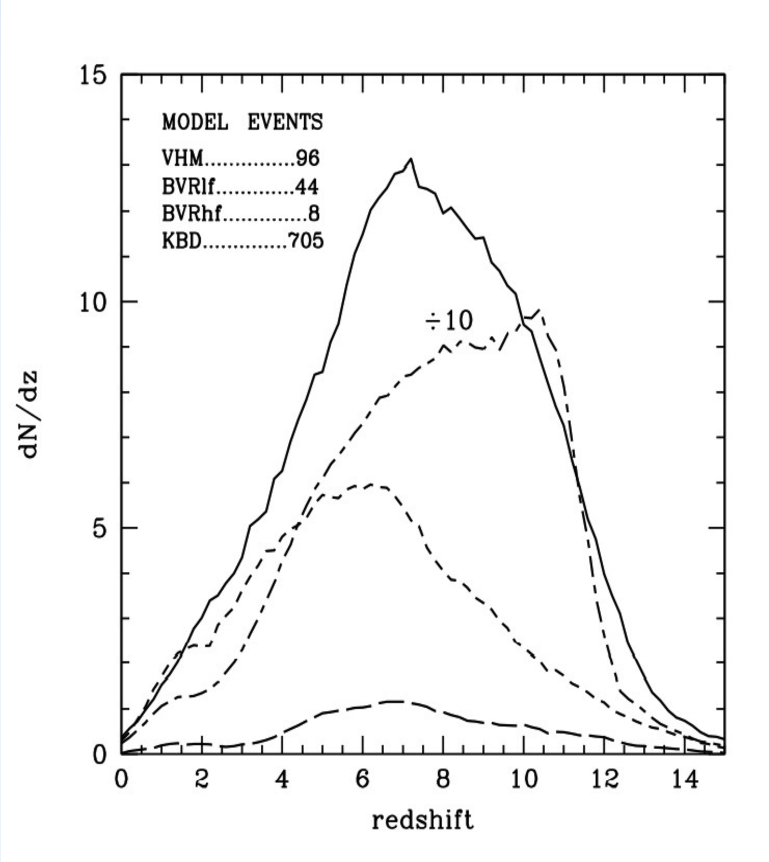


PN waveforms for BH binaries with equal masses and maximal spins (Arun et al 2009)

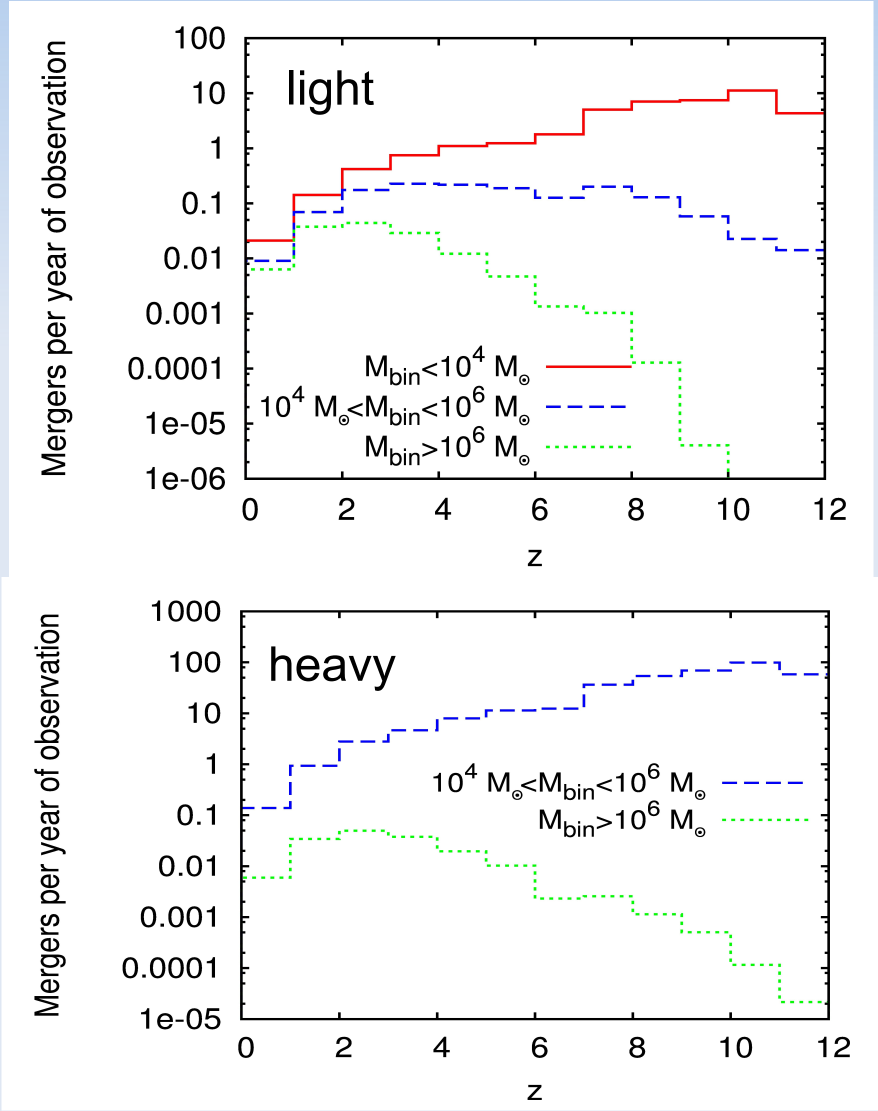
Testable with LISA or similar space-base detectors!

Galaxy formation with LISA

- Number of event rates (light vs heavy seeds)



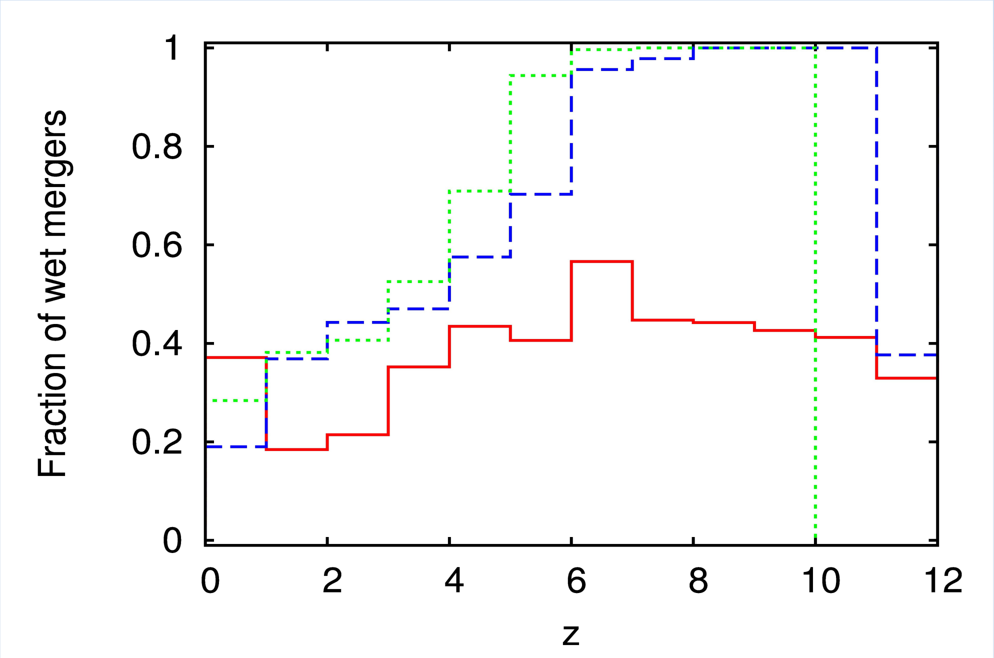
from Sesana, Volonteri and Haardt (2007)

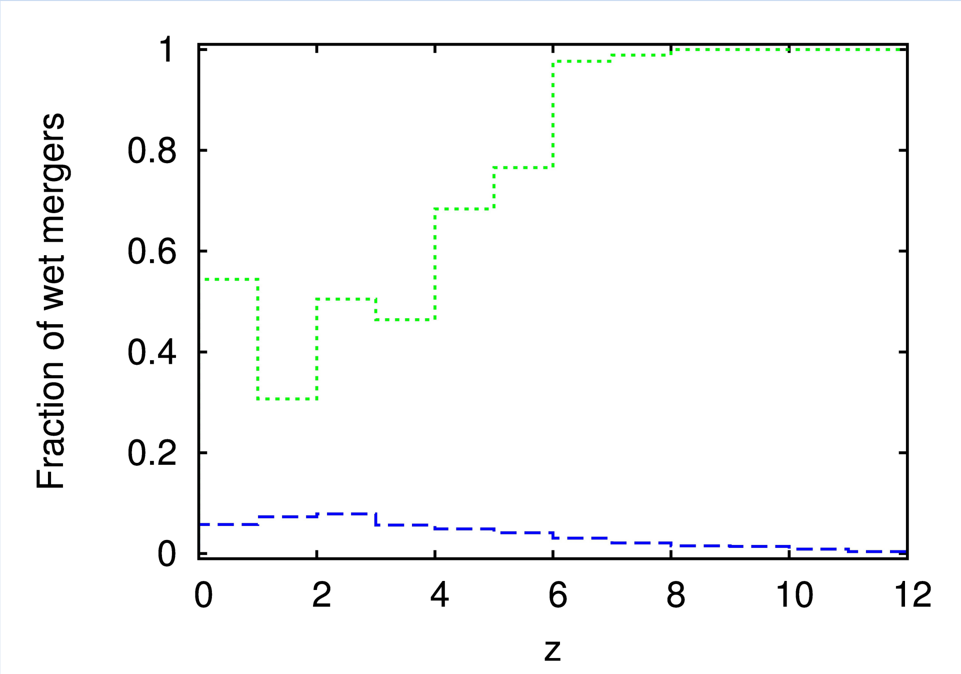


from Barausse (2012)

Galaxy formation with LISA

- Wet vs dry mergers





Red = $M_{bin} < 10^4\ M_{sun}$

Blue = $10^4\ M_{sun} < M_{bin} < 10^6\ M_{sun}$

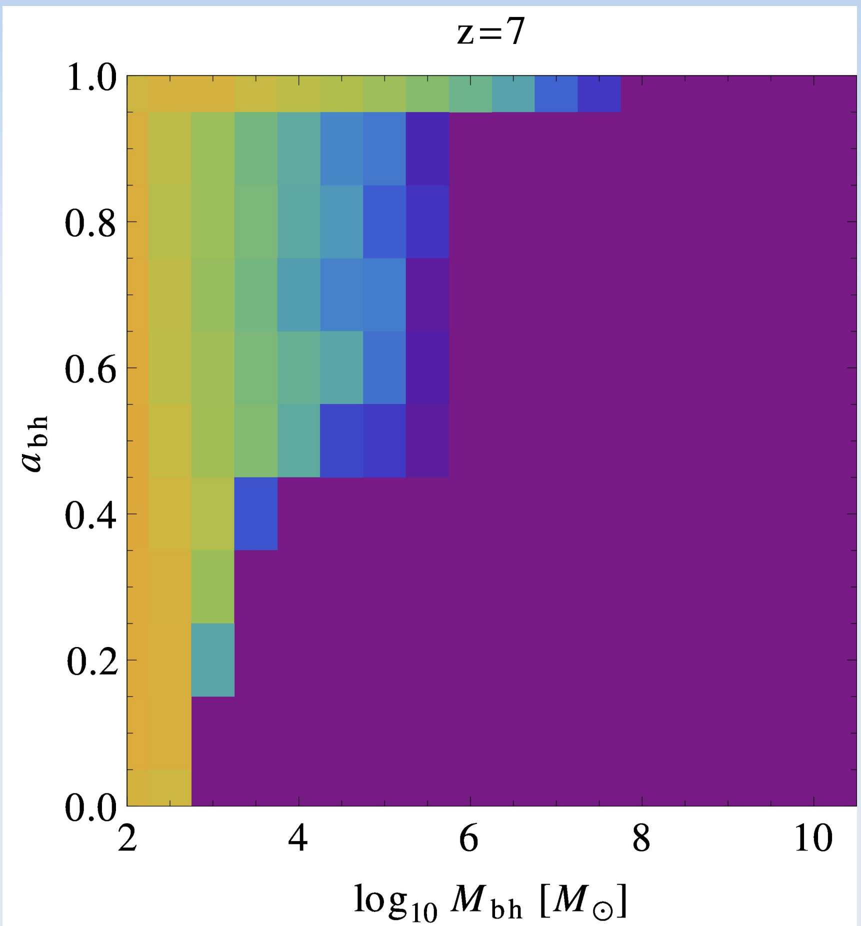
Green = $M_{bin} > 10^6\ M_{sun}$

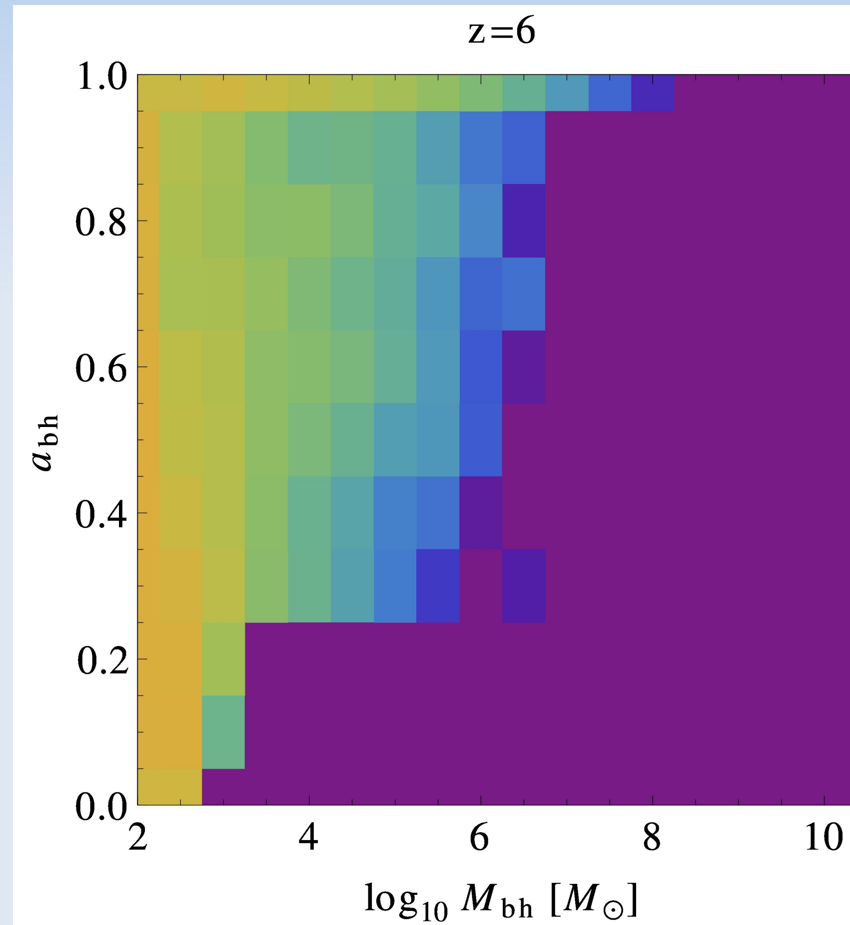
from Barausse (2012)

Galaxy formation with LISA

- The spin evolution

from Barausse (2012)





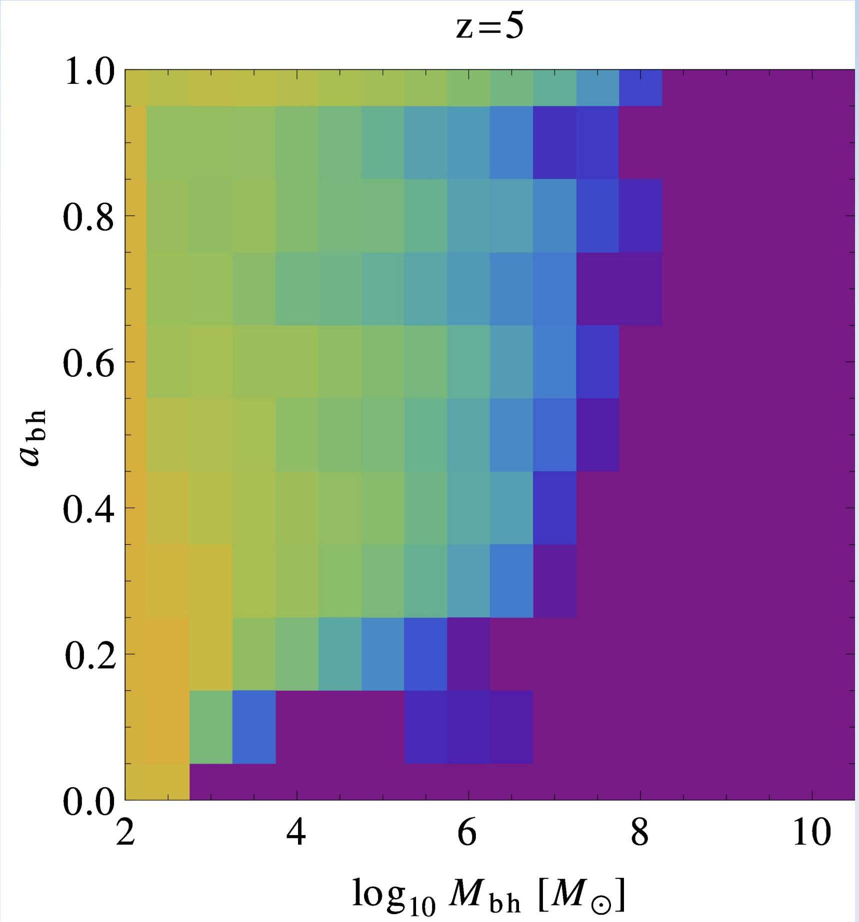
Color code = $\log_{10}$ of number density of MBHs per unit log-mass and unit spin, i.e

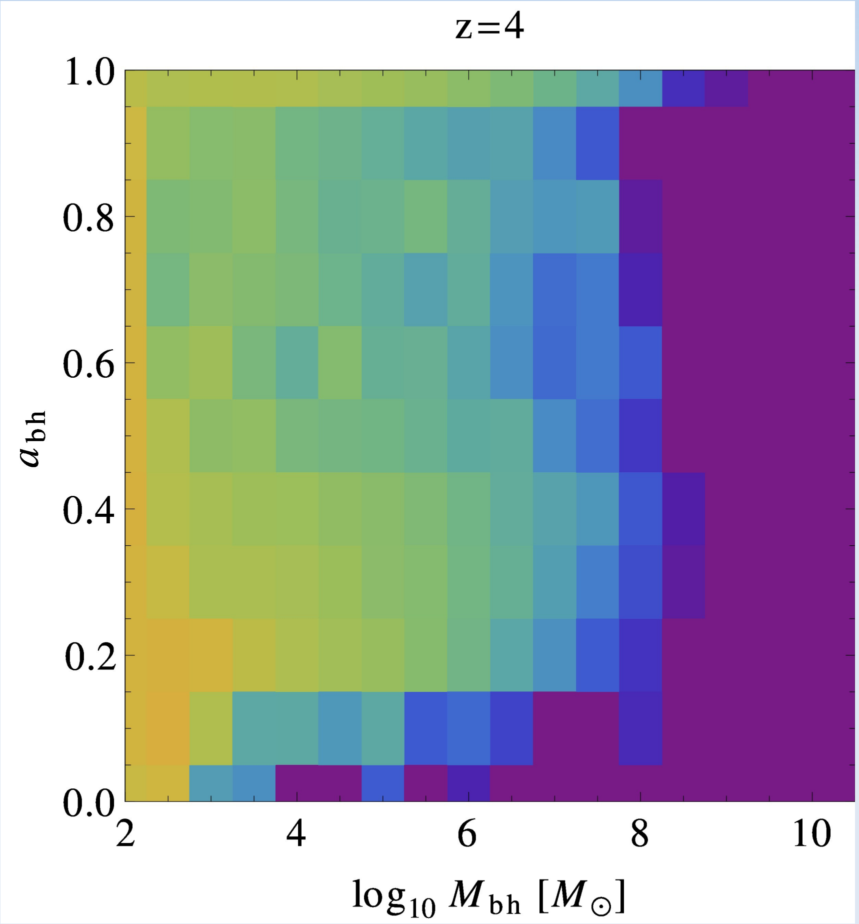
$$\log_{10}(\mathrm{d}\phi_{\rm bh}[\mathrm{Mpc}^{-3}]/\mathrm{d}a) = \log_{10}(\mathrm{d}^2 n_{\rm bh}[\mathrm{Mpc}^{-3}]/(\mathrm{d}\log_{10} M_{\rm bh}[M_\odot]\,\mathrm{d}a)))$$

Galaxy formation with LISA

- The spin evolution

from Barausse (2012)





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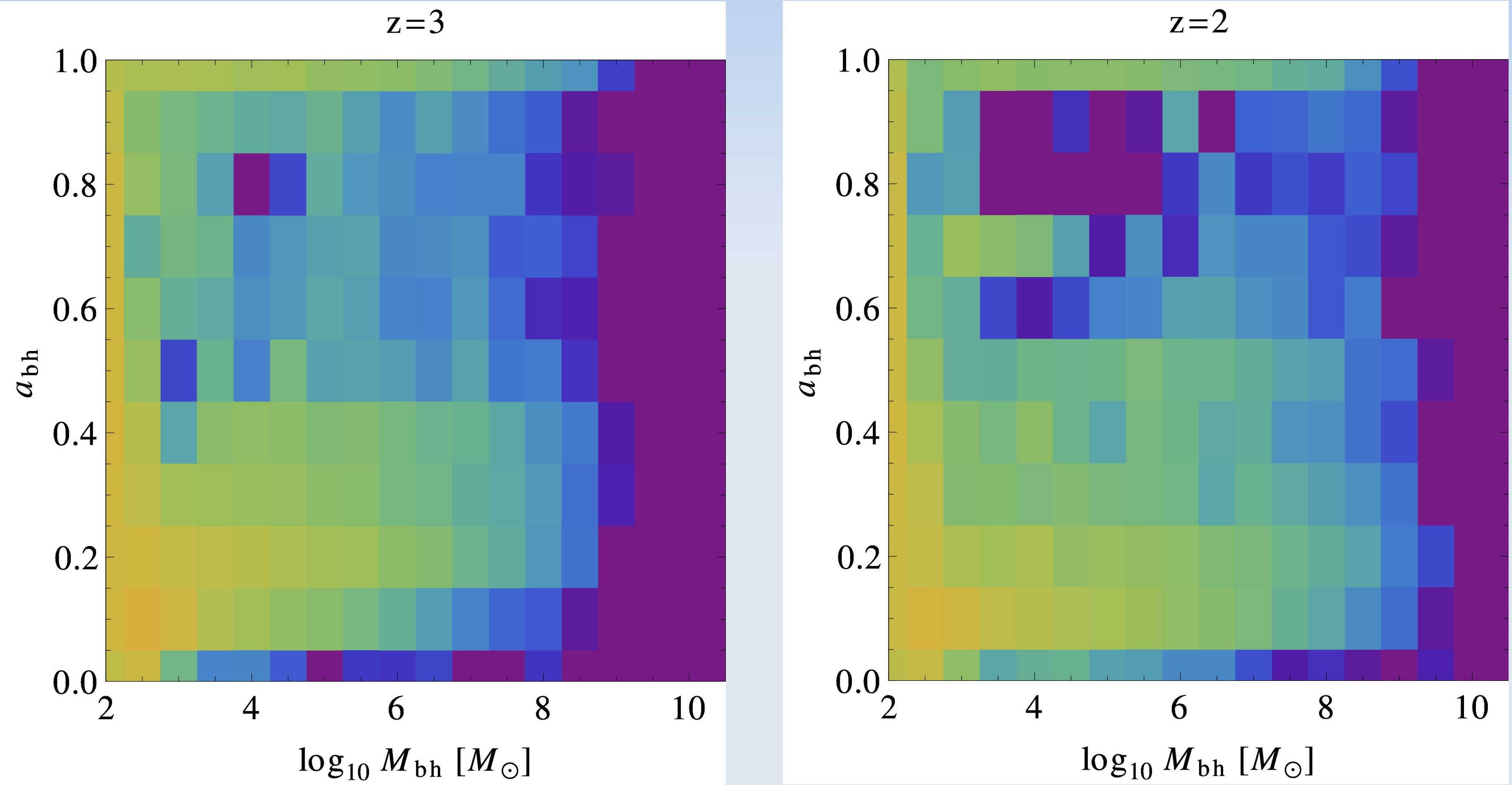
Galaxy formation with LISA

- The spin evolution

from Barausse (2012)



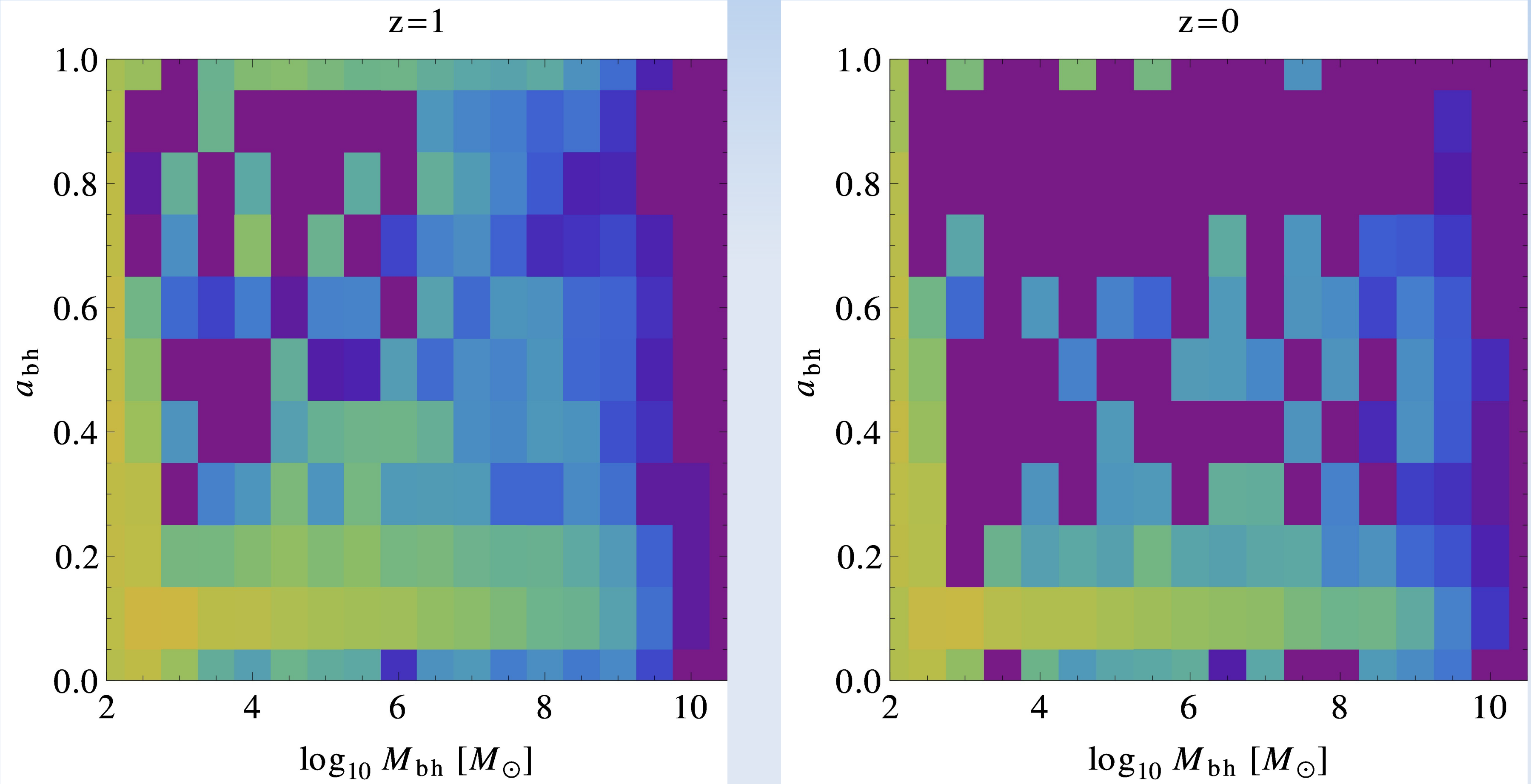
Color code = $\log_{10}$ of number density of MBHs per unit log-mass and unit spin, i.e

$$\log_{10}(\mathrm{d}\phi_{\rm bh}[\mathrm{Mpc}^{-3}]/\mathrm{d}a) = \log_{10}(\mathrm{d}^2 n_{\rm bh}[\mathrm{Mpc}^{-3}]/(\mathrm{d}\log_{10} M_{\rm bh}[M_\odot]\,\mathrm{d}a)))$$

Galaxy formation with LISA

- The spin evolution

from Barausse (2012)



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Conclusions

- GWs are a generic prediction of relativistic gravity theories (including GR!) because they follow from casual structure of such theories (nothing can propagate faster than light)
- GWs have been observed indirectly, and direct detection is imminent
- They are can be used to probe gravitational physics, astrophysics, nuclear physics, cosmology