CARGESE LECTURES 2010

Jean-Philippe Uzan

I. Recombination and background properties

II. Anisotropy of the CMB
- Sachs-Wolfe formula
- C_ℓ
- main behaviour

III. Boltzmann description
- Distribution function
- Liouville term
- Macroscopic quantities
- Brightness-Temperature
- Collision term
- moment expansion.

<u>CARGESE</u> : cours n° 1

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The cosmic microwave radiation and its temperature were predicted by Alpher and Herman [Nature <u>162</u> (1948) 774] following the arguments by Gamow [Nature <u>162</u> (1948) 680].

The CMB is part of the thermal history of our universe which arises from its cooling due to the expansion.

<u>$T \gg 1$ MeV</u> $n \cdot p \cdot e^{\pm} \cdot \gamma \cdot \nu$ weak interaction keeps n-p in equilibrium.

<u>$T \sim 1$ MeV</u> - freeze-out of the weak interaction neutrons will decay into protons.

Synthesis of light nuclei can start only when the temperature is low enough for deuterium to be synthetized

$$n + p \rightleftharpoons D + \gamma \qquad B_D = 2.2\,\text{MeV}$$

γ photodissociate D until below B_D. $\left[T_D \sim \dfrac{B_D}{-1.5 \ln \frac{T_D}{m_N} - \ln \eta} \right]$

It follows that to roughly explain $Y_p \sim 25\%$ one needs

$$T_D \sim 10^9\,\text{K} \quad \text{and} \quad \eta \sim 5.10^{-10} \; ; \; n_b \sim 10^{18}\,\text{cm}^{-3}$$

see Peter-Uzan, Primordial cosmology (OUP, 2009) chapter 4

Since today $n_{b_0} \sim 10^{-7}\,\text{cm}^{-3}$ we deduce

$$\boxed{1 + z_D = \left(\frac{n_b}{n_{b0}}\right)^{1/3} \sim 2.10^8 \qquad T_{rad}\big|_0 = \frac{T_D}{1+z_D} \sim 5\,\text{K}}$$

$\boxed{k_B = 8{,}617\ 10^{-5}\ \mathrm{eV.K^{-1}}}$

$\boxed{\hbar c = 197{,}326\ \mathrm{MeV.fm}}$

Such a background radiation has been detected by Penzias-Wilson (1964) and was then interpreted by Dicke (1965)

It is observed as a Radiation with Planck spectrum with temperature

$$T_0 = 2.725\ \mathrm{K} \qquad k_B T_0 = 2{,}345 \times 10^{-4}\ \mathrm{eV}$$

We deduce that
$$n_{\gamma_0} = \frac{2}{\pi^2}\zeta(3)\, T_0^3 = \frac{2}{\pi^2}\zeta(3)\left(\frac{k_B T_0}{\hbar c}\right)^3$$
$$= 410\ \mathrm{cm^{-3}}$$

$$\rho_{\gamma_0} = \left(\frac{\pi^2}{30}\right) 2\, T_0^4 = 2\frac{\pi^2}{30}\frac{(k_B T_0)^4}{(\hbar c)^3}\ [\mathrm{eV.cm^{-3}}]$$
$$\begin{cases} = 0{,}26\ \mathrm{eV.cm^{-3}} \\ = 4{,}96\ .10^{-13}\ \mathrm{eV^4} \\ = 4{,}6\ .10^{-34}\ \mathrm{g.cm^{-3}} \end{cases}$$

$$\boxed{\Omega_{\gamma_0} h^2 = \frac{8\pi G \rho_{\gamma_0}}{3H_0^2} = 2{,}469 \times 10^{-5}}$$

For radiation, one needs to take into account neutrinos. Assuming $N_\nu = 3$ families of massless neutrinos (each with $g_\nu = 1$), one gets that

$$\rho_\nu = \frac{7}{8} N_\nu \left(\frac{\pi^2}{30}\right) \overset{\nu\bar\nu}{2} g_\nu\, T_\nu^4 = \frac{7}{8}\cancel{N_\nu g_\nu}\left(\frac{4}{11}\right)^{4/3} \times \underbrace{2\frac{\pi^2}{30} T_0^4}_{\rho_{\gamma_0}}$$

[see lect. by J. Lesgourgues]

$$\boxed{\Omega_{\nu_0} h^2 = 1{,}68\ .10^{-5}}$$

$$\boxed{\Omega_{\mathrm{rad}\,0} h^2 = 4{,}148\ .10^{-5}}$$

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Equality radiation-matter

$$1 + z_{eq} = \frac{\Omega_{m_0}}{\Omega_{r_0}} \qquad \boxed{z_{eq} \simeq 3612\left(\frac{\Omega_{m_0} h^2}{0.15}\right)}$$

Spectrum

Because of the background spacetime symmetries, the distribution function of the CMB-γ depends only on Energy and time since otherwise it would violate either isotropy or homogeneity.

$$f(x^\alpha, p_\beta) = f(E, t)$$

After decoupling the γ propagate freely and their geodesic equation implies that

$$E = E_{LSS} \frac{a_{LSS}}{a} = E_0 \frac{a_0}{a} = E_0 (1+z)$$

Today we observe a black body spectrum with temperature T_0 so that today

$$f(E_0, T_0) = \frac{1}{e^{E_0/T_0} - 1}$$

This implies that

$$f(E, T) = \frac{1}{e^{E/\underbrace{T_0(1+z)}_{T}} - 1}$$

Because the γ have redshifted achromatically the spectrum was black body with temperature

$$\boxed{T = T_0 (1+z)}$$

- Departure from the BB spectrum are small and constrain energy injection roughly up to $z \sim 10^6$
- $T(z \sim 2.33) \sim 9.1\,K$ and observation $6 < T < 14\,K$

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No.

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Photons in FL-universe

I consider a homogeneous & isotropic spacetime with metric

$$\boxed{ds^2 = -dt^2 + \underbrace{a^2(t)}_{\text{scale factor}} \gamma_{ij}\, dx^i dx^j}$$

γ_{ij} is the spatial metric. I will restrict to spatially Euclidean spacetime:

$$\gamma_{ij}\, dx^i dx^j \begin{cases} = dr^2 + r^2 d\Omega^2 & \text{(sph.)} \\ = dx^2 + dy^2 + dz^2 & \text{(cartesian)} \end{cases}$$

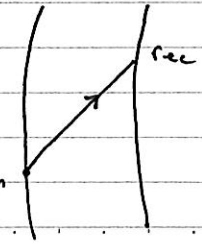
For comoving observers $u^\mu = \frac{1}{a}\delta^\mu{}_0$ is the tangent vector to their worldline

($u^\mu u_\mu = -1$ timelike) $\quad u_\mu = -a\,\delta_{\mu 0}$

Using the result of 2-ter then $g = FL$, $\hat{g} = M4$

- $\hat{k}^0 =$ constante for $M4$
- Then the energy of a γ is $E = -k^\mu u_\mu$ (as measured by an observer comoving with u_μ)

it follows that $E = -\frac{\hat{k}^0}{a^2} u_0 = \frac{\hat{k}^0}{a}$



$$\frac{E_{rec}}{E_{em}} = \frac{a_{em}}{a_{rec}} = \frac{\nu_{em}}{\nu_{rec}}$$

$$\boxed{1+z = \frac{E_{em}}{E_0} = \frac{a_0}{a}}$$ redshift

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Null geodesics of two conformal spacetimes (may be left as an exercise)

Consider $X^\mu(\lambda)$ a null geodesic with tangent vector $k^\mu = \frac{dx^\mu}{d\lambda}$

k^μ is a null-vector $\boxed{k_\mu k^\mu = 0}$

geodesic equation is $\boxed{k^\mu \nabla_\mu k^\nu = 0}$

covariant derivative associated to $g_{\mu\nu}$

Now consider another spacetime with metric $\hat{g}_{\mu\nu}$ with derivative $\hat{\nabla}$ and that the null geodesic has tangent vector $\hat{k}^\mu$

Set $g_{\mu\nu} = a^2 \hat{g}_{\mu\nu}$ & $k^\mu = \alpha \hat{k}^\mu$ $\qquad$ $a(t)$ $\quad \alpha(t)$

Then

$$k^\mu \nabla_\mu k^\nu = 0$$

$$= \hat{k}^0 \dot{\alpha} \hat{k}^\nu + \alpha \underbrace{\hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\nu}_{0} + 2\alpha H \mathcal{J}^\nu_{\mu\rho} \hat{k}^\mu \hat{k}^\rho$$

with $\mathcal{J}^\nu_{\mu\rho} = \hat{g}^{\nu\alpha} \left[\hat{g}_{\alpha\rho} \delta_{\mu 0} + \hat{g}_{\alpha\mu} \delta_{\rho 0} - \hat{g}_{\mu\rho} \delta_{\alpha 0} \right]$

So $\dot{\alpha} + 2\alpha H = 0 \iff \alpha = a^{-2}$

$$\boxed{k^\nu \text{ geodesic of } g_{\mu\nu} \iff \hat{k}^\mu = a^2 k^\mu \text{ geodesic of } \hat{g}_{\mu\nu} = a^{-2}(t)\, g_{\mu\nu}}$$

$$\mathcal{J}^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho} - \hat{\Gamma}^\nu_{\mu\rho} = g^{\nu\alpha}\left(\partial_\mu g_{\alpha\rho} + \partial_\rho g_{\alpha\mu} - \partial_\alpha g_{\mu\rho}\right) - \hat{\Gamma}^\nu_{\mu\rho}$$

$$= \frac{1}{a^2} \hat{g}^{\nu\alpha}\left(\partial_\mu a^2 \hat{g}_{\alpha\rho} + \partial_\rho a^2 \hat{g}_{\alpha\mu} - \partial_\alpha \hat{g}_{\mu\rho} a^2\right) - \hat{\Gamma}^\nu_{\mu\rho}$$

$$= \frac{1}{a^2} \times 2a\dot{a}\, \hat{g}^{\nu\alpha}\left[\hat{g}_{\alpha\rho}\delta_{\mu 0} + \hat{g}_{\alpha\mu}\delta_{\rho 0} - \hat{g}_{\mu\rho}\delta_{\alpha 0}\right] + \hat{\Gamma}^\nu_{\mu\rho} - \hat{\Gamma}^\nu_{\mu\rho} \quad \text{C.Q.F.D.}$$

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Dipole The observer is a priori not a comoving observer

To a high precision, the CMB is isotropic in its reference frame (assumed to coincide with the cosmological rest frame)

If the observer has 4-velocity $u^\mu = \frac{1}{\sqrt{1-v^2/c^2}} \begin{pmatrix} 1 \\ v^i/c \end{pmatrix}$

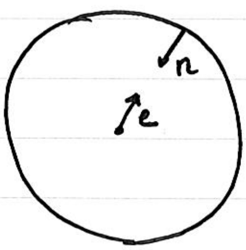
then he will observe a black-body radiation with temperature

$$T_{obs}(\vec{e}) = \frac{T_{CMB}}{\gamma(1+\vec{v}\cdot\vec{e})} \sim T_{CMB}(1-\vec{v}\cdot\vec{e})$$

we should thus observe a dipole ($v/c \ll 1$) but there is also a quadrupole in $O(v^2)$.

FIRAS observations give $v \sim 368 \pm 2$ km.s^{-1} for the velocity of the solar syst. barycenter.

Note



$u^\mu = \gamma \begin{pmatrix} 1 \\ v^i/c \end{pmatrix}$ for the observer / CMB rest frame

$k^\mu = E_{CMB} \begin{pmatrix} 1 \\ n^i \end{pmatrix}$

$$E = -k^\mu u_\mu \Rightarrow E = \gamma E_{CMB}(1-\vec{n}\cdot\vec{v})$$
$$= \gamma E_{CMB}(1+\vec{n}\cdot\vec{e}).$$

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Recombination

As long as the matter in the Universe is ionized, the γ interact strongly.

As long as the energy of the γ is high they prevent H to be formed through

$$p + e^- \rightleftharpoons H + \gamma$$

As long as the photoionisation is able to maintain equilibrium, the relative abundance of e, p, H have to satisfy

$$\frac{n_p\, n_e}{n_H} \sim e^{-\frac{E_I}{T}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}$$

with $E_I = m_p + m_e - m_H = 13{,}6$ eV

(using the Maxwell-Boltzmann distributions and $g_p = g_e = 2$; $g_H = 1$)

chemical equilibrium: $\mu_p + \mu_e = \mu_H$
electric neutrality $n_e = n_p$

Define the ionization fraction as $X_e = \frac{n_e}{n_b} = \frac{n_e}{n_p + n_H}$

Then
$$\begin{cases} n_e = n_p = X_e\, n_b \\ n_H = (1 - X_e)\, n_b \end{cases}$$

So that
$$\frac{X_e^2}{1 - X_e} = \left(\frac{m_e T}{2\pi}\right)^{3/2} \frac{e^{-E_I/T}}{n_b}$$

$$n_b = \eta\, n_\gamma = \eta\, \frac{2}{\pi^2}\, \zeta(3)\, T^3$$

see Peter & Uzan, Primordial cosmology (OUP, 2009) chapter 4

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$\boxed{1\,eV = 1{,}160 \cdot 10^4\ K}$

$\boxed{m_e = 0{,}510\ MeV}$

$$\boxed{\frac{X_e}{1-X_e^2} = \left(\frac{m_e c^2}{k_B T}\right)^{3/2} \sqrt{\frac{\pi}{2^5}}\ \frac{e^{-E_I/k_B T}}{\zeta(3)\,\eta}}$$

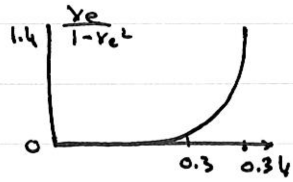
usually called the Saha equation.

For $\eta = 5 \cdot 10^{-10}$ & $T = 13{,}6\ eV = E_I$

$\frac{X_e}{1-X_e^2} = 1.39 \cdot 10^{15} \Leftrightarrow$ $X_e = 1$

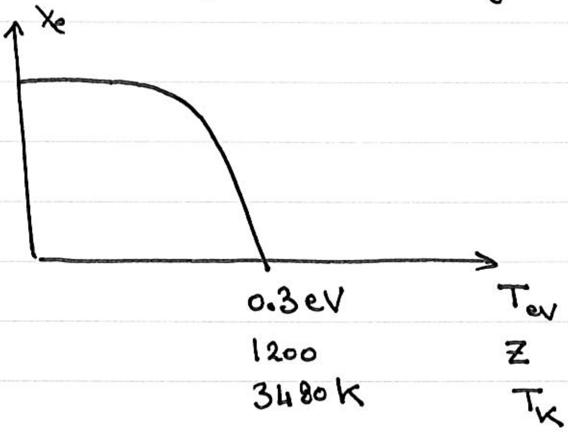
Because η is large, X_e becomes of order unity at $T < 13.6\ eV$ (significantly.

Numerically $X_e \sim \mathcal{O}(1)$ at $T \sim 0.3\ eV$



That correspond to a redshift $\boxed{1+z = \frac{0.3\ eV}{T_0} \sim 1276}$

From this analysis, assuming equilibrium we deduce



To describe the dynamics of the recombination, one needs solve a Boltzmann equation for the electron density.

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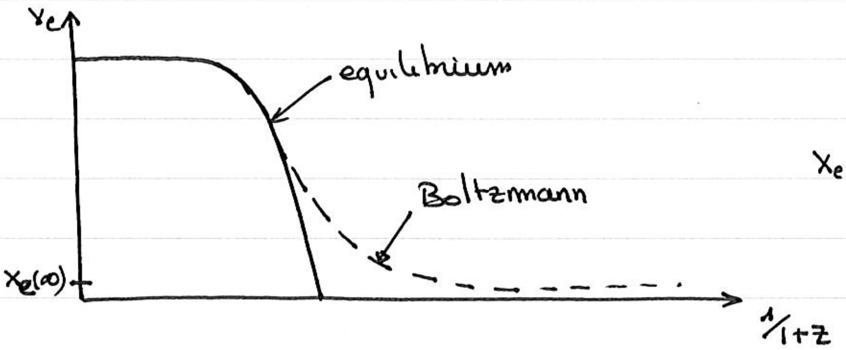
Again $n_e = n_b X_e$
$n_H = n_b (1 - X_e)$

$$\boxed{\dot{X}_e = C_r \Big[\underbrace{\beta (1 - X_e)}_{H+\gamma \to e+p} - \underbrace{\alpha^{(2)} n_b X_e^2}_{e+p \to H+\gamma} \Big]}$$

$$\begin{cases} \beta = \left(\dfrac{m_e T}{2\pi}\right)^{3/2} \alpha^{(2)} e^{-E_I/T} \\ \alpha^{(2)} = \langle \sigma_T v \rangle \end{cases}$$

σ_T : Thomson-scattering cross-section.



$X_e(\infty) \sim 7.10^{-3}$

Full description requires - a detailed description of the atomic transition
- take into account Helium.

See
Peebles Ap.J. 153 (1968) 1
Jones & Wise A&A 149 (1985) 144
Seager et al. Ap.J. Supp. 128 (2000) 407 [RECFAST]

$\boxed{\text{Photon découpling \& last scattering surface}}$

During recombination n_e varies rapidly so that the rate of $e^- - \gamma$ interaction drops rapidly

$$\Gamma_T = n_e \sigma_T c$$

$$\Gamma_T = n_b X_e \sigma_T c$$

<u>Remarque</u>

<u>Today</u>: $\Gamma_{T_0} \sim 1{,}4 \cdot 10^{-3} H_0$ i.e.
$\frac{1\gamma}{700}$ interact with a e^- en a time H_0^{-1}

<u>$z \sim 10^3$</u> $n_e \sim 10^3 n_{e0}$ $H \sim H_0 \sqrt{(1+z)^3 \Omega_{m0}}$
$\Gamma_T \sim 80 H$
1γ interagit 80 fois avec 1 e^-
if univers totally <u>ionized</u>.

Since X_e drops rapidly, the compton scattering will freeze-out soon after the recombination.

To estimate the time of recombination let us assume it happens when $\Gamma_T = H$

$$\hookrightarrow \quad n_{b_0}(1+z)^3 \sigma_T c \, X_e(z) = \sqrt{\Omega_{m_0} H_0^2 (1+z)^3 \Big[\underset{\uparrow M}{1} + \underset{\uparrow R}{\frac{1+z}{1+z_{eq}}}\Big]}$$

To simplify $X_e(z) = X_e(\infty) \sim 7 \cdot 10^{-3}$
then,

$$\boxed{(1+z_{LSS})^{3/2} \simeq \frac{280}{X_e(\infty)} \left(\frac{\Omega_{b_0} h^2}{0.02}\right)^{-1} \left(\frac{\Omega_{m_0} h^2}{0.15}\right)^{-1/2} \sqrt{1 + \frac{1+z_{LSS}}{1+z_{eq}}}}$$

clearly z_{LSS} depends mostly on Ω_{b_0} & Ω_{m_0}. and is of order 10^3.

<u>Numerical calculations</u> give the fit $\quad 1+z_{LSS} \sim 1089 \left(\frac{\Omega_{m_0} h^2}{0.14}\right)^{0.0105} \left(\frac{\Omega_{b_0} h^2}{0.024}\right)^{-0.028}$

[Hu, a-ph/0407158]

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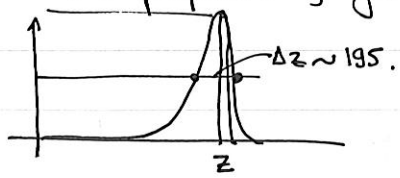
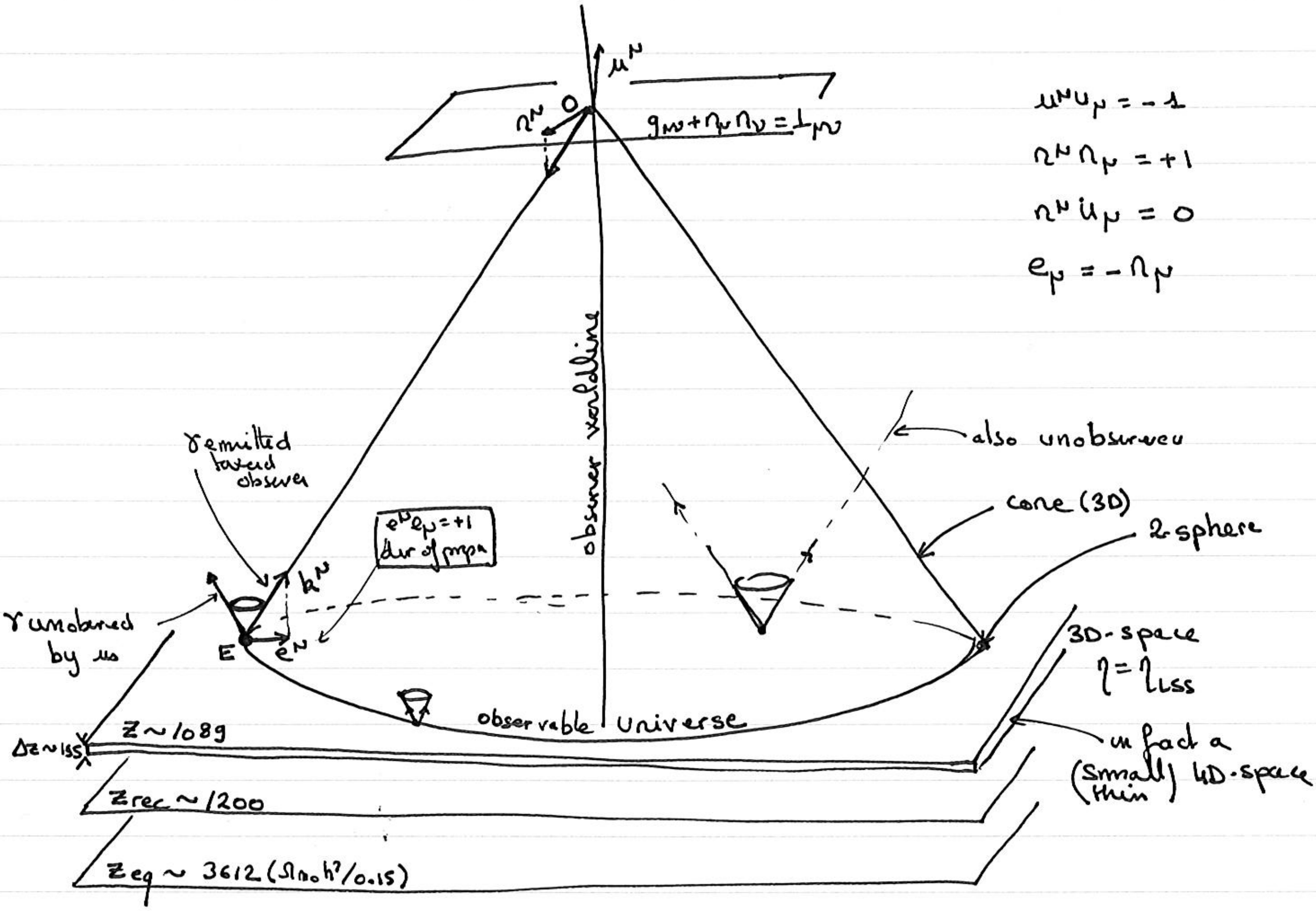
The optical depth is defined as $\tau = \int n_e X_e \sigma_T \, dx$ and the visibility function $g = e^{-\tau} \frac{d\tau}{dz}$ determines the propability for a γ to be scattered between z and $z + dz$





$$u^\mu u_\mu = -1$$
$$n^\mu n_\mu = +1$$
$$n^\mu u_\mu = 0$$
$$e_\mu = -n_\mu$$

$$T_{obs}[n] = T_E[x_E, \eta_{LSS}] \times \frac{a_E}{a_0} = \frac{T_E}{(1+z)}$$

$= T_0$ independent of the direction

E : $\eta_E = \eta_{LSS}$, $\vec{x}_E = \vec{x}_0 + (\eta_0 - \eta_{LSS})\,\vec{n}$ coord. comoving

We want to do the same but taking into account that the Universe is an almost - FL spacetime.

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$\boxed{\text{The Sachs-Wolfe formula}}$

After decoupling γ propagates freely and we can, ~~to~~ a very good approximation, assume that they are following null-geodesics.

Again, we use the trick of a conformal transformation

$g_{\mu\nu}$ is now the metric of a perturbed FL universe

$$g_{\mu\nu}\,dx^\mu dx^\nu = a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$

$$= \bar{g}_{\mu\nu} + h_{\mu\nu}$$

conformal time   spatial metric

$\boxed{\text{Note:}}$ the perturbations are decomposed in SVT modes

$$B_i = D_i B + \bar{B}_i \quad \text{with } D_i \bar{B}^i = 0$$

$$h_{ij} = 2C\gamma_{ij} + 2D_iD_jE + 2D_{(i}\bar{E}_{j)} + 2\bar{E}_{ij} \quad \text{with } D_i\bar{E}^{ij} = \bar{E}^i_i = 0$$

we thus have

$$\left.\begin{array}{lll} \text{Scalar}: & A, B, C, E & = 4 \\ \text{Vectors}: & \bar{B}_i, \bar{E}_i & = 2\times 2 = 4 \\ \text{Tensor}: & \bar{E}_{ij} & = 2 \end{array}\right\} \text{10 d.o.f of } h_{\mu\nu}$$

They are not invariant in a change of coordinates

$$x^\mu \longrightarrow x^\mu - \xi^\mu \qquad \text{with } \xi^\mu = (T, \bar{L}^i + D^iL) \quad \text{i.e. } 2S \text{ et } 2V$$

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$$

on peut donc absorber 4 d.o.f in a change of coordinates, so that only 6 d.o.f will remain from which we can construct 6 g.i variables i.e such that value remain unchanged in any change of coordinate.

$$\Psi = -C - \mathcal{H}(B-E') \qquad \bar{\Phi}^i = \bar{E}^{i\prime} - \bar{B}^i \qquad \bar{E}_{ij}$$

$$\Phi = A + \mathcal{H}(B-E') + (B-E')'$$

see Peter-Uzan, Primordial cosmology, (OUP, 2009) chapter 5

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Let us consider a null geodesic of this perturbed spacetime.

Again, we consider $\hat{k}^\mu$ null-vector of $\hat{g}_{\mu\nu} = a^{-2}(\eta)\, g_{\mu\nu}$ and we decompose it as

because of bgd eq

- $$\hat{k}^\mu = E(1 + M, e^i + \delta e^i)$$ where $\begin{cases} E \text{ is a constant} \\ e^i \text{ is a 3D-unit vector constant} \end{cases}$

$$\gamma_{ij} e^i e^j = 1$$

$$\boxed{\delta\hat{k}^\mu = E(M, \delta e^i)} \qquad \boxed{k^\mu = \frac{\hat{k}^\mu}{a^2}}$$

- It has 4 composants but only 3 are independant because

$$\hat{k}_\mu \hat{k}^\mu = 0 = \hat{g}_{\mu\nu} \hat{k}^\mu \hat{k}^\nu = (\bar{g} + h)\cdot(\bar{k} + \delta k)(\bar{k} + \delta k)$$

Thus,

remind that $\bar{g}\cdot\bar{k}\cdot\bar{k} = 0$

$$\hat{h}_{\mu\nu} \hat{\bar{k}}^\mu \hat{\bar{k}}^\nu + 2\hat{\bar{g}}_{\mu\nu} \hat{\bar{k}}^\mu \delta\hat{k}^\nu = 0$$

$$\boxed{\underbrace{e_j\, \delta e^j}_{\gamma_{ij} e^i \cdot \delta e^j} = \underline{A + M - B_i e^i - \frac{1}{2} h_{ij} e^i e^j}}$$

$\hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\nu = 0$

- The geodesic equation for $\hat{k}^\mu$ takes the simplified form

$$\hat{k}^\mu \partial_\mu \hat{k}^\nu = -\delta\hat{\Gamma}^\nu_{\mu\rho} \hat{k}^\mu \hat{k}^\rho$$

focusing on the time-component ($\nu = 0$)

$$\underbrace{(1+\cancel{M})\partial_0 M + (e^i + \cancel{\delta e^i})\partial_i M}_{\underbrace{\partial_0 M + e^i \partial_i M}_{\frac{dM}{ds}}} + \underbrace{\Gamma^0_{00}(1+M)^2}_{A'} + \underbrace{2\Gamma^0_{0i}(1+M)(e^i + \delta e^i)}_{2(D_i A)\, e^i}$$

$$+ \underbrace{\Gamma^0_{ij}(e^i + \delta e^i)(e^j + \delta e^j)}_{[\frac{1}{2} h'_{ij} - D_{(i} B_{j)}]\, e^i e^j} = 0$$

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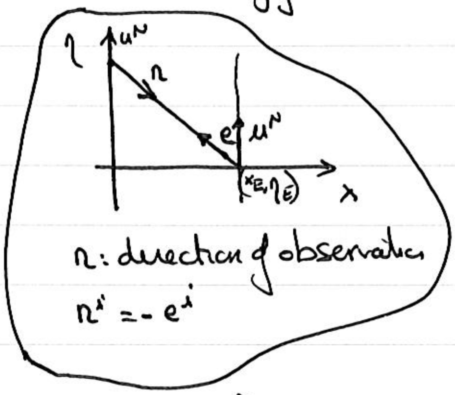
It follows that

$$\boxed{\frac{dM}{ds} = -A' - 2\, e^i \partial_i A - \frac{1}{2} h'_{ij}\, e^i e^j + (D_i B_j)\, e^i e^j}$$

We can now compare the observed energy of a γ compared to its energy at emission



$$\frac{E_{obs}}{E_{em}(x_E, \eta_E)} = \frac{(-k^\mu u_\mu)_{obs}}{(-k^\mu u_\mu)_{em}}$$

$$u_\mu = a(-1 - A,\ v_k + B_k) \qquad [u_\mu u^\mu = -1]$$

$$k^\mu u_\mu = \frac{E}{a}\left[-1 + \Pi + A + e^i(v_i + B_i)\right]$$

$$\vec{X}_E = \vec{X}_0 + \vec{n}(\eta_0 - \eta_E)$$

$$\frac{E_{obs}(\vec{n})}{E_{em}(x_E, \eta_E)} = \frac{a_{em}}{a_{obs}} \times \frac{\left[1 + \Pi + A + e^i(v_i + B_i)\right]_{obs}}{\left[1 + \Pi + A + e^i(v_i + B_i)\right]_{em}}$$

$$= \frac{a_{em}}{a_{obs}} \times \left\{1 + \left[M + A + n^i(v_i + B_i)\right]^{obs}_{em}\right\}$$

We deduce that the temperature of the Black-bodies at emission and reception are related by

$$\boxed{\frac{T_0(\vec{n})}{T_E(\eta_E, x_E)} = \frac{a(\eta_E)}{a(\eta_{obs})} \left\{1 + \left[M + a + n^i(v_i + B_i)\right]^{obs}_{em}\right\}}$$

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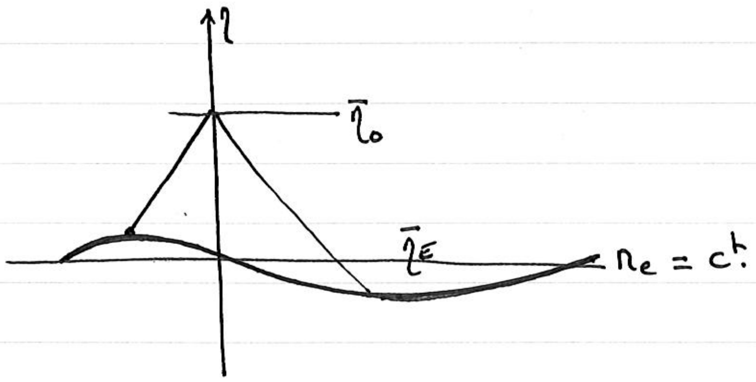
As we have seen, the last scattering process is mostly governed by $\sigma_T n_e$.

We can <u>model</u> the LSS by a 3D-hypersurface defined by $\{n_e = c^{t.}\}$ assuming it is instantaneous.



It follows that the time of decoupling is given by

$$\eta_E = \bar{\eta}_E + \delta\eta_E$$

denser region decouple later..

~~Remark, however, that~~

We decompose the temperature as

$$\begin{cases} T_0(\vec{n}) = \bar{T}_0\left[1 + \Theta_0(\vec{n})\right] \\ T_E(\vec{x}_E, \eta_E) = \bar{T}_E(\eta_E)\left[1 + \Theta_E(\vec{x}_E, \eta_E)\right] \end{cases}$$

It follows that

$$\frac{\bar{T}_0}{T_E(x_E, \eta_E)} \times \left\{1 + \Theta_0(\vec{n}) - \Theta_E\right\} = \frac{a(\eta_E)}{a_0}\left\{1 + \left[\Pi + A + n^i(v_i + B_i)\right]_{em}^{abs}\right\}$$

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Remark that

$$\bar{T}_E(\eta_E)\, a(\eta_E) = \bar{T}_E(\bar{\eta}_E)\, a(\bar{\eta}_E)$$

show this if not convinced

$$\begin{cases} \bar{T}_E(\eta_E) = \bar{T}(\bar{\eta}_E)\left[1 + \frac{T'}{T}\delta\eta_E\right] \\ a(\eta_E) = a(\bar{\eta}_E)\left[1 + \mathcal{H}\,\delta\eta_E\right] \end{cases}$$

$\frac{T'}{T} = -\mathcal{H}$ @ background level

Thus, we get that

$$\boxed{\begin{aligned} \bar{T}_0 &= \bar{T}_E \frac{a_E(\bar{\eta}_E)}{a_0} \\ \Theta_0(\vec{n}) &= \Theta_E(x_E, \bar{\eta}_E) + \left[\Psi + A + n^i(v_i + B_i)\right]^0_E \end{aligned}}$$

pure first order terms

in a Born approximation it is evaluated along the background geodesic.

<u>We need to determine Θ_E</u>

n_E is a function of ρ_γ and ρ_b.

As we have seen from the Saha equation, baryons are negligible in the process so that we can <u>approximate</u>

$$LSS = \{\rho_\gamma = c^{te}\}$$

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This implies that $\Theta_E(x_E, \bar\eta_E) = \frac{1}{4}\delta_\gamma(x_E, \bar\eta_E)$

remind Stefan-Boltzmann law

Then, we need to evaluate $[M]_E^o$

$$[M]_E^o = \int_E^o \frac{dM}{ds}\,ds$$

we use the geodesic equation to get:

$$= \int_E^o \Big(\underbrace{-A' - 2e^i\partial_i A}_{-2\frac{dA}{ds}+A'} - \frac{1}{2}h'_{ij}e^ie^j + D_iB_j\,e^ie^j\Big)\,ds$$

$$= -2[A]_E^o + \int_E^o \Big(A' - \frac{1}{2}h'_{ij}e^ie^j + D_iB_j\,e^ie^j\Big)\,ds$$

We conclude that

$\oplus$ because $[\,]_E^o = o - E$

$$\Theta_o(\vec n) = \Big[\frac{1}{4}\delta_\gamma + A - n^i(v_i + B_i)\Big]_{\vec x_E, \bar\eta_E}$$

$$+\int_E^o \big[A' - C' - n^in^j(D_iD_jE' + D_i\bar E'_j - D_iB_j + \bar E'_{ij})\big]\,ds$$

$$+ f(o)$$

function of the perturbations evaluated today in O

This can now be rewritten in terms of gauge invariant variables

$$\begin{cases} \delta_\gamma^N = \delta_\gamma - 4\mathcal{H}(B - E') \\ \Phi + \Psi = A - C + (B - E)' \end{cases}$$

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$$n^j D_j (\bar{E}'_i - B_i) = e^j D_j \bar{\Phi}_i = - \Phi'_i + \frac{d\Phi_i}{ds}$$

$$n^i D_i (V + B) = e^i D_i (\underbrace{V + E'}_{V}) + e^i D_i (B - E')$$

$$= e^i D_i V + e^i D_i (B - E')$$

$$\Theta_0(\vec{n}) = \left[\frac{1}{4}\delta^N_\gamma + \Phi - n^i \left(D_i V_b + \bar{V}_{bi} + \bar{\Phi}_i\right)\right]_{x_E, \eta_E}$$

$$+ \int_E^0 \left\{ \Phi' + \Psi' - n^i \bar{\Phi}'_i - n^i n^j \bar{E}'_{ij} \right\} ds.$$

integral along the line of sight
(γ-geodesic)
i.e

$$\int_E^0 X \, ds = \int_E^0 x(\vec{x}(\eta), \eta) \, d\eta$$

if we parametrise $\vec{x} = x_0 + \vec{n}(\eta_0 - \eta)$

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IN CONCLUSION

$$\Theta_0(\vec{n}) = \Theta_0^S(\vec{n}) + \Theta_0^V(\vec{n}) + \Theta_0^T(\vec{n})$$

$$\begin{cases} \Theta_0^S(\vec{n}) = \left[\frac{1}{4}\delta_\gamma^N + \Phi - n^i D_i V_b\right]_{x_E, \eta_E} + \int_E^0 (\Phi' + \Psi')\, d\eta \\ \Theta_0^V(\vec{n}) = \left[-n^i(\bar{V}_{bi} + \bar{\Phi}_{bi})\right]_{x_E, \eta_E} - \int_E^0 n^i \bar{\Phi}_i'\, d\eta \\ \Theta_0^T(\vec{n}) = -\int n^i n^j \bar{E}_{ij}'\, d\eta \end{cases}$$

Physical interpretation

$$\begin{cases} \text{SW}: & \frac{1}{4}\delta_\gamma^N + \Phi & = \text{hotter-denser / Einstein effect} \\ \text{dop}: & -n^i(\bar{V}_i + \bar{\Phi}_i) - n^i D_i V_b & = \text{doppler (em \& observ have not the same velocity)} \\ \text{ISW}: & -\int_E^0 n^i \cdots & = \text{evol. of geometry on the line of sight : residuals of } \Sigma \text{ Einstein effects.} \end{cases}$$

This is the Sachs-Wolfe formula (1966).

It relies on:
- instantaneous recombination
- light geodesics
- γ travel freely after LSS.

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CARGESE : cours n°2

Jean-Philippe UZAN

Yesterday, we have related $\Theta_0(\vec{n})$ to the perturbation variables.

Now, we will investigate the properties of the angular power spectrum.

For simplicity I consider only scalar modes

more details: see Peter-Uzan, Primordial cosmology (OUP, 2009) chapter 6.

ANGULAR POWER SPECTRUM

The perturbation variables are stochastic variables (see the lecture on inflation), thus so is $\Theta(\vec{n})$. It can be characterized by its angular power spectrum:

$$C(\theta) = \langle \Theta(\vec{x}_0, \eta_0, n_1) \cdot \Theta(\vec{x}_0, \eta_0, n_2) \rangle$$

ensemble average

The statistical homogeneity implies that it does depend on $\vec{x}_0$

The statistical isotropy implies that it depends only on $\vec{n}_1 \cdot \vec{n}_2$

We set $\cos\theta = \vec{n}_1 \cdot \vec{n}_2$ and can expand $C(\theta)$ as

$$C(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \, P_\ell(\cos\theta)$$

angular power spectrum

Legendre polynomials

Since P_ℓ has ℓ zero an multipole ℓ corresponds to angular scale π/ℓ

C_ℓ measures the variance of Θ_0 on this scale.

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Our goal is to relate Θ_0 and C_ℓ to the power spectrum of the perturbations which, e.g. can be predicted by inflation.

$$\Theta_0(\vec{x}_0, \eta_0, \vec{n}) = \sum_{\ell m} a_{\ell m}(\vec{x}_0, \eta_0) \, Y_{\ell m}(\vec{n})$$

spherical harmonics

It can be inverted as

$$a_{\ell m}(\vec{x}_0, \eta_0) = \int d^2\vec{n} \; \Theta_0(x_0, \eta_0, \vec{n}) \, Y^*_{\ell m}(\vec{n})$$

Tool Box

$$\begin{cases} \sum_{-\ell}^{\ell} Y_{\ell m}(n) \, Y^*_{\ell m}(n') = \frac{2\ell+1}{4\pi} P_\ell(\cos\theta) \\ \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \; Y^*_{\ell m} Y_{\ell' m'} = \delta_{\ell\ell'} \, \delta_{mm'} \end{cases}$$

Using the first relation to express P_ℓ and then the second twice, one gets that

$$\boxed{(2\ell+1) \, C_\ell = \sum_m \langle a_{\ell m}(\vec{x}_0, \eta_0) \, a^*_{\ell m}(\vec{x}_0, \eta_0) \rangle}$$

Now, we can Fourier transform Θ_0:

$$\Theta_0(\vec{x}_0, \eta_0, \vec{n}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \, \hat{\Theta}(\eta_0, \vec{k}, \vec{n}) \quad \times \left[e^{i\vec{k}\cdot\vec{x}_0} \text{ which can be included in } \hat{\Theta} \right]$$

so that

$$\boxed{a_{\ell m}(\vec{x}_0, \eta_0) = \int d^2n \, \frac{d^3\vec{k}}{(2\pi)^{3/2}} \, \hat{\Theta}(\eta_0, \vec{k}, \vec{n}) \, Y^*_{\ell m}(\vec{n})}$$

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For the scalar modes, we found that

$$\Theta_0(\vec{n}) = \left[\frac{1}{4}\delta_\gamma^N + \Phi - n^i D_i V_b\right]_{x_E, \eta_E} + \int_E^0 (\Phi' + \Psi')\, d\eta$$

Consider e.g. $\Phi(x_E, \eta_E)$

$$\int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \Phi(\vec{k}, \eta_E)\, e^{i\vec{k}\cdot\vec{x}_E} = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \Phi(\vec{k}, \eta_E)\, e^{i\vec{k}\cdot\vec{n}(\eta_0 - \eta_E)} \times \left[e^{i\vec{k}\cdot\vec{x}_0}\right]$$

and thus contributes to $\hat{\Theta}(\eta_0, \vec{k}, \vec{n})$ as $\Phi(\vec{k}, \eta_E)\, e^{ik\mu\Delta\eta_E}$

with $\boxed{\begin{aligned} \mu &\equiv \vec{k}\cdot\vec{n} \\ \Delta\eta &= \eta_0 - \eta \end{aligned}}$

Consider $n^i D_i V_b$

$$n^i D_i \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \hat{V}_b(\vec{k}, \eta_E)\, e^{i\vec{k}\cdot\vec{x}_E} = \int ik\mu\, \hat{V}_b(\vec{k}, \eta_E)\, e^{ik\mu\Delta\eta_E} \times \left[e^{i\vec{k}\cdot\vec{x}_0}\right]$$

In conclusion,

$$\boxed{\begin{aligned} \hat{\Theta}(\eta_0, \vec{k}, \vec{n}) = {} & \left[\hat{\Theta}_{SW}(\vec{k}, \eta_E) - ik\mu\, \hat{V}_b(\vec{k}, \eta_E)\right] e^{ik\mu\Delta\eta_E} \\ & + \int \left[\hat{\Phi}'(\vec{k}, \eta) + \hat{\Psi}'(\vec{k}, \eta)\right] e^{ik\mu\Delta\eta}\, d\eta \end{aligned}}$$

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Each term of the r.h.s of this expression is a random variable that we decompose as

$$X(\vec{k},\eta) = X(k,\eta)\, a(\vec{k})$$

unit random Gaussian variable

function of $|\vec{k}|$

$$\langle a(\vec{k})\, a^*(\vec{k}')\rangle = \delta^{(3)}(\vec{k}-\vec{k}')$$

It follows that

$$\hat{\Theta}(\eta_0,\vec{k},\vec{n}) = \hat{\Theta}(\eta_0,k,\vec{n}) \times a(\vec{k})$$

with

$$\boxed{\begin{aligned}\hat{\Theta}(\eta_0,k,\vec{n}) &= \left[\hat{\Theta}_{sw}(\eta_0,k) - \hat{V}_b(k)\,\partial_{\Delta\eta_E}\right] e^{ik\mu\Delta\eta_E} \\ &\quad + \int (\phi'+\psi')\, e^{ik\mu\Delta\eta}\, d\eta\end{aligned}}$$

Tool Box

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr)\, P_l(\cos\theta_{k.r})$$

$$= 4\pi \sum_{l,m} i^l j_l(kr)\, Y_{lm}^*(\hat{k})\, Y_{lm}(\hat{r})$$

If we insert this decomposition in the previous formula, then we get

$$\Theta(\eta_0,k,\mu) = 4\pi \sum_{lm} \Theta_l(k)\, Y_{lm}^*(\hat{k})\, Y_{lm}(n)$$

$$\Theta_l(k) = \hat{\Theta}_{sw}(\eta_E,k)\, j_l(k\Delta\eta_E) - \frac{\hat{V}_b(\eta_E,k)}{k}\, j_l'(k\Delta\eta_E)$$

$$+ \int \left(\hat{\Phi}'(\eta,k) + \hat{\psi}'(\eta,k)\right) j_l(k\Delta\eta)\, d\eta$$

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we can now obtain C_ℓ :

$$(2\ell+1)\, C_\ell = \sum_m \int d^2n_1\, d^2n_2\, \frac{d^3k}{(2\pi)^3}\, \underbrace{\Theta(\eta_0,k,n_1)}_{\Sigma\ell_1m_1}\, \underbrace{\Theta^*(\eta_0,k,n_2)}_{\Sigma_{\ell_2m_2}}\; Y_{\ell m}(n_1)\, Y^*_{\ell m}(n_2)$$

$$= \frac{2}{\pi}\int k^2 dk \sum_{\substack{\ell_1 m_1\\ \ell_2 m_2\\ m}} \Theta_{\ell_1}(\eta_0,k)\,\Theta_{\ell_2}(\eta_0,k)\int d^2\hat{k}\, d^2n_1\, d^2n_2\; Y_{\ell_1 m_1}(\hat{k})\, Y^*_{\ell_2 m_2}(\hat{k})\; Y_{\ell m}(n_1)\, Y_{\ell_1 m_1}(n_1)\, Y_{\ell_2 m_2}(n_2)\, Y^*_{\ell m}(n_2)$$

$\int d^2\hat{k} \to \delta_{\ell_1\ell_2}\,\delta_{m_1m_2}$

$\int d^2n_1$ ~~d^2n_2~~ $\to \delta_{\ell_1\ell}\,\delta_{m_1m}$

$\int d^2n_2 \to \delta_{\ell\ell_2}\,\delta_{mm_2}$

so that

$$\boxed{\begin{aligned} C_\ell &= \frac{2}{\pi}\int \left|\hat{\Theta}_\ell(\eta_0,k)\right|^2 k^2 dk \\ \Theta_\ell(\eta_0,k) &= \hat{\Theta}_{SW}(\eta_E,k)\, j_\ell(k\Delta\eta_E) - \frac{\hat{V}_b(\eta_E,k)}{k}\, j_\ell'(k\Delta\eta_E) \\ &\quad + \int \left[\Phi'(\eta,k)+\Psi'(\eta,k)\right] j_\ell(k\Delta\eta)\, d\eta \end{aligned}}$$

This is the first step. we now need to solve the perturbation equations to get $\Theta_{SW}(\eta_E,k)$ $V_b(\eta_E,k)$ and $\Phi'(\eta,k)+\Psi'(\eta,k)$.

Usually it has to be performed numerically. we discuss some analytical limits.

PROPERTIES OF THE ANGULAR POWER SPECTRUM

LARGE ANGULAR SCALES - SMALL ℓ

The perturbation equations are γ-b fluid.

GIVEN WITHOUT DEMONSTRATION
See Peter-Uzan, Primordial Cosmology (OUP, 2009) chapter 5

$$\begin{cases} \delta_\gamma^{N\prime} = \frac{4}{3}k^2 V_\gamma + 4\Psi' \\ V'_\gamma = -\frac{1}{4}\delta^N_\gamma - \Phi + k^2 \sigma_\gamma + a n_e \sigma_T (V_b - V_\gamma) \end{cases}$$

$\pi_\gamma/6$

$$\begin{cases} \delta_b^{N\prime} = k^2 V_b + 3\Psi' \\ V'_b = -\mathcal{H} V_b - \Phi - c_b^2 \delta^N_b + \frac{4}{3}\frac{\rho_\gamma}{\rho_b} a n_e \sigma_T (V_\gamma - V_b) \end{cases}$$

Compton scattering

$$\begin{cases} \Psi - \Phi = 6\frac{\mathcal{H}^2}{k^2}\Omega_\gamma \sigma_\gamma \quad ; \quad \Phi' + \mathcal{H}\Phi = -\frac{3\mathcal{H}^2}{2}\sum \Omega_i V_i (1+w_i) \\ -k^2\Psi = \frac{3}{2}\mathcal{H}^2 \sum_i \left[\Omega_i \delta_i^N - 3\mathcal{H}(1+w_i)\Omega_i V_i\right] \end{cases}$$

$$\sigma_\gamma' = -\frac{4}{15}V_\gamma - \frac{9}{5} a n_e \sigma_T \sigma_\gamma$$

To be accepted at this stage
[Need Boltzmann description]

Initial conditions: on super-Hubble scale @ η_i

$$\left.\begin{aligned} \delta_\gamma^N(\eta_i) &= \frac{4}{3}\delta^N_b = \frac{4}{3}\delta^N_{cdm} \\ V_\gamma(\eta_i) &= V_b(\eta_i) = V_c(\eta_i) \end{aligned}\right\} \text{adiabatique.}$$

$$\sigma_\gamma(\eta_i) = 0 \longrightarrow \Psi(\eta_i) = \Phi(\eta_i)$$

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On super-Hubble scales $\Phi(\vec{k},\eta)$ = constant so that $\Phi' = 0$

$\Rightarrow \mathcal{H}\Phi = -\frac{3}{2}\mathcal{H}^2 \frac{4}{3} V_\gamma$. deep in the RDU $\mathcal{H} = \frac{1}{\eta}$

$$\boxed{k V_r(\eta_i) = -\frac{1}{2}(k\eta_i)\,\Phi(\eta_{init})}$$

Poisson equation implies that $-k^2\eta_i^2\,\Phi = \frac{3}{2}\left[\delta_r^N - \frac{3}{\eta k}\frac{4}{3}\,kV_r\right]$

$$= \frac{3}{2}\Big[\underbrace{\delta_r^N + 2\Phi}_{\mathcal{O}(k^2\eta^2)}\Big]$$

so $\boxed{\delta_r^N(\eta_i) = -2\,\Phi_i}$

So we have that:

$$\begin{cases} \delta_r^N(\eta_i) = -2\Phi_i \\ \Psi(\eta_i) = \Phi(\eta_i) \\ kV_\gamma(\eta_i) = -\frac{1}{2}(k\eta_i)\,\Phi(\eta_i) \\ \delta^N_{matter} = \frac{3}{4}\delta_r^N \\ V_{matter} = V_\gamma \end{cases}$$

$\Rightarrow$ everything is expressed in terms of $\Phi(k,\eta_i)$.

~~For super-Hubble scales at the time of decoupling~~

~~V_γ did not have time to grow significantly [Φ cst $2k\eta \ll 1$].~~
~~This implies that $(\delta_\gamma^N - 4\Psi)' = \frac{4}{3}k^2 V_\gamma \simeq 0$~~

$$\cancel{\delta_\gamma^N - 4\Psi \sim \text{constant}}$$

$$\cancel{\frac{1}{4}\delta_\gamma^N = \frac{1}{3}\delta^N_{mat} =}$$

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The LSS is in the MDU. We neglect the radiation so that $\mathcal{H} = \frac{2}{\eta}$; $c_s^2 = w = 0$

For adiabatic perturbations $\delta_m = \frac{3}{4}\delta_\gamma$ whatever the gauge.

- The Poisson equation tells us that

$$-k^2\eta^2\Psi = 6\,\Omega_m\left(\delta_m^N - 3\mathcal{H}\,V_m\right)$$

- The Euler equation $V_b' + \mathcal{H}V_b = -\Phi$ has a particular solution

$$\mathcal{H}V_b = -\frac{2}{3}\Phi$$ on super-Hubble scales where $\Phi \sim$ constant.

The solution of the homogeneous equation is decreasing so that we neglect it

- The Poisson equation on super-Hubble scales ($k\eta \ll 1$) tells that $\delta_m^N \sim 3\mathcal{H}V_m \sim -2\,\Phi$

- It follows that

$$\Theta_{SW} \sim \frac{1}{4}\delta_\gamma^N + \Phi = \frac{1}{3}\delta_m^N + \Phi = \frac{1}{3}\Phi$$

adiabatic

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It follows that

$$\Theta_{SW} \simeq \frac{1}{4}\delta^N_\gamma + \Phi \simeq \frac{1}{3}\Phi$$

and thus

$$C_\ell = \frac{2}{\pi}\int \underbrace{\left|\frac{1}{3}\Phi(k,\eta_E)\right|^2}_{\substack{\Phi \text{ constant} \\ \frac{1}{9}P_\Phi(k)}} \underbrace{j_\ell^2(k\Delta\eta_E)}_{j_\ell^2(k\eta_0)} \underbrace{k^2\,dk}_{k^3\frac{dk}{k}}$$

$$\boxed{C_\ell = \frac{2}{\pi}\times\frac{1}{9}\int k^3 P_\Phi(k)\, j_\ell^2(k\eta_0)\,\frac{dk}{k}}$$

predictions of inflation are given in terms of curvature perturbation

$$\Phi = \frac{3}{5}\zeta \longrightarrow P_\Phi = \frac{9}{25}P_\zeta$$

$$A_s^2 \equiv \frac{4}{25}P_\zeta$$

$$\longrightarrow P_\Phi = \frac{9}{4}A_s^2(k)$$

Also $\mathcal{P}_x = \dfrac{k^3 P_x}{2\pi^2}$

$$\boxed{C_\ell = \pi\int A_s^2(k)\, j_\ell(k\eta_0)\,\frac{dk}{k}}$$

The integral is dominated by the modes such that $k\eta_0 \sim \ell$ and we thus expect $C_\ell \sim A_s^2\left(\frac{\ell}{\eta_0}\right)$

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Setting $A_S^2(k) = A_S^2(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}$ and using that

$$\int_0^\infty j_\ell^2(t)\, t^{-k}\, dt = \frac{\pi}{2} \frac{\Gamma(k+1)\,\Gamma\left(\frac{2\ell+1-k}{2}\right)}{2^{k+1}\,\Gamma^2\left(\frac{k+2}{2}\right)\Gamma\left(\frac{2\ell+k+3}{2}\right)}$$ we get

$$C_\ell \simeq \pi A_S^2(k_0)\left[\frac{\pi}{2}\,\frac{\Gamma(3-n_s)}{2^{3-n_s}}\,\frac{\Gamma\left(\ell+\frac{n_s-1}{2}\right)}{\Gamma^2\left(2-\frac{n_s}{2}\right)\Gamma\left(\ell+\frac{5-n_s}{2}\right)}\right]$$

for small n_s, $\Gamma(\ell+a) \propto \ell^a$ so that

$$\boxed{\begin{array}{ll} \ell(\ell+1)\,C_\ell \propto \ell^{n_s-1} & \\ \ell(\ell+1)\,C_\ell = \frac{\pi}{2} A_S^2(k_0) & n_s=1 \end{array}}$$

This is the SW plateau. It gives access to the power spectrum of the initial perturbation.

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INTERMEDIATE SCALES

We now focus the γ-b system. Let me recall the perturbation equations

$$\begin{cases} \delta_\gamma^{N\,\prime} = \frac{4}{3} k^2 V_\gamma + 4\Psi' \\ \delta_b^{N\,\prime} = k^2 V_b + 3\Psi' \end{cases} \qquad \begin{cases} V_\gamma' = -\frac{1}{4}\delta_\gamma^N - \Phi + \frac{1}{6} k^2 \pi_\gamma + \tau'(V_b - V_\gamma) \\ V_b' = -\mathcal{H} V_b - \Phi + \frac{\tau'}{R}(V_\gamma - V_b) \end{cases}$$

with $\tau' = a n_e \sigma_T$ & $R = \frac{3}{4}\frac{\rho_b}{\rho_\gamma}$.

The anisotropic stress is related to V_γ: $\boxed{\frac{k^2}{12}\pi_\gamma = -\frac{8}{45}\frac{k^2}{\tau'} V_\gamma}$

(we need the kinetic theory to get this)

The terms in $\tau'(V_b - V_\gamma)$ represents the interaction between b & γ.

As long as τ' is large, it imposes that $\boxed{V_\gamma = V_b}$, i.e the two fluids are <u>tightly coupled</u>.

It implies that $\left(\delta_\gamma - \frac{4}{3}\delta_b\right)' = \frac{4}{3}k^2(V_\gamma - V_b) = 0$, i.e. the γb entropy remains constant.

Under this approximation, $\pi_\gamma \propto \frac{k}{\tau'} \sim 0$ [no quadrupole] $\rightarrow \boxed{\Phi = \Psi}$

~~Then we are left with~~

$$\cancel{\boxed{V_\gamma' = -\frac{1}{4}\delta_\gamma^N - \Phi \ ; \ V_b' = -\mathcal{H} V_b - \Phi}}$$

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We work under the approximation that

$$\Pi_\gamma = 0 \quad ; \quad V_b \sim V_\gamma$$

We cannot just set $V_b = V_\gamma$ and we have to look at the dominant terms in an expansion in k/τ' ; i.e. $V_\gamma = V_b + \mathcal{O}(\frac{k}{\tau'})$

$$\tau'(V_\gamma - V_b) = (V_b' + \mathcal{H}V_b + \Phi)R$$

$$= R(V_\gamma' + \mathcal{H}V_\gamma + \Phi) + \mathcal{O}(k/\tau')$$

Inserting in the γ-Euler equation

$$V_\gamma' = -\frac{1}{4}\delta_\gamma^N - \Phi - R(V_\gamma' + \mathcal{H}V_\gamma + \Phi) + \mathcal{O}(k/\tau')$$

$$(1+R)V_\gamma' = -\frac{1}{4}\delta_\gamma^N - (1+R)\Phi - \underbrace{\mathcal{H}R}_{R'}V_\gamma$$

$$\boxed{\left[(1+R)V_\gamma\right]' = -\frac{1}{4}\delta_\gamma^N - (1+R)\Phi}$$

Now, we use this in the γ-continuity equation to eliminate V_γ. We obtain

$$\boxed{\delta_\gamma^{N\,\prime\prime} + \frac{R'}{1+R}\delta_\gamma^{N\,\prime} + k^2 c_s^2 \delta_\gamma^N = F(\Phi,\Psi) \equiv 4\left(\Psi'' + \frac{R'}{1+R}\Psi' - \frac{1}{3}k^2\Phi\right)}$$

where $c_s^2 = \frac{1}{3}\frac{1}{1+R}$ & remind $R = \frac{3}{4}\frac{\rho_b}{\rho_\gamma}$

This equation describes a <u>forced</u> <u>damped</u> oscillator in which the forcing is determined par the 2 Bardeen potentials.

As R increases Φ & Ψ are mostly determined by CDM

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Let us first consider modes such that we can neglect the slow variation of R, Φ and Ψ, the equation for δ^N_γ rewrites

$$\delta^{N\,\prime\prime}_\gamma + \frac{k^2}{3(1+R)}\delta^N_\gamma = -\frac{4}{3}k^2\Phi$$

Since $\Theta_{SW} = \frac{1}{4}\delta^N_\gamma + \Phi$, it gives

$$(1+R)\,\Theta''_{SW} + \frac{k^2}{3}\Theta_{SW} = -\frac{k^2}{3}R\Phi$$

The solution is the sum of oscillating modes of the homogeneous equation and the particular solution $\Theta_{SW} = -R\Phi$, i.e.

$$\boxed{\Theta_{SW}(k,\eta) = \left(\Theta_{SW} + R\Phi\right)_0 \cos\frac{k\eta}{\sqrt{3}} + \left(\dot{\Theta}_{SW}\right)_0 \sin\frac{k\eta}{\sqrt{3}} - R\Phi}$$

Deep in the radiation era, and for adiabatic initial conditions, we have seen that

$$\delta^N_\gamma(0) = -2\Phi(0) \qquad \& \qquad kV_\gamma = -\frac{1}{2}(k\eta_i)\,\Phi(0) \ll \Phi(0)$$

Now, the γ-continuity equation implies that

$$(\delta^N_\gamma)' \sim 4\Phi' \quad \Rightarrow \quad \delta^N_\gamma = 4\Phi(\eta) - 6\Phi(0)$$

After horizon crossing $\Phi(\eta) \longrightarrow 0$ and $\frac{1}{4}\delta^N_\gamma \longrightarrow -\frac{3}{2}\Phi(0)$ while $\delta^{N\,\prime}_\gamma \longrightarrow 0$

It implies that only the $\cos\frac{k\eta}{\sqrt{3}}$ term is excited.

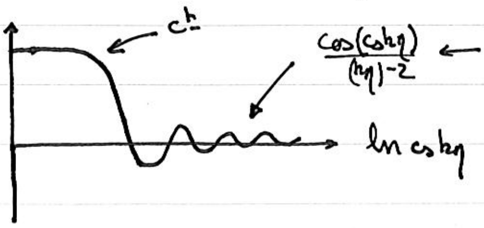
Note in a pure RDU universe

$c_s^2 = w \qquad a \propto \eta^\nu \qquad \nu = \frac{2}{1+3w} \qquad f = x^\nu\Phi \qquad x = k\eta$

$$\frac{d^2f}{dx^2} + \frac{2}{x}\frac{df}{dx} + \left[w - \frac{\nu(\nu+1)}{x^2}\right]f = 0 \longrightarrow \Phi = -\frac{3}{2}\nu^2 x^{-\nu}\left\{A\, j_\nu(c_s x) + B\, n_\nu(c_s x)\right\}$$

$\boxed{\nu=2} \Longrightarrow \Phi \propto \frac{j_1(c_s k\eta)}{k\eta}$

$\delta^c = x^{2-\nu}\Phi$



$\frac{\cos(c_s k\eta)}{(k\eta)^{-2}}$ ← Again we see the cos is excited.

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This is indeed very rough, to take into account the time evolution of R, we need to perform a WKB approach.

First we rewrite our evolution equation as

$$\left(\frac{\delta_r}{4}-\Psi\right)'' + \frac{R'}{1+R}\left(\frac{\delta_r}{4}-\Psi\right)' + k^2 c_s^2 \overbrace{\left(\frac{\delta_r}{4}-\Psi\right)}^{X} = \overbrace{-\frac{k^2}{3}\frac{2+R}{1+R}\Phi}^{F_X}$$

when $k \gg \mathcal{H} = \frac{R'}{R}$ we can neglect the term in X' and the solution of the homogeneous equation is

$$X \sim e^{i\int w_s d\eta} \sim e^{ikr_s} \qquad \text{with } r_s = \int^\eta c_s(\eta')\,d\eta'$$

we thus set $X = A\, e^{i\int w_s d\eta'}$ where A satisfies

$$\overbrace{A'' + 2iw_s A' - (w_s^2 - iw_s')A}^{X''} + \frac{R'}{1+R}\overbrace{\left(A' + iw_s A\right)}^{X'} + w_s^2 A = 0$$

that is

$$A'' + \left(\frac{R'}{1+R} + 2iw_s\right)A' + i\underbrace{\left(w_s' + \frac{R'}{1+R}w_s\right)}_{\frac{i}{2}\frac{R'}{1+R}w}A = 0$$

! $\frac{w_s'}{w_s} = -\frac{R'}{2(1+R)}$

In the WKB regime the amplitude varies slowly $\frac{A'}{A} \ll kc_s$; $\frac{A''}{A} \ll k^2c_s^2$ so that $\frac{R'}{R} \ll w_s$ and

$$A' = -\frac{R'}{4(1+R)}A \Rightarrow A \propto \frac{1}{(1+R)^{1/4}}$$

The homogeneous solution is thus

$$\boxed{\begin{gathered} X = C_a\,\theta_a + C_b\,\theta_b \\ w_s = kr_s \;;\; r_s = \int^\eta c_s\,d\eta' \;;\; \theta_a = \frac{1}{(1+R)^{1/4}}\cos[w_s] \;;\; \theta_b = \frac{1}{(1+R)^{1/4}}\sin[w_s] \end{gathered}}$$

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one then needs to determine the particular solution, which can be done in terms of the Green function

$$\mathcal{G}(\eta,\eta') = \frac{\theta_a(\eta')\theta_b(\eta) - \theta_a(\eta)\theta_b(\eta')}{\theta_a(\eta')\theta_b'(\eta') - \theta_a'(\eta')\theta_b(\eta')} = \frac{\sqrt{3}}{k}\frac{[1+R(\eta')]^{3/4}}{[1+R(\eta)]^{1/4}} \sin k[r_s(\eta) - r_s(\eta')]$$

The general solution is thus

$$[1+R(\eta)]^{1/4} X = X(0)\cos kr_s + \frac{\sqrt{3}}{k}\left[\dot{X}(0) + \frac{1}{4}\dot{R}(0)X(0)\right]\sin kr_s$$

$$+ \frac{\sqrt{3}}{k}\int_0^{\eta} [1+R(\eta')]^{3/4} \sin k(r_s(\eta) - r_s(\eta'))\, F_X(\eta')\, d\eta'$$

on subble-scale, using $\Phi(k\eta)$ from RDU, initial conditions = adiabatic and $\Theta = \frac{1}{4}\delta_r + \Phi = X + 2\Phi$, one gets

$$\boxed{\Theta_{SW} = \frac{[\Theta_{SW} + R\Phi]_0}{(1+R)^{1/4}}\cos[kr_s(\eta)] - R\Phi}$$

* The Euler equation of the γ allows to deduce that

$$\frac{kV_\gamma}{3} \sim -c_s\frac{[\Theta_{SW} + R\Phi]_0}{(1+R)^{1/4}}\sin[kr_s(\eta)] \sim \frac{\Theta_{dop}}{\sqrt{3}}$$

$\Theta_{dop} \sim \frac{kV_\gamma}{\sqrt{3}}$ by isotropy

The Doppler terms oscillates in quadrature with Θ_{SW} and with a zero mean. Its amplitude is also suppressed by a factor $c_s \sim \frac{1}{\sqrt{1+R}}$

The peaks of Θ_{dop} fill the throughs of Θ_{SW}.

when $R \to 0$ they have the same amplitude so that the oscillations are damped.

when $R \to$ large $\Theta_{dop} \ll \Theta_{SW}$ and we have oscillations.

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For adiabatic initial conditions, there is an excess of power for
$k r_s(\eta_{LSS}) \sim p\pi$ (Θ_{SW} max) $p = 1, 2..$

Since $C_\ell \sim \int \Theta_{SW}(k, \eta_{LSS})\, j_\ell(k \Delta\eta_{LSS})\, dk$, the oscillation

in k will transfer in oscillation in ℓ

Typically, the physical scales associated to $k_{(p)}$ is $d = a(\eta_{LSS}) \frac{\pi}{k_{(p)}}$

It is observed under the angle $\Theta_{(p)} = \frac{d_{(p)}}{D_A(\eta_{LSS})} \simeq \frac{\pi}{k_{(p)} f_K(z_{LSS})}$

angular distance
$D_A = a_0 \frac{f_K(\chi(z))}{1+z}$

The angular scale Θ roughly corresponds to $\ell \sim \pi/\Theta$ so that the C_ℓ should be peaked at

$$\ell_{(p)} \sim \frac{f_K(z_{LSS})}{r_s(z_{LSS})}\, p$$

Neglecting Λ - $f_K(z_{LSS}) \sim \frac{2}{H_0 \Omega_0}$

$a_0 \chi(z) = \frac{1}{H_0} \int_0^{z_{LSS}} \frac{dz}{E(z)}$ $\quad f_K(\chi) = \begin{cases} K^{-1/2} \sin \sqrt{K}\chi \\ \chi \\ (-K)^{-1/2} \,\mathrm{sh}\, \sqrt{-K}\chi \end{cases}$

$$\begin{cases} k_{eq} r_s = \frac{2}{3}\sqrt{\frac{6}{R_{eq}}} \ln \frac{\sqrt{1+R} + \sqrt{R + R_{eq}}}{1 + \sqrt{R_{eq}}} \\ k_{eq} \equiv \sqrt{2 \Omega_0 H_0^2 (1+z_{eq})} \end{cases}$$

$R = \frac{3}{4}\frac{\rho_b}{\rho_\gamma}$

for $(\ell + \Pi)$ universe.

$\Downarrow$

$r_s \propto \Omega_0^{-1/2}$

For Einstein de Sitter $\ell_{(1)} \sim 220$ so that $\boxed{\ell_{(1)} \sim \frac{220}{\sqrt{\Omega_0}}}$

<u>Note @ LSS</u> $\quad R_* = \frac{3}{4}\frac{\rho_b}{\rho_\gamma} \sim 0.729 \left(\frac{\Omega_b h^2}{0.024}\right)\left(\frac{1+z_*}{10^3}\right)$

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This approximation holds for intermediate scales:

$$\begin{cases} k/\tau' \ll 1 & \text{(tight coupling)} \\ k \gg \frac{R'}{R} & \text{(WKB regime)} \end{cases}$$

At larger k, one needs to go beyond tight coupling approximation.

SILK DAMPING

The mean free path for the Compton scattering is $\lambda_c \sim 1/\tau'$.

In a time η, the γ undergo a random-walk so that the typical diffusion length is

$$\lambda_D \sim \sqrt{N}\,\lambda_c \quad \text{and} \quad N \sim \frac{\eta}{\lambda_c} \Rightarrow \lambda_D \sim \sqrt{\eta \lambda_c} \sim \sqrt{\eta/\tau'}$$

λ_D is of the order of geometric mean of λ_c (mean free path) and horizon scale (η)

Because of the diffusion we expect modes with $k > k_D \sim \sqrt{\tau'/\eta_{LSS}}$ to be exponentially damped.

$$\begin{cases} \lambda_c^{-1} = \overset{1{,}12\ 10^{-5} x_e (1-Y_p)\,\Omega_b h^2 (1+z)^3\ \text{cm}^{-3}}{n_e}\ \underset{6{,}65\cdot 10^{-25}\ \text{cm}^2}{\sigma_T}\ a \quad \Big\}\ \lambda_c \sim 2.5\ \text{Mpc} \\ \lambda_D \sim 64.5\ \text{Mpc} \left(\frac{\Omega_m h^2}{0.14}\right)^{-0.278} \left(\frac{\Omega_b h^2}{0.024}\right)^{-0.18} \quad \text{[see Hu]} \end{cases}$$

$\Omega_b h^2 \sim 0.02$
$Y_p \sim 0.24$
$z \sim 10^3$
$x_e \sim 1$

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To estimate the amplitude of the effect, we go back to the perturbation equation:

$$\begin{cases} V'_\gamma = -\frac{1}{4}\delta^{N\,\prime}_\gamma + \frac{1}{6}k^2\Pi_\gamma - \tau'(V_\gamma - V_b) \\ V'_b = \frac{1}{R}\tau'(V_\gamma - V_b) \end{cases}$$

$\frac{k^2}{12}\Pi_\gamma = -\frac{8}{45}\frac{k^2}{\tau'}V_\gamma$

REQUIRES KINETIC DESCRIPTION!
(+ polarisation)

To go beyond TCA, we need to find the contribution of $V_b - V_\gamma$ scaling as $1/\tau'$

$$\frac{R+1}{R}\tau'(V_\gamma - V_b) = -\frac{1}{4}\delta^N_\gamma$$

Then, combined the 2 Euler equations to get

$$V'_\gamma + RV'_b = (R+1)V'_\gamma + R(V'_b - V'_\gamma) \quad \text{and then}$$

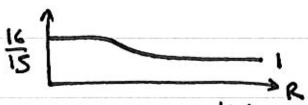
$$(R+1)V'_\gamma = -\frac{1}{4}\delta_\gamma + \frac{1}{6}k^2\Pi_\gamma - \frac{1}{4}\frac{R^2}{1+R}\frac{\delta'_\gamma}{\tau'}$$

we conclude that. (neglecting R', Φ', Ψ' ...)

$$\boxed{\delta''_\gamma + \frac{k^2 c_s^2}{\tau'}\left(\frac{16}{15} + \frac{R^2}{1+R}\right)\delta'_\gamma + k^2 c_s^2\,\delta_\gamma = 0}$$

It follows that $\delta_\gamma \propto e^{-k^2/k_D^2}\, e^{\pm i k r_s}$

with $$k_D^{-2} = \frac{1}{6}\int_0^\eta \underbrace{\left[\frac{1}{1+R}\left(\frac{16}{15} + \frac{R^2}{1+R}\right)\right]}\frac{d\eta'}{\tau'}$$



varies very little.

$$\Rightarrow \quad k_D^{-2} \sim \frac{1}{6}\int\frac{d\eta}{\tau'}$$

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k_D can be evaluated as

$$k_D \eta_{LSS} \sim (1+z_{LSS}) \sqrt{\frac{\sigma_T}{m_p} \frac{3H_0}{8\pi G_N} \Omega_{b_0} h}$$

The effect of the damping starts at $\ell \sim 140 \sqrt{\frac{\Omega_{b_0} h^2}{0.019}}$ i.e. before the first peak.

$\hookrightarrow$ Kinetic theory is required to get a good accuracy and to include all effects at high ℓ.

(in particular polarisation)

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Global Picture

SW plateau

- initial conditions
  $\Theta_{SW} \sim \frac{1}{3}\Phi$
- normalisation of P_Φ, n_S

⊖ ISW to be included
(late time: K, Λ ...)

Peaks

- oscillation of γ-b plasma
- WKB approx $\frac{1}{(1+R)^{1/4}} e^{\pm i k r_S}$

  phase related to r_S !
- Effect of Nature of CI (adiabatic) ↙ cos ↘ $\ell_{(p)}$
- Θ_{dop} in quadrature with Θ_{SW}

⊖ Silk damping
kinetic theory.

Main parameters $\begin{cases} P_\Phi, n_S + \text{adiabatic/isocurvature.} \\ \Omega_K, \Omega_b \end{cases}$

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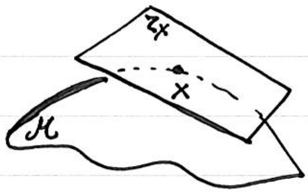
CARGESE : cours n°3 Jean-Philippe UZAN

Yesterday we have computed the C_ℓ in different approximations assuming a fluid description of the radiation.

While a good description on large scales, it misses several physical effects. We thus now turn to a more precise approach based on a kinetic description.

Boltzmann Equation

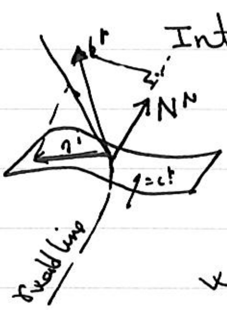
We describe the radiation by its distribution function $f(x^\alpha, p_\beta)$ that depends on spacetime position AND momentum.



It follows that f leaves the tangent space $T\mathcal{M}$

$$T\mathcal{M} = \left\{ (x^\mu, p_\mu)\ ;\ x^\mu \in \mathcal{M},\ p_\mu \in T_x \right\}$$

T_x being the tangent space at x.



Introducing the vector field normal to $\Sigma_\eta = \{\eta = \text{const.}\}$ hypersurfaces

$$N_\mu = -a(1+A, 0) \quad ; \quad N^\mu = \frac{1}{a}(1-A, -B^i),$$

we can decompose γ tangent vector as

$$\begin{cases} k^\mu = \dfrac{E}{a}\left[1-A,\ n^i - \left(B_j n^j + \dfrac{1}{2} h_{jk} n^j n^k\right) n^i\right] \\ \text{w. } \gamma_{ij} n^i n^j = +1 \end{cases}$$

$k^\mu N_\mu = -E$

Then, the distribution function can be split as

$$f(x^\mu, p_\mu) = f(\eta, x^i, E, n^j) = \underset{\text{PC}}{\bar{f}(\eta, E)} + \delta f(\eta, x^i, E, n^j)$$

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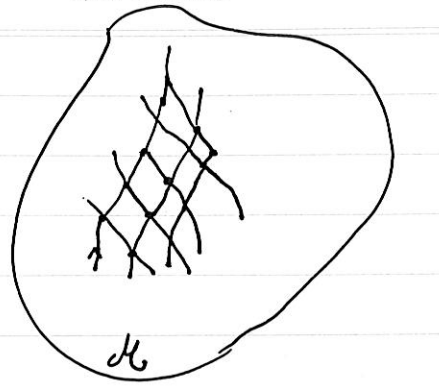
The evolution of f is dictated by the Boltzmann equation



$$L[f] = C[f]$$

$$\underbrace{L[f]}_{\text{Liouville}} = \underbrace{C[f]}_{\text{collision}}$$

LIOUVILLE EQUATION

$$L[f] = \frac{df}{d\eta} = \left(\underbrace{\frac{dx^\mu}{d\lambda}}_{k^\mu} \frac{\partial f}{\partial x^\mu} + \underbrace{\frac{dk^\mu}{d\lambda}}_{-\Gamma^\mu_{\alpha\beta} k^\alpha k^\beta} \frac{\partial f}{\partial k^\mu} \right) \cdot \underbrace{\frac{d\lambda}{d\eta}}_{\frac{1}{k^0}} = 0$$

$$\boxed{L[f] = \frac{1}{k^0} \left(k^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta \frac{\partial f}{\partial k^\mu} \right)}$$

Let us now use the decomposition of the previous page

$$L[f] = \frac{\partial f}{\partial \eta} + \underbrace{\frac{dx^i}{d\eta}}_{\frac{k^i}{k^0} \to \mathcal{O}(0) \to n^i} \underbrace{\partial_i f}_{\mathcal{O}(1)} + \frac{dE}{d\eta} \frac{\partial f}{\partial E} + \underbrace{\frac{dn^i}{d\eta}}_{\mathcal{O}(0) = -{}^{(3)}\Gamma^i_{jk} n^j n^k} \underbrace{\frac{\partial f}{\partial n^i}}_{\mathcal{O}(1)}$$

$$\frac{dx^i}{d\eta}\partial_i f \to n^i \partial_i f \qquad \frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i} \to -{}^{(3)}\Gamma^i_{jk} n^j n^k \frac{\partial f}{\partial n^i}$$

$$\frac{dE}{d\eta} = -E\left[\mathcal{H} + n^i \partial_i A + \left(\frac{1}{2} h'_{ij} - D_{(i} B_{j)} \right) n^i n^j \right]$$

from geodesic equation $\frac{dk^0}{d\eta} = -\Gamma^0_{\alpha\beta} \frac{k^\alpha k^\beta}{k^0}$ & $k^0 = \frac{E}{a}(1-A)$

so that

$$\boxed{L[f] = f' + n^k \partial_k f - \left[\mathcal{H} + n^k \partial_k A + \left(\frac{1}{2} h'_{ij} - D_{(i} B_{j)} \right) n^i n^j \right] E \frac{\partial f}{\partial E} - {}^{(3)}\Gamma^k_{ij} n^i n^j \frac{\partial f}{\partial n^k}}$$

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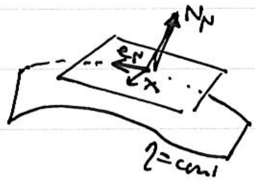
This then be split as background + $\mathcal{O}(1)$:

$$\begin{cases} L[\bar{f}] = \bar{f}' - \mathcal{H} E \dfrac{\partial \bar{f}}{\partial E} \\ L[\delta f] = \delta f' + n^k \partial_k \delta f - \mathcal{H} E \dfrac{\partial \delta f}{\partial E} - \left[n^k \partial_k A + \left(\frac{1}{2} h'_{ij} - D_{(i} B_{j)}\right) n^i n^j \right] E \dfrac{\partial \bar{f}}{\partial E} \\ \qquad - {}^{(3)}\Gamma^k_{ij} n^i n^j \dfrac{\partial \delta f}{\partial n^k} \end{cases}$$

MACROSCOPIC QUANTITIES

The Einstein equations involve $T_{\mu\nu}$ the stress-energy tensor that should be constructed from f.

consider the tetrad $(e^a_\mu)_{a=0..3}$ defined by



$$e^0_\mu = N_\mu \qquad e^i_\mu e^j_\nu g^{\mu\nu} = \delta^{ij} \qquad e^i_\mu N^\mu = 0$$

k^μ can then be decomposed as $k^a = k^\mu e^a_\mu$
and $k^2 = \delta_{ab} k^a k^b$; $E_k = \sqrt{k^2 + m^2} = k$.

Locally, we are in a M_4 spacetime so that

$$T^{\mu\nu} = \int_{P_m(x)} k^\mu k^\nu f \cdot \frac{d^3\vec{k}}{(2\pi)^3 E_k}$$

$$\boxed{T^{\mu\nu} = \int k^\mu k^\nu \frac{f}{(2\pi)^3} E \,.\, dE \,.\, d^2 n}$$

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For an observer with 4-velocity u^μ, this corresponds to

$$\begin{cases} \rho = \int (k^\mu u_\mu)^2 f \, E \, dE \, d^2 n \\ P = \int \frac{1}{3} (k^\mu k^\nu \perp_{\mu\nu}) f \, E \, dE \, d^2 n \\ \Pi^{\mu\nu} = \int k^\alpha k^\beta \left(\perp^\mu_\alpha \perp^\nu_\beta - \frac{1}{3} \perp_{\alpha\beta} \perp^{\mu\nu} \right) f \, E \, dE \, d^2 n \end{cases}$$

$\perp_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$

that can be split as background + $O(1)$ and are related to $\bar\rho_\gamma$, δ_γ, v_γ and π_γ

Brightness and Temperature

One needs to define a temperature from the distribution function.

Usually, one consider the distribution function to be of the form

$$f = \bar f \left[\frac{E}{T(x, n^j)} \right] \quad \text{and expand } T = \bar T \left[1 + \Theta(\eta, x^i, n^j) \right]$$

So that $f = \bar f(\eta, E) - E \frac{\partial \bar f}{\partial E} \Theta(\eta, x^i, n_j)$.

Such an ansatz assumes that Θ does not depend on E and that both f and $\bar f$ are black-body.

A cleaner way to define Θ is to consider the brightness

$$\boxed{I(\eta, x^i, n_j) \equiv 4\pi \int E^3 f(\eta, x^i, n^j) \, dE}$$

i.e. energy density per unit solid angle that propagates along n^i at (η, x^i).

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Again, it can be split as $I = \bar{I} + \delta I$ and we note that by definitions $\rho_\gamma = \int \frac{d^2n}{4\pi} I$

$$\bar{I} = 4\pi \int E^3 \bar{f}\, dE \qquad \delta I = 4\pi \int E^3\, \delta f\, dE$$

<u>background level</u>

$C = 0$ and the Liouville equation can be integrated to give

$$\int \left(\bar{f}' - \mathcal{H} E \frac{\partial \bar{f}}{\partial E} \right) E^3 dE = 0$$

$$\bar{I}' \qquad \underbrace{- \mathcal{H} \int E^4 \frac{\partial \bar{f}}{\partial E}}_{+4\mathcal{H}\bar{I}} \quad \longrightarrow \quad \boxed{\bar{I}' + 4\mathcal{H}\bar{I} = 0}$$

Because of isotropy $\bar{I} = \bar{\rho}_\gamma$ and the Liouville equation gives, as expected, the conservation equation

<u>Linear order</u>:

we define $C[\delta I] = 4\pi \int C[\delta f]\, E^3 dE$

$$4\pi \int E^3 L[\delta f]\, dE = C[\delta I]$$

$$\delta I' + n^k \partial_k I + 4\mathcal{H}\, \delta I + 4\left[n^k \partial_k A + \left(\tfrac{1}{2} h'_{ij} - D_{(i} B_{j)}\right) n^i n^j\right] \bar{I} - {}^{(3)}\Gamma^k_{ij}\, n^i n^j \frac{\partial \delta I}{\partial n^k} = C[\delta I]$$

we now define the <u>brightness temperature</u> by

$$\Theta(\eta, x^i, n^i) = \frac{1}{4} \frac{\delta I}{\bar{I}}$$

It is clear that the monopole of Θ is $\Theta_0 = \int \Theta \frac{d^2n}{4\pi} = \frac{1}{4}\delta_\gamma$

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The evolution of Θ is deduced easily from the one of δI

$$\boxed{\Theta' + n^k \partial_k \Theta - {}^{(3)}\Gamma^i_{jk} n^j n^k \frac{\partial \Theta}{\partial n^i} + \left[n^k \partial_k A + \left(\frac{1}{2} h'_{ij} - D_{(i} B_{j)} \right) n^i n^j \right] = C(\Theta)}$$

$$\boxed{C(\Theta) \equiv \frac{C[\delta I]}{4 I}}$$

collision term

In general, it can be decomposed as $C[f] = \frac{df_+}{d\eta} - \frac{df_-}{d\eta}$ with $f_{+/-}$ the in/out-going distribution functions.

The collision is Thompson scattering of γ on e^-/p^+.

In the rest-frame of $e^- p^+$, we have

$$\begin{cases} \frac{df_+}{d\eta} = \tau' \int f \; \omega(n, n') \frac{d^2 n'}{4\pi} \qquad \frac{df_-}{d\eta} = \tau' f \\ \tau' = a \sigma_T n_e \quad \text{as before} \end{cases}$$

n_e → free e^- density
σ_T → Thomson scattering cross-section

$$\omega(\vec{n}, \vec{n}') = \frac{3}{4}\left[1 + (\vec{n}\cdot\vec{n}')^2\right] = 1 + \frac{1}{2} P_2(\vec{n}\cdot\vec{n}')$$
$$= 1 + \frac{3}{4} n_{ij} n'^{ij}$$

Legendre

<u>Note</u>: in this limit C does not depend on E

⇒ no spectral distortion.

This not true @ higher energies where compton scattering involves E/m_e terms.

⇒ brightness correspond to temperature

we also recover that $C[\bar{f}] = 0$ by symmetry of bg'round.

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background $\quad C[\bar{f}] = 0$

Linear perturbation

energy in e^-/p^+ restframe

$$C[\delta f] = \tau' \int \delta f(\eta, x^i, \bar{E}, \vec{n}') \frac{d^2 n'}{4\pi} - \delta f(\eta, x^i, \bar{E}, \vec{n})$$

$$+ \frac{3}{4} n^{ij} \int \delta f(\eta, x^i, \bar{E}, n') \, n'_{ij} \frac{d^2 n'}{4\pi}$$

$n_i n_j - \frac{1}{3}\gamma_{ij} \equiv n_{ij}$

$\bar{E}$ is related to E by

$$\bar{E} = -k_\mu u_b^\mu = E\left(1 - (B^i + v^i_b) n_i\right) \quad \text{so that}$$

$$E^3 dE = \left[1 + 4(B^i + v_b^i) n_i\right] \bar{E}^3 d\bar{E}$$

we can then integrate over E to get

$$C[\delta I] = \tau' \Big[\underbrace{\int \delta I \frac{d^2 n'}{4\pi}}_{\text{monopole of } \delta I} - \delta I + \frac{3}{4} n^{ij} \underbrace{\int \delta I \, n'_{ij} \frac{d^2 n'}{4\pi}}_{\text{anisotropic stress}} + 4\bar{I}(B^i + v^i_b) n_i \Big]$$

$$\boxed{C(\Theta) = \tau' \left[\Theta_0 - \Theta + (B^i + v^i_b) n_i + \frac{1}{16} n^i n^j \Pi_{ij} \right]}$$

Boltzmann equation

we gather the 2 terms to get

$$\boxed{\begin{aligned} &\Theta' + n^k \partial_k \Theta - {}^{(3)}\Gamma^i_{jk} n^j n^k \frac{\partial \Theta}{\partial n^i} + \left[n^k \partial_k A + \left(\frac{1}{2} h'_{ij} - D_{(i} B_{j)}\right) n^i n^j \right] \\ &\qquad = \tau' \left[\Theta_0 - \Theta + (B^i + v^i_b) n_i + \frac{1}{16} n^i n^j \Pi_{ij} \right] \end{aligned}}$$

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This gives the equation of evolution of $\Theta(\eta, x^i; n^i)$ in an arbitrary gauge.

We should discuss the transformation of under an arbitrary gauge transformation and the construction of a gauge invariant distribution function.

This is beyond what I can do in 3 lectures. I will thus pick up the Newtonian gauge from now on.

We thus have

$$\boxed{\begin{aligned}\Theta' + n^k \partial_k(\Theta + \Phi) - {}^{(3)}\Gamma^k_{ij} n^i n^j \frac{\partial \Theta}{\partial n_k} - \Psi' + \left(E'_{ij} + D_{(i}\Phi_{j)}\right) n^i n^j \\ = \tau'\left(\Theta_0 - \Theta + n_i V^i + \frac{1}{16} n^i n^j \pi_{ij}\right)\end{aligned}}$$

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Multipole expansion

$\Theta(\eta, x^i, n^i)$ can be decomposed in Fourier modes as

$$\Theta(\eta, x^i, n^i) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \, \Theta(\vec{k}, \eta) \, e^{i\vec{k}\cdot\vec{x}}$$

as usual. It follows that the Boltzmann eq. takes the form

$$\frac{\vec{n}\cdot\vec{k}}{k} \equiv \mu$$

$$(*) \quad \Theta' + ik\mu(\Theta + \Phi) = \Psi' + \tau'\left(\Theta_0 - \Theta + ik\mu V_b + \frac{1}{16} n_i n_j \pi^{ij}\right)$$

Then, $\Theta(\vec{k}, n^i)$ can be decomposed in P_ℓ as

$$\boxed{\Theta(\vec{k}, \eta, n^i) = \sum_\ell (-i)^\ell \Theta_\ell(k, \eta) \, P_\ell(\mu)}$$

Now

$$\begin{cases} * \ \Psi' + \tau'\Theta_0 = (\Psi' + \tau'\Theta_0) P_0(\mu) \\ * \ i\tau' k\mu V_b = -k V_b (-i)^1 P_1(\mu) \\ * \ ik\mu\Theta = k \sum_\ell \left(\frac{\ell+1}{2\ell+3}\Theta_{\ell+1} - \frac{\ell}{2\ell-1}\Theta_{\ell-1}\right)(-i)^\ell P_\ell(\mu) \\ \qquad (\ell+1)P_{\ell+1} = (2\ell+1)\mu P_\ell - \ell P_{\ell-1} \\ * \ \frac{1}{16}\pi^{ij} n_i n_j = (-i)^2 P_2 \frac{\Theta_2}{10} \end{cases}$$

using this into $(*)$ we get a hierarchy for Θ_ℓ:

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$$\begin{cases} \Theta_0' = -\frac{k}{3}\Theta_1 + \Psi' \\ \Theta_1' = k\left(\Theta_0 - \frac{2}{5}\Theta_2 + \Phi\right) - \tau'\left(kV_b + \Theta_1\right) \\ \Theta_2' = k\left(\frac{2}{3}\Theta_1 - \frac{3}{7}\Theta_3\right) - \frac{9}{10}\tau'\Theta_2 \\ \Theta_\ell' = k\left(\frac{\ell}{2\ell-1}\Theta_{\ell-1} - \frac{\ell+1}{2\ell+3}\Theta_{\ell+1}\right) - \tau'\Theta_\ell \end{cases}$$

The 2 first moments are just the fluid equation and we have

$$4\Theta_0 = \delta^N_\gamma \quad ; \quad \Theta_1 = -kV_\gamma \qquad \Theta_2 = \frac{5}{12}k^2\pi_\gamma$$

$\boxed{\text{RELATIONS TO } C_\ell}$

The temperature field today is $\Theta(\eta_0, \vec{x}_0, n^i)$

$$\langle \Theta(x_0,\vec{n})\,\Theta^*(x_0,\vec{n}')\rangle = \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) C_\ell\, P_\ell(\vec{n}\cdot\vec{n}') \quad \text{by definition}$$

$$= \sum_{\ell\ell'} \int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^3} \langle \Theta_\ell(\vec{k},\eta_0)\,\Theta^*_{\ell'}(\vec{k}',\eta_0)\rangle$$

$$P_\ell(k\cdot n)\, P_{\ell'}(k'\cdot n')$$

Since $\langle \Theta_\ell(\vec{k},\eta_0)\,\Theta^*_{\ell'}(\vec{k}',\eta_0)\rangle = \Theta_\ell(k,\eta_0)\,\Theta^*_{\ell'}(k',\eta_0)\,\delta^D(\vec{k}-\vec{k}')$ we integrate on $\vec{k}'$ to get

$$\langle \Theta\,\Theta\rangle = \sum_{\ell\ell'} \int \frac{d^3\vec{k}}{(2\pi)^3}\, \Theta_\ell(k,\eta_0)\,\Theta_{\ell'}(k',\eta_0)\, P_\ell(k\cdot n)\, P_{\ell'}(k'\cdot n')$$

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The next step is to decompose the P_ℓ

$$P_\ell(\hat{k}\cdot n) = \sum_m Y_{\ell m}(\hat{k})\, Y^*_{\ell m}(n)\, \frac{4\pi}{2\ell+1}$$

to get

$$\langle \Theta\Theta \rangle = \sum_{\ell\ell'} \int \frac{k^2 dk}{(2\pi)^3}\, \Theta_\ell(k,\eta_0)\, \Theta^*_{\ell'}(k,\eta_0) \left(\frac{4\pi}{2\ell+1}\right)\left(\frac{4\pi}{2\ell'+1}\right)$$

$$\sum_{mm'} \underbrace{\int d^2\hat{k}\; Y_{\ell m}(\hat{k})\, Y^*_{\ell' m'}(\hat{k})}_{\delta_{\ell\ell'}\,\delta_{mm'}} \quad Y^*_{\ell m}(n)\, Y_{\ell' m'}(n')$$

$$= \sum_\ell \underbrace{\int \frac{k^2 dk}{(2\pi)^3} \left(\frac{4\pi}{2\ell+1}\right)}_{\frac{2\ell+1}{4\pi} C_\ell} P_\ell(n\cdot n')$$

we identify to get

$$\boxed{(2\ell+1)^2 C_\ell = \frac{2}{\pi} \int k^2 dk\, |\Theta_\ell(k,\eta_0)|^2}$$

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$\boxed{\text{Developments}}$

- include V and more important T modes

- Polarisation : - Stokes parameters Q, U to be added
  ↳ 3 hierarchies of equations

  ↳ change the $C(\theta)$ term

- Neutrinos : - Boltzmann eq. with $C_\nu = 0$

  - add Π_ν

- C_ℓ describes the 2-point statistics
  ok if Gaussian.
  Need to be tested → higher order statistics
  → 2^{nd} order Boltzmann equation.

- $\langle a_{\ell m} a^*_{\ell' m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$ if isotropy satisfied

  . may not be the case if topology / anisotropic cosmology

  . data - cut

- foregrounds ; reionisation, ISW

These developments should be covered by the coming lectures.

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