Weak Gravitational Lensing

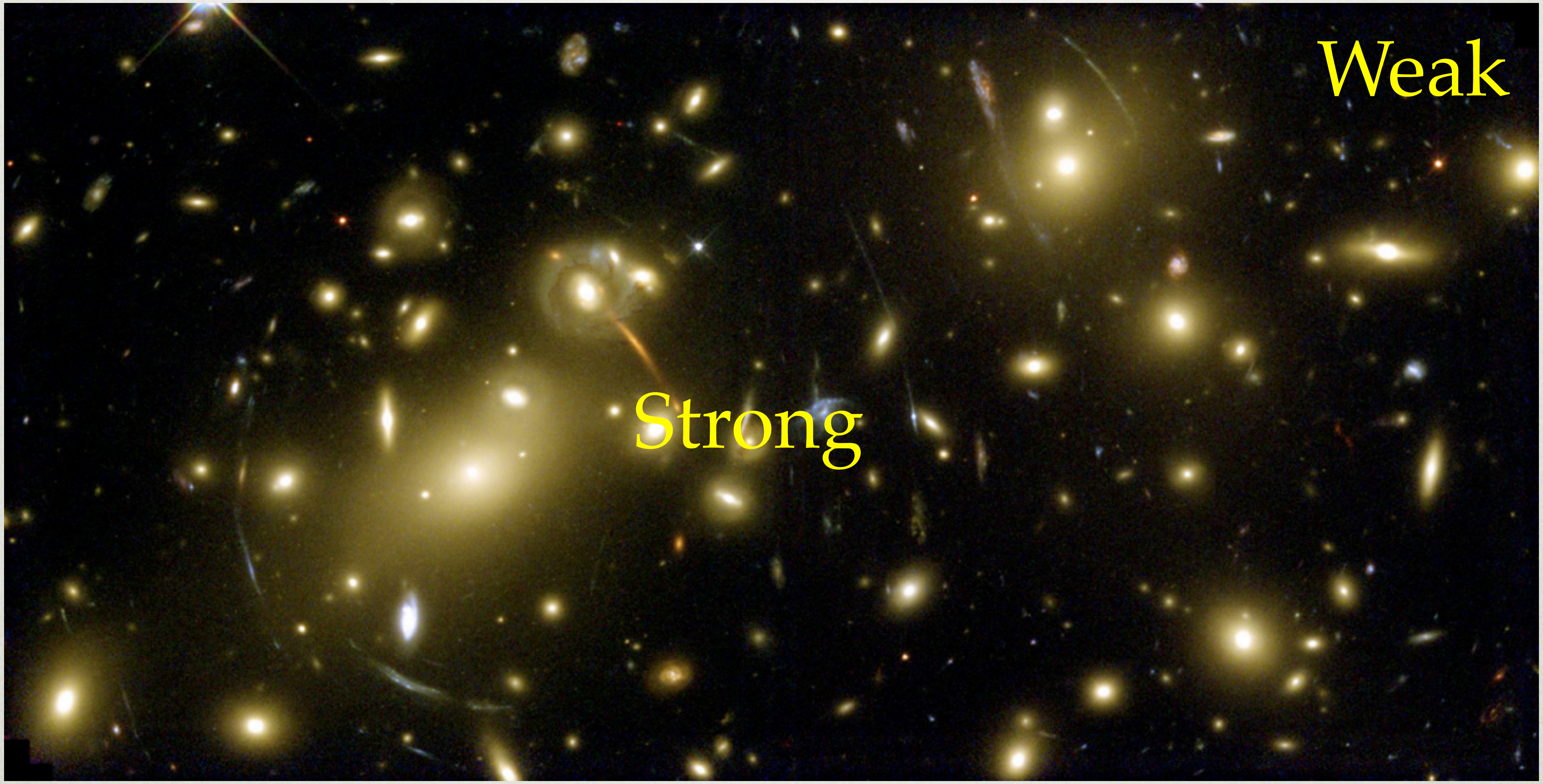
Cargèse 2012
David Bacon

Institute of Cosmology and Gravitation

Centre for cosmology in Portsmouth

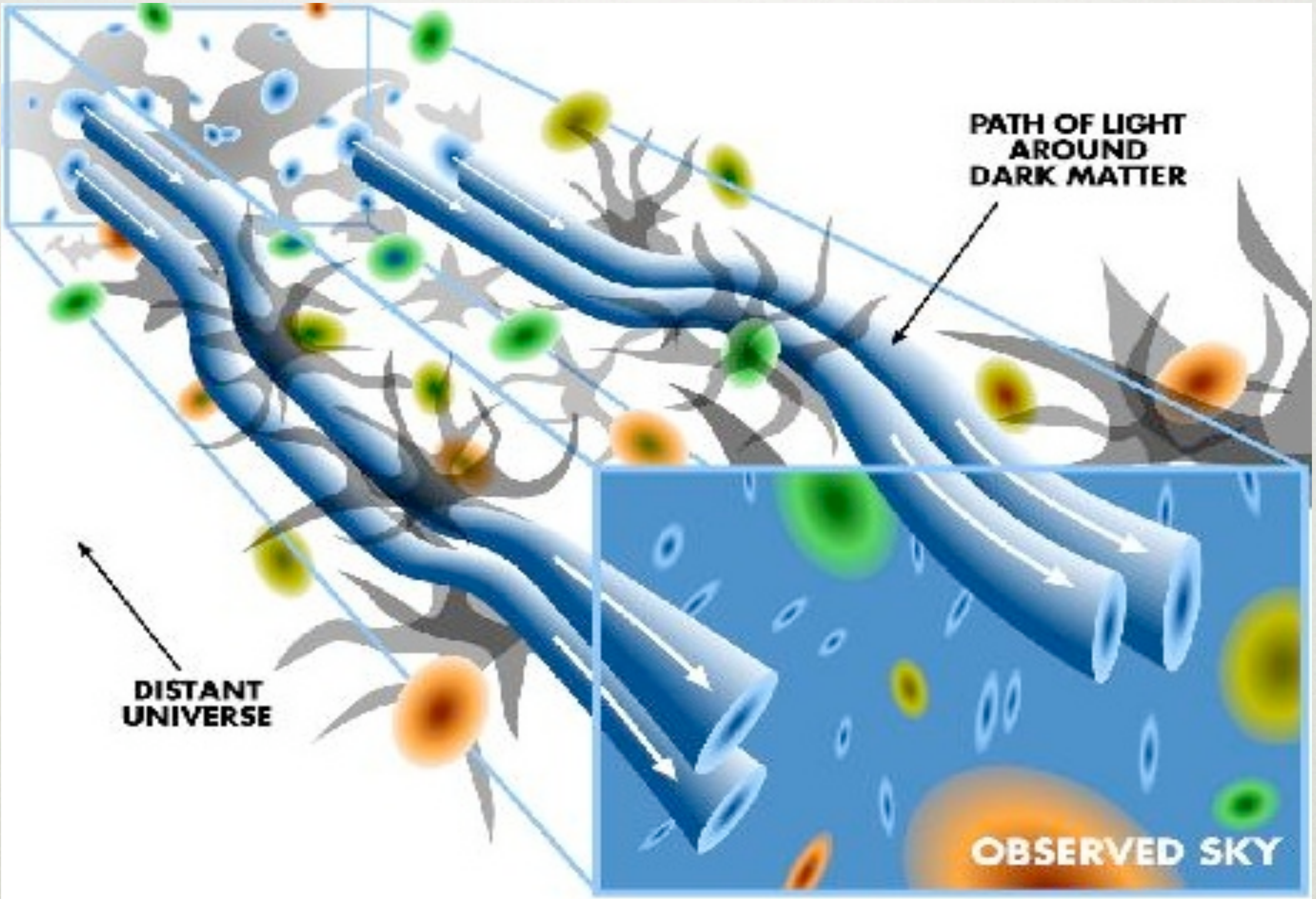
Interesting projects such as SDSS-III, DES, LOFAR, Galaxy Zoo

Gravitational Lensing as we all know it



Abell 2218 (Draco)
Cluster of galaxies 2 billion light years away, background galaxies 6 billion light years away

A Large Scattering Experiment



Wittman et al 2000

Direct measure of mass and geometry

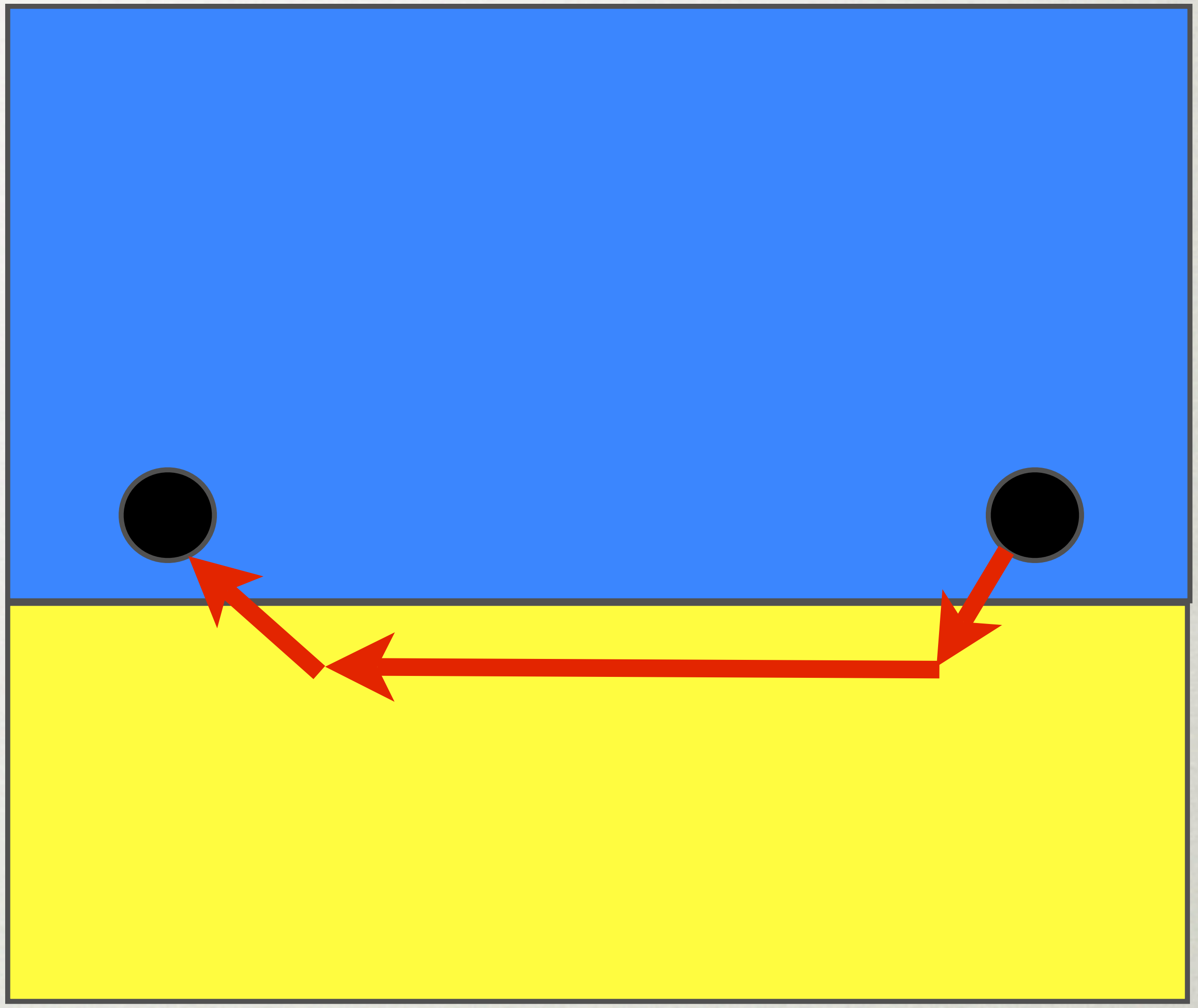
These Lectures:

- How does the light bend?
- How can we describe weak lensing?
- What are the observational challenges?
- Mapping dark matter
- Cosmological statistics

Mirages

Utah

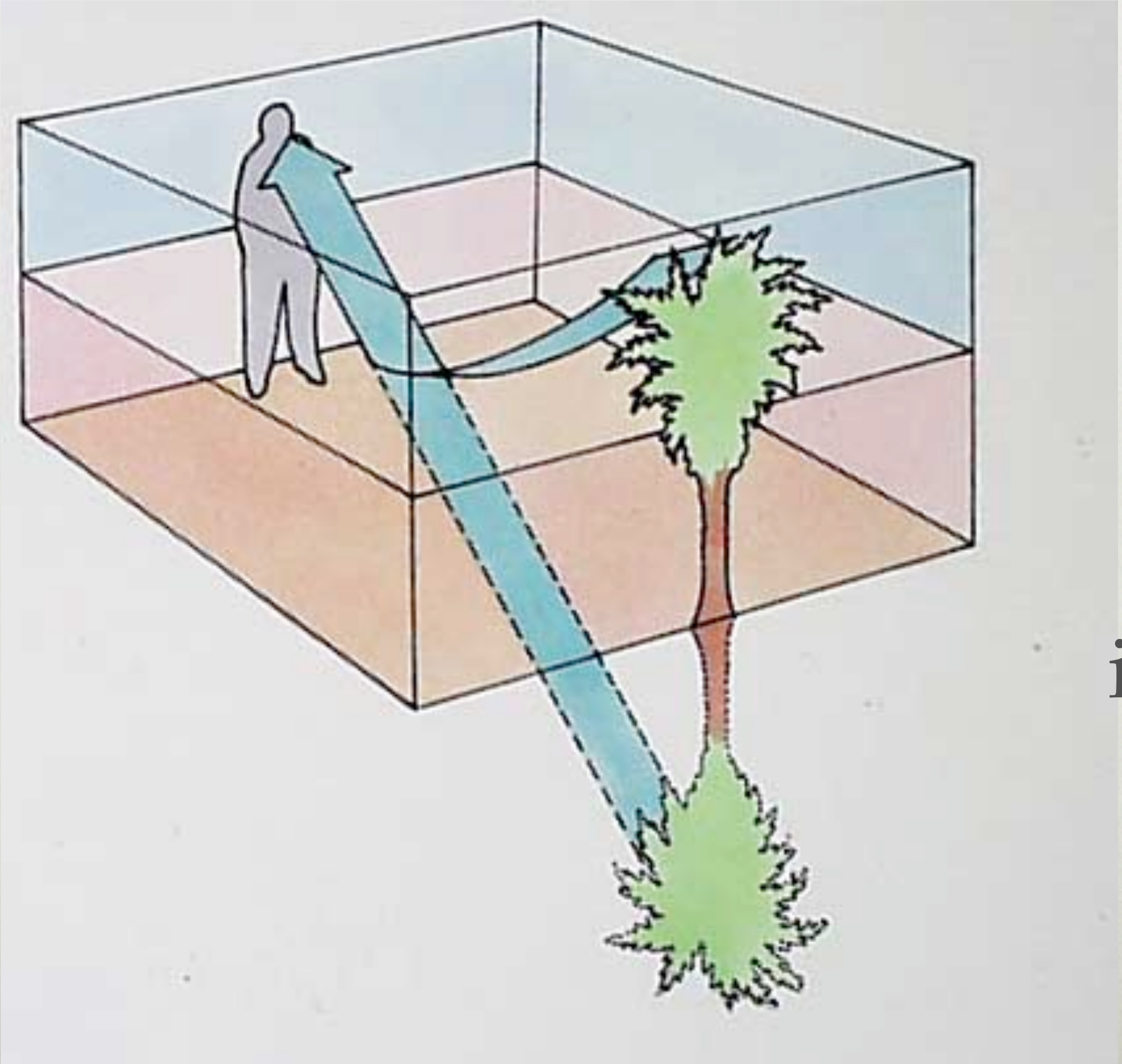
Mirages - why?



Hasselhoff should run along the beach to save the drowning woman

Mirages - why?

Fermat: light behaves like Hasselhoff



Refractive
index
is function of
temperature

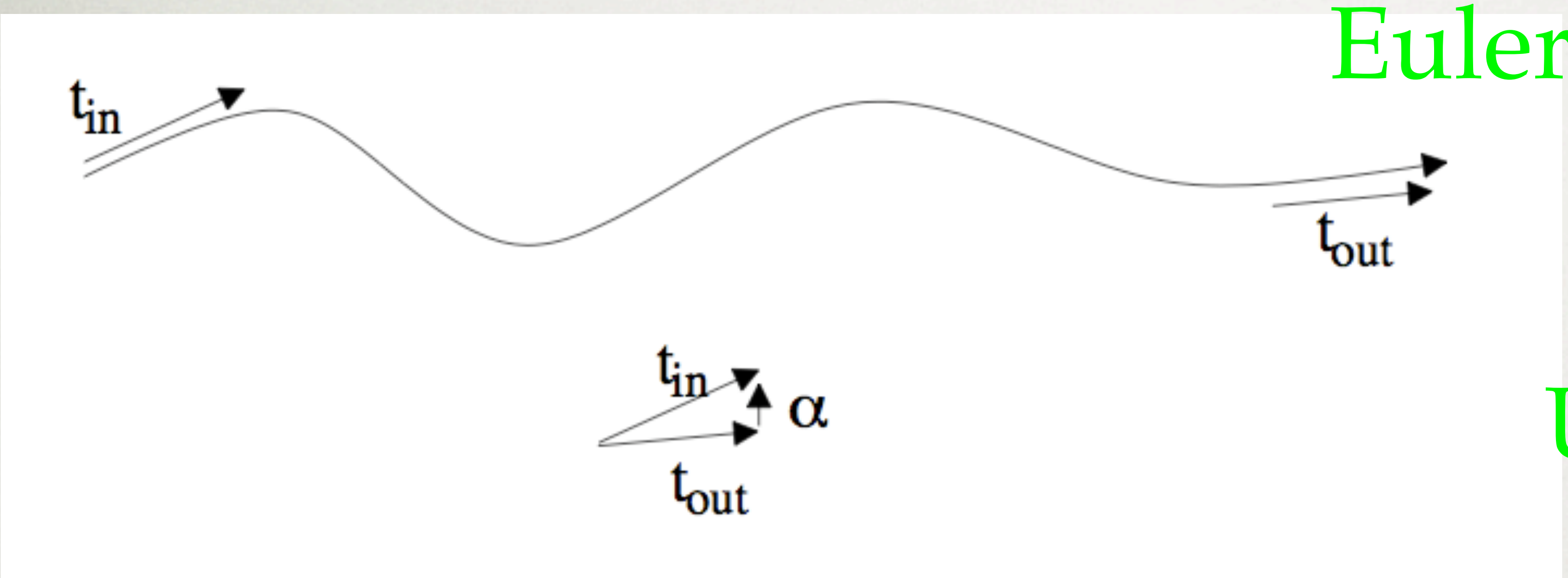
Fermat in GR

In GR, the **arrival time** is stationary with respect to nearby candidate rays.

Full proof is involved (e.g. Perlick 1990)

We'll see that geodesic equation leads to same results.

How do we describe paths with varying refractive index?



Euler (from Fermat):

$$\frac{\partial n}{\partial x_i} = \frac{d}{dl}\left(n\frac{\partial x_i}{\partial l}\right)$$

Unit tangent vector:

$$t_i = \frac{\partial x_i}{\partial l}$$

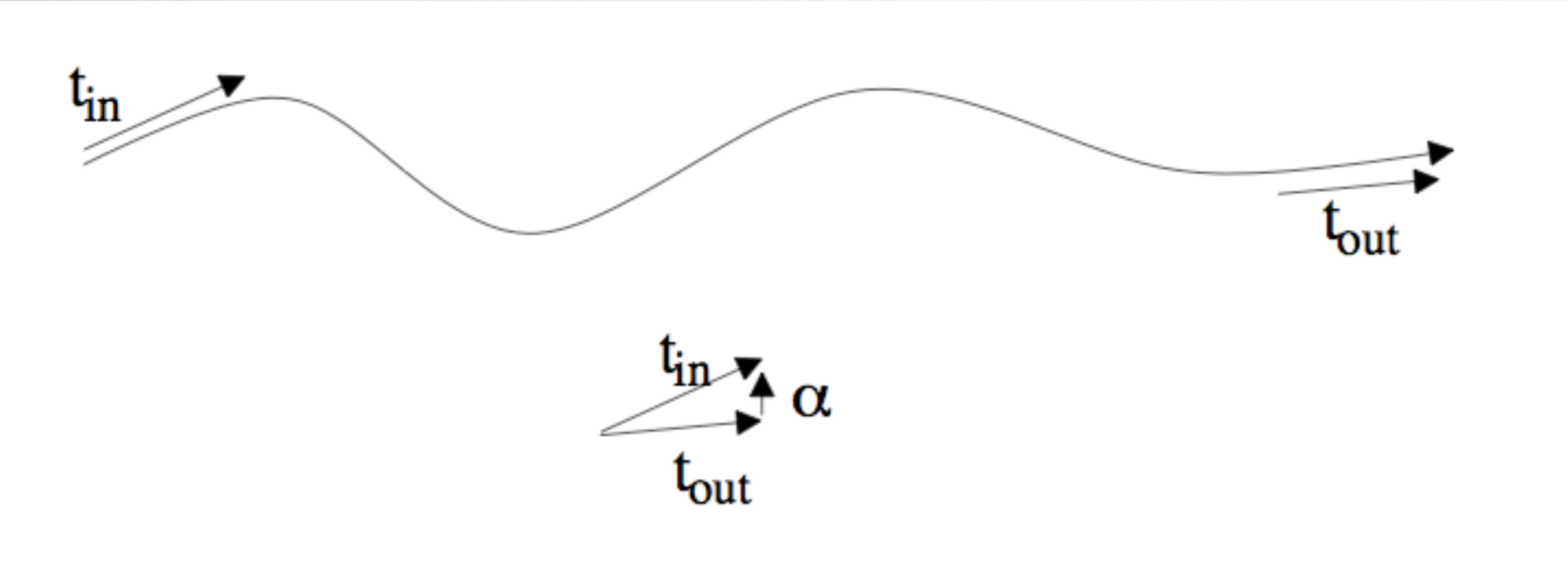
Rearrange Euler: $\frac{d\mathbf{t}}{dl} = \frac{1}{n}\nabla n - \frac{1}{n}\frac{dn}{dl}\mathbf{t}$ (1)

Take dot product with **t**, noting $\mathbf{t}\cdot d\mathbf{t} = 0$

$$\frac{1}{n}\frac{dn}{dl} = \frac{1}{n}\mathbf{t}\cdot\nabla n$$

Substitute into (1):

How do we describe paths with varying refractive index?



We obtain

$$\frac{d\mathbf{t}}{dl} = \frac{1}{n}(\nabla n - (\mathbf{t}\cdot\nabla n)\mathbf{t})$$

This is just $\frac{d\mathbf{t}}{dl} = \frac{\nabla_\perp n}{n}$ where $\nabla_\perp$ is perpendicular to ray.

The bend angle is $\hat{\vec{\alpha}} = \mathbf{t}_{\text{in}} - \mathbf{t}_{\text{out}} = -\int dl \frac{d\mathbf{t}}{dl}$

So $\boxed{\hat{\vec{\alpha}} = -\int dl \frac{\nabla_\perp n}{n}}$ Wonderful!

Approximate metric

Einstein field equations: $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Rough analogues:

Newton	Einstein
Φ	$g_{\mu\nu}$
$\mathbf{F}$	$\Gamma^{\sigma}_{\mu\nu}$
$\nabla^2\Phi$	$R_{\mu\nu}$
ρ	$T^{\mu\nu}$

So Einstein's field equations contain an analogue to Poisson, $\nabla^2\Phi = 4\pi G\rho$

Approximate metric

For galaxies and clusters, $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} \ll 1$

Putting this into field equations, one finds

$$\Box^2 \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}$$

Wave equation - if $\partial^2/\partial t^2$ is small, then like Poisson.

Then to secure the roll-over to Newton, we need

$h_{\mu\mu} = 2\Phi/c^2$ so

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) dl^2$$

Effective Refractive Index

ds^2=0, so we can rearrange this to give

$$ct = \int \left(1 - \frac{2\Phi}{c^2}\right)^{\frac{1}{2}} \left(1 + \frac{2\Phi}{c^2}\right)^{-\frac{1}{2}} dl \simeq \int \left(1 - \frac{2\Phi}{c^2}\right) dl$$

n!

So space is acting like a refractive medium with

$$n = 1 - \frac{2\Phi}{c^2}$$

So bend angle $\hat{\vec{\alpha}} = -\int dl \frac{\nabla_\perp n}{n}$ is $\hat{\vec{\alpha}} = \frac{2}{c^2} \int \nabla_\perp \Phi \, dl$ Very important.

Geodesic equation

Ray: $x^a(l)$

Basis vectors $\mathbf{e}_a$

Tangent vector $\mathbf{t} = \frac{dx^a}{dl}\mathbf{e}_a$

This is parallel transported along ray, so

$$\frac{dt_a}{dl} - \Gamma^b_{ac} t_b \frac{dx^c}{dl} = 0$$

And we can rewrite this as $\boxed{\frac{dt_a}{dl} = \frac{1}{2} g_{cd,a} t^c t^d}$ Nice form.

Geodesic equation

$$\frac{dt_a}{dl} = \frac{1}{2} g_{cd,a} t^c t^d$$

So e.g. ray in z direction, $\boldsymbol{t}$=[1,0,0,1]:

$$\frac{dt_x}{dl} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x} - \frac{1}{2}\frac{\partial g_{33}}{\partial x} = -\frac{2}{c^2}\frac{\partial \Phi}{\partial x}$$

which is what we got earlier for 'glass'!

Recall $\frac{d\mathbf{t}}{dl} = \frac{\nabla_\perp n}{n}$, $n = 1 - \frac{2\Phi}{c^2}$

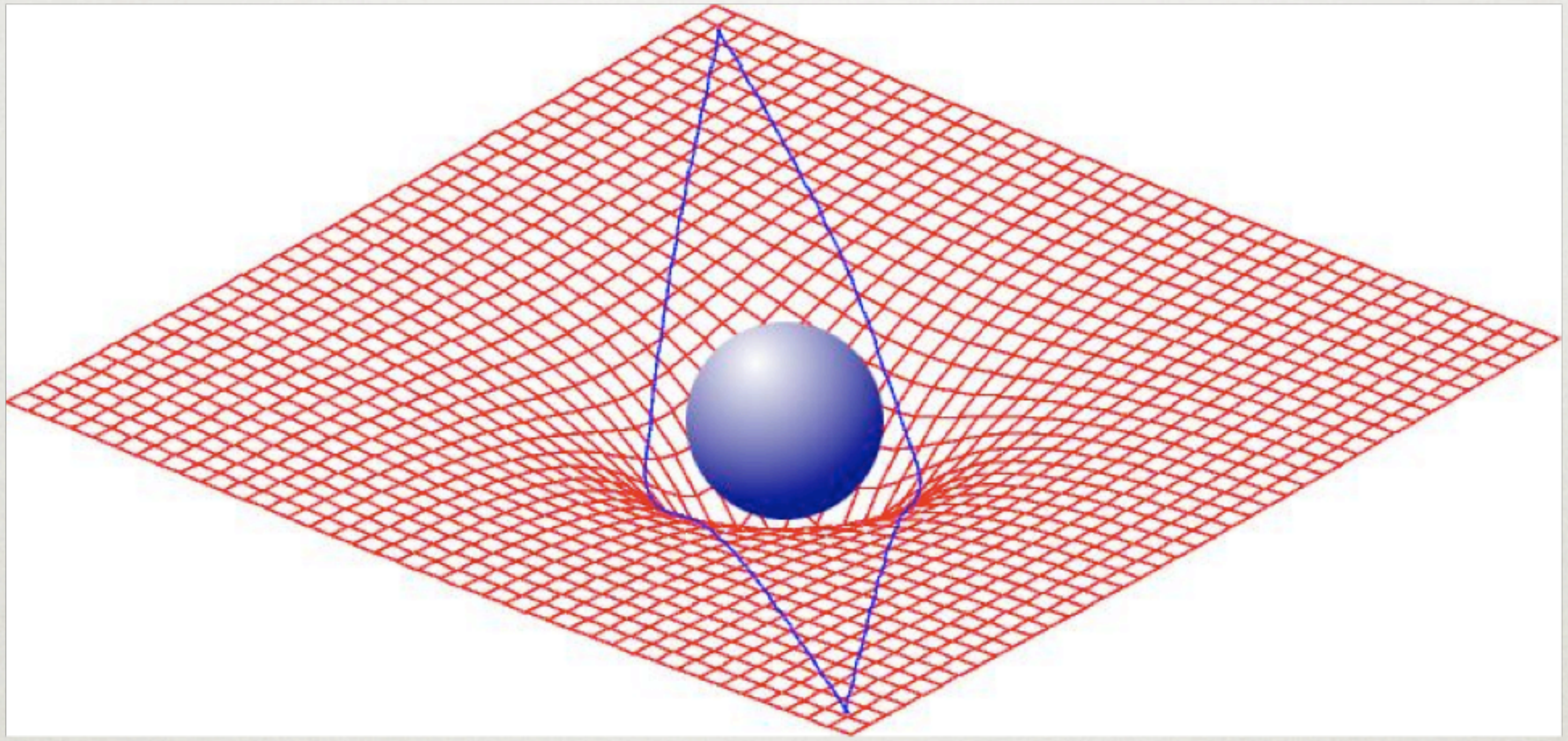
Comparison with Newton

Suppose we treated light's motion the same as low-velocity motion. Then we'd have $\boldsymbol{t} \simeq [1,0,0,0]$:

$$\frac{dt_x}{dl} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial x}$$

ie we miss the spatial curvature (g_{33}) part, so factor of two smaller.

A potentially misleading diagram:



This is OK as long as it's understood that the light rays are not locally straight in this bent 3-space; time distortion also needs to be taken into account.

Modified Gravity

An important class of modified gravities are metric theories with two perturbations:

$$ds^2 = \left(1 + \frac{2\Psi}{c^2}\right) dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) dl^2$$

As before we set this =0, then

$$t = \int \left(1 - \frac{2\Phi}{c^2}\right)^{\frac{1}{2}} \left(1 + \frac{2\Psi}{c^2}\right)^{-\frac{1}{2}} dl \simeq \int \left(1 - \frac{\Phi + \Psi}{c^2}\right) dl$$

so bend angle $$\hat{\vec{\alpha}} = \frac{1}{c^2} \int \nabla_\perp (\Phi + \Psi) dl$$

i.e. lensing responds to the combination $\Phi + \Psi$.

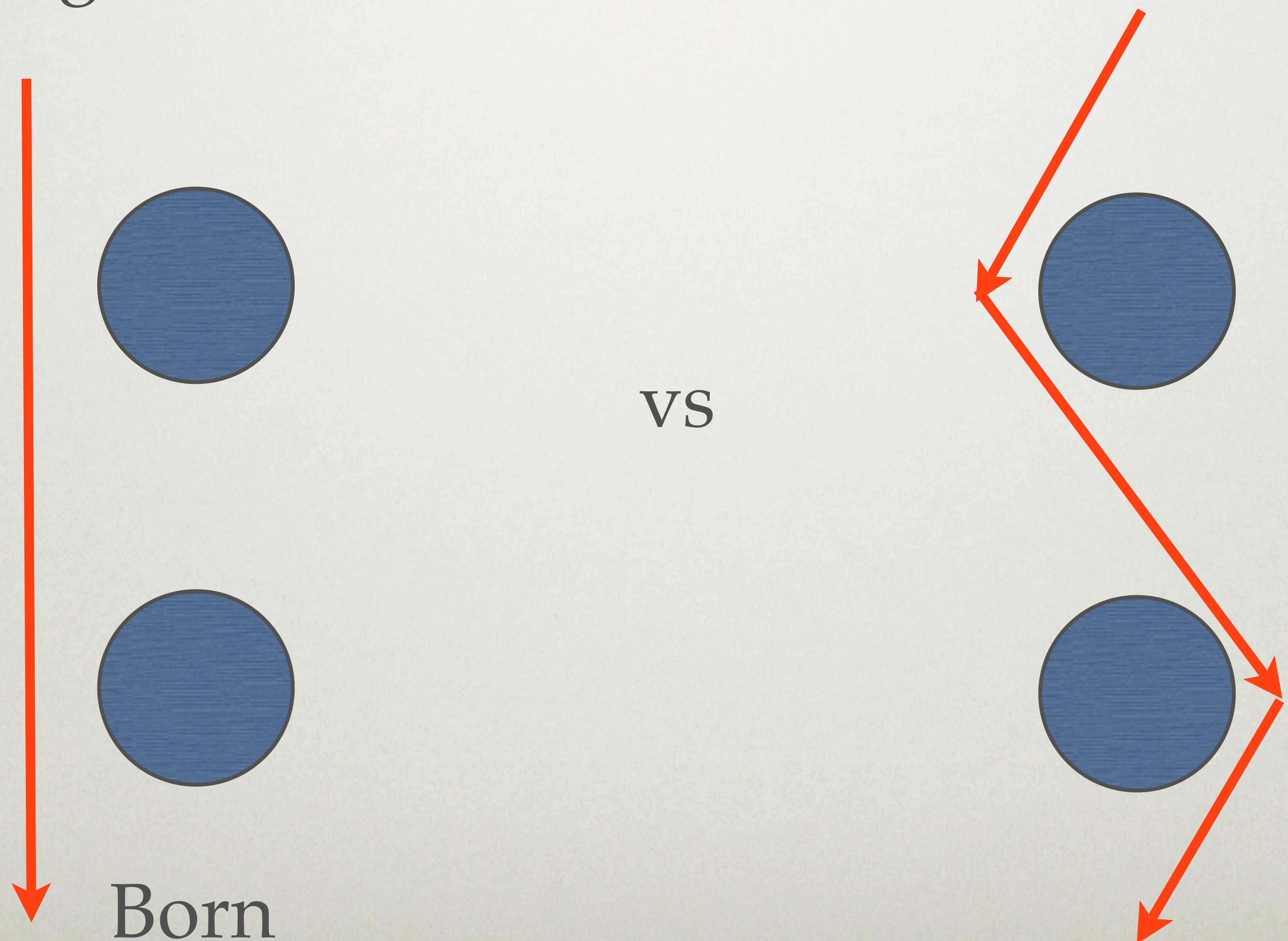
Moving beyond the Bend Angle...

Earlier we found the GR lensing bend angle:

$$\boxed{\hat{\vec{\alpha}} = \frac{2}{c^2}\int \nabla_\perp \Phi \, dl}$$

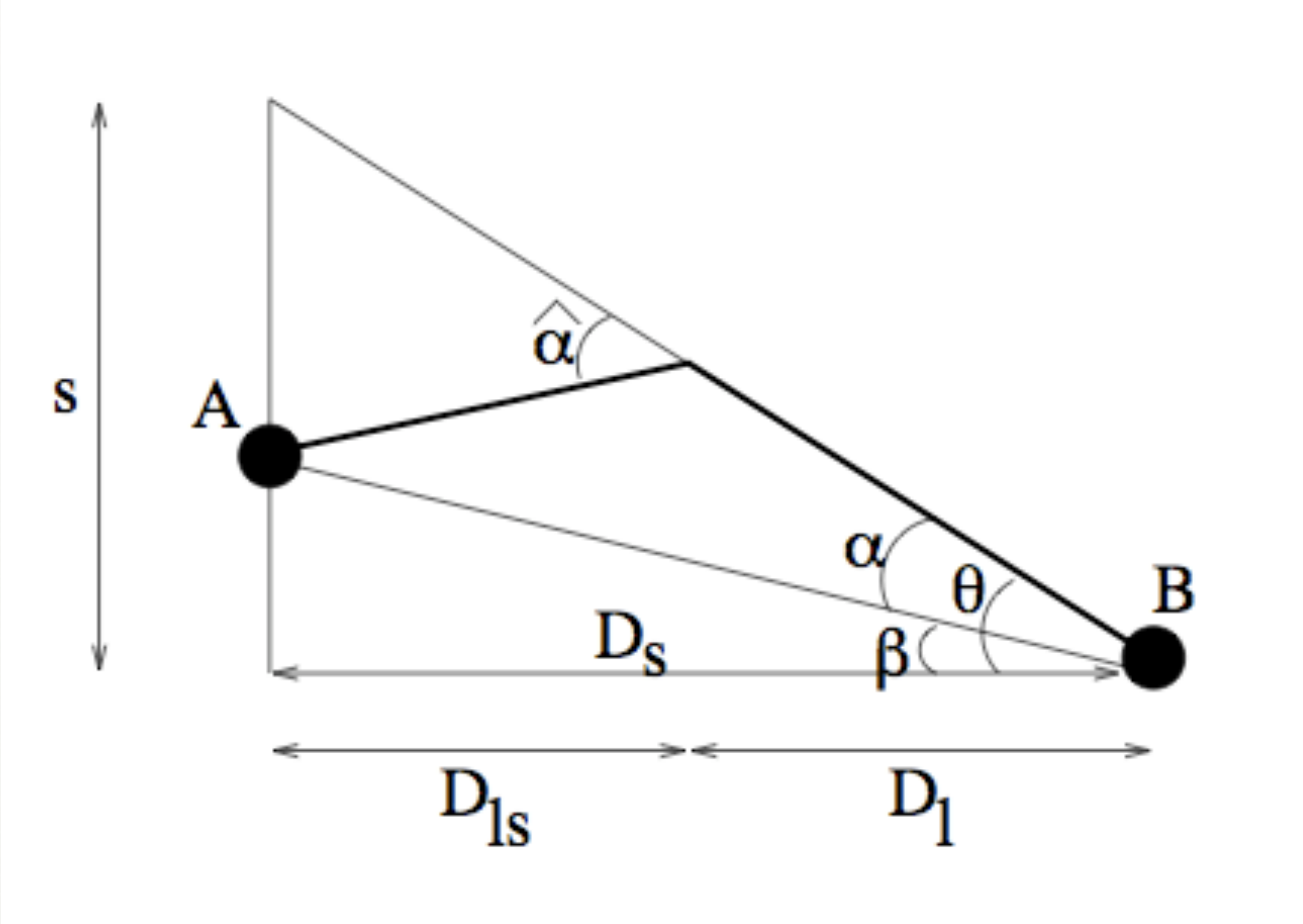
Born Approximation

We typically make the approximation that the integral can be taken in the radial direction:



Lens geometry

Suppose we have the following:



$$\vec{\theta} = \vec{\beta} + \vec{\alpha}$$

Lens equation

Small angles: $\vec{\alpha} D_s = \hat{\vec{\alpha}} D_{ls}$ so $\vec{\alpha} = \frac{D_{ls}}{D_s} \hat{\vec{\alpha}}$

RAYTRACING EXAMPLE

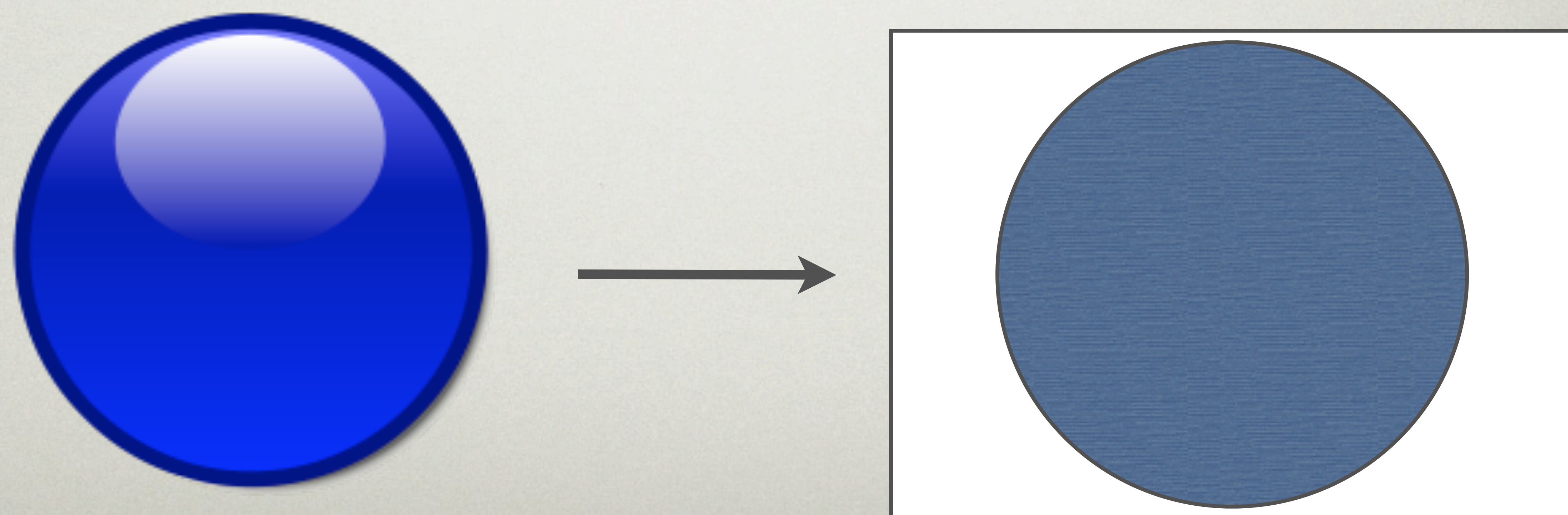
$$\vec{\theta} = \vec{\beta} + \vec{\alpha}$$

LENSING POTENTIAL

We'll find it very useful to squash (project) the gravitational potential into 2D on the sky.

Gravitational potential well in 3D

Lensing potential in 2D



Lensing potential

How do we do this?

Chosen for cleanness in a moment!

$$\psi = \frac{2D_{ls}}{c^2 D_s D_l} \int dr\, \Phi$$

Recall $\hat{\vec{\alpha}} = \frac{2}{c^2} \int \nabla_\perp \Phi \, dl$ and $\vec{\alpha} = \frac{D_{ls}}{D_s} \hat{\vec{\alpha}}$

Then $\vec{\alpha} = \nabla_\theta \psi$

Derivative on the sky

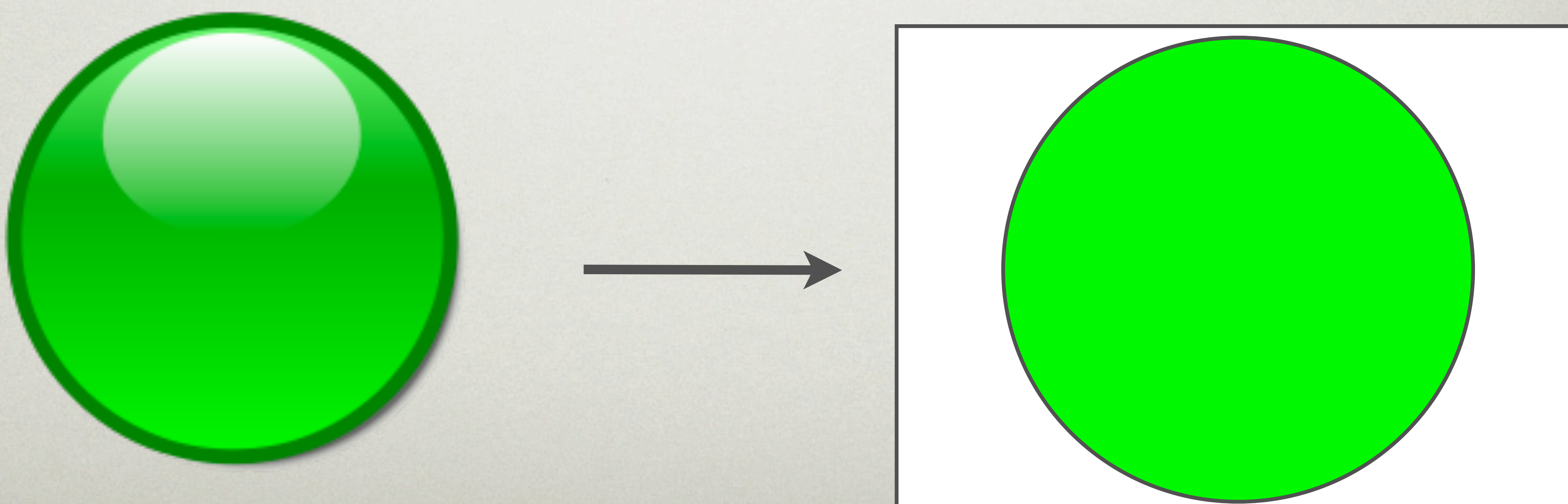
$$\nabla_\theta = \left(\frac{\partial}{\partial \theta_x}, \frac{\partial}{\partial \theta_y} \right)$$

Surface Mass Density

We'll also find it very useful to squash (project) the density into 2D on the sky.

Density in 3D

Surface density in 2D



Surface Density

How do we do this?

$$\Sigma = \int dr\, \rho$$

Let's also introduce a quantity containing all the constants and distance ratios:

$$\Sigma_c = \frac{c^2}{4\pi G}\frac{D_s}{D_{ls} D_l}$$

Then $$\nabla_\theta \cdot \vec{\alpha} \sim \int dr \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi \sim \int dr\, \rho$$

and we find $$\nabla_\theta \cdot \vec{\alpha} = \nabla_\theta^2 \psi = \frac{2\Sigma}{\Sigma_c}$$ - a 2D Poisson!

Near-observables: 1) Magnification

Interesting: lensing conserves surface brightness

(Why? Because I/ν^3 is conserved along rays in GR, and ν change is almost all from cosmological redshift)

Call ratio of lensed to unlensed luminosity, the magnification

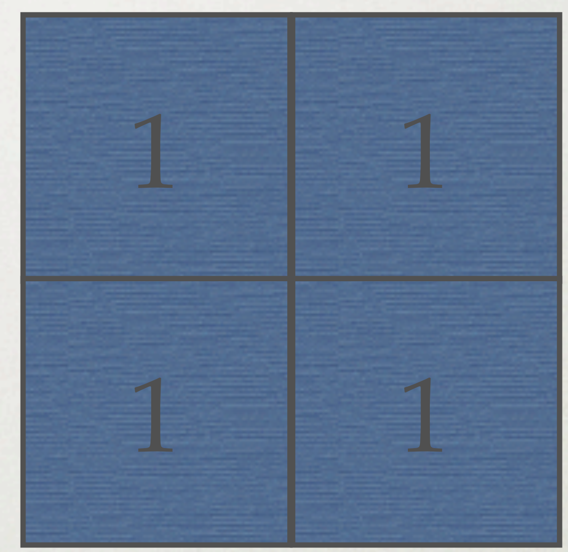
So since *I* is conserved, magnification is the ratio of lensed to unlensed area of image.

Near-observables: 1) Magnification

e.g. Unlensed

Total=1

Lensed



Total=4

Magnification=4, which is ratio of areas

Jacobian

To go further, it will help to describe the lensing with the Jacobian matrix

$$\mathcal{A}_{ij} = \frac{\partial \beta_i}{\partial \theta_j}$$

i.e. how a position in the image plane maps to a position in the source plane.

The lens equation says $\beta_i = \theta_i - \alpha_i$

so $$\mathcal{A}_{ij} = \delta_{ij} - \frac{\partial \alpha_i}{\partial \theta_j} = \delta_{ij} - \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j}$$

2) Convergence

This includes the term $\nabla^2_\theta \psi$ which we found $= \frac{2\Sigma}{\Sigma_c}$

So let's call $\boxed{\kappa = \frac{\Sigma}{\Sigma_c}}$ the convergence.

The convergence is proportional to the projected density.

Then the mapping is just $\mathcal{A} = \begin{pmatrix} 1-\kappa & 0 \\ 0 & 1-\kappa \end{pmatrix} + \ldots$

which is an expansion / contraction.

The convergence sets the amount of isotropic expansion of the image.

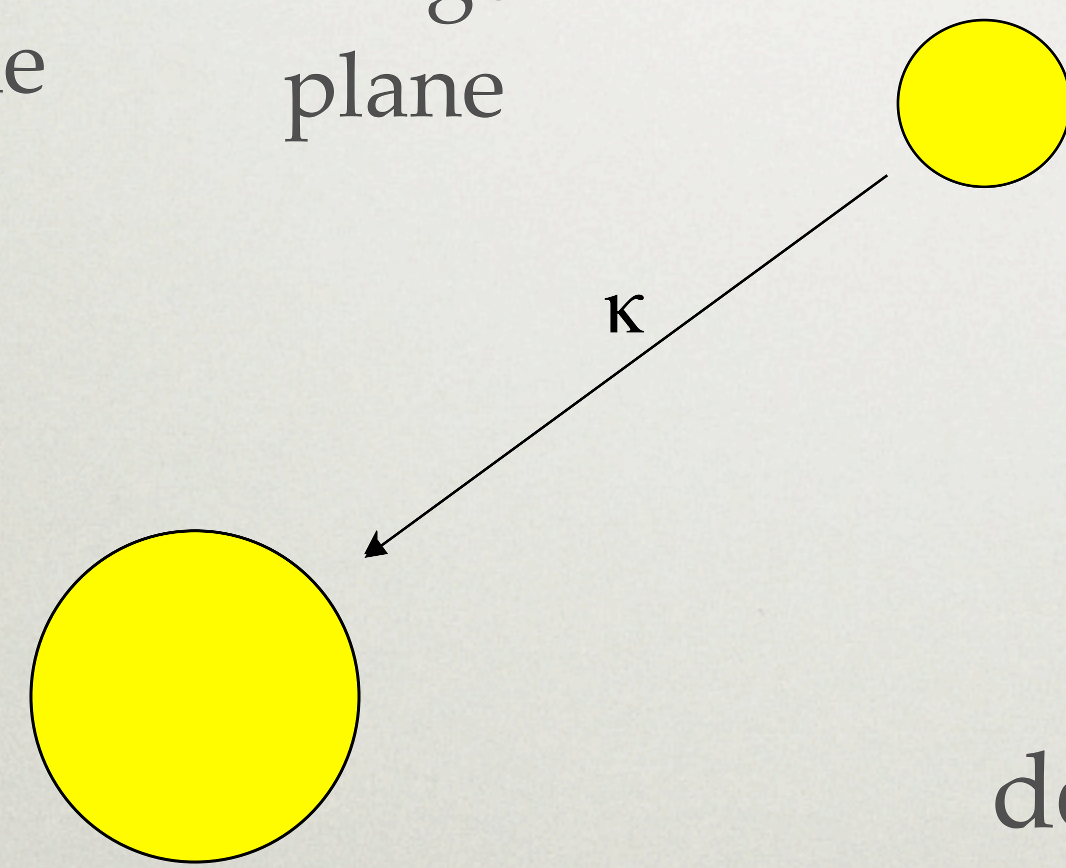
The Distortions

$$\beta_i = A_{ij}\theta_j$$

Source plane

Image plane

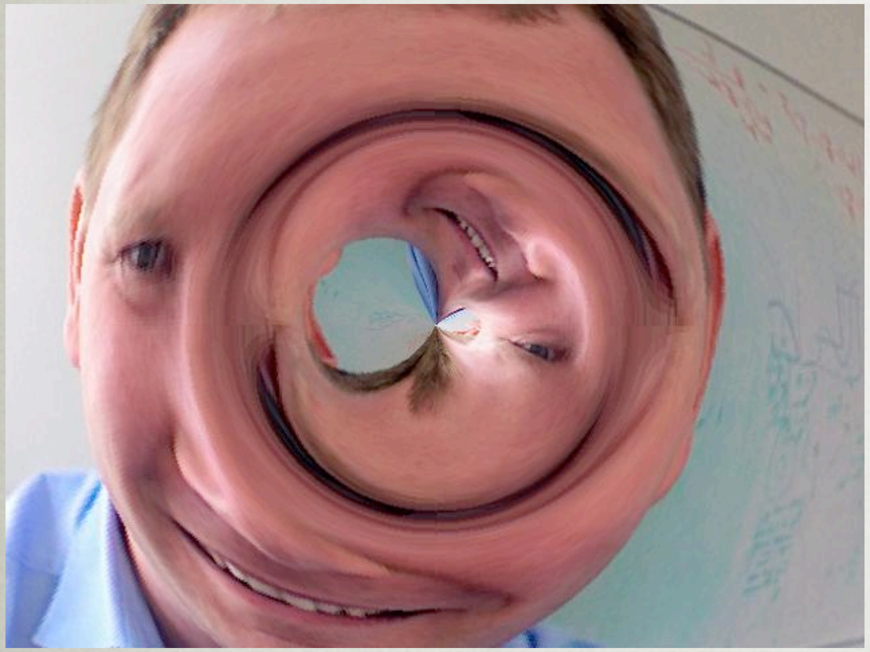
$$\mathcal{A} = \begin{pmatrix} 1-\kappa & 0 \\ 0 & 1-\kappa \end{pmatrix} + \ldots$$



Convergence

Extraordinarily close relationship between density and optical effect!

Shear



What about the other terms in

$$\mathcal{A}_{ij} = \delta_{ij} - \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j} \quad ?$$

We can write these as

$$\gamma_1 = \frac{1}{2}(\partial_1^2 - \partial_2^2)\psi, \quad \gamma_2 = \partial_1 \partial_2 \psi$$

then

$$\boxed{\mathcal{A} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

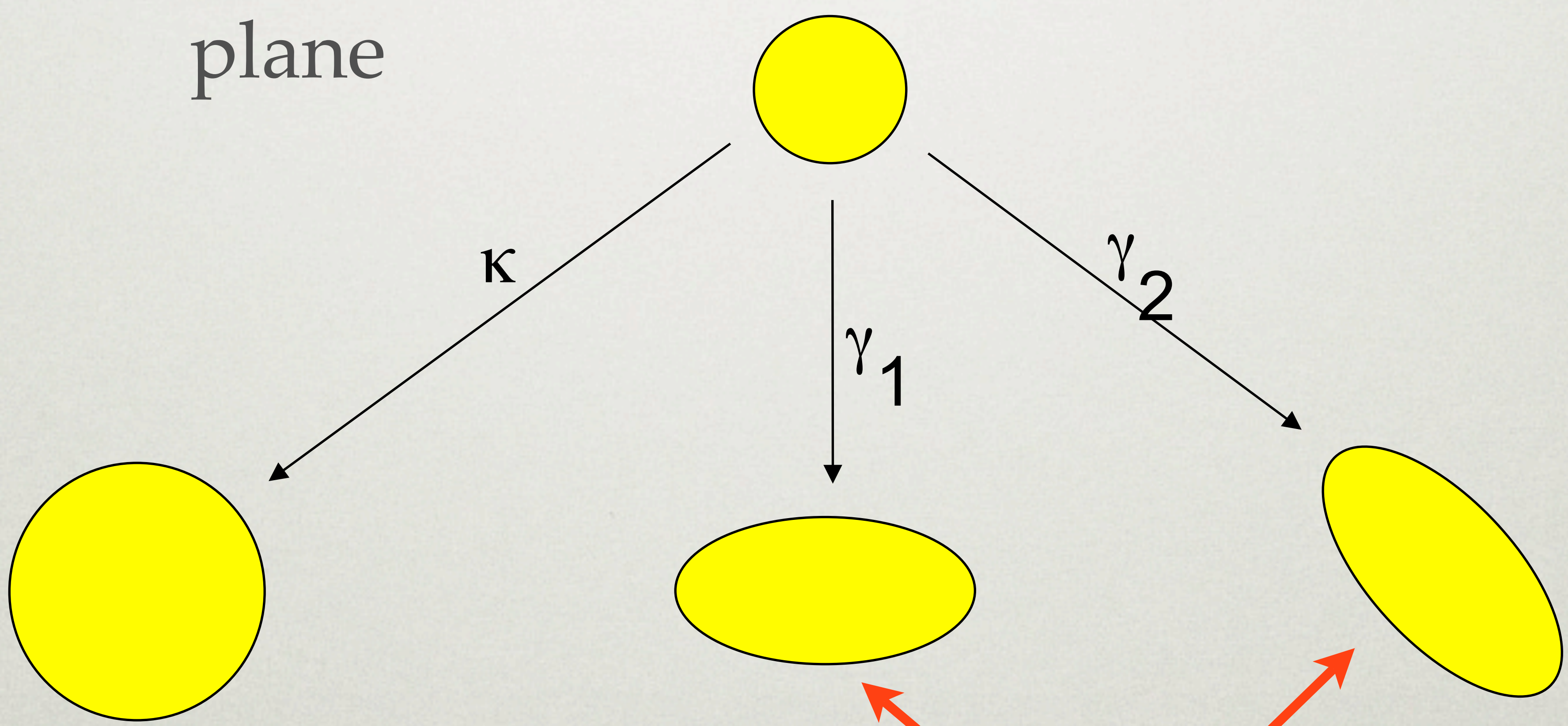
The Distortions

$$\beta_i = A_{ij}\theta_j$$

Source plane

Image plane

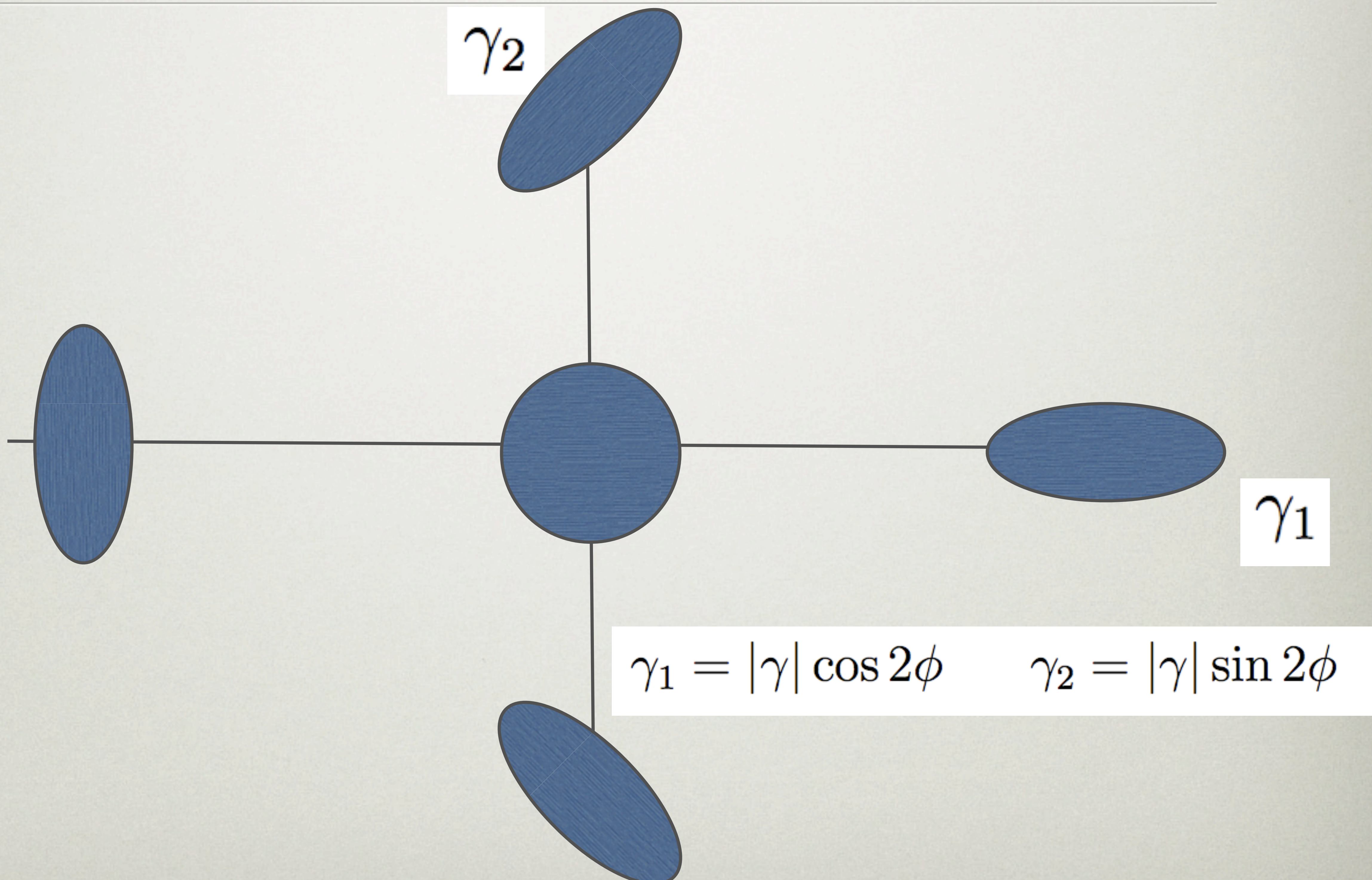
$$A = \begin{pmatrix} 1-\kappa & 0 \\ 0 & 1-\kappa \end{pmatrix} + \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix}$$



Convergence

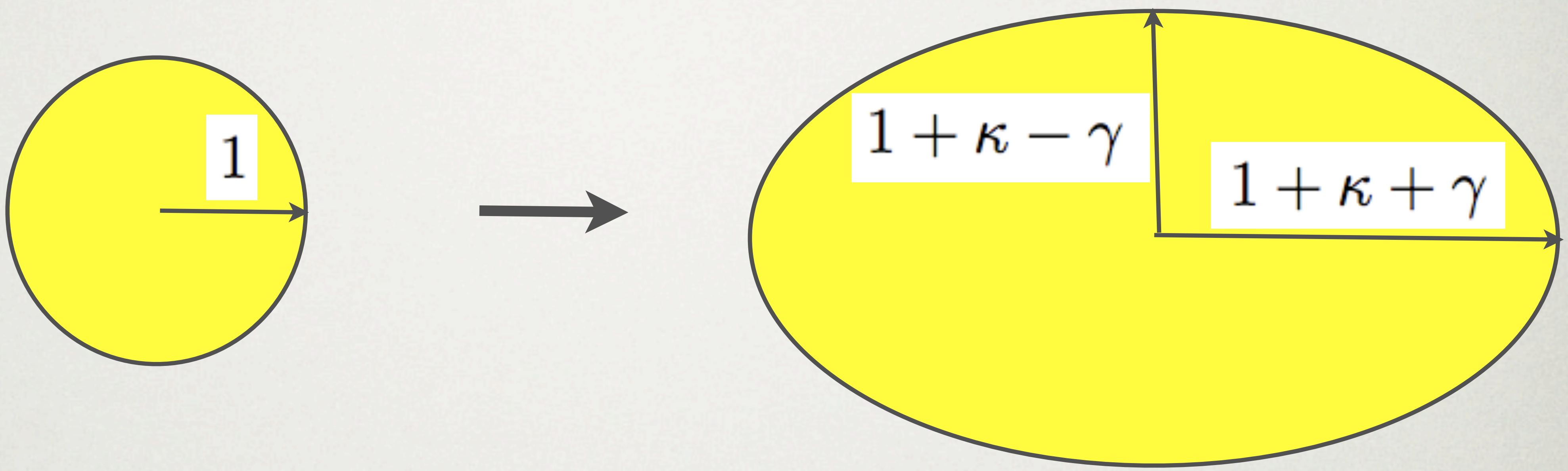
Shear

Shear



γ_2
γ_1
$\gamma_1 = |\gamma| \cos 2\phi \qquad \gamma_2 = |\gamma| \sin 2\phi$

...But we can't measure it.



So at best can only measure a combination of convergence and shear.

We can rewrite

$$\mathcal{A} = (1-\kappa)\begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}$$

where g is the reduced shear, $g_i = \dfrac{\gamma_i}{1-\kappa}$

NB for very weak shear, $\gamma \sim \kappa \sim 0.01$ then $\gamma \sim g$.

Perfect

So now we just go to a telescope and measure

the magnification
the convergence
the shear

and we perfectly understand cosmology.

No?