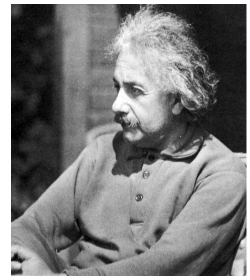
XIIth School of Cosmology

IESC Cargèse

Averaging Inhomogeneous Cosmologies

September 2014



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Contents

Further Reading

In the following *Reviews* you may also find all relevant references.

[1] Dark Energy from Structure: A Status Report.
Gen. Rel. Grav. **40**: 467-527, 2008. e–Print: arXiv:0707.2153

[2] Toward physical cosmology: focus on inhomogeneous geometry and its non-perturbative effects. *Class. Quant. Grav.* **28**: 164007, 2011. e–Print: arXiv:1103.2016

[3] Backreaction in late–time cosmology. (with S. Räsänen)
Ann. Rev. Nucl. & Part. Phys. **62**: 57, 2012. e–Print: arXiv:1112.5335

Chapter 1 in the following popular book provides the basic ideas as well as an appendix of the relevant formulae:

[4] Cosmic Update: dark puzzles, the arrow of time, future history.
(with F. Adams and L. Mersini–Houghston)
Springer New York: Multiversal Journeys II, ed. by F. Nekogaar, 2011.

The following *Reviews* on the subject have been written by collegues:

[5] Inhomogeneity effects in cosmology.
by George F.R. Ellis
Class. Quant. Grav. **28**, 164001 (2011). e–print: arXiv:1103.2335

[6] Does the growth of structure affect our dynamical models of the Universe? The averaging, backreaction, and fitting problems in cosmology.
by C. Clarkson, G.F.R. Ellis, J. Larena and O. Umeh
Rep. Prog. Phys. **74**, 112901 (2011). e–print: arXiv:1109.2314

[7] Backreaction: directions of progress.
by Syksy Räsänen
Class. Quant. Grav. **28**, 164008 (2011). e–print: arXiv:1102.0408

[8] What is dust? – Physical foundations of the averaging problem in cosmology.
by David L. Wiltshire
Class. Quant. Grav. **28**, 164006 (2011). e–print: arXiv: 1106.1693

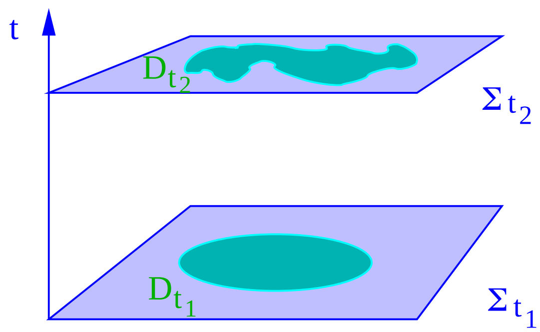

Figure 1: Two spatial domains embedded into consecutive, three–dimensional Eulerian space sections Σ_t, parametrized by the absolute Newtonian time t, are shown.

1 Newtonian Gravity

1.1 Galilei–Newton spacetime and the Eulerian picture

The stage on which we shall first study the motion of a continuum is the Galilei–Newton spacetime (Fig. 1), which itself is viewed as a continuum, foliated into Euclidian 3–spaces that are indexed by values of the Newtonian *absolute time* $t > 0$. Within the space sections, *observables* are represented by continuous fields as functions of (non–rotating and usually *Cartesian*) Eulerian coordinates $\mathbf{x}$ and time t.

As a basic observable we introduce the *particle number density* $n(\mathbf{x},t)$ for which we require that the total number N of fluid elements within a spatial domain $\mathcal{D}_t$ be conserved:

$$N := \int_{\mathcal{D}_t} n(\mathbf{x},t)\; d^3x \quad = \quad \text{const.} \tag{1a}$$

Assigning to each 'particle' an elementary mass m, we can instead consider the restmass density field $\varrho(\mathbf{x},t) = m\, n(\mathbf{x},t)$ and the conservation of the total restmass:

$$M := \int_{\mathcal{D}_t} \varrho(\mathbf{x},t)\; d^3x \quad = \quad \text{const.} \tag{1b}$$

A simple *kinematically* complete description for the evolution of a system of fluid elements is provided by the restmass density field $\varrho(\mathbf{x},t)$ and a vector field for the mean velocities of the fluid elements $\mathbf{v}(\mathbf{x},t)$. The general solution of evolution equations for these fields should be *uniquely* determined by giving the initial data $\varrho_0(\mathbf{x},t_0)$ and $\mathbf{v}_0(\mathbf{x},t_0)$. Such a description we call *Eulerian picture*.

1.2 Lagrangian picture: trajectories in space and mass conservation

We localize a collection of fluid elements in space by the Eulerian *position vector field* **f**:

$$\mathbf{x} = \mathbf{f}(\mathbf{X},t) \quad ; \quad \mathbf{f}(\mathbf{X},t_0) =: \mathbf{X} \quad . \tag{2}$$

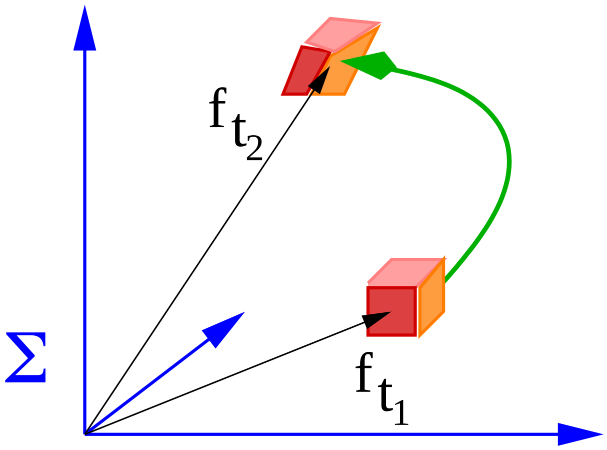

Figure 2: A trajectory in Eulerian space is shown that connects two time–consecutive fluid elements. The volume elements are deformed during the motion of the fluid. Eulerian position vectors $\mathbf{f}_t$ locate them.

In the *Lagrangian picture* of continuum mechanics the Eulerian positions at time t_0 are held fixed along trajectories of the fluid elements. The *Lagrangian coordinates*[1] X_i just index the fluid elements, while the components of the position field f_i locate the fluid element with Lagrangian coordinates $\mathbf{X}$ in Eulerian space (Figs. 2 and 3).

The position field $\mathbf{f}$ defines a time–dependent *diffeomorphism* in the Galilei–Newton space-time:

$$\Phi_t : \mathbf{X} \quad \mapsto \quad \mathbf{x} = \mathbf{f}(\mathbf{X}, t) \quad . \tag{3}$$

At fixed time $t = t'$, the flow $\Phi_{t'}$ can be viewed as a coordinate transformation from Lagrangian to Eulerian coordinates:

$$\begin{aligned} \mathbb{R}^3 \quad &\rightarrow \quad \mathbb{R}^3 \\ \mathbf{X} \quad &\mapsto \quad \mathbf{x} = \mathbf{f}(\mathbf{X}; t') \quad , \end{aligned} \tag{4a}$$

with the Jacobian[2]

$$J_{ik}(\mathbf{X}, t') := \frac{\partial f_i}{\partial X_k} \quad . \tag{4b}$$

The volume elements of the continuum transform according to

$$d^3x = J d^3X \quad ; \quad J(\mathbf{X}, t') := \det(J_{ik}) \quad . \tag{4c}$$

Consider now the total temporal change of the position field defining the mean velocity field of fluid elements $\mathbf{v}$:

$$v_i(\mathbf{X}, t) := \frac{d}{dt} f_i(\mathbf{X}, t) = \frac{\partial}{\partial t}\Big|_{\mathbf{X}} f_i(\mathbf{X}, t) + \frac{\partial f_i}{\partial X_k}\frac{dX_k}{dt} = \frac{\partial}{\partial t}\Big|_{\mathbf{X}} f_i(\mathbf{X}, t) \quad . \tag{5}$$

We have used the fact that the Lagrangian coordinates are, by their very definition, conserved in time. It is also said that the position field, providing the trajectories $\mathbf{f}$ labelled by $\mathbf{X}$, defines the *integral curves* of the vector field $\mathbf{v}$ (Fig. 3).

[1] We shall maintain the notion *Lagrangian coordinates*, although it is known that their introduction is due to Euler, and Lagrange himself obviously preferred Eulerian coordinates.

[2] Hereafter, latin indices run through $1, 2, 3$; we distinguish, as a guide for the eye, Lagrangian indices, e.g., in derivatives, $\partial/\partial X_k$ from Eulerian $\partial/\partial x_j$. Summation over repeated indices is understood.

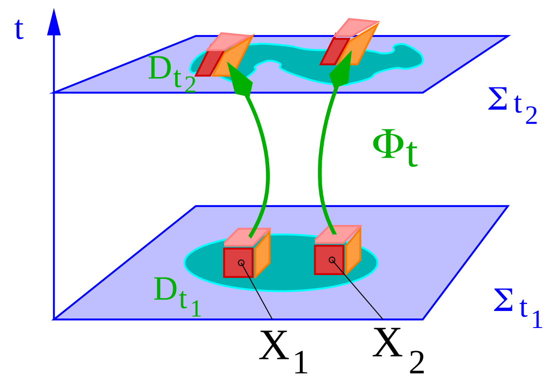

Figure 3: Two volume elements at initial time t_0 are indexed by their Lagrangian coordinates $\mathbf{X}_1$ and $\mathbf{X}_2$. The trajectory field (represented by two flow lines) defines a diffeomorphism Φ_t in the Galilei–Newton spacetime.

Accordingly, we define the mean acceleration field of the fluid $\mathbf{b}$ by the total time–derivative of the mean velocity field:

$$b_i(\mathbf{X},t) := \frac{d}{dt}\,v_i(\mathbf{X},t) = \frac{\partial}{\partial t}\Big|_{\mathbf{X}}\,v_i(\mathbf{X},t) + \frac{\partial\,v_i}{\partial X_k}\frac{dX_k}{dt} = \frac{\partial}{\partial t}\Big|_{\mathbf{X}}\,v_i(\mathbf{X},t)\quad . \tag{6}$$

For comparison we look at the mean velocity field as a function of Eulerian coordinates $\mathbf{v}(\mathbf{x},t)$. Its total time–derivative

$$\frac{d}{dt}v_i(\mathbf{x},t) = \frac{\partial}{\partial t}\Big|_{\mathbf{x}}v_i(\mathbf{x},t) + \frac{\partial v_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial}{\partial t}\Big|_{\mathbf{x}}v_i(\mathbf{x},t) + v_j\frac{\partial v_i}{\partial x_j} \tag{7}$$

features a second term, the so–called *convective time–derivative*, which takes into account that in the Eulerian picture the fluid elements move with respect to the Eulerian space. The convective time–derivative of $\mathbf{v}$ is the *directional derivative* of $\mathbf{v}$ along $\mathbf{v}$. (In other words, the spatial variation of $\mathbf{v}$ is projected onto $\mathbf{v}$.) In summary, we define the total time–derivative with respect to a trajectory field (a collection of integral curves to $\mathbf{v}$) as the *Lagrangian time–derivative*:

$$\frac{d}{dt} := \frac{\partial}{\partial t}\Big|_{\mathbf{X}} = \frac{\partial}{\partial t}\Big|_{\mathbf{x}} + \mathbf{v}\cdot\boldsymbol{\nabla}\quad , \tag{8}$$

with the nabla operator $\boldsymbol{\nabla} := \partial/\partial x_j$.

1.3 Remark: independent variables

In the Eulerian picture $\mathbf{x}$ and t are the independent variables, and $\partial/\partial t|_{\mathbf{x}}$ commutes with $\partial/\partial x_j$ and d^3x, while in the Lagrangian picture $\mathbf{X}$ and t are the independent variables, and $d/dt = \partial/\partial t|_{\mathbf{X}}$ commutes with $\partial/\partial X_k$ and d^3X.

As an example we consider the conservation of the total restmass within a spatial domain $\mathcal{D}_t$:

$$0 = \frac{d}{dt}M = \frac{d}{dt}\int_{\mathcal{D}_t}\varrho(\mathbf{x},t)\;d^3x = \int_{\mathcal{D}_{t_0}}\frac{d}{dt}\left(\varrho(\mathbf{X},t)J(\mathbf{X},t)\right)\;d^3X\quad . \tag{9a}$$

Since this equation holds for any Lagrangian domain $\mathcal{D}_{t_0}$, we conclude that

$$\frac{d}{dt}(\varrho J) = 0 \quad \Rightarrow \quad \varrho J = C(\mathbf{X}) \ . \tag{9b}$$

For initial data $J(\mathbf{X}, t_0) = \det(\delta_{ik}) = 1$ and $\varrho(\mathbf{X}, t_0) =: \varrho_0(\mathbf{X})$ we obtain a *general integral* for the density field:

$$\varrho(\mathbf{X}, t) = \varrho_0(\mathbf{X}) \ J^{-1}(\mathbf{X}, t) \ . \tag{9c}$$

1.4 Exercise: continuity equation

Give a local proof of the fact that the integral (9c) solves the Eulerian continuity equation

$$\frac{\partial}{\partial t}\varrho + \boldsymbol{\nabla} \cdot (\varrho \mathbf{v}) \ = \ \frac{d}{dt}\varrho + \varrho \boldsymbol{\nabla} \cdot \mathbf{v} \ = \ 0 \ . \tag{10}$$

Solution: Employ the notation in terms of functional determinants and prove the identity $d/dt\, J = J \nabla \cdot \mathbf{v}$. First, we write:

$$J = \det\left(\frac{\partial f_i}{\partial X_k}\right) = \frac{\partial(f_1, f_2, f_3)}{\partial(X_1, X_2, X_3)} \ . \tag{11}$$

For the total (Lagrangian) time–derivative (denoted by an overdot) we then obtain:

$$\frac{d}{dt}J = \frac{\partial(\dot{f}_1, f_2, f_3)}{\partial(X_1, X_2, X_3)} + \frac{\partial(f_1, \dot{f}_2, f_3)}{\partial(X_1, X_2, X_3)} + \frac{\partial(f_1, f_2, \dot{f}_3)}{\partial(X_1, X_2, X_3)} \ ,$$

and, using $\mathbf{v} = \dot{\mathbf{f}}$, $\det(J^{-1}_{ik}) = J^{-1}$, and the multiplication rule for determinants:

$$\frac{\frac{d}{dt}J}{J} = \frac{\partial(v_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} + \frac{\partial(f_1, v_2, f_3)}{\partial(x_1, x_2, x_3)} + \frac{\partial(f_1, f_2, v_3)}{\partial(x_1, x_2, x_3)} = \boldsymbol{\nabla} \cdot \mathbf{v} \ . \tag{12}$$

With this identity, the time–derivative of (9c) yields the continuity equation.

1.5 Remark: dependent variables

In the *Eulerian picture* $\varrho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ are the dependent variables (fields), whereas in the *Lagrangian picture* $\mathbf{f}(\mathbf{X}, t)$ is the only dependent field variable. For given $\mathbf{f}(\mathbf{X}, t)$ the evolution of the continuum is determined, and the Eulerian fields can be calculated from it provided the transformation from Lagrangian to Eulerian coordinates can be inverted (Φ_t is a diffeomorphism), i.e.,

$$\varrho[\mathbf{x}, t] = \varrho_0(\mathbf{X} = \mathbf{h}(\mathbf{x}, t)) J^{-1}(\mathbf{X} = \mathbf{h}(\mathbf{x}, t), t) \ ; \ \ \mathbf{v}[\mathbf{x}, t] = \dot{\mathbf{f}}(\mathbf{X} = \mathbf{h}(\mathbf{x}, t), t) \ , \tag{13}$$

where $\mathbf{h}$ denotes the inverse transformation $\mathbf{f}^{-1}$. (Rectangular brackets indicate that the functional dependence on the coordinates changes, but *not* the value of the field.)

1.6 The Euler–Newton system

In the sequel we think of the acceleration as solely caused by the gravitational field strength as the dominating force in a cosmological and astrophysical context,

$$\mathbf{b} =: \mathbf{g}(\mathbf{x}, t) \quad . \tag{14}$$

Eq. (14) forms the content of Einstein's *equivalence principle* that asserts the equality of *inertial mass* m and *gravitational mass* m_G in the Newtonian force balance condition:

$$\mathbf{F} = m\,\mathbf{b} = m_G\,\mathbf{g} = \mathbf{F}_G \;\; ; \;\; m_G \;\equiv\; m \quad . \tag{15}$$

The Newtonian gravitational force field is assumed velocity–independent and derived from a potential. With (14) the evolution equations (6) and (10) for the mean velocity field and the density field supplemented by the Newtonian field equations form a closed system of coupled, nonlinear partial differential equations, the *Euler–Newton system*:

$$\frac{d}{dt}\varrho \,+\, \varrho\boldsymbol{\nabla}\cdot\mathbf{v} \;=\; 0 \qquad ; \qquad \frac{d}{dt}\mathbf{v} \;=\; \mathbf{g} \;\; ; \tag{16a}$$

$$\boldsymbol{\nabla}\times\mathbf{g} \;=\; \mathbf{0} \quad ; \quad \boldsymbol{\nabla}\cdot\mathbf{g} \;=\; \Lambda - 4\pi G\varrho \;\; . \tag{16b}$$

Instead of this system of eight equations for the seven variables ϱ, v_i, g_i we can write down a closed system of five equations for the five variables ϱ, v_i, Φ, the *Euler–Poisson system*:

$$\frac{d}{dt}\varrho \,+\, \varrho\boldsymbol{\nabla}\cdot\mathbf{v} \;=\; 0 \;\; ; \;\; \frac{d}{dt}\mathbf{v} \;=\; -\boldsymbol{\nabla}\Phi \;\; ; \;\; \Delta\Phi = 4\pi G\varrho - \Lambda \;\; . \tag{17}$$

The above systems involve *boundary value problems* that have to be well–posed and solved for all times. Note that the *initial value problem* for both systems has to be posed in terms of *evolution equations*, in our case four equations for four independent variables, which are ϱ and the three components of $\mathbf{v}$. The field–strength $\mathbf{g}$ or the gravitational potential Φ, respectively, are only subjected to *constraints*.

We end this section with an example of a simple exact class of solutions to the Euler–Poisson system (1.6). These solutions lie at the basis of the *standard model of cosmology*. As Fig. 4 shows, already this simple equation admits a wide variety of possible expansion laws. It is a striking historical fact that the choice fell on models that correspond to singular solutions reflected in the notion *Big Bang*.

1.7 Excursion: homogeneous–isotropic solutions

For a homogeneous–isotropic trajectory field $\mathbf{f}_H = a(t)\mathbf{X}$ the evolution equation for the *scale–factor* $a(t)$ obeys *Friedmann's differential equation*:

$$H^2 - \frac{8\pi G}{3}\varrho_H - \frac{\Lambda}{3} + \frac{k}{a^2} \;=\; 0 \;\; , \tag{18}$$

with the *Hubble function* $H := \dot{a}/a$, and the constant of integration k.

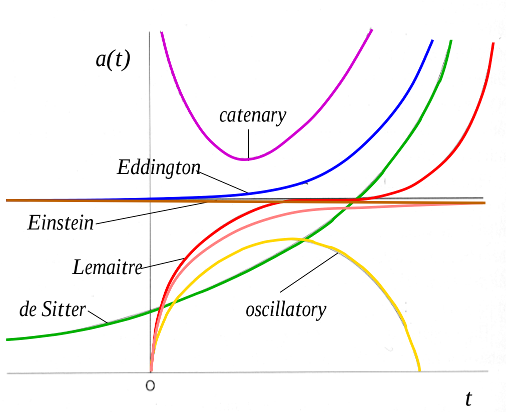

Figure 4: The time–evolution of the scale–factor $a(t)$ is shown, exemplifying the wealth of possible homogeneous–isotropic cosmological models. Standing out are: the *Eddington model*, which evolves out of the static *Einstein cosmos* and approaches asymptotically the exponential expansion of the *de Sitter model*. The *Lemaître models* start out from a singularity (hence, the notion *Big Bang*), either having an inflection point (gradually entering the exponential expansion phase), or oscillating back into a *Big Crunch*. The simplest model within this class (not drawn) is the *Einstein–de Sitter model* that has no cosmological constant and euclidian space sections. Models with positive constant curvature and no cosmological constant also oscillate back into a *Big Crunch*, while models with no cosmological constant and negative constant curvature cross the static line and expand forever (not drawn). There are also *catenary models* that contract to a minimum and expand thereafter.

We can obtain this equation as follows. The assumption of a homogeneous matter distribution $\varrho = \varrho_H(t)$ implies, by the continuity equation, that the (irrotational) velocity field is a linear function of $\mathbf{x}$, $(\mathbf{v}_H)_i = H_{ij}(t)x_j$. In general, a velocity model linear in $\mathbf{x}$ can be derived from the *homogeneous deformation* $(\mathbf{f}_H)_i = a_{ij}(t)X_j$. The assumption of isotropy reduces the tensor function $a_{ij}(t) = a(t)\delta_{ij}$, so that the velocity and acceleration fields may be written as $\mathbf{v}_H = H\mathbf{x}$, $\mathbf{g}_H = (\ddot{a}/a)\,\mathbf{x}$, while the density is given by the general integral, $\varrho_H = \varrho_H(t_0)a^{-3}$. Inserting this ansatz into the Euler–Newton system results in

$$3\frac{\ddot{a}}{a} + 4\pi G\varrho_H - \Lambda \;=\; 0 \;, \tag{19}$$

which is the time–derivative of Friedmann's differential equation. (To integrate (19) we multiply it by $\dot{a}a$ and, using the integral $\varrho_H = \varrho_H(t_0)a^{-3}$, we cast it into the form:

$$3\left(\frac{\dot{a}^2}{2}\right)^{\cdot} - 4\pi G\varrho_H(t_0)\left(\frac{1}{a}\right)^{\cdot} - \Lambda\left(\frac{a^2}{2}\right)^{\cdot} \;=\; 0 \;,$$

which can be readily integrated.

Note that Friedmann's differential equation has been first derived in the framework of general relativity. It has, however, the same form in the Newtonian framework. (The integration constant in Newton's theory is related to the constant curvature $\mathcal{R}$ of the hypersurface by $k/a^2 = c^2\mathcal{R}/6$; Friedmann's equation can be directly obtained from the *Hamiltonian constraint*, which we shall encounter later.

1.8 Kinematics of Newtonian models

We start with Euler's equation and write it down in index notation,

$$\frac{\partial}{\partial t} v_i + v_k v_{i,k} = g_i \quad . \tag{20a}$$

Performing the spatial Eulerian derivative of this equation we obtain:

$$\frac{d}{dt} v_{i,j} = -v_{i,k} v_{k,j} + g_{i,j} \quad , \tag{20b}$$

which is an evolution equation for the velocity gradient $v_{i,j}$ along the flow lines $\mathbf{f}$. It is convenient for the discussion of fluid motion to split the velocity gradient into its symmetric part, $v_{(i,j)} = 1/2\,(v_{i,j} + v_{j,i}) =: \theta_{ij}$ (the expansion tensor), its anti–symmetric part $v_{[i,j]} = 1/2\,(v_{i,j} - v_{j,i}) =: \omega_{ij}$ (the vorticity tensor), and to separate the symmetric part into a trace–free part (the shear tensor σ_{ij}) and the trace (the rate of expansion) $\theta := v_{i,i}$:

$$v_{i,j} = v_{(i,j)} + v_{[i,j]} =: \theta_{ij} + \omega_{ij} =: \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij} \quad . \tag{21}$$

1.9 Exercise: Lagrangian evolution equations for kinematical variables

Insert the kinematical decomposition of the velocity gradient (21) into Eq. (20b) to obtain Lagrangian evolution equations for the kinematical variables.

Solution: We obtain:

$$\frac{d}{dt}\theta = -\frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) + g_{i,i} \; ; \tag{22a}$$

$$\frac{d}{dt}\omega_{ij} = -\frac{2}{3}\theta\omega_{ij} - \sigma_{ik}\omega_{kj} - \omega_{ik}\sigma_{kj} + g_{[i,j]} \; ; \tag{22b}$$

$$\frac{d}{dt}\sigma_{ij} = -\frac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} - \omega_{ik}\omega_{kj} + \frac{2}{3}(\sigma^2 - \omega^2)\delta_{ij} + g_{(i,j)} - \frac{1}{3}g_{k,k}\delta_{ij} \; , \tag{22c}$$

where $\sigma^2 := 1/2\,\sigma_{ij}\sigma_{ij}$ and $\omega^2 := 1/2\,\omega_{ij}\omega_{ij}$ denote the *rate of shear* and the *rate of vorticity*.

We can read the above equations in the sense that they reconstruct the acceleration gradient $g_{i,j}$ in terms of kinematical variables. More precisely, its trace, its anti–symmetric part, and its trace–free symmetric part. The latter is known as the *Newtonian tidal tensor*:

$$E_{ij} := g_{i,j} - \frac{1}{3}\, g_{k,k}\delta_{ij} \quad . \tag{23}$$

We may now establish a system of equations for the variables ϱ, θ, ω_{ij} and σ_{ij} by using the

Euler–Newton system (16). We arrive at[7]:

$$\frac{d}{dt}\varrho = -\varrho\theta \quad ; \tag{24a}$$

$$\frac{d}{dt}\theta = -\frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) + \Lambda - 4\pi G\varrho \quad ; \tag{24b}$$

$$\frac{d}{dt}\omega_i = -\frac{2}{3}\theta\omega_i + \sigma_{ij}\omega_j \quad ; \tag{24c}$$

$$\frac{d}{dt}\sigma_{ij} = -\frac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} - \omega_{ik}\omega_{kj} + \frac{2}{3}(\sigma^2 - \omega^2)\delta_{ij} + E_{ij} \quad . \tag{24d}$$

This system only constrains the trace and the anti–symmetric part of $(g_{i,j})$, but *not* the tidal tensor E_{ij}. Note here that Newton's theory of gravity is a *vector theory* of the gravitational field; a vector field is, up to harmonic functions, determined by the sources of its curl and its divergence. Hence, the trace–less symmetric part of the field strength gradient needs to be determined only in a tensor theory of gravity.

2 Integral Properties and Effective Dynamics in Newtonian Gravity

We are now going to study the *effective* (spatially averaged) dynamics of observables without the assumption that the system within $\mathcal{D}_t$ is isolated from the environment. We shall, however, exploit an advantage of the Lagrangian description and define, as before, the spatial domain to be restmass–preserving, so that the collection of fluid elements is confined within $\mathcal{D}_t$ during the evolution, and consequently, there is no matter in– or outflow through the boundary $\partial\mathcal{D}_t$. The domain changes its shape while sweeping out a tube in the Galilei–Newton spacetime, as shown in Fig. 1.

2.1 Averaging and evolving observables: the commutation rule

The key–operation to which we subject integral properties of the self–gravitating continuum contained within $\mathcal{D}_t$ is *spatial averaging* of local tensor fields $\mathbf{A}(\mathbf{x},t)$ as measured by their Euclidian volume integral,

$$\langle \mathbf{A} \rangle_{\mathcal{D}_t} := \frac{1}{V_{\mathcal{D}_t}} \int_{\mathcal{D}_t} \mathbf{A} \; d^3x \quad . \tag{25}$$

We then consider their *total change in time*, e.g., for the domain's volume, $V_{\mathcal{D}_t} = \int_{\mathcal{D}_t} d^3x$, we obtain:

$$\frac{d}{dt}\int_{\mathcal{D}_t} d^3x = \frac{d}{dt}\int_{\mathcal{D}_0} J \; d^3X = \int_{\mathcal{D}_0} \frac{dJ}{dt} \; d^3X = \int_{\mathcal{D}_t} \frac{1}{J}\frac{dJ}{dt} \; d^3x = \int_{\mathcal{D}_t} \theta \; d^3x \quad .$$

Hence, the total rate of change of the domain's volume is the averaged rate of expansion:

$$\langle \theta \rangle_{\mathcal{D}_t} = \frac{1}{V_{\mathcal{D}_t}}\frac{dV_{\mathcal{D}_t}}{dt} \quad . \tag{26}$$

[7]We have expressed the vorticity tensor in terms of the vector $\boldsymbol{\omega} = 1/2\,\boldsymbol{\nabla}\times\mathbf{v}$ by means of the formula $\omega_{ij} = -\epsilon_{ijk}\omega_k$.

This quantity is a typical observable within $\mathcal{D}_t$. We are interested in the total change of such quantities in time, e.g., we want to calculate:

$$\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}_t} = \frac{d}{dt}\left(\frac{1}{V_{\mathcal{D}_t}}\right)\int_{\mathcal{D}_t}\theta\; d^3x + \frac{1}{V_{\mathcal{D}_t}}\frac{d}{dt}\int_{\mathcal{D}_t}\theta\; d^3x$$

$$= -\langle\theta\rangle^2_{\mathcal{D}_t} + \frac{1}{V_{\mathcal{D}_t}}\int_{\mathcal{D}_t}(\frac{d}{dt}\theta + \theta^2)\; d^3x \;\;,\;\; \text{hence} \;\;,$$

$$\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}_t} - \langle\frac{d}{dt}\theta\rangle_{\mathcal{D}_t} = \langle\theta^2\rangle_{\mathcal{D}_t} - \langle\theta\rangle^2_{\mathcal{D}_t} = \langle(\theta - \langle\theta\rangle_{\mathcal{D}_t})^2\rangle_{\mathcal{D}_t} \;\;. \tag{27}$$

This equation can be interpreted by saying that the operations of spatially averaging the local field θ and the total evolution in time *do not commute*, their difference being given by the averaged fluctuation of the local field with respect to its spatial average. We can easily prove a more general statement, establishing the *commutation rule*.

2.2 Exercise: commutation rule

Show that, for any tensor field $\mathbf{A}(\mathbf{x},t)$, the following *commutation rule* holds:

$$\frac{d}{dt}\langle\mathbf{A}\rangle_{\mathcal{D}_t} - \langle\frac{d}{dt}\mathbf{A}\rangle_{\mathcal{D}_t} = \langle\mathbf{A}\theta\rangle_{\mathcal{D}_t} - \langle\mathbf{A}\rangle_{\mathcal{D}_t}\langle\theta\rangle_{\mathcal{D}_t} \;\;. \tag{28}$$

Solution: The proof follows easily by employing the transformation to the Lagrangian domain as done for the expansion rate.

The *commutation rule* is a purely kinematical relation. We have not employed any dynamical theory so far. If we consider as dynamical equations the hierarchy of hydrodynamical equations, e.g., the *Euler–Newton system*, we can average the evolution equations for the variables ϱ, θ, etc. Let us confine our study to the two fields ϱ and θ. Setting $\mathbf{A} = \varrho$ in the *commutation rule* we get:

$$\frac{d}{dt}\langle\varrho\rangle_{\mathcal{D}_t} - \langle\frac{d}{dt}\varrho\rangle_{\mathcal{D}_t} = \langle\varrho\theta\rangle_{\mathcal{D}_t} - \langle\varrho\rangle_{\mathcal{D}_t}\langle\theta\rangle_{\mathcal{D}_t} \;\;, \tag{29a}$$

which we can also write as follows:

$$\frac{d}{dt}\langle\varrho\rangle_{\mathcal{D}_t} + \langle\theta\rangle_{\mathcal{D}_t}\langle\varrho\rangle_{\mathcal{D}_t} = \langle\frac{d}{dt}\varrho + \theta\varrho\rangle_{\mathcal{D}_t} \;\;. \tag{29b}$$

If the local continuity equation holds, the right–hand–side of Eq. (29b) vanishes and we obtain the result that also the averaged density obeys a continuity equation.

2.3 Raychaudhuri's equation

Setting $\mathbf{A} = \theta$ we already obtained Eq. (27), which needs a dynamical specification of the temporal change of the local expansion rate. To calculate this we employ the *Euler equation*, which we write explicitly as follows:

$$\frac{\partial}{\partial t}v_i + v_k v_{i,k} = g_i \;\;. \tag{30}$$

Performing the spatial derivative as we did before, we arrive at

$$\frac{d}{dt}v_{i,j} + v_{i,k}v_{k,j} = g_{i,j} \quad . \tag{31}$$

Forming the trace of this equation and employing the field equation $g_{i,i} = \Lambda - 4\pi G\varrho$, we obtain *Raychaudhuri's equation* for the time–evolution of θ:

$$\frac{d}{dt}\theta = \Lambda - 4\pi G\varrho + 2II - \theta^2 \quad . \tag{32}$$

(We have used the second scalar invariant of the velocity gradient, see Subsect. 2.5.)
Inserting the local dynamical equation (32) into the *commutation rule* we obtain the evolution equation for the averaged expansion rate:

$$\frac{d}{dt}\langle\theta\rangle_{\mathcal{D}_t} = \Lambda - 4\pi G\langle\varrho\rangle_{\mathcal{D}_t} + 2\langle II\rangle_{\mathcal{D}_t} - \langle\theta\rangle^2_{\mathcal{D}_t} \quad . \tag{33}$$

It is remarkable that also the spatial averages (expressed in terms of scalar invariants of the velocity gradient) obey *Rauchaudhuri's equation*, although we have pointed out the non–commutativity of averaging and evolution. This property is a consequence of the special nonlinearity (quadratic in θ) featured by *Raychaudhuri's equation*.
Note that the averaged continuity equation together with the averaged Raychaudhuri equation do not provide a closed system of equations. A further equation is needed for the evolution of $\langle II\rangle_{\mathcal{D}_t}$, which invokes another hierarchy of ordinary differential equations that has to be closed.
We are now going to discuss an application of the foregoing formalism.

2.4 Effective cosmological equations and backreaction

Let us introduce, as a measure of the effective expansion of a spatial domain, the scale–factor

$$a_{\mathcal{D}_t} := V_{\mathcal{D}_t}^{1/3} \quad . \tag{34}$$

Eq. (71a) then furnishes an evolution equation for this scale–factor:

$$3\frac{\ddot{a}_{\mathcal{D}_t}}{a_{\mathcal{D}_t}} + 4\pi G\langle\varrho\rangle_{\mathcal{D}_t} - \Lambda = \mathcal{Q}_{\mathcal{D}_t} \quad , \tag{35a}$$

where the source term

$$\mathcal{Q}_{\mathcal{D}_t} := 2\langle II\rangle_{\mathcal{D}_t} - \frac{2}{3}\langle I\rangle^2_{\mathcal{D}_t} \tag{35b}$$

is named *backreaction*, since it measures the departure from the standard Friedmann model described by Eq. (19) due to the influence of inhomogeneities.
Upon integrating the expansion law Eq. (35a), we obtain:

$$3\frac{\dot{a}^2_{\mathcal{D}_t}}{a^2_{\mathcal{D}_t}} + 3\frac{k_{\mathcal{D}_t}}{a^2_{\mathcal{D}_t}} - 8\pi G\langle\varrho\rangle_{\mathcal{D}_t} - \Lambda = \frac{1}{a^2_{\mathcal{D}_t}}\int_{t_0}^{t} dt'\ \mathcal{Q}_{\mathcal{D}_{t'}}\frac{d}{dt'}a^2_{\mathcal{D}_{t'}}(t') \quad , \tag{35c}$$

where $k_{\mathcal{D}_t}$ enters as a domain–dependent integration constant. The effective Hubble–parameter

$$H_{\mathcal{D}_t} := \frac{\dot{a}_{\mathcal{D}_t}}{a_{\mathcal{D}_t}} \tag{35d}$$

is determined by the (now domain–dependent) components known in the standard model of cosmology, and additionally the *backreaction* term, integrated over the whole history of the evolving inhomogeneities.

We are now making explicit that the *backreaction* term consists of surface terms describing fluxes through the domain's boundary. For this end we quote a useful result related to principal scalar invariants.

2.5 Remark: Divergence property of principal scalar invariants

We note the following properties of the *principal scalar invariants* (here written for the invariants of the mean velocity gradient):

$$I = \nabla \cdot \mathbf{v} \quad ; \quad II = \nabla \cdot \Upsilon_{II} \quad ; \quad III = \nabla \cdot \Upsilon_{III} \ , \ \text{with}$$
$$\Upsilon_{II} := \tfrac{1}{2}\left(\mathbf{v}\nabla \cdot \mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v}\right) \ ;$$
$$\Upsilon_{III} := \tfrac{1}{3}\left(\tfrac{1}{2}\nabla\cdot(\mathbf{v}\nabla\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v})\mathbf{v} - (\mathbf{v}\nabla\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v})\cdot\nabla\mathbf{v}\right) \ . \tag{36}$$

For completeness we give their relation in terms of kinematical variables:

$$I = \theta \ ; \ II = \omega^2 - \sigma^2 + \frac{1}{3}\theta^2 \ ; \ III = \frac{1}{27}\theta^3 - \frac{1}{3}\theta(\sigma^2 - \frac{11}{3}\omega^2) + \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} - \frac{1}{3}\sigma_{ij}\omega_i\omega_j \ , \tag{37}$$

where we have introduced the components of the vorticity vector, $(\boldsymbol{\omega})_i = \omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}$. (Note: the velocity gradient has, in general, six independent scalar invariants: θ, σ, ω, $\tau := \frac{1}{6}\sigma_{ij}\sigma_{jk}\sigma_{ki}$, $\sigma_{ij}\omega_i\omega_j$, and $\sigma_{ij}\sigma_{jk}\omega_i\omega_k$.)

Using the latter expressions, $\mathcal{Q}_{\mathcal{D}_t}$ may be written in terms of the kinematical scalars, the expansion rate θ, the rate of shear σ, and the rate of vorticity ω, featuring three positive–definite fluctuation terms:

$$\mathcal{Q}_{\mathcal{D}_t} = \frac{2}{3}\left(\langle\theta^2\rangle_{\mathcal{D}_t} - \langle\theta\rangle^2_{\mathcal{D}_t}\right) + 2\langle\omega^2\rangle_{\mathcal{D}_t} - 2\langle\sigma^2\rangle_{\mathcal{D}_t} \ . \tag{38}$$

The first two of the divergence formulas (36) show that $\mathcal{Q}_{\mathcal{D}_t}$ can be written as a sum of flux integrals (using Gauss' theorem):

$$\mathcal{Q}_{\mathcal{D}_t} = \frac{2}{V_{\mathcal{D}_t}}\int_{\partial\mathcal{D}_t}\Upsilon_{II}\cdot\mathbf{dA} \ - \frac{2}{3V^2_{\mathcal{D}_t}}\left(\int_{\partial\mathcal{D}_t}\mathbf{v}\cdot\mathbf{dA}\right)^2 \ . \tag{39}$$

We conclude that, for an isolated system, $\mathcal{Q}_{\mathcal{D}_t}$ would vanish and the domain would expand like a homogeneous distribution of matter. However, especially for small spatial domains, this is unrealistic.

2.6 Global properties of Newtonian models

There are alternative cases in which these boundary terms vanish. First, one would think of periodic boundary conditions imposed on the mean velocity: for periodic fields we can think of the domain as being a box whose opposite faces are identified. The domain would be topologically equivalent to a 3−torus having no boundary and, hence, surface integrals over the boundary would vanish. The problem here is that, for vanishing $\mathcal{Q}_{\mathcal{D}_t}$, the evolution on the domain would

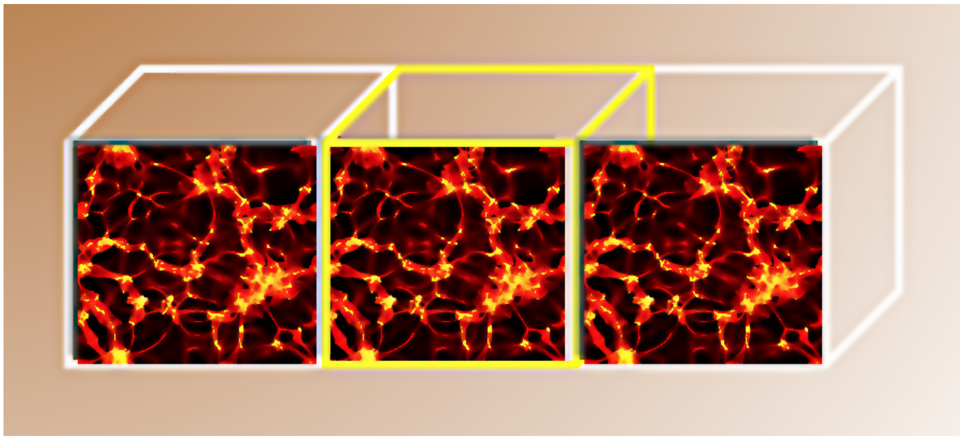

Figure 5: Current simulations of structure formation in the Universe are based on a simple architecture: a box is introduced that follows the expansion of the standard model of cosmology. Structures are described to be periodic on the scale of the box that implies that the average over the matter distribution in the box always vanishes. As a consequence the global evolution of the universe model is not influenced by the structure formation process by construction. The Universe is seen as a replica of identical boxes, justified by the argument that the box should be large enough to describe typical properties of the matter distribution in a statistical sense. The cosmological principle demands that every observer would see the same statistical properties of the matter distribution, which allows extrapolation of the properties seen in a single box. The simulations are Newtonian, i.e. the geometry is Eudlidean in space and remains so.

be described by a *Hubble flow*, i.e., the velocity would be a linear function of $\mathbf{x}$ and cannot be periodic. We can, however, extract a Hubble flow by introducing a *peculiar–velocity* $\mathbf{u}$ as follows:

$$\mathbf{v} =: \mathbf{v}_H + \mathbf{u} \;\; ; \;\; \mathbf{v}_H = H(t)\mathbf{x} \;\; , \tag{40a}$$

where we have split off a standard Hubble flow field $\mathbf{v}_H$ corresponding to a homogeneous–isotropic matter distribution. Now, we can impose periodic boundary conditions on the mean peculiar–velocity field. We now calculate the *backreaction* for this separation ansatz. We first obtain[8]:

$$I(v_{i,j}) = 3H(t) + I(u_{i,j}) \;\; ;$$
$$II(v_{i,j}) = 3H^2(t) + 2H(t)I(u_{i,j}) + II(u_{i,j}) \;\; , \tag{40b}$$

so that the *backreaction* term becomes:

$$\mathcal{Q}_{\mathcal{D}_t} = 2\langle II(u_{i,j})\rangle_{\mathcal{D}_t} - \frac{2}{3}\langle I(u_{i,j})\rangle^2_{\mathcal{D}_t} \;\; . \tag{40c}$$

We see that $\mathcal{Q}_{\mathcal{D}_t}$ does not contain any term due to the Hubble expansion, but is solely determined by the inhomogeneities encoded in the deviations $\mathbf{u}$ from the Hubble field.

Using these results, we may assume that a global Hubble flow exists on some large scale on which periodic boundary conditions can be imposed on $\mathbf{u}$; the Universe is then composed of replica of a periodic box (see Fig. 5), the regional scale–factor describing the global evolution,

[8]Principal scalar invariants refer to the tensor field that follows in brackets.

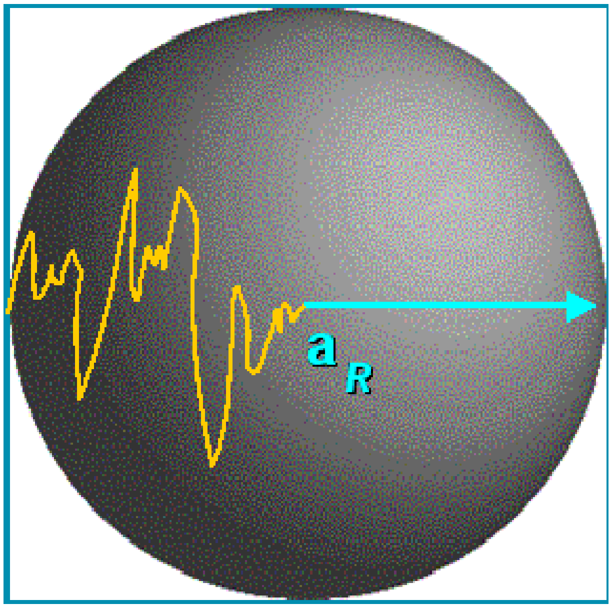

Figure 6: Newton's theorem of the *Iron Spheres* allows that a spherical domain that remains spherical in its time–evolution contains radially symmetric inhomogeneities. The radius of the sphere $a_{\rm R}$ follows the scale–factor of a homogeneous density distribution with the same total mass contained in the inhomogeneous distribution.

$a_{\mathcal{D}_t}(t) \equiv a(t)$, and the global value of *backreaction* vanishes. On every smaller scale, however, the fluctuations in the fields within and outside the domain influence the dynamics, and boundary terms may be important.

A comment about the non–locality of the gravitational evolution of observables calculated within the domain $\mathcal{D}_t$ is in order: suppose we have a solution for the evolution of observables integrated over an arbitrary domain. Although, for given initial data *within* the domain, the evolution of averaged observables is then determined, the influence on the domain from outer regions is included in the solution, since initial data have to be constructed *non–locally* by also taking into account matter outside the domain (Poisson's equation has to be solved on the global scale).

Another case in which $\mathcal{Q}_{\mathcal{D}_t} = 0$ is related to an old theorem proved by Newton (see Fig. 6): let us consider spherical domains. If the dynamics of the fluid preserves the spherical shape of the domain, which is the case for spherically symmetric dynamical fields, then a test particle on the boundary of the sphere would be attracted by the matter inside as if that matter were concentrated in a mass point at the center of the sphere. In other words, one finds that the radius of the sphere obeys Friedmann's differential equation with the parameters (the total restmass etc.) of the spherical region. Here, we do not give Newton's proof of his *theorem of the iron spheres*, but instead show his result within the framework of the generalized Friedmann equations in the following *Exercise*.

2.7 Exercise: Newton's iron spheres

Show that, for spherically symmetric mean velocity field, the *backreaction* term $\mathcal{Q}_{\mathcal{B}_t}$ vanishes on a spherical domain $\mathcal{B}_t$ with radius $r = r(\mathcal{R}, t)$.

Solution: The mean velocity field inside $\mathcal{B}_t$ is only depending on the radial distance r to the origin and is always parallel to the radial unit vector $\mathbf{e}_r$: $v = \mathcal{S}(r)\mathbf{e}_r$ (therefore, we exclude rotational velocity fields). The averaged first principal scalar invariant may be obtained using Gauss' theorem:

$$\langle I \rangle_{\mathcal{B}_t} = \frac{3}{4\pi r^3}\int_{\partial\mathcal{B}_t} \mathcal{S}(r)\mathbf{e}_r \cdot \mathbf{dA} = 3\frac{\mathcal{S}(r(\mathcal{R},t))}{r(\mathcal{R},t)} \quad . \tag{41a}$$

A similar calculation, using the representation of the second principal scalar invariant as a full divergence, leads to:

$$\langle II \rangle_{\mathcal{B}_t} = 3\frac{\mathcal{S}^2(r(\mathcal{R},t))}{r^2(\mathcal{R},t)} = \frac{1}{3}\langle I \rangle^2_{\mathcal{B}_t} \quad , \tag{41b}$$

so that the *backreaction* term vanishes, and $a_{\mathrm{R}}(t)$, *cf.* Fig. **??**, obeys Friedmann's differential equation (18) for the parameters of the spherical mass distribution.

For completeness, note that a calculation of the averaged third scalar invariant results in the following relation:

$$\langle III \rangle_{\mathcal{B}_t} = \frac{1}{27}\langle I \rangle^3_{\mathcal{B}_t} \quad . \tag{41c}$$

2.8 Morphological and statistical interpretation

The expansion law discussed above is built on the rate of change of a simple morphological quantity, the volume content of a domain. Although functionally it depends on other morphological characteristics of a domain, it does not explicitly provide information on their evolution. An evolution equation for the backreaction term $\mathcal{Q}_{\mathcal{D}_t}$ is missing.

We shall, in this subsection, provide a morphological interpretation of $\mathcal{Q}_{\mathcal{D}_t}$ that is possible in the Newtonian framework (the following considerations substantially rely on the Euclidian geometry of space). This will improve our understanding of what $\mathcal{Q}_{\mathcal{D}_t}$ actually measures, if geometry is not considered as a dynamical variable. The dynamical coupling of $\mathcal{Q}_{\mathcal{D}_t}$ to the geometry of space sections in the relativistic case will change this picture.

Let us focus our attention on the boundary of the spatial domain $\mathcal{D}_t$. A priori, the location of this boundary in a non–evolving background space enjoys some freedom which we may constrain by saying that the boundary coincides with a velocity front of the fluid (hereby restricting attention to irrotational flows). This way we employ the Legendrian point of view of velocity fronts that is dual to the Lagrangian one of fluid trajectories. Let $S(x,y,z,t) = s(t)$ define a velocity front at Newtonian time t, $\mathbf{v} = \nabla S$.

Defining the unit normal vector $\mathbf{n}$ on the front, $\mathbf{n} = \pm\nabla S/|\nabla S|$ (the sign depends on whether the domain is expanding or collapsing), the average expansion rate can be written as a flux integral using Gauss' theorem,

$$\langle \Theta \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}_t}}\int_{\mathcal{D}_t} \nabla\cdot\mathbf{v}\; d^3x = \frac{1}{V_{\mathcal{D}_t}}\int_{\partial\mathcal{D}_t} \mathbf{v}\cdot\mathbf{dS} \quad , \tag{42}$$

with the Euclidian volume element d^3x, and the surface element $d\sigma$, $\mathbf{dS} = \mathbf{n}\,d\sigma$. We obtain the intuitive result that the average expansion rate is related to another morphological quantity of the domain, the total area of the enclosing surface:

$$\langle\Theta\rangle_{\mathcal{D}} = \pm\frac{1}{V_{\mathcal{D}_t}}\int_{\partial\mathcal{D}_t}|\nabla S|\,d\sigma \quad . \tag{43}$$

We now use the fact that the principal scalar invariants of the velocity gradient $v_{i,j} =: S_{,ij}$ can be transformed into complete divergences of vector fields (see Subsect. 2.5). (With our assumptions ω in those expressions vanishes identically.)
In obtaining these expressions, the flatness of space is used essentially. Inserting the velocity potential and performing the spatial average, we obtain:

$$\langle II\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}_t}}\int_{\mathcal{D}_t} II\;d^3x = \int_{\partial\mathcal{D}_t} H\,|\nabla S|^2 d\sigma\;; \tag{44}$$

$$\langle III\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}_t}}\int_{\mathcal{D}_t} III\;d^3x = \pm\int_{\partial\mathcal{D}_t} G\,|\nabla S|^3 d\sigma \quad , \tag{45}$$

where H is the local mean curvature and G the local Gaussian curvature at every point on the $2-$surface bounding the domain (see Subsect. 2.9). $|\nabla S| = \frac{ds}{dt}$ equals 1, if the instrinsic arc–length s of the trajectories of fluid elements is used instead of the extrinsic Newtonian time t. The averaged invariants comprise, together with the volume, a complete set of morphological characteristics known as the *Minkowski Functionals* $\mathcal{W}_\alpha$ of a body:

$$\mathcal{W}_0(s) := \int_{\mathcal{D}_t} d^3x = V_{\mathcal{D}_t} \quad ; \quad \mathcal{W}_1(s) := \frac{1}{3}\int_{\partial\mathcal{D}_t} d\sigma \;\; ;$$
$$\mathcal{W}_2(s) := \frac{1}{3}\int_{\partial\mathcal{D}_t} H\;d\sigma \quad ; \quad \mathcal{W}_3(s) := \frac{1}{3}\int_{\partial\mathcal{D}_t} G\;d\sigma = \frac{4\pi}{3}\chi \;\; . \tag{46}$$

The Euler–characteristic χ determines the topology of the domain and is assumed to be an integral of motion ($\chi = 1$), if the domain remains simply–connected.

Thus, we have gained a morphological interpretation of the backreaction term: it can be entirely expressed through three of the four Minkowski Functionals:

$$Q_{\mathcal{D}_t}(s) = 6\left(\frac{\mathcal{W}_2}{\mathcal{W}_0} - \frac{\mathcal{W}_1^2}{\mathcal{W}_0^2}\right) \quad . \tag{47}$$

The $\mathcal{W}_\alpha\;;\;\alpha = 0,1,2,3$ have been introduced into cosmology in order to statistically assess morphological properties of cosmic structure.

For a ball with radius R we have for the Minkowski Functionals:

$$\mathcal{W}_0^{\mathcal{B}_R}(s) := \frac{4\pi}{3}R^3 \;\;;\;\; \mathcal{W}_1^{\mathcal{B}_R}(s) := \frac{4\pi}{3}R^2 \;\;;\;\; \mathcal{W}_2^{\mathcal{B}_R}(s) := \frac{4\pi}{3}R \;\;;\;\; \mathcal{W}_3^{\mathcal{B}_R}(s) := \frac{4\pi}{3} \quad . \tag{48}$$

Inserting these expressions into the backreaction term, Eq. (47), shows that $Q_{\mathcal{D}_t}^{\mathcal{B}_R}(s) = 0$, and we have proved Newton‘s ‘Iron Sphere Theorem’, i.e. the fact that a spherically–symmetric configuration features the expansion properties of a homogeneous–isotropic model. Moreover, we can understand now that the backreaction term encodes the deviations of the domain‘s morphology from that of a ball, a fact that we shall illustrate now with the help of Steiner‘s formula of integral geometry.

Let $d\sigma^0$ be the surface element on the unit sphere, then (according to the Gaussian map) $d\sigma = R_1 R_2 d\sigma^0$ is the surface element of a $2-$surface with radii of curvature R_1 and R_2. Moving the surface a distance ε along its normal we get for the surface element of the parallel velocity front:

$$d\sigma^{\varepsilon} = (R_1+\varepsilon)(R_2+\varepsilon)d\sigma^0 = \frac{R_1R_2+\varepsilon(R_1+R_2)+\varepsilon^2}{R_1R_2}d\sigma = (1+\varepsilon 2H+\varepsilon^2 G)d\sigma \; , \tag{49}$$

where

$$H = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad ; \quad G = \frac{1}{R_1R_2} \quad , \tag{50}$$

are the mean curvature and Gaussian curvature of the front as before.

Integrating Eq. (49) over the whole front we arrive at a relation between the total surface area $A_{\mathcal{D}_t}$ of the front and $A_{\mathcal{D}_{t_\varepsilon}}$ of its parallel front. The gain in volume may then be expressed by an integral of the resulting relation with respect to ε (which is known as *Steiner's formula* defining the Minkowski Functionals of a (convex) body in three spatial dimensions):

$$V_{\mathcal{D}_{t_\varepsilon}} = V_{\mathcal{D}_t} + \int_0^{\varepsilon} d\varepsilon' A_{\mathcal{D}_{t_{\varepsilon'}}} = V_{\mathcal{D}_t} + \varepsilon A_{\mathcal{D}_t} + \varepsilon^2 \int_{\partial\mathcal{D}_t} H \, d\sigma + \frac{1}{3}\varepsilon^3 \int_{\partial\mathcal{D}_t} G \, d\sigma \quad . \tag{51}$$

An important lesson that can be learned here is that the backreaction term $\mathcal{Q}_{\mathcal{D}_t}$ obviously encodes *all* orders of the N–point correlation functions, since the Minkowski Funktionals have this property; it is not merely a two–point term as the form of $\mathcal{Q}_{\mathcal{D}_t}$ as an averaged variance would suggest. In other words, a complete measurement of fluctuations must take into account that the domain is Lagrangian and the shape of the domain is an essential expression of the full N–point statistics of the matter enclosed within $\mathcal{D}_t$. Kinematically, Steiner's formula shows that the volume scale factor $a_{\mathcal{D}_t}$, being defined through the volume, also depends on other morphological properties of $\mathcal{D}_t$ in the course of evolution. In a comoving relativistic setting, the domain $\mathcal{D}_t$ is frozen into the metric of spatial sections, so that we also understand that an evolving geometry in general relativity takes the role of this shape–dependence in the Newtonian framework.

2.9 Excursion: curvature invariants of surfaces

Let $S(x,y,z;t) = s(t)$ define velocity fronts at Newtonian time t as in the main text. In what follows we consider a fixed instant of time and we shall derive explicit expressions for the mean and Gaussian curvatures of the front, which in differential geometry text books are usually given for surfaces in the form $z = \chi(x,y)$.
At points $P = (x_0, y_0, z_0)$, where the representation of the front in terms of χ is nonsingular, $\nabla S \neq \mathbf{0}$, we have for the mean curvature H and the Gaussian curvature G (here, indexed letters denote partial derivatives with respect to the coordinates):

$$2H := \frac{(1+\chi_x^2)\chi_{yy} - 2\chi_x\chi_y\chi_{xy} + (1+\chi_y^2)\chi_{xx}}{(1+\chi_x^2+\chi_y^2)^{3/2}} \quad ; \tag{52}$$

$$G := \frac{\chi_{xx}\chi_{yy} - \chi_{xy}^2}{(1+\chi_x^2+\chi_y^2)^2} \quad . \tag{53}$$

Using the implicit definition $S(x,y,\chi(x,y)) = s$ of the velocity front, we calculate the derivatives of χ:

$$
\begin{aligned}
\chi_x = & -\frac{S_x}{S_z} \quad , \qquad \chi_y = -\frac{S_y}{S_z} \quad , \\
\chi_{xx} = & -\frac{S_{xx}}{S_z} + 2\frac{S_x S_{xz}}{S_z^2} - \frac{S_x^2 S_{zz}}{S_z^3} \quad , \\
\chi_{xy} = \chi_{yx} = & -\frac{S_{xy}}{S_z} + \frac{S_x S_{yz}}{S_z^2} + \frac{S_y S_{xz}}{S_z^2} - \frac{S_x S_y S_{zz}}{S_z^3} \quad , \\
\chi_{yy} = & -\frac{S_{yy}}{S_z} + 2\frac{S_y S_{yz}}{S_z^2} - \frac{S_y^2 S_{zz}}{S_z^3} \quad ,
\end{aligned}
\tag{54}
$$

and obtain for the curvature invariants of the front:

$$
\begin{aligned}
2H = \frac{1}{|\nabla S|^3} \Big[& 2S_x S_y S_{xy} + 2S_x S_z S_{xz} + 2S_y S_z S_{yz} - S_{xx}(S_y^2 + S_z^2) \\
& - S_{yy}(S_x^2 + S_z^2) - S_{zz}(S_x^2 + S_y^2) \Big] \quad ;
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
G = \frac{1}{|\nabla S|^4} \Big[& S_x^2 (S_{yy} S_{zz} - S_{yz}^2) + S_y^2 (S_{xx} S_{zz} - S_{xz}^2) + S_z^2 (S_{xx} S_{yy} - S_{xy}^2) \\
& - 2S_x S_y (S_{xy} S_{zz} - S_{xz} S_{yz}) - 2S_x S_z (S_{xz} S_{yy} - S_{xy} S_{yz}) \\
& - 2S_y S_z (S_{yz} S_{xx} - S_{xy} S_{xz}) \Big] \ .
\end{aligned}
\tag{56}
$$

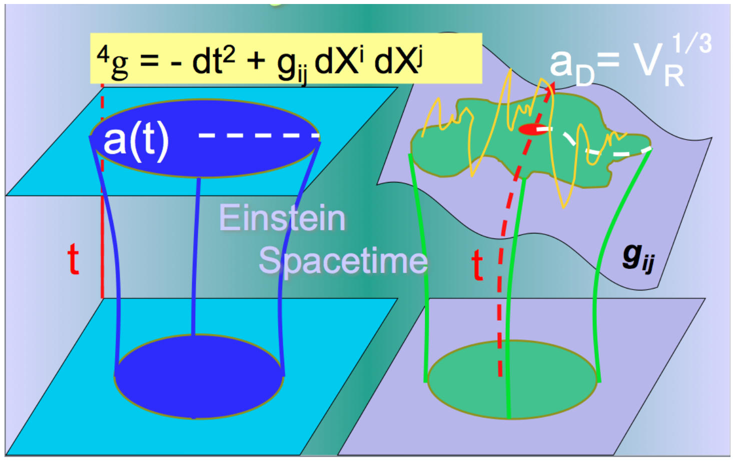

Figure 7: Two simply–connected spatial domains are drawn in the Einstein–Riemann spacetime. Starting with everywhere Euclidian space sections and spherical domains, the evolution on the left displays a homogeneous–isotropic distribution on a flat space section that leaves the geometry (conformally) Euclidian and the domain spherical. The scale factor $a(t)$ (the geodesic radius of the sphere depicted as a dashed line) obeys Friedmann's differential equation (18) for $k = 0$, since such an evolution corresponds to the cosmological models of Friedmann, Eddington and Lemaître, introduced previously. On the right, both the geometry of the space section (represented by the 3–metric g_{ij}) and the morphology of the domain change in the course of evolution. The picture is tentatively drawn in such a way that the direction of 'time' is orthogonal to the (three–dimensional) hypersurfaces of constant time, and the domain's radius (the dashed line) follows geodesics within the 3–geometry. We shall encounter this situation when the fluid continuum is identified with the space continuum, and fluid elements are all at rest in the so constructed spacetime.

3 Einstein Gravity

In general relativity we may consider a very close analogy to the Newtonian picture of the previous sections: we may introduce spatial (local) coordinates that are attached to each fluid element, so that this latter is *at rest* in this coordinate system; with this choice of coordinates we guarantee that the evolved fluid elements are again *at rest* in the same coordinate system at the evolved time. The realization of this picture, as shown in Fig. 7, is a set of four local coordinates $(\mathbf{X}, t)$ (taken to be a local patch of *Minkowski spacetime*, made up of a local *Lagrangian coordinate system* $\mathbf{X}$ that stays attached to the fluid elements in the course of evolution, and the *eigentime* of fluid elements, i.e. the normalized length of the fluid element's *worldline*[10] that connects two fluid elements (identified by the same coordinate label). Putting this length to an equal value for all the fluid elements defines a *foliation of a spacetime* that entirely consists of a four–dimensional *tube* of the simply–connected fluid domains; in other words, the fluid's evolution itself defines the foliation of our spacetime. Upon quantifying the geometry of this tube by *Riemannian geometry* we may speak about an *Einstein–Riemann spacetime*. In the framework of general relativity the so construed intrinsic coordinate system of the spacetime foliation is called *comoving synchronous coordinate system*. Viewing the fluid evolution in this way, we can

[10]The term *trajectory* was used previously; here we use *worldline* to express the corresponding notion in the relativistic setting.

also say that fluid elements are *free–falling* in their own gravitational field, tracing geodesics in spacetime.

We are now going to have a look at Einstein's equations. The following *Excursion* presents the $3{+}1$ form of Einstein's equations, written for the special matter model *irrotational dust*. Note that the irrotationality is a restriction required for our construction of spacetime (the expansion tensor is defined as a temporal change of the metric, and has therefore to be symmetric.

3.1 Excursion: Einstein's equations for irrotational dust in $3+1$ form

Einstein's equations read:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R + \frac{\Lambda}{c^2}\, g_{\mu\nu} \;=\; \frac{8\pi G}{c^4}\, T_{\mu\nu} \quad , \tag{57a}$$

with the 4–metric $g_{\mu\nu}$, the 4–Ricci tensor $R_{\mu\nu}$, describing the *intrinsic curvature* of spacetime, the cosmological constant Λ, and the *energy–momentum tensor* for a pressure–less continuum (*dust*),

$$T^{\rm dust}_{\mu\nu} := \varrho\, u_\mu\, u_\nu \quad , \tag{57b}$$

where the 4–velocity u^μ is normalized,

$$u^\mu\, u_\mu \;=\; -c^2 \quad . \tag{57c}$$

Substituting the trace of Einstein's equations, $R = R^{\alpha}_{\;\;\alpha} = -8\pi G T/c^4 + 4\Lambda/c^2$; $T = T^{\alpha}_{\;\;\alpha} = -c^2\varrho$, again into Einstein's equations yields the compact relation between the intrinsic curvature tensor and the energy density source $c^2\varrho$:

$$c^2\, R_{\mu\nu} = \frac{8\pi G\varrho}{c^2}\left(u_\mu\, u_\nu + \frac{c^2}{2}\, g_{\mu\nu}\right) + \Lambda\, g_{\mu\nu} \quad . \tag{57d}$$

In the spacetime setting that we thus far developed the 4–metric and 4–velocity have a simple form:

$$g_{\mu\nu} = {\rm diag}(-c^2\, ,\, g_{ij}) \;\; ; \;\; u_\mu = (-c^2, 0, 0, 0) \;\; , \;\; u^\mu = (1, 0, 0, 0)\;, \tag{58a}$$

i.e., the 4–velocity is chosen to be normal to the hypersurfaces of constant time with 3–metric g_{ij}; the four–dimensional line–element in local coordinates simplifies to:

$$ds^2 \;=\; -c^2\, dt^2 + g_{ij}\, dX^i\, dX^j \quad . \tag{58b}$$

Given this particular split into space and time (that is only possible for *irrotational dust*), we can compute the components of the symmetric 4–Ricci tensor:

$$R_{00} = 4\pi G\varrho - \Lambda \;\; ; \;\; R_{0i} = 0 \;\; ; \;\; c^2\, R_{ij} = 4\pi G\varrho\, g_{ij} + \Lambda\, g_{ij} \quad . \tag{59}$$

The above components of the 4–Ricci tensor can be expressed geometrically through the three–dimensional quantitites *extrinsic curvature* K_{ij} and *instrinsic curvature* $\mathcal{R}_{ij}$ of the hypersurfaces at constant t as follows:

$$R_{00} = \dot{K} - K^{ij}\,K_{ij} \;\; ; \;\; R_{0i} = K_{|i} - K^k_{\;i||k} \;\; ; \;\; c^2\,R_{ij} = c^2\,\mathcal{R}_{ij} - \dot{K}_{ij} - 2K_{ik}K^k_{\;j} + K K_{ij} \;\; , \tag{60}$$

where a vertical slash denotes partial spatial derivative with respect to X^i, and a double vertical slash *covariant spatial differentiation* with respect to the 3−metric. Thus, we end up with the following set of equations (supplemented by the *continuity equation* and the defining equation for the extrinsic curvature:

$$\dot{\varrho} = K\varrho \;\; ; \tag{61a}$$
$$\dot{g}_{ij} = -2\,K_{ij} \;\; ; \tag{61b}$$
$$-\dot{K} + K^{ij}\,K_{ij} \; = \; \Lambda - 4\pi G\varrho \;\; ; \tag{61c}$$
$$K_{|i} - K^k_{\;i||k} \; = \; 0 \;\; ; \tag{61d}$$
$$-\dot{K}_{ij} + K K_{ij} - 2K_{ik}K^k_{\;j} \; = \; -c^2\,\mathcal{R}_{ij} + (\,4\pi G\varrho + \Lambda\,)\,g_{ij} \;\; . \tag{61e}$$

The third equation is *Raychaudhuri's equation*, the fourth set of equations are called *momentum constraints*. Forming the trace of the last equation and inserting it into Raychaudhuri's equation, we obtain *Hamilton's constraint*:

$$c^2\,\mathcal{R} + K^2 - K^i_{\;j}\,K^j_{\;i} \; = \; 16\pi G\varrho + 2\Lambda \;\; . \tag{62}$$

Since the 3–Ricci tensor is determined by the 3–metric and its spatial derivatives, the system of equations (61) is closed. A common representation of this system is provided by considering six second–order *evolution equations* for the metric (obtained by inserting the second equation into the last one), and four *constraint equations* consisting of the three equations of the *momentum constraints* and *Hamilton's constraint*. For convenience and for the purpose of comparing with variables that correspond to the Newtonian theory, we write down the above system with upper and lower indices and replace the extrinsic curvature K_{ij} by the expansion tensor $-\Theta_{ij}$; also note: $\dot{g}_{ij} = -2K_{ij}$, but $\dot{g}^{ij} = +2K^{ij}$, and $\dot{K}^i_{\;j} - 2K^i_{\;k}K^k_{\;j} = g^{mi}\dot{K}_{mj}$:

$$\dot{\varrho} = -\theta\varrho \;\; ; \tag{63a}$$
$$\dot{g}_{ij} = 2\,g_{ik}\Theta^k_{\;j} \;\; ; \tag{63b}$$
$$\Theta_{[ij]} = 0 \;\; ; \tag{63c}$$
$$\dot{\theta} = -\Theta^i_{\;j}\,\Theta^j_{\;i} + \Lambda - 4\pi G\varrho \;\; ; \tag{63d}$$
$$\dot{\Theta}^i_{\;j} = -c^2\,\mathcal{R}^i_{\;j} - \theta\Theta^i_{\;j} + (\,4\pi G\varrho + \Lambda\,)\,\delta^i_{\;j} \;\; . \tag{63e}$$

$$\theta_{|i} - \Theta^k_{\;i||k} \; = \; 0 \;\; ; \tag{63f}$$
$$c^2\,\mathcal{R} + \theta^2 - \Theta^i_{\;j}\,\Theta^j_{\;i} \; = \; 16\pi G\varrho + 2\Lambda \;\; . \tag{63g}$$

As an example we have a look at the relativistic homogeneous–isotropic models of cosmology.

3.2 Exercise: relativistic form of Friedmann's differential equation

Show that *Hamilton's constraint*, Eq. (62), furnishes the relativistic form of Friedmann's differential equation (18) in the case of a homogeneous–isotropic matter distribution.

Solution: Indeed, putting $\varrho = \varrho_H(t)$ and expressing the second scalar invariant of the extrinsic curvature through kinematical invariants (recall that vorticity has to vanish in the present setting), $II := 1/2(K^2 - K^i_{\ j}\,K^j_{\ i}) = 1/3\theta^2 - \sigma^2$, then for $\sigma = 0$ we have:

$$c^2\,\mathcal{R}_H + \frac{2}{3}\theta_H^2 \;=\; 16\pi G\varrho_H + 2\Lambda \ , \tag{64}$$

which is Friedmann's differential equation (18) by defining $\theta_H = 3H(t) = 3\dot{a}(t)/a(t)$, and employing the expression for a *constant curvature space* $c^2\mathcal{R}_H = 6k/a(t)^2$.

Note that a homogeneous–isotropic deformation $\eta^a = a(t)\eta^a(t_0)$ implies that the spatial metric is *conformal* to a time–independent constant curvature metric G_{ij}: $g_{ij} = \delta_{ab}\,\eta^a_{\ i}\eta^b_{\ j} = a^2(t)\,G_{ij}$ with $G_{ij} = \delta_{ab}\,\eta^a_{\ i}(t_0)\eta^b_{\ j}(t_0)$.

We are now going to spatially average the scalar parts of Einstein's equations.

4 Integral Properties and Effective Dynamics in Einstein Gravity

Finding solutions to Einstein's equations for generic initial conditions requires approximations or numerical integration. In relativistic cosmology we would be happy to know some global properties for generic initial data, e.g. the evolution of the cosmological parameters governing solutions of Einstein's equations *on average*. We are now going to develop a formalism that integrates the scalar parts of Einstein's equations and so provides a framework to discuss scalar average characteristics of an inhomogeneous cosmology. This provides an example that shows, how we can easily work with Einstein's equations based on our Newtonian knowledge.

4.1 Averaging the scalar parts of Einstein's equations

For the purpose of averaging we shall consider a compact and simply–connected domain contained within spatial hypersurfaces that are specified below. This domain will be followed along the flow lines of the fluid elements; thus we require that the total restmass of the fluid within the domain be conserved. We shall denote this domain by $\mathcal{D}_g$ or, for notational ease[23], $\mathcal{D}$.

We define the averaging operation in terms of Riemannian volume integration, restricting attention to scalar functions $\Psi(X^i,t)$,

$$\langle\Psi(X^i,t)\rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}}\Psi(X^i,t)\;\sqrt{g}d^3X \quad , \tag{65}$$

[23] Compared with the previous Newtonian investigation, this domain corresponds to a Lagrangian domain; it is now implicitly time–dependent due to the evolution of the 3–metric. We may say that the domain is 'frozen' into the metric.

with the Riemannian volume element $d\mu_g := \sqrt{g} d^3X$, $g := \det(g_{ij})$, and the volume of an arbitrary compact domain, $V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \sqrt{g} d^3X$; X^i are coordinates in a $t = \mathrm{const.}$ hypersurface (with $3-$metric g_{ij}) that are comoving with fluid elements of dust (i.e. they are constant along spacetime geodesics here; note that we henceforth set $c = 1$):

$$ds^2 = -dt^2 + g_{ij} dX^i dX^j \quad . \tag{66}$$

It is evident from the above setting that we predefine a simple time–orthogonal foliation, which restricts the matter to an irrotational dust continuum. (The formalism below can be carried over to more general settings.)

As in the Newtonian case we define an *effective scale factor* by the volume of a simply–connected domain $\mathcal{D}$ in a t–hypersurface, here normalized by the volume of the initial domain $\mathcal{D}_{\mathbf{i}}$,

$$a_{\mathcal{D}_t} := \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}_{\mathrm{i}}}} \right)^{1/3} \quad . \tag{67}$$

For a restmass preserving domain $\mathcal{D}$, volume averaging of a scalar function Ψ does not commute with its time–evolution (you can easily proof this along the lines of the Newtonian proof by setting $J = \sqrt{g}$ and noting the identity $\dot{J} = \theta J$):

$$\dot{\langle \Psi \rangle}_{\mathcal{D}} - \langle \dot{\Psi} \rangle_{\mathcal{D}} = \langle \Psi \theta \rangle_{\mathcal{D}} - \langle \Psi \rangle_{\mathcal{D}} \langle \theta \rangle_{\mathcal{D}} = \langle \Psi \delta\theta \rangle_{\mathcal{D}} = \langle \theta \delta\Psi \rangle_{\mathcal{D}} = \langle \delta\Psi \delta\theta \rangle_{\mathcal{D}} \quad , \tag{68}$$

where θ denotes the local expansion rate (as minus the trace of the extrinsic curvature of the hypersurfaces $t = \mathrm{const.}$). We have rewritten the right–hand–side of the first equality in terms of the deviations of the local fields from their spatial averages, $\delta\Psi := \Psi - \langle \Psi \rangle_{\mathcal{D}}$ and $\delta\theta := \theta - \langle \theta \rangle_{\mathcal{D}}$.

The key–statement of the *commutation rule* (68) is that the operations *spatial averaging* and *time evolution* do not commute. In cosmology we may think of initial conditions at the epoch of last scattering, when the fluctuations imprinted on the Cosmic Microwave Background are considered to be averaged–out on a restframe of a standard Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology. In this picture the evolution of the Universe is described by first averaging–out (or ignoring) inhomogeneities and then evolving the average distribution by a homogeneous (in the above case homogeneous–isotropic) universe model. A realistic model would first evolve the inhomogeneous fields and, at the present epoch, the resulting fields would have to be evaluated by spatial averaging to obtain the final values of, e.g., the averaged density field. In particular, this comment applies to all cosmological parameters.

4.2 Effective relativistic cosmologies

First, note that for the averaged expansion rate $\langle \theta \rangle_{\mathcal{D}}$ we have as before:

$$\langle \theta \rangle_{\mathcal{D}} = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} =: 3H_{\mathcal{D}} \quad . \tag{69}$$

The latter equality demonstrates that this quantity may be regarded as an *effective Hubble function*.

We are now interested to find an evolution equation for the averaged expansion of a general inhomogeneous model. For this purpose we can exploit the *commutation rule* (68) by setting $\Psi \equiv \theta$. For the spatially averaged expansion, Eq. (69) we first obtain:

$$\dot{\langle \theta \rangle}_{\mathcal{D}} - \langle \dot{\theta} \rangle_{\mathcal{D}} = \langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2 = \langle (\delta\theta)^2 \rangle_{\mathcal{D}} \quad . \tag{70}$$

Inserting *Raychaudhuri's evolution equation*, $\dot{\theta} = \Lambda - 4\pi G\varrho - \frac{1}{3}\theta^2 - 2\sigma^2$ (with the rate of shear $\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}$), into (70) and using the effective scale–factor $a_{\mathcal{D}}$ we obtain:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{M_{\mathcal{D}}}{V_{\mathbf{i}}a_{\mathcal{D}}^3} - \Lambda = \mathcal{Q}_{\mathcal{D}} \ . \tag{71a}$$

The first integral of the above equation is directly given by averaging the *Hamiltonian constraint*:

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - \frac{8\pi G}{3}\frac{M_{\mathcal{D}}}{V_{\mathbf{i}}a_{\mathcal{D}}^3} + \frac{\langle R\rangle_{\mathcal{D}}}{6} - \frac{\Lambda}{3} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6} \ , \tag{71b}$$

where the total restmass $M_{\mathcal{D}}$, the averaged spatial Ricci scalar $\langle R\rangle_{\mathcal{D}}$ and the *kinematical back-reaction term* $\mathcal{Q}_{\mathcal{D}}$ are domain–dependent and, except the mass, time–dependent functions. The backreaction source term is given by

$$\mathcal{Q}_{\mathcal{D}} := 2\langle II\rangle_{\mathcal{D}} - \frac{2}{3}\langle I\rangle_{\mathcal{D}}^2 = \frac{2}{3}\langle(\theta - \langle\theta\rangle_{\mathcal{D}})^2\rangle_{\mathcal{D}} - 2\langle\sigma^2\rangle_{\mathcal{D}} \ ; \tag{71c}$$

here, $I = \Theta^i_{\ i}$ and $II = \frac{1}{2}[(\Theta^i_{\ i})^2 - \Theta^i_{\ j}\Theta^j_{\ i}]$ denote the principal scalar invariants of the expansion tensor, defined as minus the extrinsic curvature tensor $K_{ij} := \Theta_{ij}$. In the second equality above it was split into kinematical invariants through the decomposition $\Theta_{ij} = \frac{1}{3}g_{ij}\theta + \sigma_{ij}$, with the rate of expansion $\theta = \Theta^i_{\ i}$, and the shear tensor σ_{ij}. (Note that vorticity is absent in the present model, unlike the situation in the corresponding Newtonian model.)

The time–derivative of the averaged Hamiltonian constraint (71b) agrees with the averaged Raychaudhuri equation (71a) by virtue of the following *integrability condition*:

$$\partial_t\mathcal{Q}_{\mathcal{D}} + 6\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\mathcal{Q}_{\mathcal{D}} + \partial_t\langle R\rangle_{\mathcal{D}} + 2\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle R\rangle_{\mathcal{D}} = 0 \ , \tag{72a}$$

which we may write in the more compact form:

$$\frac{1}{a_{\mathcal{D}}^6}\partial_t\left(\mathcal{Q}_{\mathcal{D}}a_{\mathcal{D}}^6\right) + \frac{1}{a_{\mathcal{D}}^2}\partial_t\left(\langle R\rangle_{\mathcal{D}}a_{\mathcal{D}}^2\right) = 0 \ . \tag{72b}$$

Formally integrating this condition yields:

$$\frac{k_{\mathcal{D}_{\mathbf{i}}}}{a_{\mathcal{D}}^2} - \frac{1}{3a_{\mathcal{D}}^2}\int_{t_i}^t dt'\ \mathcal{Q}_{\mathcal{D}}\frac{d}{dt'}a_{\mathcal{D}}^2(t') = \frac{1}{6}\left(\langle R\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}\right) \ , \tag{72c}$$

i.e., besides the total material mass $M_{\mathcal{D}}$ we have a further integral of motion given by the domain–dependent integration constant $k_{\mathcal{D}_{\mathbf{i}}}$.

Eq. (72b), having no Newtonian analogue, shows that the averaged intrinsic curvature and the averaged extrinsic curvature (encoded in the backreaction term) are dynamically coupled. Stating this genuinly relativistic property, we also note the surprising fact that, inserting (72c) into (71b) results in an equation that is formally equivalent to its Newtonian counterpart, Eq. (35c):

$$\frac{\dot{a}_{\mathcal{D}}^2 + k_{\mathcal{D}_{\mathbf{i}}}}{a_{\mathcal{D}}^2} - \frac{8\pi G\langle\varrho\rangle_{\mathcal{D}}}{3} - \frac{\Lambda}{3} = \frac{1}{3a_{\mathcal{D}}^2}\int_{t_{\mathbf{i}}}^t dt'\ \mathcal{Q}_{\mathcal{D}}\frac{d}{dt'}a_{\mathcal{D}}^2(t') \ . \tag{73}$$

The effective scale–factor obeys the same equation as in Newtonian theory similar to the situation known for the homogeneous–isotropic case. Note that these effective equations also cover anisotropic inhomogeneous cosmologies.

4.3 The production of information in the Universe

The above considerations on effective expansion properties can be essentially traced back to the non–commutativity of averaging and time–evolution, lying at the root of backreaction. The same reasoning underlies the following entropy argument. Applying the *commutation rule* (68) to the density field, $\Psi = \varrho$,

$$\langle \partial_t \varrho \rangle_{\mathcal{D}} - \partial_t \langle \varrho \rangle_{\mathcal{D}} \; = \; \frac{\partial_t \, S\{\varrho || \langle \varrho \rangle_{\mathcal{D}}\}}{V_{\mathcal{D}}} \;\; , \tag{74}$$

we derive, as a source of non–commutativity, the (for positive–definite density) positive–definite Lyapunov functional (known as *Kullback–Leibler functional* in information theory):

$$S\{\varrho || \langle \varrho \rangle_{\mathcal{D}}\} : = \int_{\mathcal{D}} \varrho \ln \frac{\varrho}{\langle \varrho \rangle_{\mathcal{D}}} \, J d^3 X \;\; ; \;\; J \equiv \sqrt{g} \;\; . \tag{75}$$

This measure vanishes for Friedmannian cosmologies ('zero structure'). It attains some *positive* time–dependent value otherwise. The source in (74) shows that relative entropy production and volume evolution are competing: commutativity can be reached, if the volume expansion is faster than the production of information contained within the same volume.

Generally, information entropy is produced, i.e. $\partial_t \, S > 0$ with

$$\frac{\partial_t \, S\{\varrho || \langle \varrho \rangle_{\mathcal{D}}\}}{V_{\mathcal{D}}} = - \langle \delta\varrho \Theta \rangle_{\mathcal{D}} = - \langle \varrho \delta \Theta \rangle_{\mathcal{D}} = - \langle \delta\varrho \delta \Theta \rangle_{\mathcal{D}} \;\; , \tag{76}$$

(and with the deviations of the local fields from their average values, e.g. $\delta\varrho := \varrho - \langle \varrho \rangle_{\mathcal{D}}$), if the domain $\mathcal{D}$ contains more *expanding underdense* and *contracting overdense* regions than the opposite states *contracting underdense* and *expanding overdense* regions. The former states are clearly favoured in the course of evolution, as can be seen in simulations of large–scale structure.

There are essentially three lessons relevant to the origin of backreaction that can be learned here. First, structure formation (or 'information' contained in structures) installs a positive–definite functional as a potential to increase the deviations from commutativity; it can therefore not be statistically 'averaged away' (the same remark applies to the averaged variance of the expansion rate discussed before). Second, gravitational instability acts in the form of a negative feedback that enhances structure (or 'information'), i.e. it favours contracting clusters and expanding voids. This tendency is opposite to the thermodynamical interpretation within a closed system where such a relative entropy would decrease and the system would tend to thermodynamical equilibrium. This is a result of the long–ranged nature of gravitation: the system contained within $\mathcal{D}$ must be treated as an open system. Third, backreaction is a genuinely non–equilibrium phenomenon, thus, opening this subject also to the language of non–equilibrium thermodynamics, general questions of gravitational entropy, and observational measures using distances to equilibrium. 'Near–equilibrium' can only be maintained (not established) by a simultaneous strong volume expansion of the system.

In particular, we conclude that the standard model may be a good description for the averaged variables only when information entropy production is *over*–compensated by volume expansion (measured in terms of a corresponding adimensional quantity). This latter property is realized by linear perturbations at a FLRW background. Thus, the question is whether this remains true in the nonlinear regime, where information production is strongly promoted by structure formation and expected to be more efficient. Particular exact nonlinear solutions of general relativity show that the average model indeed deviates from the FLRW model.

4.4 Effective Friedmannian framework

We may also recast the general averaged equations by appealing to the Friedmannian framework. This amounts to re–interpret geometrical terms, that arise through averaging, as effective *sources* within a Friedmannian setting.

In the present case the averaged equations may be written as standard zero–curvature Friedmann equations for an *effective perfect fluid energy momentum tensor* with new effective sources:

$$\varrho^{\mathcal{D}}_{\rm eff} = \langle\varrho\rangle_{\mathcal{D}} - \frac{1}{16\pi G}\mathcal{Q}_{\mathcal{D}} - \frac{1}{16\pi G}\langle R\rangle_{\mathcal{D}} \;\; ; \qquad p^{\mathcal{D}}_{\rm eff} = -\frac{1}{16\pi G}\mathcal{Q}_{\mathcal{D}} + \frac{1}{48\pi G}\langle R\rangle_{\mathcal{D}} \;\; . \tag{77}$$

$$3\left(\frac{\dot a_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - 8\pi G\varrho^{\mathcal{D}}_{\rm eff} - \Lambda \;=\; 0 \;\; ;$$
$$3\frac{\ddot a_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G(\varrho^{\mathcal{D}}_{\rm eff} + 3p^{\mathcal{D}}_{\rm eff}) - \Lambda \;=\; 0 \;\; ;$$
$$\dot\varrho^{\mathcal{D}}_{\rm eff} + 3\frac{\dot a_{\mathcal{D}}}{a_{\mathcal{D}}}\left(\varrho^{\mathcal{D}}_{\rm eff} + p^{\mathcal{D}}_{\rm eff}\right) \;=\; 0 \;\; . \tag{78}$$

We notice that $\mathcal{Q}_{\mathcal{D}}$, if interpreted as a source, introduces a component with 'stiff equation of state', $p^{\mathcal{D}}_{Q} = \varrho^{\mathcal{D}}_{Q}$, suggesting a correspondence with a free scalar field (discussed in the next subsection), while the averaged scalar curvature introduces a component with 'curvature equation of state' $p^{\mathcal{D}}_{R} = -1/3\varrho^{\mathcal{D}}_{R}$. Although we are dealing with dust matter, we appreciate a 'geometrical pressure' in the effective energy–momentum tensor.

4.5 Relation to scalar field theories

In the above–introduced framework we distinguish the averaged matter source on the one hand, and averaged sources due to geometrical inhomogeneities stemming from extrinsic and intrinsic curvature (kinematical backreaction terms) on the other. As shown above, the averaged equations can be written as standard Friedmann equations that are sourced by both. Thus, we have the choice to consider the averaged model as a (scale–dependent) 'standard model' with matter source evolving in a *mean field* of backreaction terms. This form of the equations is closest to the standard model of cosmology. It is a 'morphed' Friedmann cosmology, sourced by matter and 'morphed' by a (minimally coupled) scalar field, the *morphon field*[1]. We write (recall that we have no matter pressure source here):

$$\varrho^{\mathcal{D}}_{\rm eff} =: \langle\varrho\rangle_{\mathcal{D}} + \varrho^{\mathcal{D}}_{\Phi} \quad ; \quad p^{\mathcal{D}}_{\rm eff} =: p^{\mathcal{D}}_{\Phi} \;\; , \tag{79}$$

with

$$\varrho^{\mathcal{D}}_{\Phi} = \epsilon\frac{1}{2}\dot\Phi^2_{\mathcal{D}} + U_{\mathcal{D}} \quad ; \quad p^{\mathcal{D}}_{\Phi} = \epsilon\frac{1}{2}\dot\Phi^2_{\mathcal{D}} - U_{\mathcal{D}} \;\; , \tag{80}$$

where $\epsilon = +1$ for a standard scalar field (with positive kinetic energy), and $\epsilon = -1$ for a phantom scalar field (with negative kinetic energy)[2]. Thus, in view of Eq. (77), we obtain the

[1] This name is motivated by our previous morphological interpretation of the backreaction term.

[2] We have chosen the letter U for the potential to avoid confusion with the volume functional; if ϵ is negative, a 'ghost' can formally arise on the level of an effective scalar field, although the underlying theory does not contain one.

following correspondence:

$$-\frac{1}{8\pi G}\mathcal{Q}_{\mathcal{D}} \;=\; \epsilon\dot{\Phi}^2_{\mathcal{D}} - U_{\mathcal{D}} \quad ; \quad -\frac{1}{8\pi G}\langle R\rangle_{\mathcal{D}} = 3U_{\mathcal{D}} \;\; . \tag{81}$$

Inserting (81) into the integrability condition (72b) then implies that $\Phi_{\mathcal{D}}$, for $\dot{\Phi}_{\mathcal{D}} \neq 0$, obeys the (scale–dependent) Klein–Gordon equation[3]:

$$\ddot{\Phi}_{\mathcal{D}} + 3H_{\mathcal{D}}\dot{\Phi}_{\mathcal{D}} + \epsilon\frac{\partial}{\partial\Phi_{\mathcal{D}}}U(\Phi_{\mathcal{D}},\langle\varrho\rangle_{\mathcal{D}}) \;=\; 0 \;\; . \tag{82}$$

The above correspondence allows us to interpret the kinematical backreaction effects in terms of properties of scalar field cosmologies, notably quintessence or phantom–quintessence scenarii that are here routed back to models of inhomogeneities. *Dark Energy emerges as unbalanced kinetic and potential energies due to structural inhomogeneities*[4].

4.6 Examples of inhomogeneous cosmologies

In this subsection we give some examples of inhomogeneous relativistic cosmologies that qualitatively allow to trace the problems of Dark Energy, Dark Matter but also Inflation back to inhomogeneous geometrical spatial properties.

4.6.1 Cosmic equation of state and Dark Energy equation of state

We can characterize a solution of the averaged equations by a *cosmic equation of state* $p^{\mathcal{D}}_{\rm eff} = \beta(\varrho^{\mathcal{D}}_{\rm eff}, a_{\mathcal{D}})$ with $w^{\mathcal{D}}_{\rm eff} := p^{\mathcal{D}}_{\rm eff}/\varrho^{\mathcal{D}}_{\rm eff}$. Now, we may separately discuss (i.e. without matter source) the *morphon equation of state* that plays the role of the *Dark Energy equation of state*,

$$w^{\mathcal{D}}_{\Phi} := \frac{\mathcal{Q}_{\mathcal{D}} - 1/3\langle R\rangle_{\mathcal{D}}}{\mathcal{Q}_{\mathcal{D}} + \langle R\rangle_{\mathcal{D}}} \;\; . \tag{83}$$

The above expression has the advantage that one can immediately infer the case of a constant Dark Energy equation of state, so–called *quiessence models*, that here correspond to scaling solutions of the morphon field with a constant fraction of kinetic to potential energies:

$$\frac{2E^{\mathcal{D}}_{\rm kin}}{E^{\mathcal{D}}_{\rm pot}} = \frac{\varepsilon\dot{\Phi}^2_{\mathcal{D}}V_{\mathcal{D}}}{-U_{\mathcal{D}}V_{\mathcal{D}}} = -1 - \frac{3\mathcal{Q}_{\mathcal{D}}}{\langle R\rangle_{\mathcal{D}}} = 2\frac{w^{\mathcal{D}}_{\Phi}+1}{w^{\mathcal{D}}_{\Phi}-1} \;\; , \tag{84}$$

where the case $\mathcal{Q}_{\mathcal{D}} = 0$ (no kinematical backreaction), or $w^{\mathcal{D}}_{\Phi} = -1/3$ (i.e. $\varrho^{\mathcal{D}}_{\Phi} + 3p^{\mathcal{D}}_{\Phi} = 0$) corresponds to the 'virial condition'

$$2\,E^{\mathcal{D}}_{\rm kin} \,+\, E^{\mathcal{D}}_{\rm pot} \,=\, 0 \;\; , \tag{85}$$

[3]Note that the potential is not restricted to depend only on $\Phi_{\mathcal{D}}$ explicitly. An explicit dependence on the averaged density and on other variables of the system (that can, however, be expressed in terms of these two variables) is generic.

[4]More precisely, kinematical backreaction appears as excess of kinetic energy density over the 'virial balance', $\mathcal{Q}_{\mathcal{D}} = 0$, while the averaged scalar curvature of space sections is directly proportional to the potential energy density; e.g. a void (a 'classical vacuum') with on average negative scalar curvature (a positive potential) can be attributed to a negative potential energy of a morphon field ('classical vacuum energy').

obeyed by the scale–dependent Friedmannian model[5]. As has been already remarked, a non–vanishing backreaction is associated with violation of 'equilibrium'. Again it is worth emphasizing that the above–defined equations of state are scale–dependent. With the help of these dimensionless parameters an inhomogeneous, anisotropic and scale–dependent state can be effectively characterized.

4.6.2 Backreaction as a constant curvature or a cosmological constant

Kinematical backreaction terms can model a constant–curvature term as is already evident from the integrability condition (72b). Also, a cosmological constant need not be included into the cosmological equations, since $\mathcal{Q}_\mathcal{D}$ can play this role and can even provide a *constant* exactly. The exact condition reads:

$$\frac{2}{a_\mathcal{D}^2}\int_{t_\mathrm{i}}^{t} \mathrm{d}t' \; \mathcal{Q}_\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}t'}a_\mathcal{D}^2(t') \; \equiv \; \mathcal{Q}_\mathcal{D} \; , \tag{86}$$

which implies $\mathcal{Q}_\mathcal{D} = \mathcal{Q}_\mathcal{D}(t_i) = const.$ as the only possible solution. Such a 'cosmological constant' installs, however, via Eq. (72c), a non–vanishing averaged scalar curvature (even for $k_{\mathcal{D}_\mathrm{i}} = 0$):

$$\langle R\rangle_\mathcal{D} \; = \; \frac{6k_{\mathcal{D}_\mathrm{i}}}{a_\mathcal{D}^2} \; - \; 3\mathcal{Q}_\mathcal{D}(t_i) \; . \tag{87}$$

4.6.3 The Universe in an out–of–equilibrium state: a fluctuating Einstein cosmos

Following Einstein's thought to construct a globally static model, we may require the effective scale–factor a_Σ on a simply–connected 3–manifold Σ without boundary to be constant on some time–interval, hence $\dot{a}_\Sigma = \ddot{a}_\Sigma = 0$ and the averaged equations assume the form:

$$\mathcal{Q}_\Sigma \; = \; 4\pi G\frac{M_\Sigma}{V_\mathrm{i} a_\Sigma^3} - \Lambda \quad ; \quad \langle R\rangle_\Sigma \; = \; 12\pi G\frac{M_\Sigma}{V_\mathrm{i} a_\Sigma^3} + 3\Lambda \; , \tag{88}$$

with the global *kinematical backreaction* $\mathcal{Q}_\Sigma$, the globally averaged scalar 3–Ricci curvature $\langle R\rangle_\Sigma$, and the total restmass M_Σ contained in Σ.

Let us now consider the case of a vanishing cosmological constant: $\Lambda = 0$. The averaged scalar curvature is, for a non–empty Universe, always *positive*, and the balance conditions (88) replace Einstein's balance conditions that determined the cosmological constant in the standard homogeneous Einstein cosmos. A globally static inhomogeneous cosmos without a cosmological constant is conceivable and characterized by the cosmic equation of state:

$$\langle R\rangle_\Sigma \; = \; 3\mathcal{Q}_\Sigma \; = \; const. \quad \Rightarrow \quad p_\mathrm{eff}^\Sigma \; = \; \varrho_\mathrm{eff}^\Sigma \; = \; 0 \; . \tag{89}$$

Eq. (89) is a simple example of a strong coupling between curvature and fluctuations. Note that, in this cosmos, the effective Schwarzschild radius is larger than the radius of the Universe,

$$a_\Sigma = \frac{1}{\sqrt{4\pi G\,\langle\varrho\rangle_\Sigma}} \; = \; \frac{1}{\pi}2GM_\Sigma = \frac{1}{\pi}a_\mathrm{Schwarzschild} \; , \tag{90}$$

[5] In the case of vanishing kinematical backreaction, the scalar field is present for our definition of the correspondence and it models a constant–curvature term $\langle R\rangle_\mathcal{D} = 6k_{\mathcal{D}_\mathrm{i}}/a_\mathcal{D}^2$. Alternatively, we could associate a morphon with the deviations $W_\mathcal{D}$ from the constant–curvature model only.

hence confirms the cosmological relevance of curvature on the global scale Σ. The term 'out–of–equilibrium' refers to our measure of relative information entropy: in the above example volume expansion cannot compete with information production because the volume is static, while information is produced.

Such examples of global restrictions imposed on the averaged equations do not refer to a specific inhomogeneous metric, but should be thought of in the spirit of the virial theorem that also specifies integral properties but without a guarantee for the existence of inhomogeneous solutions that would satisfy this condition. (It can be shown that a possible stabilization mechanism of the stationarity condition by backreaction is possible, as opposed to the global instability of the classical Einstein cosmos.)

4.6.4 A globally stationary inhomogeneous cosmos

Suppose that the Universe indeed is hovering around a non–accelerating state on the largest scales. A wider class of models that balances the fluctuations and the averaged sources can be constructed by introducing *globally stationary effective cosmologies*: the vanishing of the second time–derivative of the scale–factor would only imply $\dot{a}_\Sigma = const. =: C$, i.e., $a_\Sigma = a_S + C(t-t_i)$, where the integration constant a_S is generically non–zero, e.g. the model may emerge from a globally static cosmos, $a_S := 1$, or from a 'Big–Bang', if a_S is set to zero. In this respect this cosmos does not appear very different from the standard model, since it evolves at an effective Hubble rate $H_\Sigma \propto 1/t$. (There are, however, substantial differences in the evolution of cosmological parameters.)

The averaged equations deliver a dynamical coupling relation between $\mathcal{Q}_\Sigma$ and $\langle R \rangle_\Sigma$ as a special case of the integrability condition (72b)[6]:

$$-\partial_t \mathcal{Q}_\Sigma + \frac{1}{3}\partial_t \langle R \rangle_\Sigma \;=\; \frac{4C^3}{a_\Sigma^3} \; . \tag{91}$$

The cosmic equation of state of the $\Lambda-$free stationary cosmos and its solutions read:

$$p_{\rm eff}^\Sigma \;=\; -\frac{1}{3}\,\varrho_{\rm eff}^\Sigma \quad ; \quad \mathcal{Q}_\Sigma \;=\; \frac{\mathcal{Q}_\Sigma(t_i)}{a_\Sigma^3} \; ; \tag{92}$$

$$\langle R \rangle_\Sigma = \frac{3\mathcal{Q}_\Sigma(t_i)}{a_\Sigma^3} - \frac{3\mathcal{Q}_\Sigma(t_i) - \langle R \rangle_\Sigma (t_i)}{a_\Sigma^2} \; . \tag{93}$$

The total kinematical backreaction $Q_\Sigma V_\Sigma = 4\pi G M_\Sigma$ is a conserved quantity in this case.

The stationary state tends to the static state only in the sense that, e.g. in the case of an expanding cosmos, the rate of expansion slows down, but the steady increase of the scale factor allows for a global change of the sign of the averaged scalar curvature. As Eq. (93) shows, an initially positive averaged scalar curvature would decrease, and eventually would become negative as a result of backreaction. This may not necessarily be regarded as a signature of a global topology change, as a corresponding sign change in a Friedmannian model would suggest.

The above two examples of globally non–accelerating universe models evidently violate the cosmological principle, while they would imply a straightforward explanation of Dark Energy on regional (Hubble) scales: in the latter example the averaged scalar curvature has acquired a

[6] The constant C is determined, for the normalization $a_\Sigma(t_i) = 1$, by:
$6C^2 = 6\Lambda + 3\mathcal{Q}_\Sigma(t_i) - \langle R \rangle_\Sigma (t_i)$.

piece $\propto a_\Sigma^{-3}$ that, astonishingly, had a large impact on the backreaction parameter, changing its decay rate from $\propto a_\Sigma^{-6}$ to $\propto a_\Sigma^{-3}$, i.e. the same decay rate as that of the averaged density. This is certainly enough to produce sufficient 'Dark Energy' on some regional patch due to the presence of strong fluctuations[7]. However, solutions that respect the cosmological principle and, at the same time, satisfy observational constraints can also be constructed. Scaling solutions that we shall discuss now, have been exploited for such a more conservative approach.

4.6.5 The solution space explored by scaling solutions

A systematic classification of scaling solutions of the averaged equations can be given. Like the averaged dust matter density $\langle\varrho\rangle_{\mathcal{D}}$ that evolves, for a restmass preserving domain $\mathcal{D}$, as $\langle\varrho\rangle_{\mathcal{D}} = \langle\varrho\rangle_{\mathcal{D}_\mathrm{i}}\, a_{\mathcal{D}}^{-3}$, we can look at the case where also the backreaction term and the averaged scalar curvature obey scaling laws,

$$\mathcal{Q}_{\mathcal{D}} = \mathcal{Q}_{\mathcal{D}_\mathrm{i}}\, a_{\mathcal{D}}^{n} \quad ; \quad \langle R\rangle_{\mathcal{D}} = R_{\mathcal{D}_\mathrm{i}}\, a_{\mathcal{D}}^{p} \ , \tag{94}$$

where $\mathcal{Q}_{\mathcal{D}_\mathrm{i}}$ and $R_{\mathcal{D}_\mathrm{i}}$ denote the initial values of $\mathcal{Q}_{\mathcal{D}}$ and $\langle R\rangle_{\mathcal{D}}$, respectively. The integrability condition (72b) then immediately provides as a first scaling solution:

$$\mathcal{Q}_{\mathcal{D}} = \mathcal{Q}_{\mathcal{D}_\mathrm{i}}\, a_{\mathcal{D}}^{-6} \quad ; \quad \langle R\rangle_{\mathcal{D}} = R_{\mathcal{D}_\mathrm{i}}\, a_{\mathcal{D}}^{-2} \ . \tag{95}$$

This is the only solution with $n \neq p$. In the case $n = p$, we can define a coupling parameter $r_{\mathcal{D}}$ (that can be chosen differently for a chosen domain of averaging[8]) such that $\mathcal{Q}_{\mathcal{D}_\mathrm{i}} \propto R_{\mathcal{D}_\mathrm{i}}$; the solution reads:

$$\mathcal{Q}_{\mathcal{D}} \;=\; r\;\langle R\rangle_{\mathcal{D}} \;=\; r\;R_{\mathcal{D}_\mathrm{i}}\, a_{\mathcal{D}}^{n} \quad ; \quad n = -2\frac{(1+3r)}{(1+r)} \;\; ; \;\; r = -\frac{(n+2)}{(n+6)} \ , \tag{96}$$

(with $r \neq -1$ and $n \neq -6$). The mean field description of backreaction defines a scalar field evolving in a *positive* potential, if $R_{\mathcal{D}_\mathrm{i}} < 0$ (and in a *negative* potential if $R_{\mathcal{D}_\mathrm{i}} > 0$), and a *real* scalar field, if $\epsilon R_{\mathcal{D}_\mathrm{i}}(r+1/3) < 0$. In other words, if $R_{\mathcal{D}_\mathrm{i}} < 0$ we have a priori a phantom field for $r < -1/3$ and a standard scalar field for $r > -1/3$; if $R_{\mathcal{D}_\mathrm{i}} > 0$, we have a standard scalar field for $r < -1/3$ and a phantom field for $r > -1/3$.

For the scaling solutions the explicit form of the self–interaction potential of the scalar field can be reconstructed:

$$U(\Phi_{\mathcal{D}}, \langle\varrho\rangle_{\mathcal{D}_\mathrm{i}}) = \frac{2(1+r)}{3}\left((1+r)\frac{\Omega_R^{\mathcal{D}_\mathrm{i}}}{\Omega_m^{\mathcal{D}_\mathrm{i}}}\right)^{\frac{3}{n+3}} \langle\varrho\rangle_{\mathcal{D}_\mathrm{i}}\, \sinh^{\frac{2n}{n+3}}\left(\frac{(n+3)}{\sqrt{-\epsilon n}}\sqrt{2\pi G}\Phi_{\mathcal{D}}\right) , \tag{97}$$

where $\langle\varrho\rangle_{\mathcal{D}_\mathrm{i}}$ is the initial averaged restmass density of dust matter, introducing a natural scale into the scalar field dynamics. This potential is well–known in the context of phenomenological quintessence models for Dark Energy. The scaling solutions correspond to specific scalar field models with a constant fraction of kinetic and potential energies of the scalar field, i.e. with Eq. (84),

$$E_{\mathrm{kin}}^{\mathcal{D}} + \frac{(1+3r)}{2\epsilon}E_{\mathrm{pot}}^{\mathcal{D}} \;=\; 0 \ . \tag{98}$$

We previously discussed the case $r = 0$ ('zero backreaction') for which this condition agrees with the standard scalar virial theorem.

[7] A conservative estimate, based on currently discussed numbers for the cosmological parameters, shows that such a cosmos provides room for at least 50 Hubble volumes.

[8] For notational ease we henceforth drop the index $\mathcal{D}$ and simply write r.

4.6.6 Inhomogeneous Inflation

In the same spirit we may prescribe a potential to mimic not only Dark Energy, but also scalar Dark Matter or Inflation. We shall give below a typical example for an inflationary potential. Backreaction thus creates inflation out of curvature inhomogeneities of the Einstein vacuum, if we do not include energy sources. It can be shown that the specific example below leads to a smoothening out of inhomogeneities after inflation and, thus, provides a natural mechanism of what is needed to exit inflation.

One of the simplest examples, which has been extensively studied in the context of so–called chaotic inflation, is a potential of the Ginzburg–Landau form:

$$U_{\mathcal{D}}^{GL} = U_0 \left(\Phi_{\mathcal{D}}^2 - \Phi_0^2\right)^2 / \Phi_0^4 \, . \tag{99}$$

This quartic potential can also be related to a fundamental Higgs field. However, even if such a scalar field is fundamental, there is the possibility that no Higgs particle is involved – as in our case.

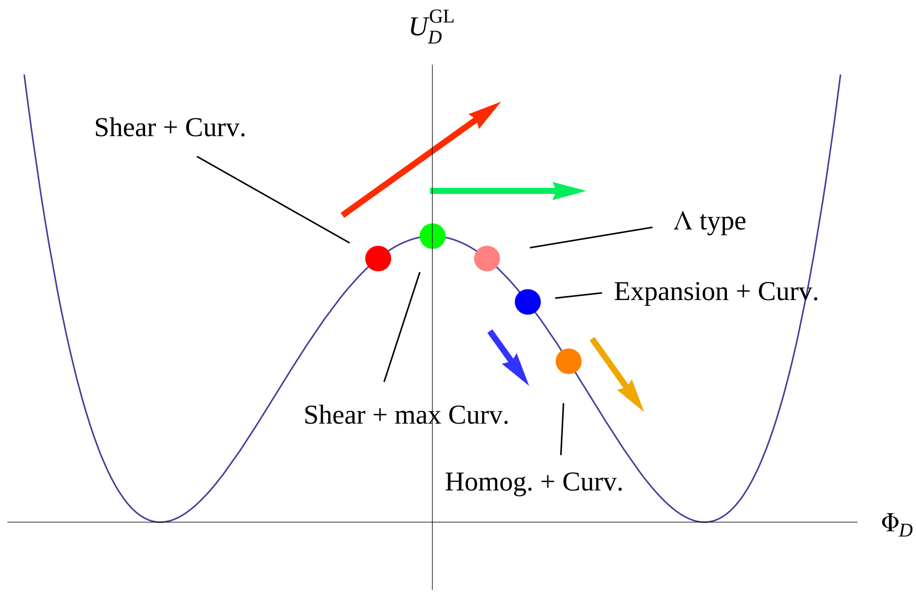

Figure 8: The Ginzburg–Landau potential in arbitrary units and the possible initial conditions as well as their physical meaning. All conditions possess some curvature $\mathcal{W}_{\mathcal{D}_{\rm i}} < 0$. The arrows schematically indicate the amplitude of the morphon's initial speed $\dot\Phi_{\mathcal{D}_{\rm i}}$. In the order of the points (from left to right): the first two points dominated by shear fluctuations (red, green) are obtained for $\mathcal{Q}_{\mathcal{D}_{\rm i}} < 0 \Leftrightarrow \dot\Phi^2_{\mathcal{D}_{\rm i}} > 2(H^2_{\mathcal{D}_{\rm i}} + k_{\mathcal{D}_{\rm i}})$; the next points dominated by expansion fluctuations (pink, blue) for $\dot\Phi^2_{\mathcal{D}_{\rm i}} < 2(H^2_{\mathcal{D}_{\rm i}} + k_{\mathcal{D}_{\rm i}})$, where the de Sitter–$\Lambda$ equivalent case has a stiff morphon $\dot\Phi_{\mathcal{D}_{\rm i}} = 0$; the homogeneous case (last point, orange) is obtained for $\dot\Phi^2_{\mathcal{D}_{\rm i}} = 2(H^2_{\mathcal{D}_{\rm i}} + k_{\mathcal{D}_{\rm i}})$. (For details see: T. Buchert, N. Obadia, *Class. Quant. Grav.* **28**, 162002 (2011). e–print: arXiv:1010.4512.)

Once the minimum Φ_0 is fixed, the evolution of the morphon, given the integrability condition, is practically independent of the initial conditions. In Fig. 8 we show how all acceptable initial conditions are reinterpreted in terms of the curvature, and expansion/shear fluctuations.