Autocorrelation exponent of conserved spin systems in the scaling regime following a critical quench

Clément Sire
Laboratoire de Physique Théorique (UMR 5152 du CNRS), Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex 4, France
E-mail: clement.sire@irsamc.ups-tlse.fr

We study the autocorrelation function of a conserved spin system following a quench at the critical temperature. Defining the correlation length $L(t) \sim t^{1/z}$, we find that for times t' and t satisfying $L(t') \ll L(t) \ll L(t')^{\phi}$ well inside the scaling regime, the autocorrelation function behaves like $\langle s(t)s(t')\rangle \sim L(t')^{-(d-2+\eta)}\,[L(t')/L(t)]^{\lambda'_c}$. For the $O(n)$ model in the $n \to \infty$ limit, we show that $\lambda'_c = d+2$ and $\phi = z/2$. We give a heuristic argument suggesting that this result is in fact valid for any dimension d and spin vector dimension n. We present numerical simulations for the conserved Ising model in $d = 1$ and $d = 2$, which are fully consistent with this result.

The quench of a ferromagnetic spin system [1], from high temperature ($T_0 > T_c$) to low temperature (typically $T = 0$ or $T = T_c$) is characterized by the growth of a correlation length scale (or domain length scale when domains can be identified), $L(t) \sim t^{1/z}$. In the non conserved case, z depends on the final temperature of the quench ($z = 2$ for $T < T_c$, while z is the dynamical critical exponent for $T = T_c$ [2]). If the order parameter $s(\mathbf{x}, t)$ (possibly a vector) is locally conserved, $z = 3$ (scalar) or $z = 4$ (vector) for a quench below T_c [1], while $z = 4 - \eta$ [2] for a quench at T_c. Another interesting and fundamental quantity is the spin autocorrelation $A(t, 0) = \langle s(\mathbf{x}, t)s(\mathbf{x}, 0)\rangle \sim L(t)^{-\lambda}$ [3–5]. For non conserved dynamics, whatever the temperature of the quench, λ is non trivial (except in $d = 1$ [1]) and only approximate theories are available for $T = 0$ [1, 4], while for $T = T_c$ [3], the ε-expansion of λ can be calculated. In the case of conserved dynamics, it is now well established that $\lambda = d$ for quenches at and below T_c [6–8]. Hence for fixed t' and $t \to +\infty$, $A(t, t') \sim L(t)^{-d}$. However, for t' and $t > t'$ both in the scaling regime (in a sense to be defined later), several authors have observed numerically [9–11] and experimentally [12] a faster power law decay of the autocorrelation as a function of $L(t)$. More precisely, in the case of a quench of an Ising system at T_c (critical quench), the authors of [11] obtained numerically the following form

$$A(t, t') \sim L(t')^{-(d-2+\eta)} \left[\frac{L(t')}{L(t)}\right]^{\lambda'_c}, \tag{1}$$

in $d = 1$ (where formally $\eta = 1$ and $T_c = 0$) and $d = 2$. They respectively found $\lambda'_c \approx 2.5$ in $d = 1$ and $\lambda'_c \approx 3.5$ in $d = 2$. They also suggested a general scaling relation

$$A(t, t') \sim L(t)^{-d} C\left[\frac{L(t)}{L(t')^{\phi}}\right], \tag{2}$$

where $C(x)$ goes to a non zero constant for $x \to +\infty$, $C(x) \sim x^{-(\lambda'_c - d)}$ for $x \to 0$, and

$$\phi = 1 + \frac{2-\eta}{\lambda'_c - d}. \tag{3}$$

As noticed in [11], this scaling implies the existence of a new relevant length scale $L(t')^{\phi}$ for conserved critical dynamics, which is the crossover length between the two observed regimes. Its physical meaning has yet to be elucidated.

In the present Letter, we address the problem of the actual analytical derivation of λ'_c in the case of the $O(n)$ model in the limit of infinite n. Within this model, the nature of this new length scale can be understood and one finds $\lambda'_c = d + 2$ and $\phi = 2$. By generalizing the interpretation of this crossover length scale to any $O(n)$ spin system, we conjecture that the result $\lambda'_c = d + 2$ holds and that $\phi = 2 - \eta/2 = z/2$.

We first examine the exactly solvable $O(n)$ model in the limit $n \to \infty$ and for dimensions $d > 2$. This model is known to be pathological for a quench at zero temperature, displaying multiscaling [13], whereas normal scaling should be restored at finite n [1, 14]. However, after a quench at T_c, the structure factor obeys standard scaling even for $n \to \infty$ [7], and it is natural to expect that this model should give a better insight concerning the existence and nature of the exponent λ'_c. In the standard Cahn-Hilliard equation describing the evolution of the magnetization field $\mathbf{s}(\mathbf{x}, t)$, $\mathbf{s}^2(\mathbf{x}, t)/n$ can be replaced by its average in the limit $n \to \infty$. Thus, any given component of $\mathbf{s}(\mathbf{x}, t)$ satisfies

$$\frac{\partial s}{\partial t} = -\Delta\left[\Delta s + k_0^2 s - \langle s^2\rangle s\right] + \eta, \tag{4}$$

where k_0^2 is a constant, $\eta(\mathbf{x}, t)$ is a conserved delta-correlated noise satisfying $\langle \eta(\mathbf{k}, t)\eta(\mathbf{k}', t')\rangle = 2T_c k^2 \delta(\mathbf{k} + \mathbf{k}')\delta(t - t')$, and $\langle s^2\rangle$ has to be computed self-consistently. Although the derivation of the structure factor has already appeared in the literature [7], we briefly repeat it as it furnishes a useful basis for our final derivation.

Eq. (4) can be readily solved in Fourier space, leading to

$$s(\mathbf{k},t) = \left[s(\mathbf{k},0) + \int_0^t \mathrm{e}^{q(k,\tau)}\eta(\mathbf{k},\tau)\,d\tau\right]\mathrm{e}^{-q(k,t)}, \quad (5)$$

where

$$q(k,t) = k^4 t - k^2\int_0^t [k_0^2 - \langle s^2(\mathbf{x},\tau)\rangle]\,d\tau. \quad (6)$$

Assuming an uncorrelated initial condition such that $\langle s(-\mathbf{k},0)s(\mathbf{k},0)\rangle = s_0^2$, we then find the structure factor $S(k,t) = \langle s(-\mathbf{k},t)s(\mathbf{k},t)\rangle$

$$S(k,t) = \left[s_0^2 + 2T_c\int_0^t \mathrm{e}^{2q(k,\tau)}\,d\tau\right]\mathrm{e}^{-2q(k,t)}. \quad (7)$$

We now express the self-consistent condition $\langle s^2(\mathbf{x},t)\rangle = \int^\Lambda S(k,t)\,\frac{d^d\mathbf{k}}{(2\pi)^d}$, where Λ is the inverse of a lattice cut-off. T_c is such that $S(k,t\to\infty)\sim k^{2-\eta}$, where η is the usual critical exponent controlling the decay of the static correlation function ($\eta = 0$ for $n\to\infty$). This leads to

$$T_c\int^\Lambda k^{-2}\,\frac{d^d\mathbf{k}}{(2\pi)^d} = k_0^2. \quad (8)$$

Finally, if the above condition is satisfied, we find that $q(k,t)$ obeys a scaling relation for large t

$$q(k,t) = q(kt^{1/4}) = k^4 t - c_d k^2 t^{1/2}, \quad (9)$$

where c_d is a universal constant determined by a simple integral relation ($c_d = 0$ for $d > 4$) [7], and $q(u) = u^4 - c_d u^2$. We thus find $L(t) = t^{1/z}$, with $z = 4$, in agreement with the general result $z = 4-\eta$ [2]. We hence reproduce the general form of the structure factor

$$S(k,t) = s_0^2\mathrm{e}^{-2q(kt^{1/z})} + t^{(2-\eta)/z}F(kt^{1/z}). \quad (10)$$

For the $O(n=\infty)$ model, we have $z = 4$, $\eta = 0$, and

$$F(u) = 2T_c u^2\int_0^1 \mathrm{e}^{-u^4(1-v)+c_d u^2(1-v^{1/2})}\,dv. \quad (11)$$

Note the following asymptotics for $F(u)$

$$\begin{aligned} F(u) &\sim 2T_c u^2, \quad u\to 0, & (12)\\ F(u) &\sim T_c u^{-2}, \quad u\to +\infty. & (13)\end{aligned}$$

In the scaling limit, the first term of the right hand side (RHS) of Eq. (10) is negligible compared to the second term. In real space, Eq. (10) illustrates the fact that conventional (critical) scaling is obeyed

$$\langle \mathbf{s}(\mathbf{x},t)\mathbf{s}(\mathbf{0},t)\rangle = L(t)^{-(d-2+\eta)}f[x/L(t)], \quad (14)$$

where f is simply the inverse Fourier transform of F.

We now move to the calculation of the two-time correlation function, focusing on the case where both considered times t' and $t > t'$ are in the scaling regime, a notion which will be made more precise hereafter. Using Eq. (5), and working along the line of the derivation of $S(k,t)$, we find the following expression for $C(k,t) = \langle s(-\mathbf{k},t')s(\mathbf{k},t)\rangle$

$$\begin{aligned} C(k,t,t') &= \mathrm{e}^{q[kt'^{1/4}]-q[kt^{1/4}]}S(k,t'), & (15)\\ &= s_0^2\mathrm{e}^{-q[kL(t')]-q[kL(t)]} + & (16)\\ &\quad L(t')^2\mathrm{e}^{q[kL(t')]-q[kL(t)]}F[kL(t')]. & (17)\end{aligned}$$

For a fixed t' and $t\to\infty$, the contribution of Eq. (17) becomes negligible, as for large t and hence $L(t)$, only the contribution of small wave vector $k\sim L(t)^{-1}$ matters. Using the result of Eq. (12), we indeed find that that this term is of order $k^2\sim L(t)^{-2}$, whereas the main contribution of Eq. (16) is of order s_0^2 which is a constant. Contrary to what occurs in $S(k,t)$, it is now the term depending on the initial conditions via s_0^2 which dominates. Hence in this limit of fixed t' and $t\to\infty$, we find

$$C(k,t,t')\sim C(k,t,0)\sim s_0^2\mathrm{e}^{-q[kL(t)]} = G[kL(t)], \quad (18)$$

and in real space

$$\langle \mathbf{s}(\mathbf{x},t)\mathbf{s}(\mathbf{0},t')\rangle \sim L(t)^{-d}g[x/L(t)], \quad (19)$$

where g is the inverse Fourier transform of G. One recovers, in the limit $t\gg t'$ to be made more precise later, that the large time autocorrelation exponent is $\lambda_c = d$, which is so far observed in all conserved models including thermal fluctuations [6, 7]. Again, in this limit, conventional scaling holds. However, we will now show that the contribution of Eq. (17) which has not so far been considered is in fact the leading term in a well defined time regime, and will prompt us to introduce another autocorrelation exponent λ_c'.

For general t' and $t > t'$, we now proceed to calculate the autocorrelation for a spin on a given lattice site. Defining $A(t,t') = \langle s(\mathbf{x},t')s(\mathbf{x},t)\rangle$, we finally find $A(t,t') = A_1(t,t') + A_2(t,t')$, where

$$A_1(t,t') = s_0^2\int^\Lambda \mathrm{e}^{-q[kL(t')]-q[kL(t)]}\,\frac{d^d\mathbf{k}}{(2\pi)^d}. \quad (20)$$

After a change of variable and noting that the region of $k\gg L(t)^{-1}$ barely contributes to the integral, we find

$$A_1(t,t') = L(t)^{-d}a_1[L(t')/L(t)]. \quad (21)$$

Thus, $A_1(t,t')$ obeys conventional scaling for any t' and $t > t'$. We explicitly find

$$a_1(u) = s_0^2\int^\infty \mathrm{e}^{-k^4(1+u^4)+c_d k^2(1+u^2)}\,\frac{d^d\mathbf{k}}{(2\pi)^d}, \quad (22)$$

where this integral is now over the entire space. $a_1(u)$ remains bounded and of order s_0^2 for any value of $u =$

$L(t')/L(t) \leq 1$. Keeping the notation $u = L(t')/L(t) \leq 1$, the expression for $A_2(t,t')$ can be written in the rescaled form

$$A_2(t,t') = L(t')^{-(d-2)} u^d \times \int^{L(t)\Lambda} \mathrm{e}^{-k^4(1-u^4)+c_d k^2(1-u^2)} F(ku) \frac{d^d\mathbf{k}}{(2\pi)^d}. \tag{23}$$

Let us analyze the different asymptotics for $A_2(t,t')$. First of all, for large $t = t'$ ($u = 1$), the integral is dominated by the region of large k's. Using Eq. (17), we find the expected result $A_2(t,t') \approx k_0^2$, which is the equilibrium value of $\langle s^2 \rangle$. Note that if $t - t' \ll 1$, we obtain $A_2(t,t') = A_2(t,t) - K_d(t-t') + ...$, where K_d is a computable constant. We now assume that $1 \ll L(t) - L(t') \ll L(t')$, which ensure that u is very close to 1. In this regime, we find that

$$A_2(t,t') \sim J_d L(t)^{-(d-2)} \left[1 - \frac{L(t')}{L(t)}\right]^{-(d-2)/4}, \tag{24}$$

$$\sim J'_d \, (t-t')^{-(d-2)/4}, \tag{25}$$

where J_d and J'_d can be written exactly as simple integrals. Finally, and this constitutes the central result of this Letter, we consider the new scaling regime behavior corresponding to $1 \ll L(t') \ll L(t)$. In this case, $u \ll 1$, and the integral of Eq. (23) is dominated by the region of k of order unity, so that the small argument asymptotics can be taken for $F(ku)$ in Eq. (23). We find

$$A_2(t,t') \sim \kappa_d \, L(t')^{-(d-2)} \left[\frac{L(t')}{L(t)}\right]^{d+2}, \tag{26}$$

$$\kappa_d = 2T_c \int^{\infty} k^2 \mathrm{e}^{-k^4+c_d k^2} \frac{d^d\mathbf{k}}{(2\pi)^d}. \tag{27}$$

Eq. (26) takes exactly the expected form of Eq. (1), with

$$\lambda'_c = d + 2. \tag{28}$$

$A_1(t,t')$ will prevail over $A_2(t,t')$ for $L(t) \gg L_0(t')$, with $L_0(t') \sim L(t')^{\phi}$ and $\phi = 2$. Eq. (23) shows that instead of Eq. (2), the correct scaling is rather

$$A(t,t') = A(t,0) + L(t')^{-(d-2+\eta)} D[L(t)/L(t')], \tag{29}$$

with $D(1/u) \sim u^{\lambda_c}$ for $u \ll 1$. Both scaling are equivalent only for $u \ll 1$. We now present an heuristic argument, based on dimensional analysis which suggests that the result $\lambda'_c = d+2$ may be of general validity for conserved spin systems. Indeed, the occurrence of a new length scale bigger than $L(t)$ could have been inferred from the small k behavior of $S(k,t)$. In the $n \to \infty$ limit and for $k \to 0$, Eq. (10) leads to

$$S(k,t) \approx s_0^2 + 2T_c k^2 L(t)^4 + ... \tag{30}$$

A natural momentum scale $k_0(t) \sim L_0(t)^{-1}$ arises by matching the two terms of the RHS of Eq. (30), which leads to $\phi = 2$ and hence $\lambda'_c = d+2$.

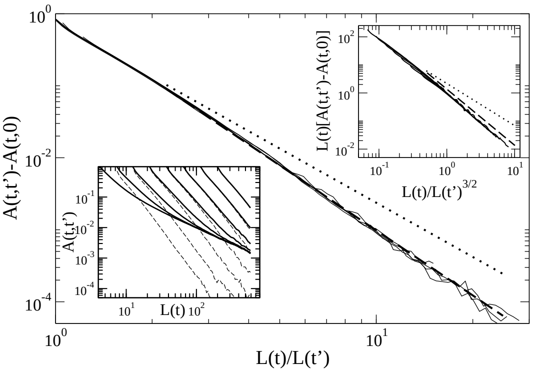

FIG. 1: Illustrating the result of Eq. (29), we plot $A(t,t_k) - A(t,0) = D[L(t)/L(t_k)]$, for $L(t_k) \approx L(0)r^k$, with $r = 1.75$ and $k = 1,...,7$ (40000 samples of length $N = 5000$). Although the initial slope is smaller ($\lambda'_c \approx 2.5$ [11]; dotted line), the asymptotic exponent is very close to $\lambda'_c = 3$ (dashed line fit). The bottom inset shows the original data for $A(t,t_k)$ and $A(t,t_k) - A(t,0)$ (dashed lines). The top inset shows $L(t)[A(t,t_k) - A(t,0)]$ as a function of $L(t)/L(t_k)^{\phi}$ (with $\phi = 3/2$). Lines of slope $\lambda'_c - 1$ are shown for $\lambda'_c = 3$ (dashed line) and $\lambda'_c \approx 2.5$ (dotted line).

In the general case, for short-range correlated initial conditions, we expect the following general form to hold

$$S(k,t) = F_1[kL(t)] + L(t)^{2-\eta} F_2[kL(t)], \tag{31}$$

with $F_1(0) = s_0^2$ being a non zero constant (equal to the variance of the initial total magnetization normalized by the volume), while the scaling contribution should vanish for $k = 0$, implying $F_2(0) = 0$. Imposing $F_2(p) \sim p^{\gamma}$, γ is necessarily an even integer. If γ were not integer, the correlation function scaling function f defined in Eq. (14) would have a power law decay for large distance, which is unphysical as such correlations cannot develop in a finite time starting from short-range ones. γ cannot be an odd integer as space isotropy guarantees that f should be an even function. Contrary to the case of a quench at $T = 0$, for which convincing theoretical arguments for $d \geq 2$ [15] and experiments [16] show that $F_2(p) \sim p^4$ (so that the second moment of f vanishes), there is no reason to expect the same for critical quenches. Generically, we expect $F_2(p) \sim p^2$ as found for the $d = 1$ conserved Ising model [6, 7], and in the present Letter for the $O(n)$ model for $n \to \infty$. Finally, the small k behavior of the structure factor should be of the form

$$S(k,t) \approx s_0^2 + C_0 k^2 L(t)^{4-\eta} + ..., \tag{32}$$

where $C_0 > 0$ is a constant. Assuming that the length scale obtained by matching both terms of the RHS of Eq. (32) is the same as the crossover length between the two observed regimes for the autocorrelation, and using the general result of Eq. (3), we obtain

$$\phi = 2 - \eta/2 = 1 + \frac{2-\eta}{\lambda'_c - d}, \tag{33}$$

which implies $\lambda'_c = d+2$. This result also extends to $d=1$ after formally taking $\eta=1$, leading to $\lambda'_c = 3$ and $\phi = 3/2$. Note that the crossover scale can also be written $L_0(t) \sim t^{\phi/z} \sim t^{1/2}$, which behaves like a diffusion scale. At least in $d=1$, this scale can be associated to the equilibrium diffusion of tagged spins observed in [17].

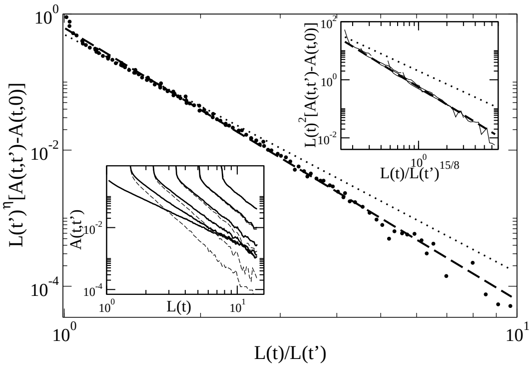

FIG. 2: In the bottom inset, we plot $A(t,t_k)$ and $A(t,t_k) - A(t,0)$ (dashed lines) as a function of $L(t)$, for $L(t_k) \approx r^k$, with $r = 1.5$ and $k = 1, ..., 5$ (16 samples of size $N = 500\times500$, $L(0) = 1/\sqrt{2}$). The main plot shows $L(t_k)^\eta[A(t,t_k) - A(t,0)] = D[L(t)/L(t_k)]$. Although the initial slope is consistent with $\lambda'_c \approx 3.5$ [11] (dotted line), the effective exponent certainly increases and the asymptotic slope is more compatible with $\lambda'_c = 4$ (dashed line fit). The top inset shows $L(t)^2[A(t,t_k) - A(t,0)]$ as a function of $L(t)/L(t_k)^\phi$ (with $\phi = 15/8$). Although not as clean as in $d=1$, the scaling plot is better described by the line corresponding to $\lambda'_c = 4$ (dashed line) rather than $\lambda'_c \approx 3.5$ (dotted line).

We now present simulations of the Ising model Kawasaki dynamics in $d=1$ and $d=2$ after a quench at T_c. In the $d=1$ case, we use the accelerated algorithm introduced in [7], which is faster than that used in [11] (but does not permit to compute simply the response function as was needed in [11]). By fitting $A(t,t')$ in the scaling regime, the authors of [11] found $\lambda'_c \approx 2.5$ lower than our prediction $\lambda'_c = 3$. However, for the moderately large numerically accessible times, the contribution of $A_1(t,t') \approx A(t,0)$ is significant. When plotting $A(t,t') - A(t,0)$ as a function of $L(t)$, one actually finds $\lambda'_c \approx 3$ instead of $\lambda'_c \approx 2.5$. In Fig. 1, we plot $L(t')[A(t,t') - A(t,0)]$ as a function of $L(t)/L(t')^{3/2}$, leading to an almost perfect scaling plot. The hull scaling function is well fitted by $0.9\,x^{-2}$. Since we also find that $A(t,0) \sim 0.92/L(t)$, we conclude that in $d=1$, the scaling function introduced in Eq. (2) is well approximated by $C(u) = 1+u^{-2}$. Result of simulations for the $d=2$ Ising model evolving with Kawasaki dynamics at T_c are shown on Fig. 2. Considering the very slow growth of $L(t) \sim t^{4/15}$, it is difficult to obtain data spanning more than one decade in $L(t)$. Hence, the regime of interest $1 \ll L(t') \ll L(t)$ cannot be reached and the separation of scales properly achieved. Still, subtracting $A(t,0)$ from $A(t,t')$ leads to $\lambda'_c \approx 4$, significantly greater than the value $\lambda'_c \approx 3.5$ found in [11].

In conclusion, in view of the exact result for the $O(n=\infty)$ model, a general argument for any n and d, and convincing simulations in $d=1$ (and consistent in $d=2$), we have strongly suggested that $\lambda'_c = d+2$ and $\phi = z/2$ generally holds. We also find that the scaling form of Eq. (29) is more appropriate than Eq. (2). The compelling generalization of our heuristic argument to a quench at $T < T_c$ (in $d \geq 2$, and admitting $F_2(p) \sim p^4$) leads to $A(t,t') \sim [L(t')/L(t)]^{\lambda'}$ for $L(t') \ll L(t) \ll L(t')^\phi$, with $\lambda' = d+4$ and $\phi = 1+d/4$. In $d=2$, the prediction $\lambda' = 6$ is significantly larger than the numerical result $\lambda' \approx 4$ [9]. However, the fit in [9] was performed in the short scaling regime over less than a decade in $L(t)$, and subtracting $A(t,0)$ before performing the fit could lead to a significantly higher value for λ', as noted in the two examples treated in this Letter.

[1] A.J. Bray, Adv. Phys. **43**, 357 (1994).

[2] P. Hohenberg and B. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[3] H.K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B **73**, 539 (1989); D.A. Huse, Phys. Rev. B **40**, 304 (1989).

[4] D.S. Fisher and D.A. Huse, Phys. Rev. B **38**, 373 (1988); G. F. Mazenko, Phys. Rev. B **42**, 4487 (1990); C. Sire and S. N. Majumdar, Phys. Rev. Lett. **74**, 4321 (1995).

[5] N. Mason, A.N. Pargellis, and B. Yurke, Phys. Rev. Lett. **70**, 190 (1993).

[6] S.N. Majumdar, D.A. Huse, and B.D. Lubachevsky, Phys. Rev. Lett. **73**, 182 (1994).

[7] S.N. Majumdar and D.A. Huse, Phys. Rev. E **52**, 270 (1995).

[8] F.J. Alexander, D.A. Huse, and S.A. Janowsky, Phys. Rev. B **50**, 663 (1994).

[9] C. Yeung, M. Rao, and R.C. Desai, Phys. Rev. E **53**, 3073 (1996).

[10] J.F. Marko and G.T. Barkema, Phys. Rev. E **52**, 2522 (1995).

[11] C. Godrèche, F. Krzakala, and F. Ricci-Tersenghi, JSTAT P04007 (2004).

[12] T. Nagaya and J.-M. Gilli, Phys. Rev. Lett. **92**, 145504 (2004).

[13] A. Coniglio and M. Zannetti, Europhys. Lett. **10**, 575 (1989); A. Coniglio, P. Ruggiero, and M. Zannetti, Phys. Rev. E **50**, 1046 (1994).

[14] A.D. Rutenberg and A.J. Bray, Phys. Rev. E **51**, 5499 (1995); S. Puri, A.J. Bray, and F. Rojas, Phys. Rev. E **52**, 4699 (1995); F. Rojas and A.J. Bray, Phys. Rev. E **51**, 188 (1995).

[15] C. Yeung, Phys. Rev. Lett. **61**, 1135 (1988); H. Furukawa, Phys. Rev. Lett. **62**, 2567 (1989); H. Furukawa, J. Phys. Soc. Jpn. **58**, 216 (1989).

[16] P. Wiltzius, F.S. Bates, and W.R. Heffner, Phys. Rev. Lett. **60**, 1538 (1988).

[17] C. Godrèche and J.M. Luck, J. Phys. A **36**, 9973 (2003).