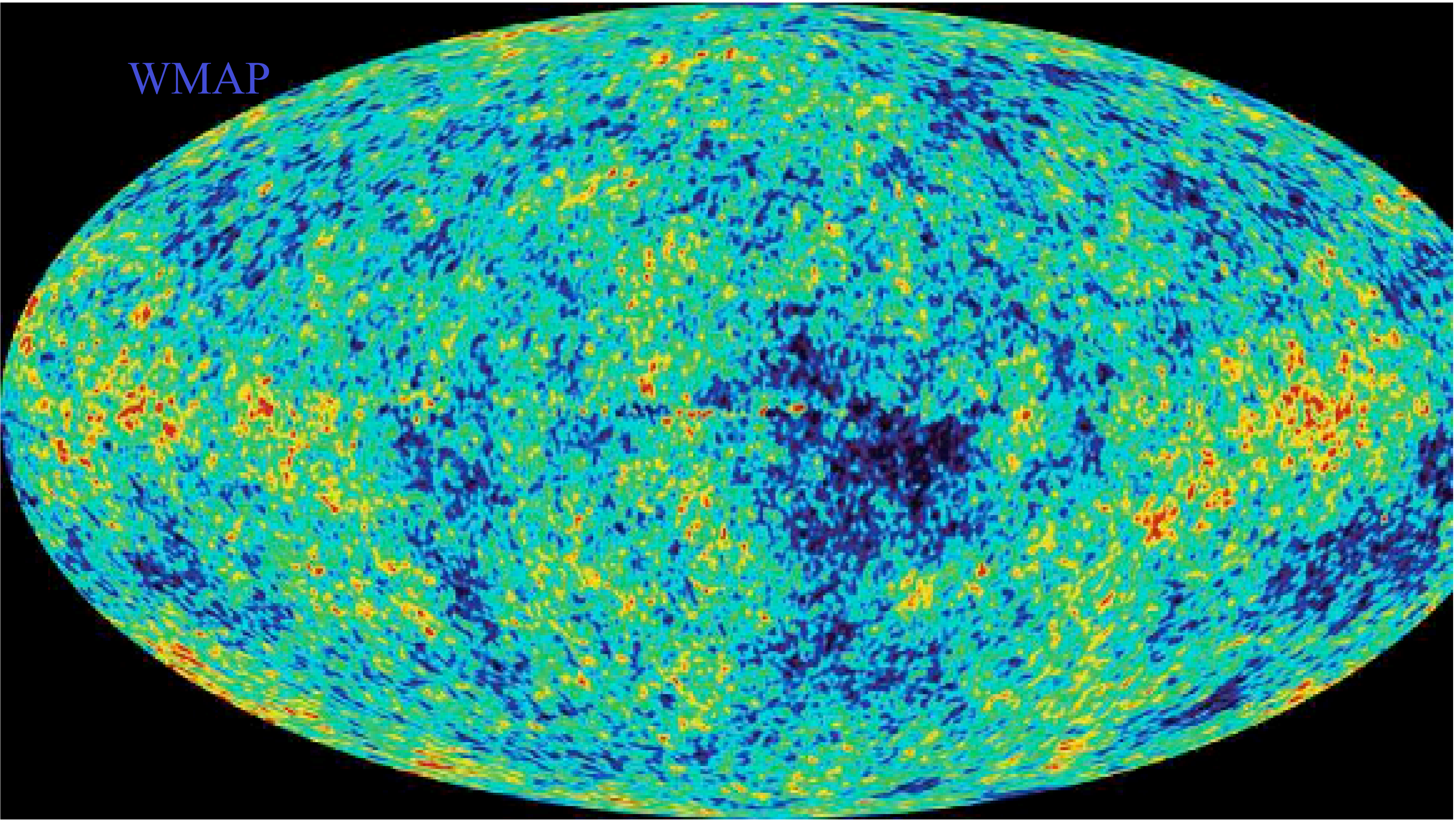
# Data on the Sphere



# Wavelet on the Sphere

- P. Schroder and W. Sweldens (Orthogonal Haar WT), 1995.
- M. Holschneider, Continuous WT, 1996.
- **W. Freeden and T. Maier, OWT, 1998**.
- J.P. Antoine, Continuous WT, 1999.
- L. Tenerio, A.H. Jaffe, Haar Spherical CWT, (CMB), 1999.
- L. Cayon, J.L Sanz, E. Martinez-Gonzales, Mexican Hat CWT, 2001.
- J.P. Antoine and L. Demanet, Directional CWT, 2002.

## Wavelet transform in the spherical harmonics space

We assume that

$$c_0(\theta, \phi) = c_{-1}(\theta, \phi) * \phi_{l_{max}}(\theta, \phi)$$

where $\phi_{l_{max}}$ is a low pass filter (scaling function) with a frequency cut-off $l_{max}$.
$c_0$ the input map, and $c_{-1}$ an unknown function. A resolution level $j$ is related to the HEALPix definition ($N_{side} = 2^j$).

The image at different resolutions is given by:

$$\begin{aligned} c_1(\theta,\phi) &= c_{-1}(\theta,\phi) * \phi_{l_{max}/2}(\theta,\phi) \\ c_2(\theta,\phi) &= c_{-1}(\theta,\phi) * \phi_{l_{max}/4}(\theta,\phi) \\ &\vdots \\ c_j(\theta,\phi) &= c_{-1}(\theta,\phi) * \phi_{l_{max}/2^j}(\theta,\phi) \end{aligned}$$

and the wavelet coefficients of the image are:

$$\begin{aligned} w_1(\theta,\phi) &= c_{-1}(\theta,\phi) * \psi_1(\theta,\phi) \\ w_2(\theta,\phi) &= c_{-1}(\theta,\phi) * \psi_2(\theta,\phi) \\ &\vdots \\ w_j(\theta,\phi)) &= c_{-1}(\theta,\phi) * \psi_j(\theta,\phi) \end{aligned}$$

where $\psi_j$ is the wavelet function at scale $j$.

## Choice of Scale Function and Wavelet Function

We define $\phi_{l_{max}}$ in the Spherical Harmonics space as a B-spline function. The B-spline function is

$$B(x) = \frac{1}{12}(| x - 2 |^3 - 4| x - 1 |^3 + 6| x |^3 - 4| x + 1 |^3 + | x + 2 |^3)$$

and the scaling function $\phi_{l_{max}}$ is:

$$\hat{\phi}_{l_{max}}(l, m) = \frac{3}{2} B(\frac{l}{l_{max}})$$

The wavelet can be chosen as the difference between two resolutions:

$$\psi_j(\theta, \phi) = \phi_{l_{max}/2^{j-1}}(\theta, \phi) - \phi_{l_{max}/2^j}(\theta, \phi)$$

# From one resolution to the next one

The image at the first scale is given by: $c_0(\theta, \phi) = c_{-1}(\theta, \phi) * \phi_{l_{max}}(\theta, \phi)$, which gives in space of spherical harmonics (as $\phi_{l_{max}}$ is azimuthally symmetric):

$$\hat{c_0}(l, m) = \sqrt{\frac{2l+1}{4\pi}} \hat{c}_{-1}(l, m) \hat{\phi}_{l_{max}}(l, m)$$

The coefficients which allow us to go from one resolution to the next one are obtained with the discrete filters $h$ and $h$:

$$\hat{h}(l, m) = \begin{cases} \frac{\hat{\phi}_{l_{max}}(2l, m)}{\hat{\phi}_{l_{max}}(l, m)} & \text{if } l < l_{max} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{g}(l, m) = \begin{cases} \frac{\hat{\psi}_1(2l, m)}{\hat{\phi}_{l_{max}}(l, m)} & \text{if } l < l_{max} \\ 0 & \text{otherwise} \end{cases}$$

We have:

$$\begin{aligned} \hat{c}_{j+1}(l, m) &= \hat{c}_j(l, m) \hat{h}(2^j l, m) \\ \hat{w}_{j+1}(l, m) &= \hat{c}_j(l, m) \hat{g}(2^j l, m) \end{aligned}$$

The cut-off frequency is reduced by a factor 2 at each step.

## Reconstruction

If the wavelet is the difference between two resolutions, an evident reconstruction for a wavelet transform $\mathcal{W} = \{w_1, \dots, w_J, c_J\}$ is:

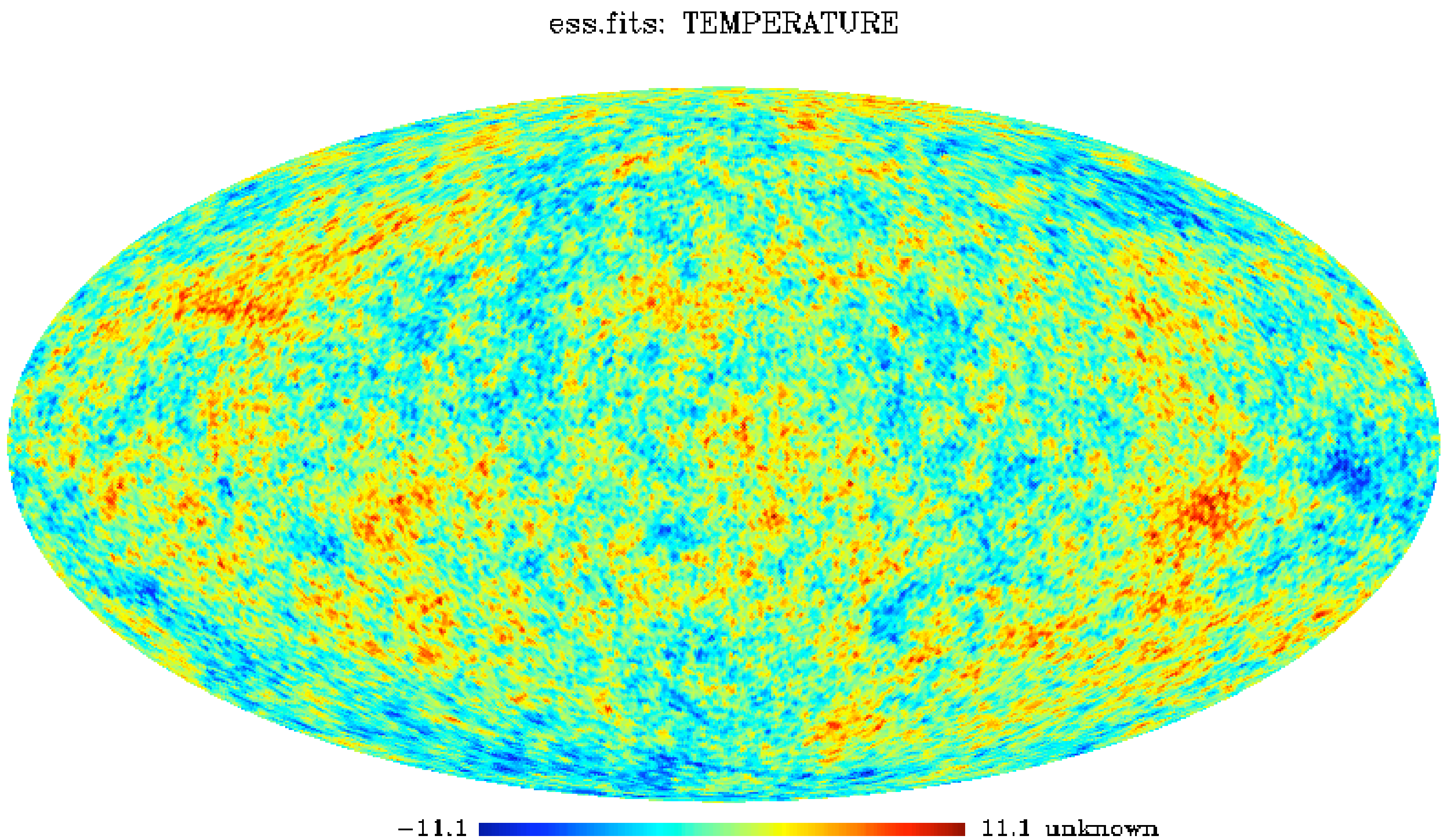
$$c_0(\theta, \phi) = c_J(\theta, \phi) + \sum_j w_j(\theta, \phi)$$

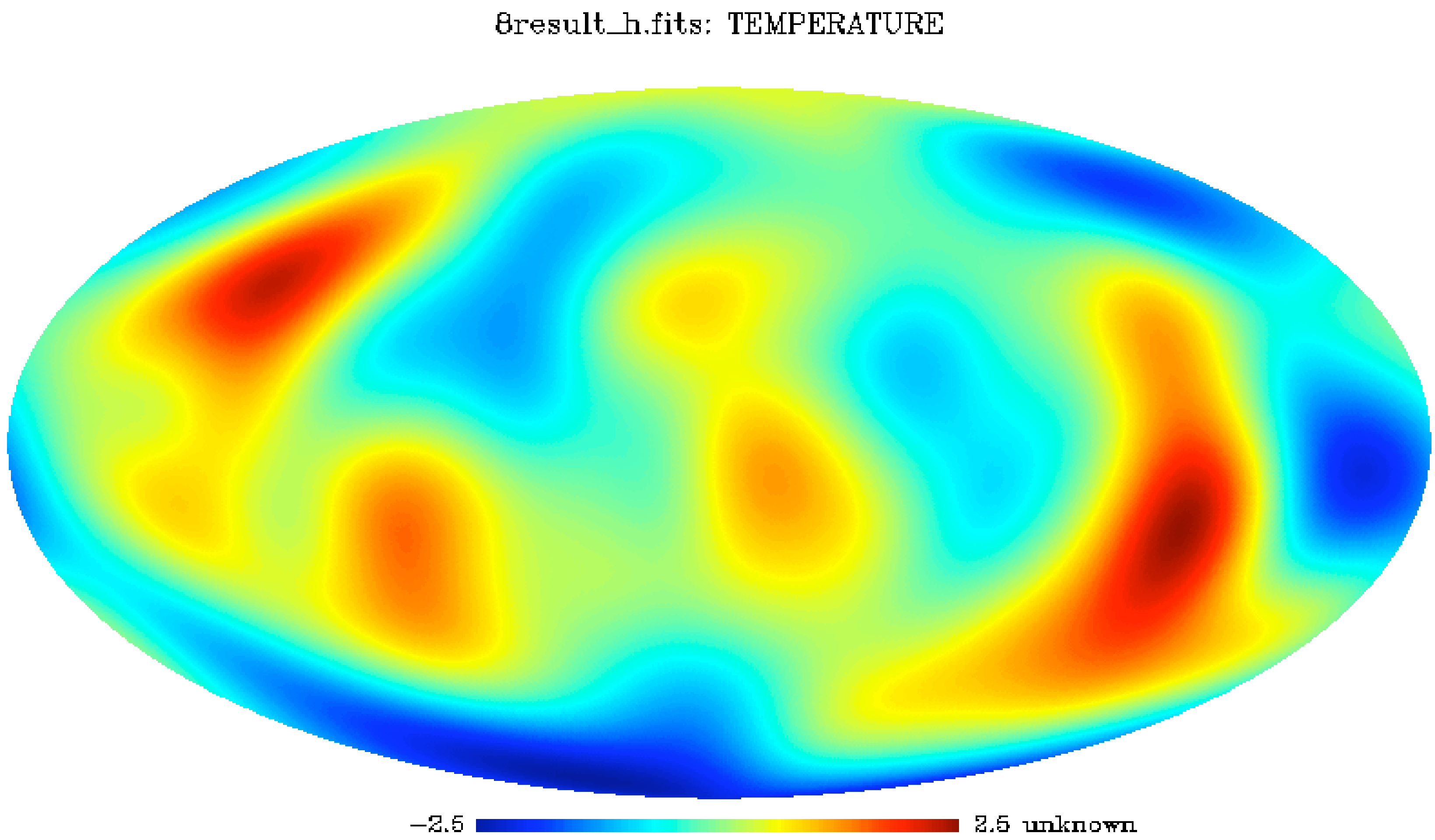
An alternative is to use the conjugate filters defined by

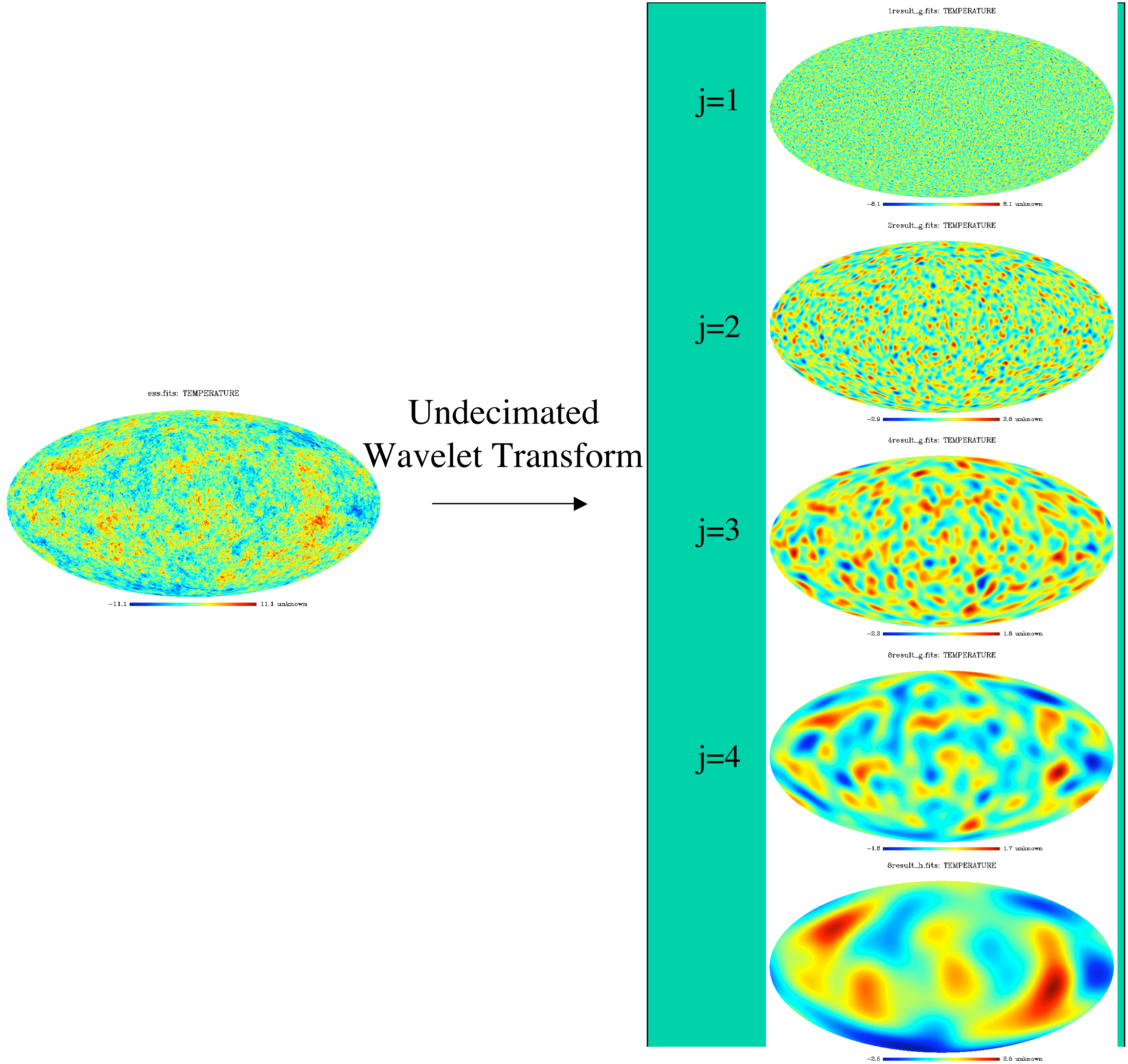
$$\begin{aligned} \hat{\tilde{h}} &= \hat{h}^* / (\mid \hat{h} \mid^2 + \mid \hat{g} \mid^2) \\ \hat{\tilde{g}} &= \hat{g}^* / (\mid \hat{h} \mid^2 + \mid \hat{g} \mid^2) \end{aligned}$$

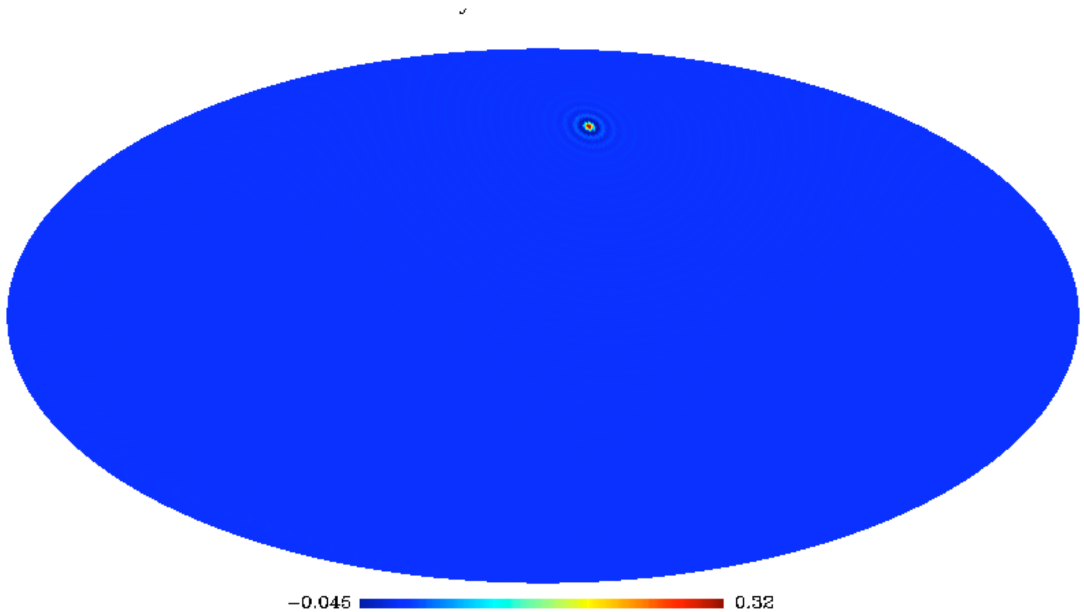
And the reconstruction is obtained by:

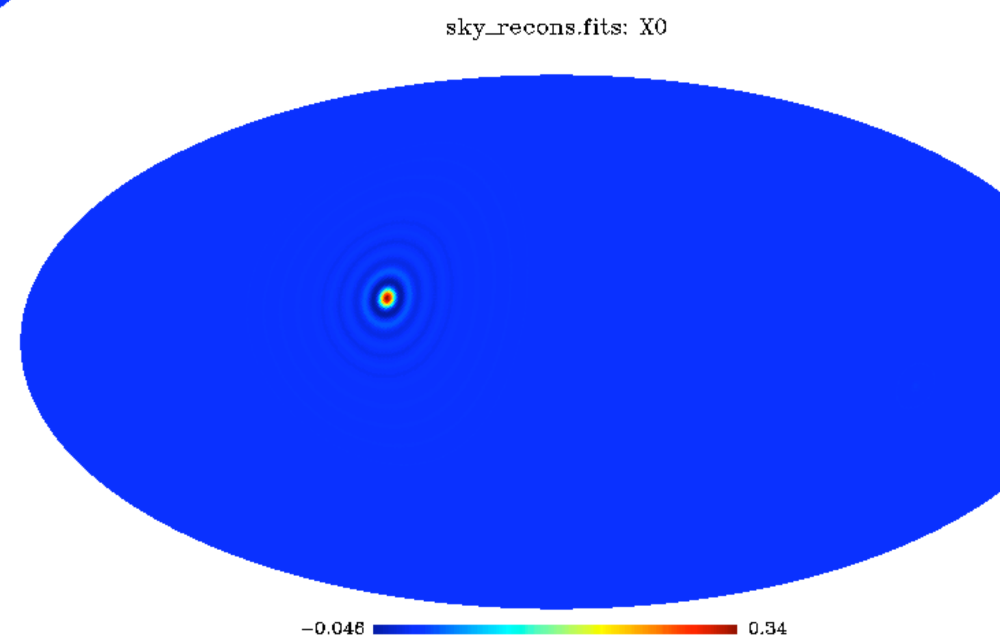
$$\hat{c}_j(l, m) = \hat{c}_{j+1}(l, m) \hat{\tilde{h}}(l_{max}/2^j, m) + \hat{w}_{j+1}(l, m) \hat{\tilde{g}}(l_{max}/2^j, m)$$

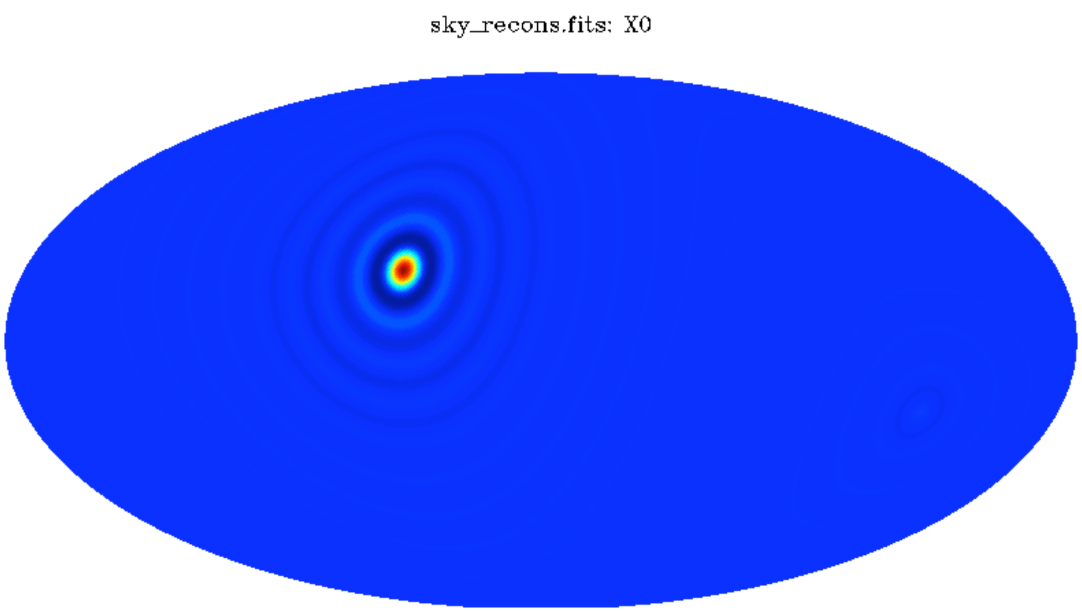

ess.fits: TEMPERATURE
−11.1
11.1 unknown


8result_h.fits: TEMPERATURE
−2.5
2.5 unknown

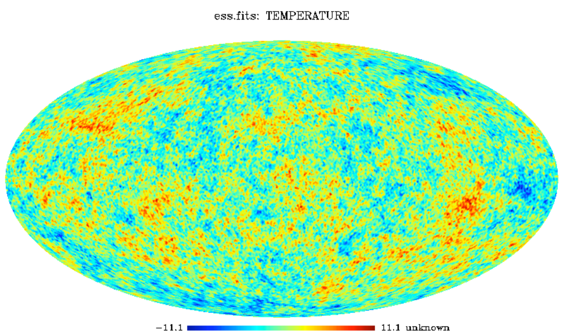

ess.fits: TEMPERATURE
−11.1
11.1 unknown
Undecimated
Wavelet Transform
j=1
1result_g.fits: TEMPERATURE
−8.1
8.1 unknown
j=2
2result_g.fits: TEMPERATURE
−2.9
2.6 unknown
j=3
4result_g.fits: TEMPERATURE
−2.2
1.8 unknown
j=4
8result_g.fits: TEMPERATURE
−1.6
1.7 unknown
8result_h.fits: TEMPERATURE
−2.5
2.5 unknown

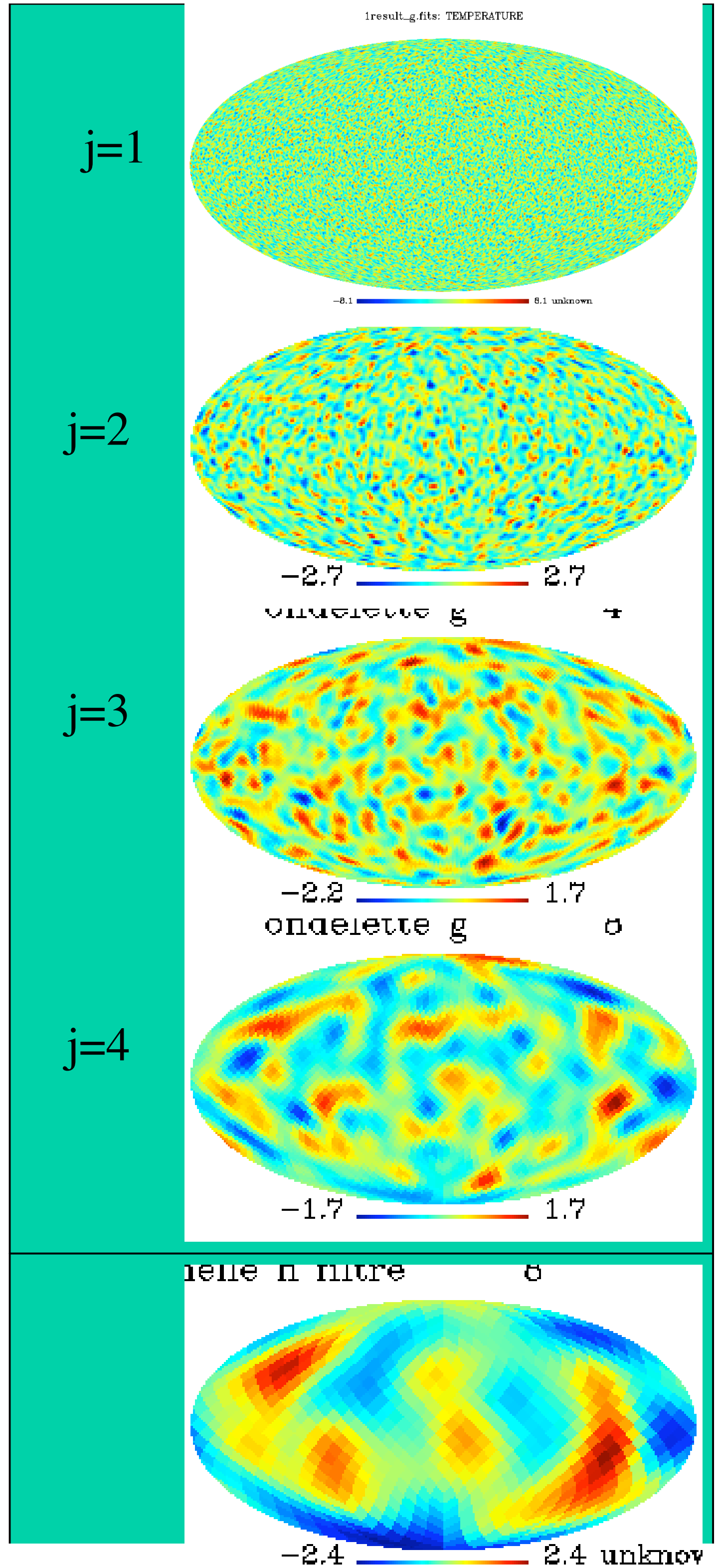

−0.045
0.32


sky_recons.fits: X0
−0.046
0.34

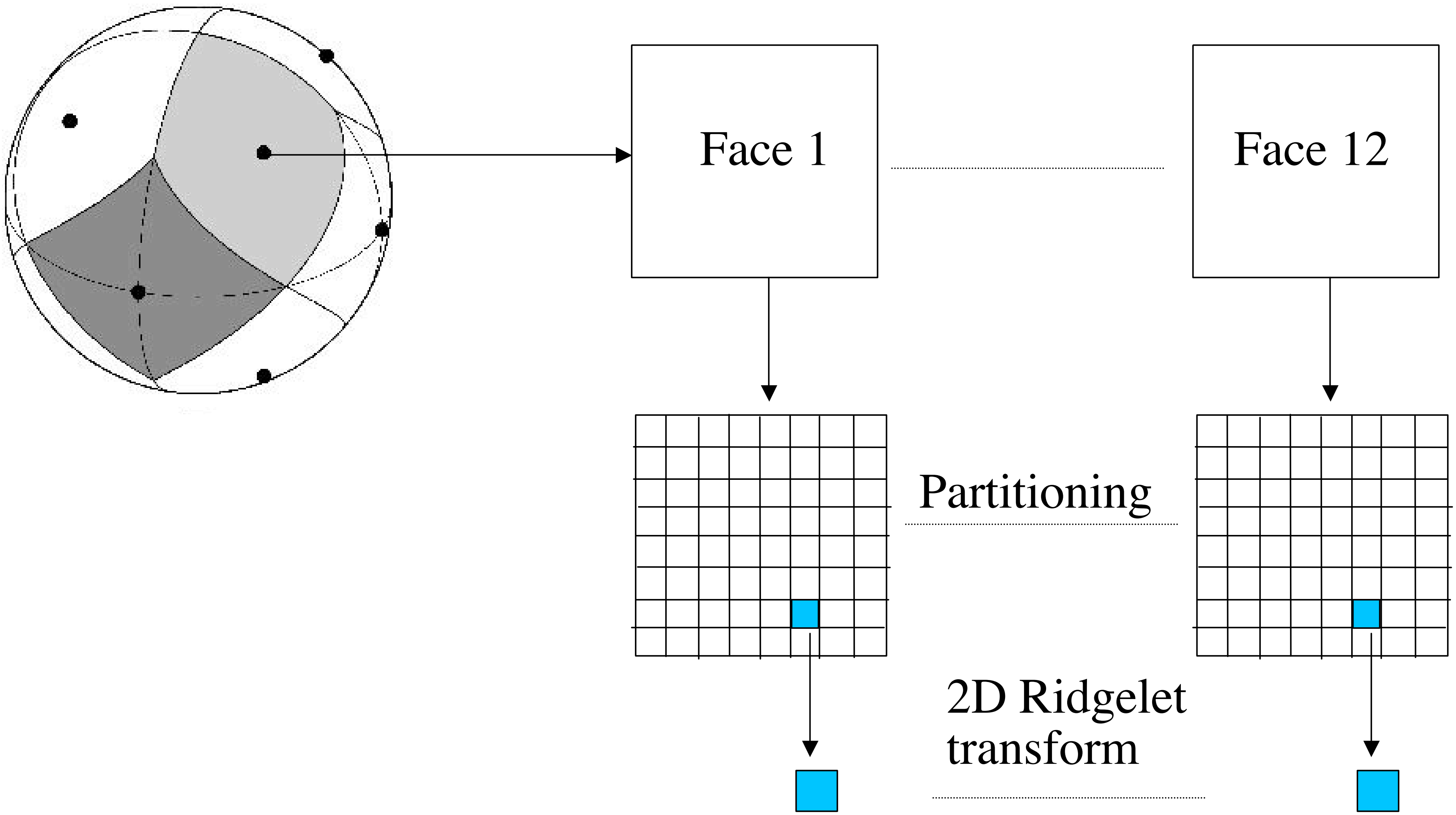

sky_recons.fits: X0

# Pyramidal Wavelet Transform
# On the Sphere

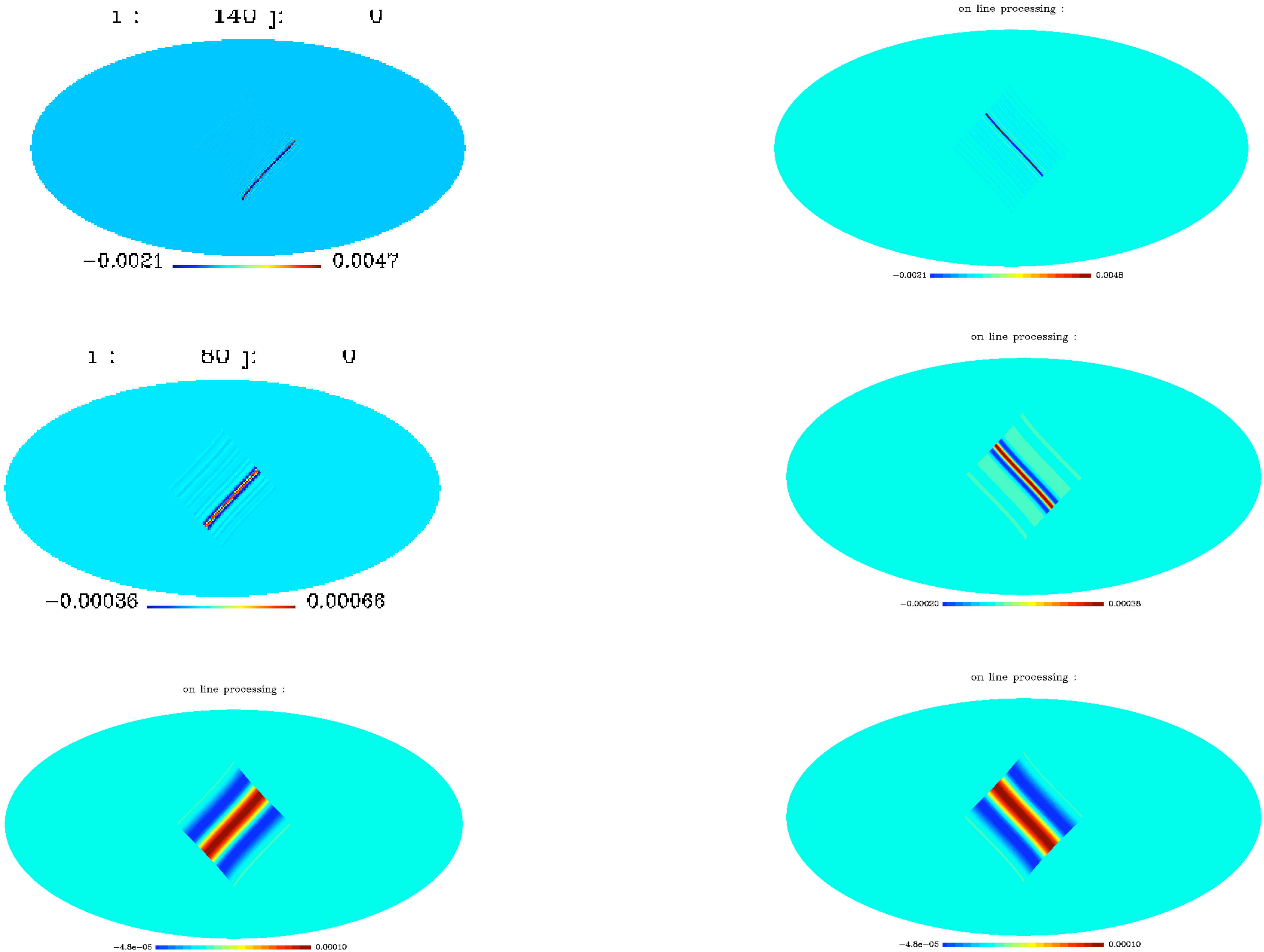


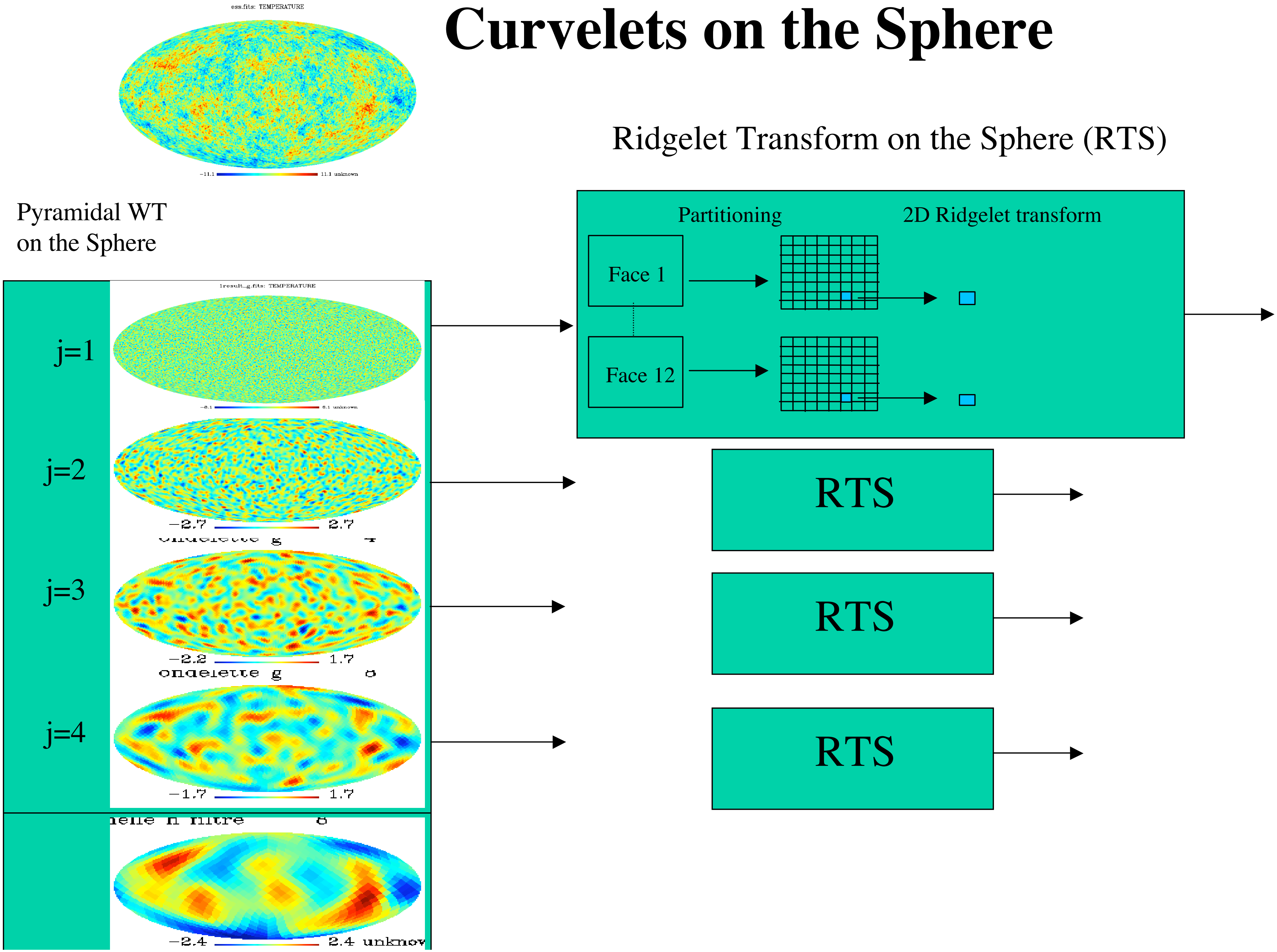
# Ridgelets on the Sphere



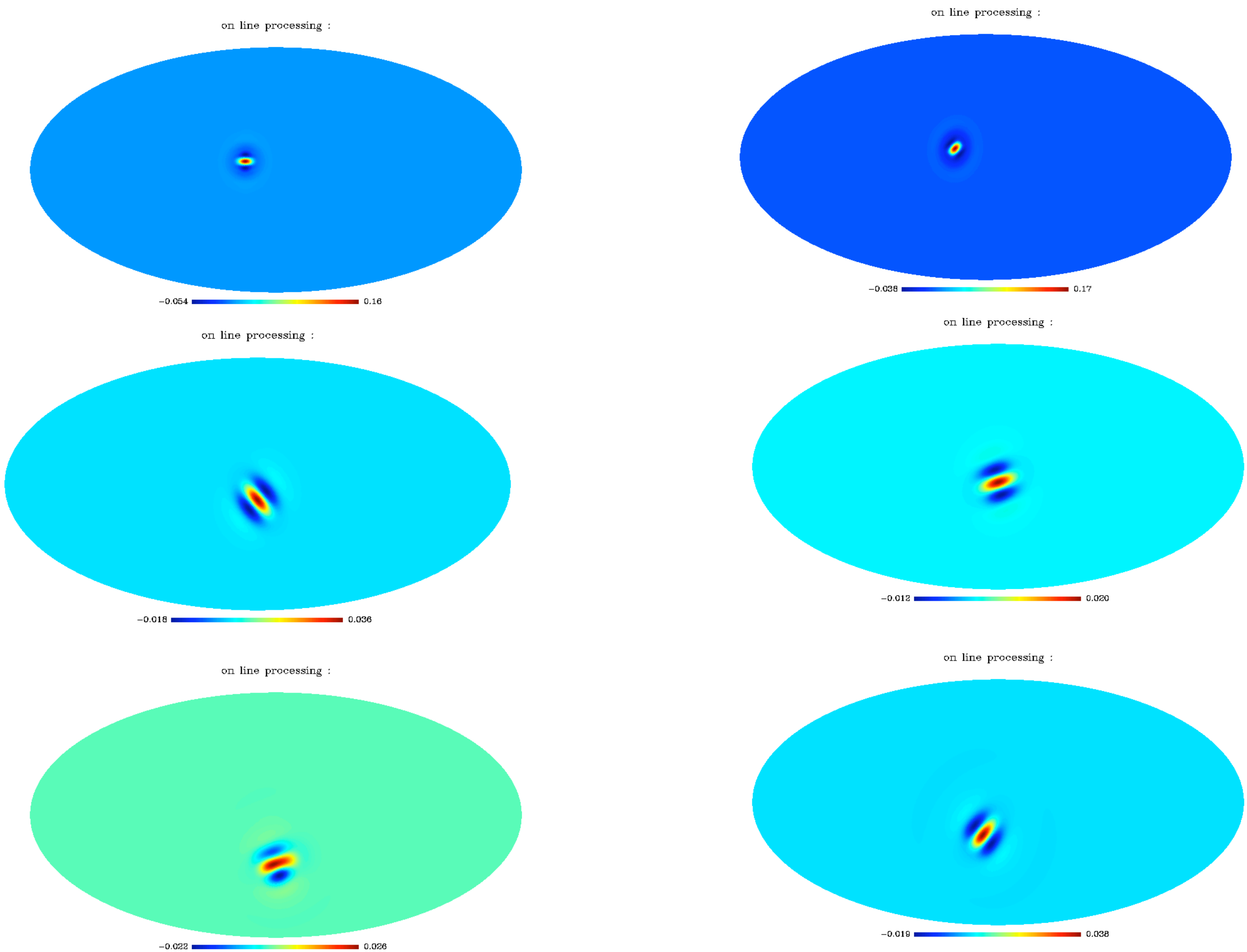
# Example of  ridgelet functions on the sphere

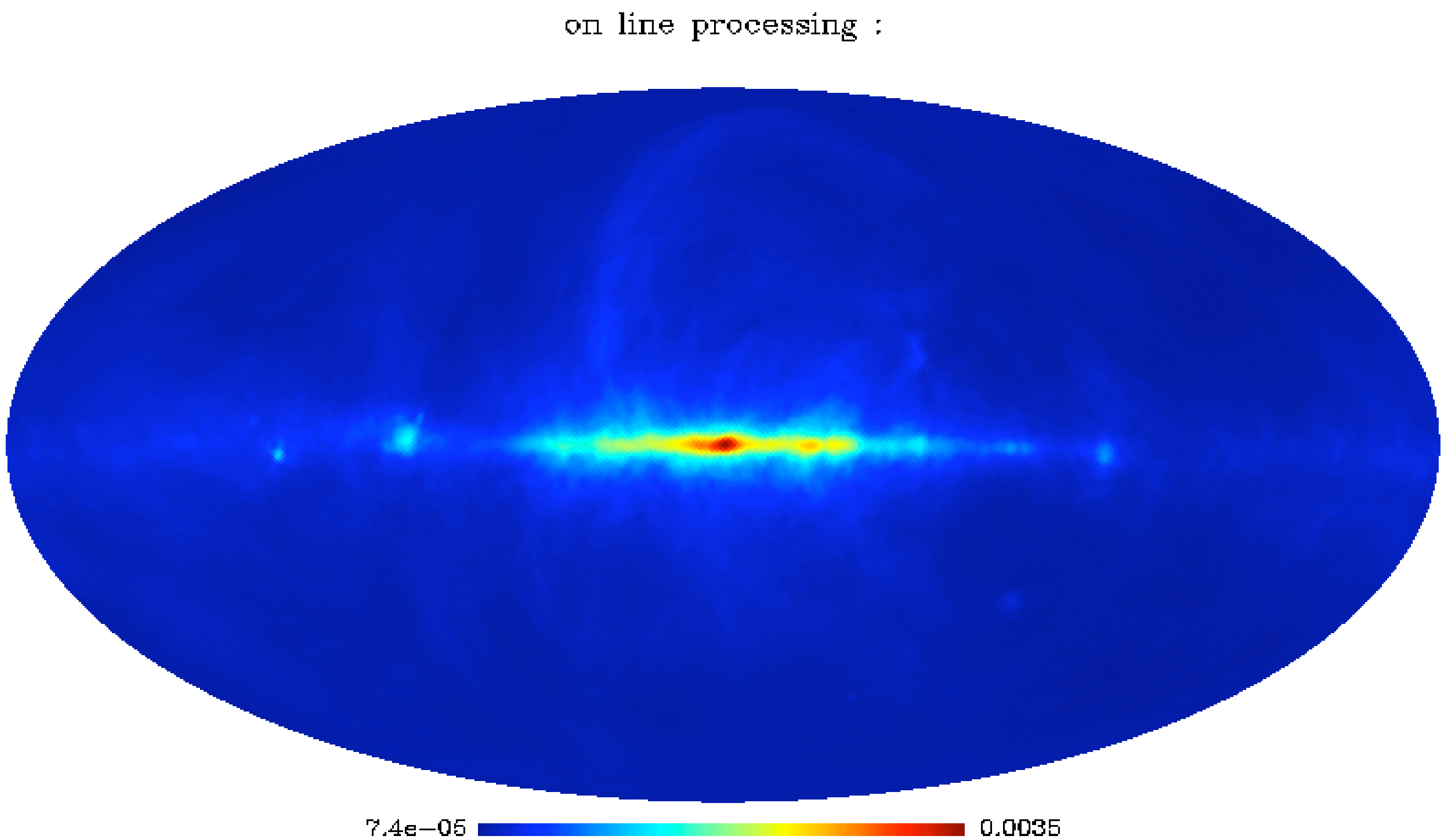


# Curvelets on the Sphere

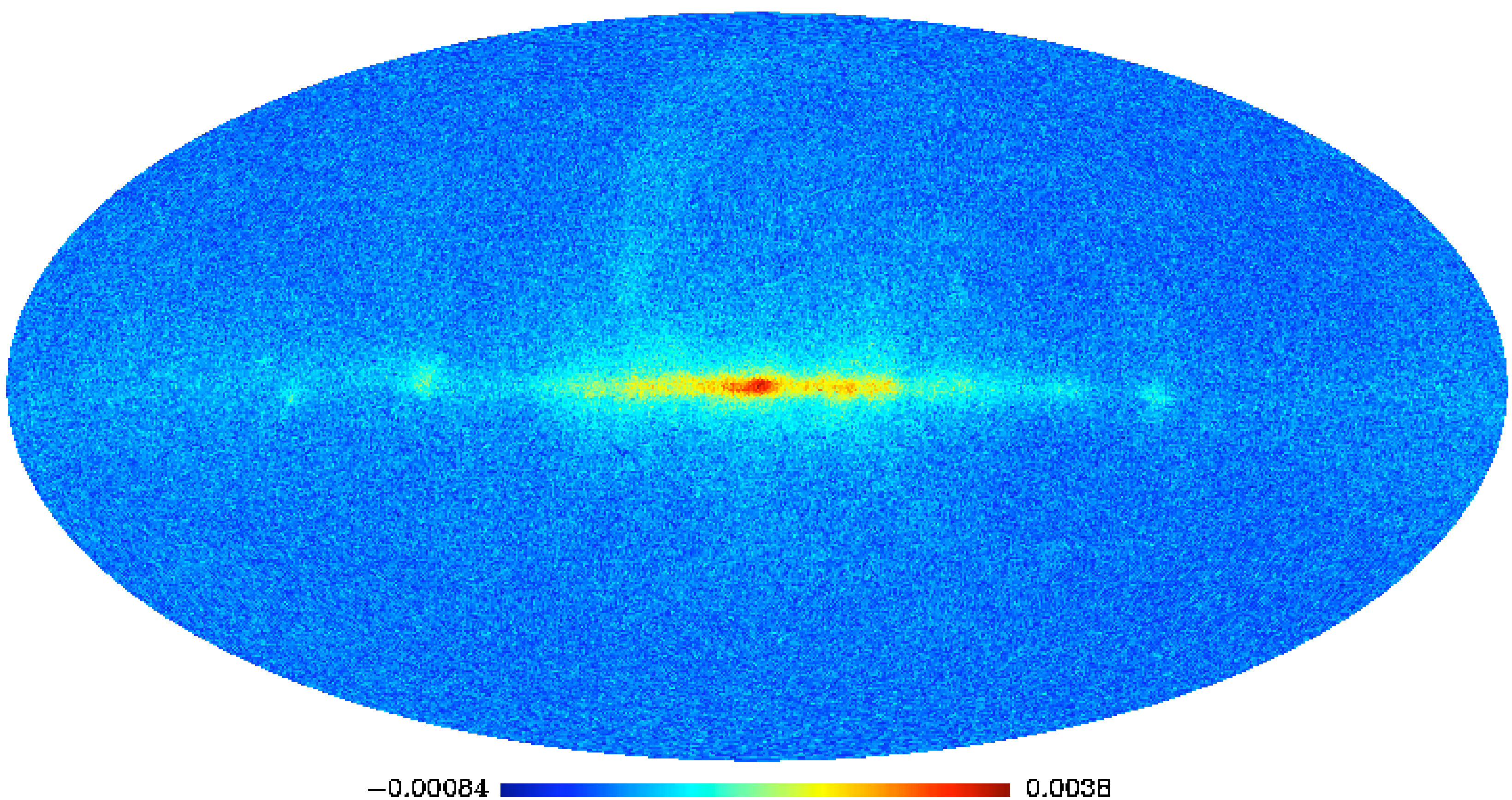


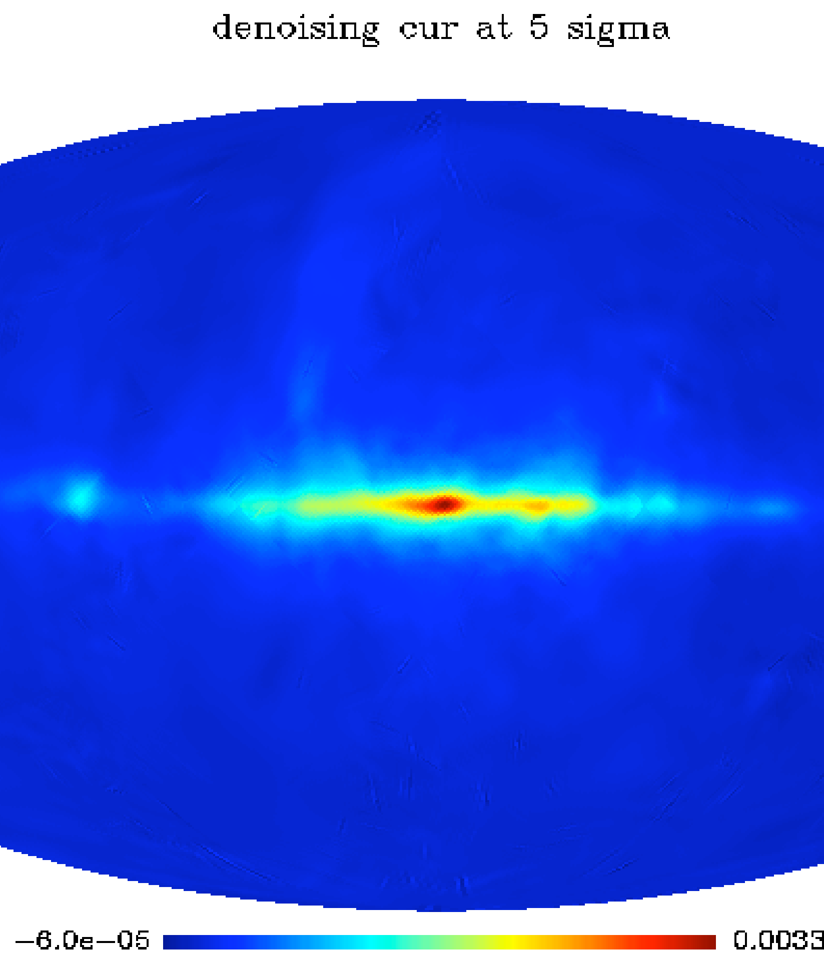
# Example of curvelet functions on the sphere

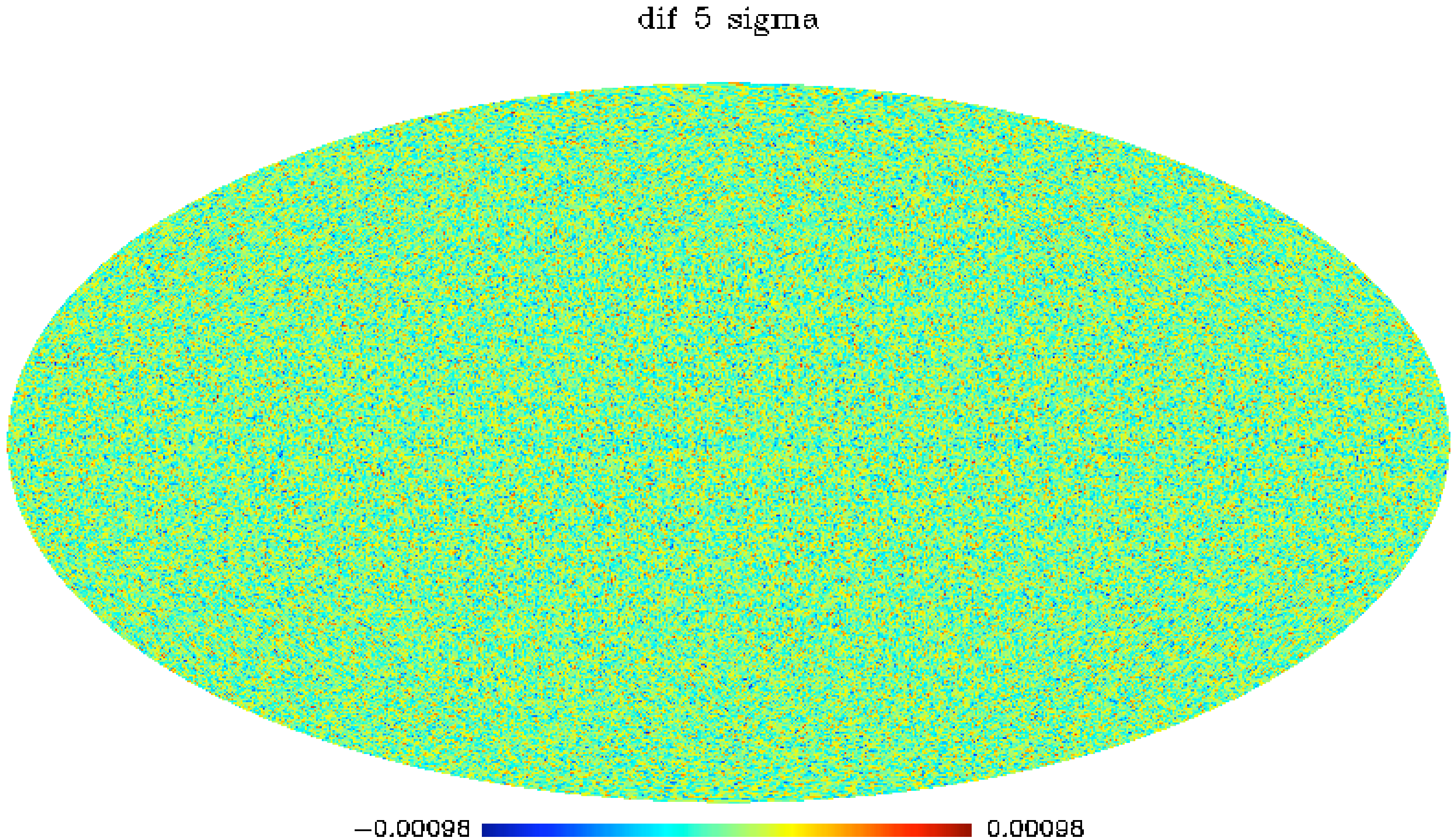



on line processing :
7.4e−05
0.0035

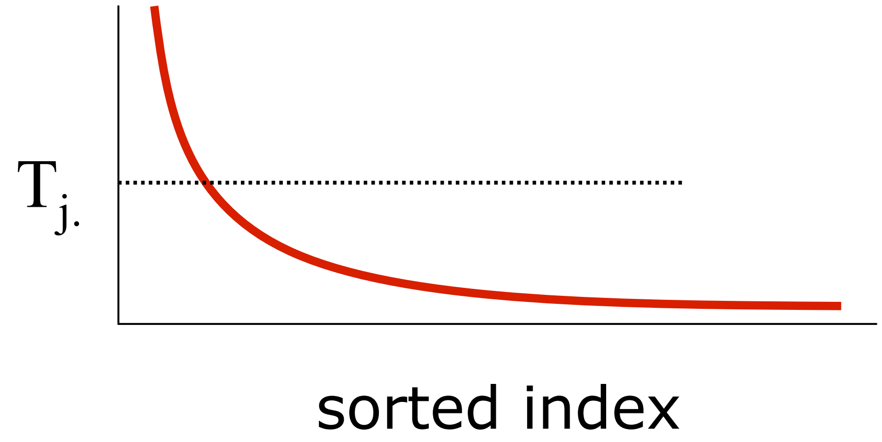
bruit1




denoising cur at 5 sigma
−6.0e−05
0.0033


dif 5 sigma
−0.00098
0.00098

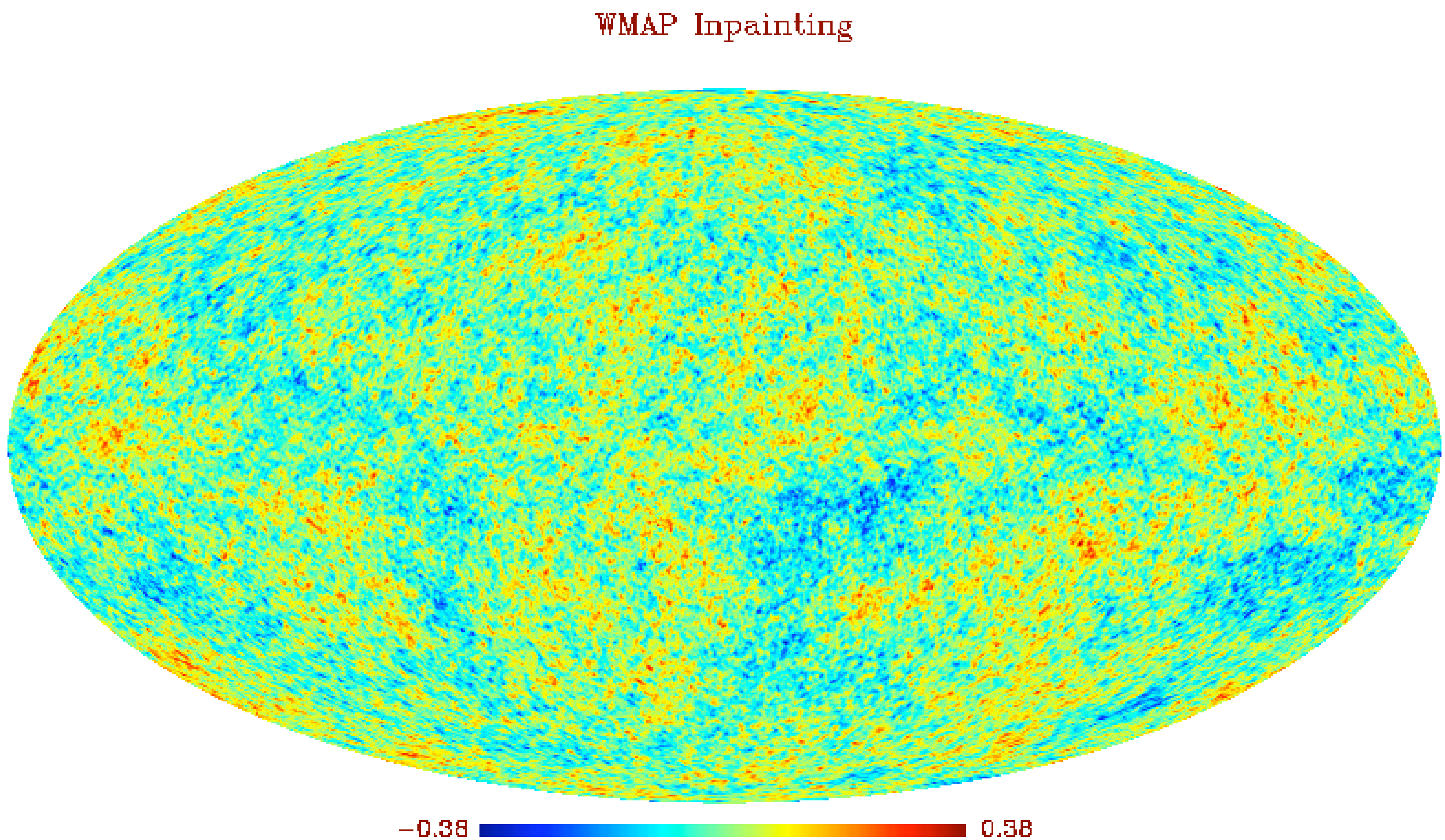
# Remove the CMB

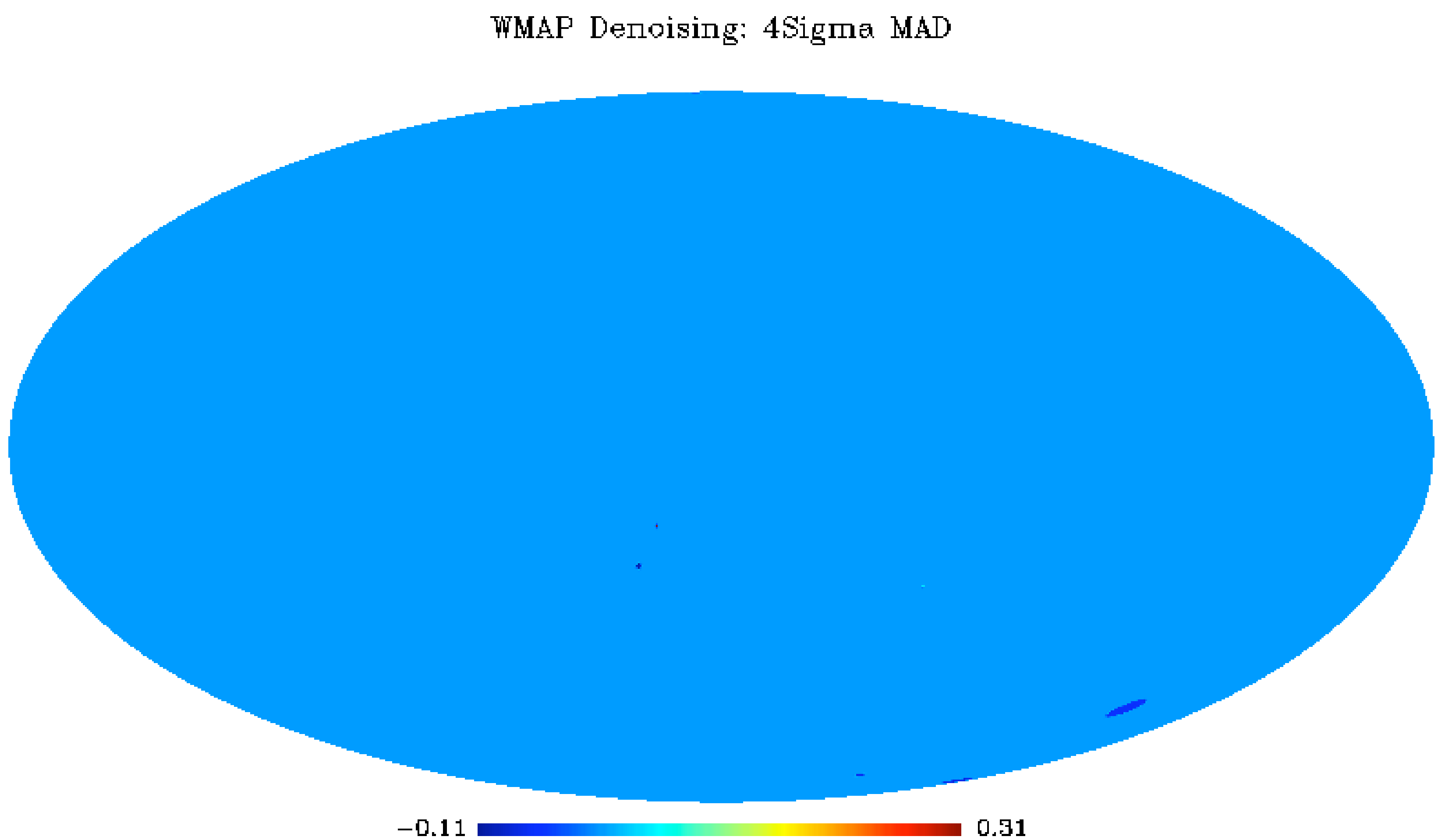


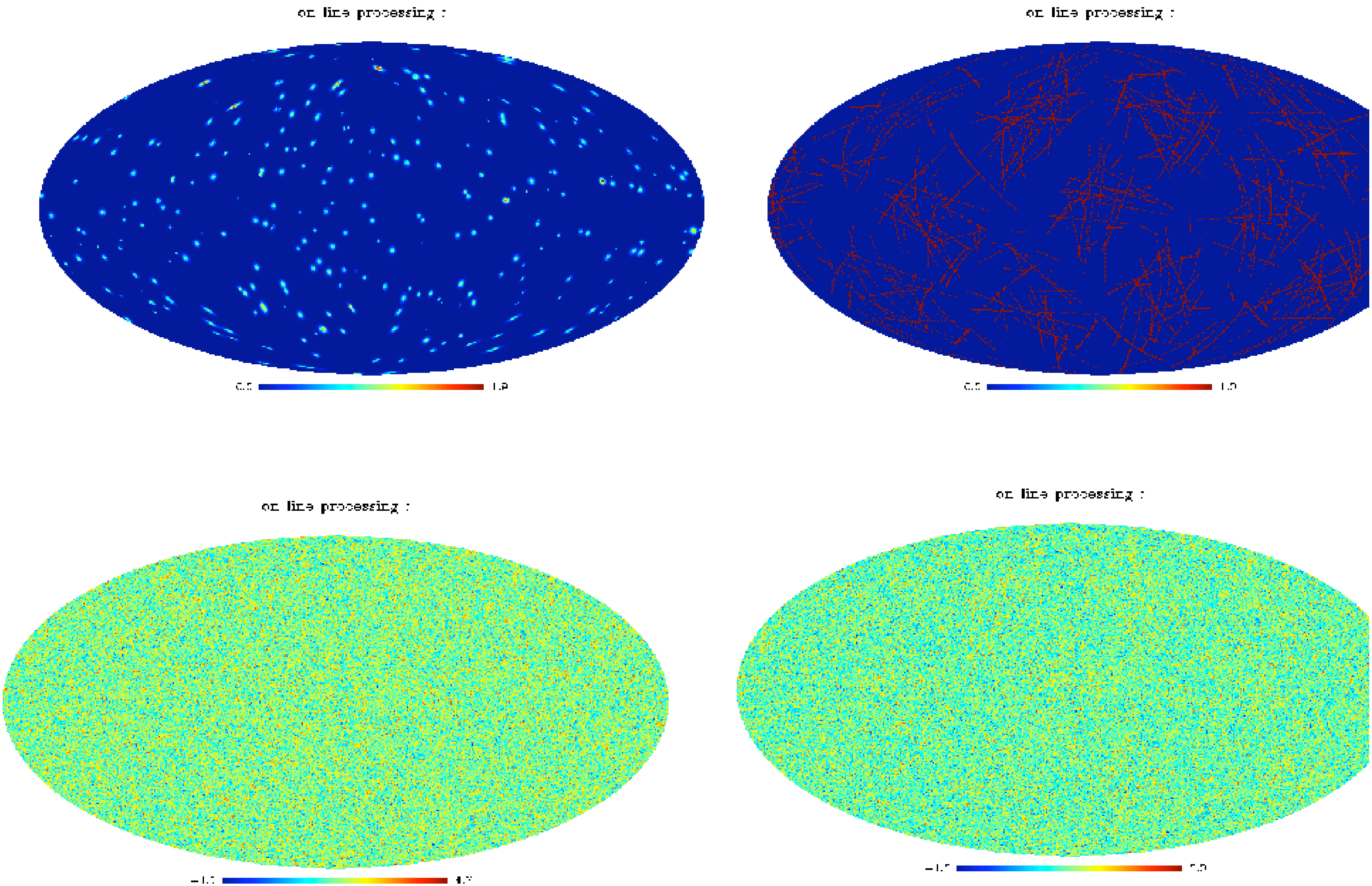

- Take the wavelet transform of the data

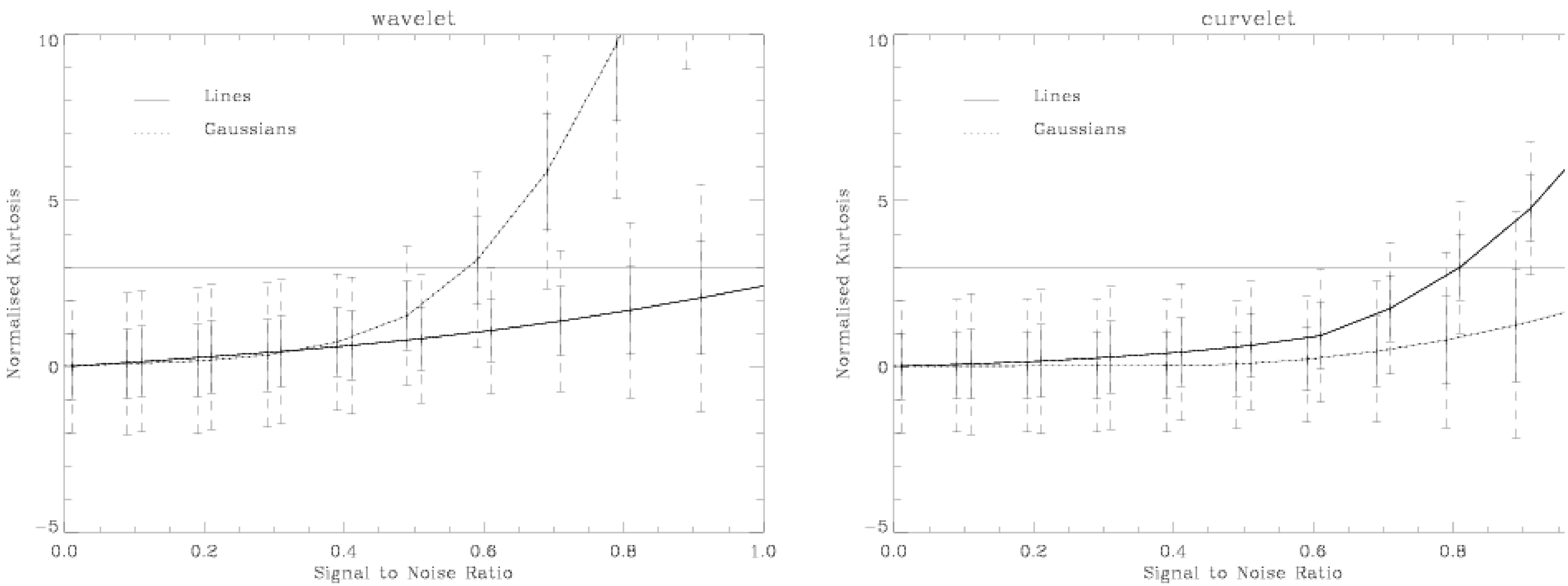
- For each wavelet scale j
  - Compute the noise at scale j using the Median Absolute Deviation: $\sigma_j$ = MAD( | coeff | )/ 0.67
  - Set to zero all coefficients with an absolute value lower than $T_j$ derived from $\sigma_j$ .

- Apply the inverse wavelet transform to the thresholded coefficients.

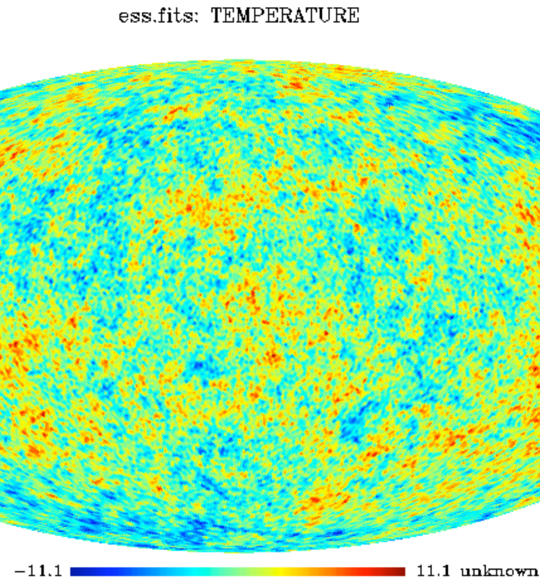

WMAP Inpainting
−0.38
0.38


WMAP Denoising: 4Sigma MAD
−0.11
0.31


on line processing :
on line processing :
on line processing :
on line processing :


wavelet
Lines
Gaussians
Normalised Kurtosis
Signal to Noise Ratio
curvelet
Lines
Gaussians
Normalised Kurtosis
Signal to Noise Ratio

# Wavelet, Ridgelet and Curvelet on the Sphere :



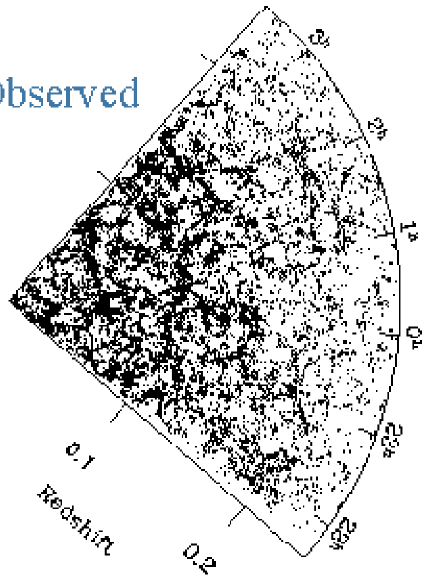

***Wavelets, Ridgelets and Curvelets on the Sphere***, *Astronomy & Astrophysics*, 446, 1191-1204, 2006.

Software available at: http://jstarck.free.fr/mrs.html

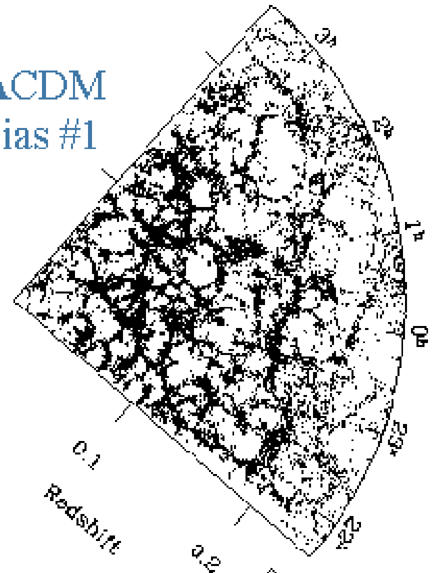
Multiscale transforms, Gaussianity tests
Denoising using Wavelets and Curvelets
Astrophysical Component Separation (ICA on the Sphere)

# Models vs observations

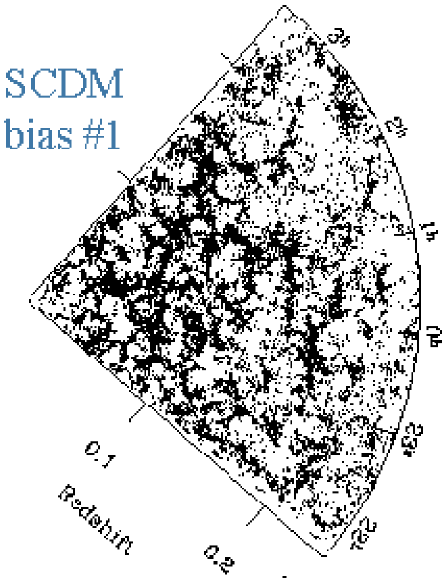
Observed



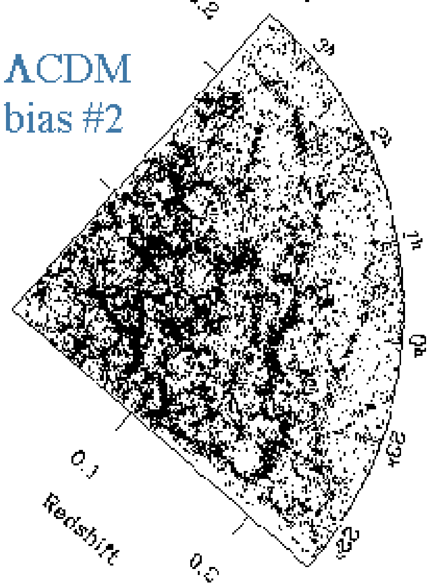

ΛCDM
bias #1



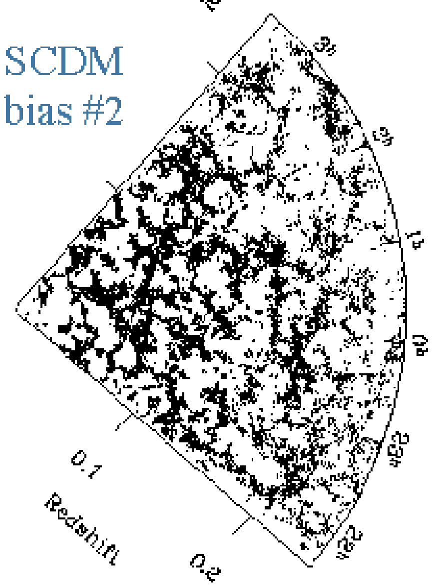

SCDM
bias #1




ΛCDM
bias #2




SCDM
bias #2



# Methods

.Two or three point correlation function

. Genus curve

. Voronoi Tessellation

. Minimal spanning trees

. Power spectrum

. Fractals

# The Two-Point Correlation Function

A measure of the deviation from randomnes:

$$\xi(r) = \frac{n_{DD}(r)}{n_{RR}(r)} - 1$$

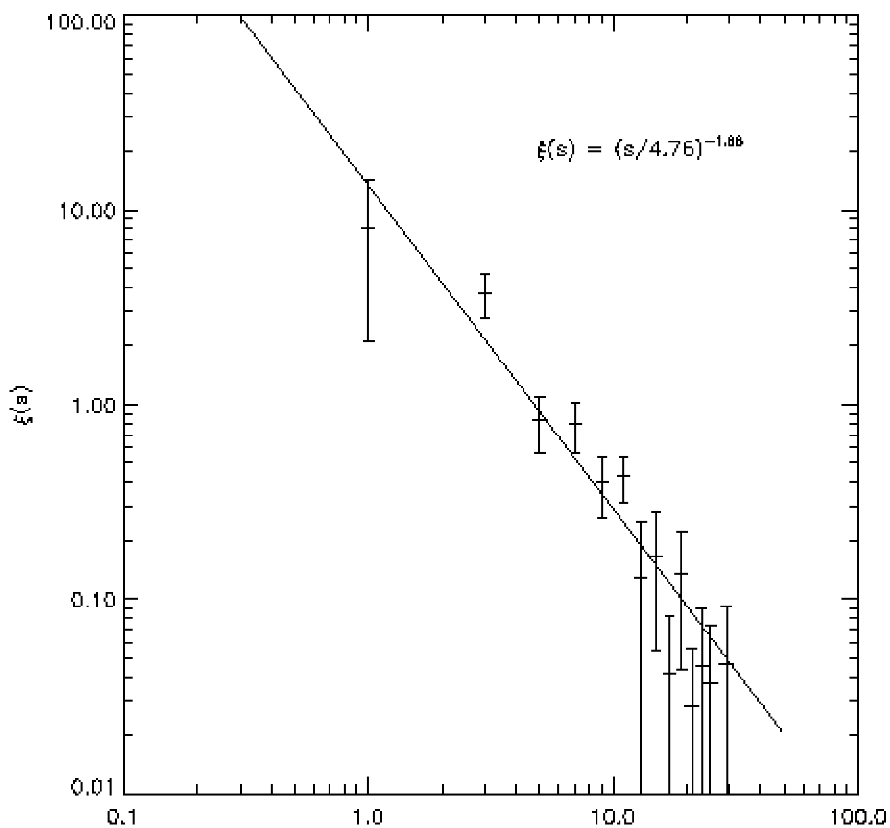
$n_{DD}(r)$ = number of pairs with a separation of r in the data

$n_{RR}(r)$ = number of pairs with a separation of r for a ramdomly distributed data set

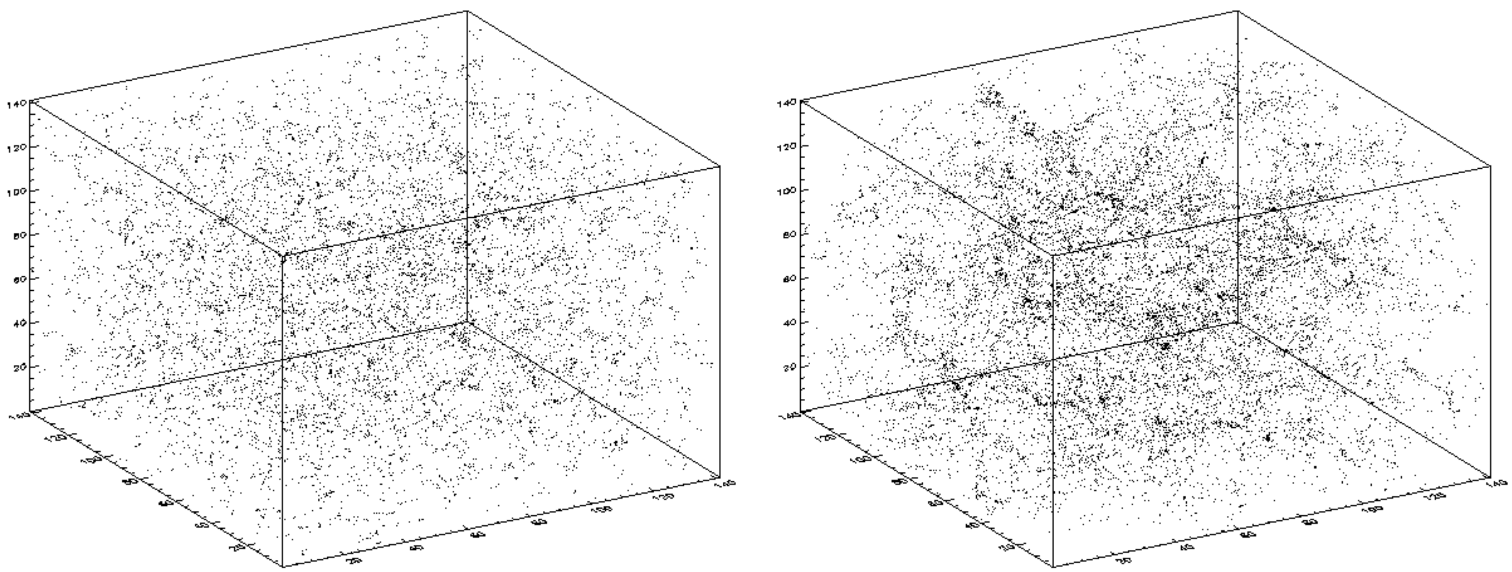
Estimates of the correlation function of the galaxies indicate that it is power law function of the form,

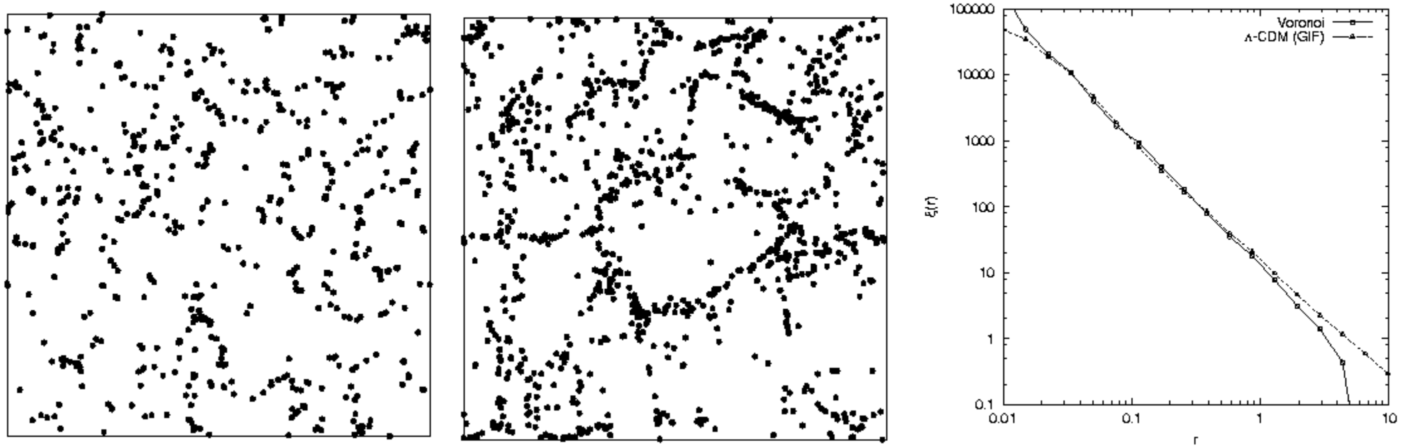
$$\xi(r) = \left(\frac{r}{r_0}\right)^{-1.8}, r_0 = 5\,Mpc$$

Where $r_0$ is called the correlation length.



# Simulations





Voronoi
Λ-CDM (GIF)
ξ(r)
r
100000
10000
1000
100
10
1
0.1
0.01
0.1
1
10

# GENUS FUNCTION

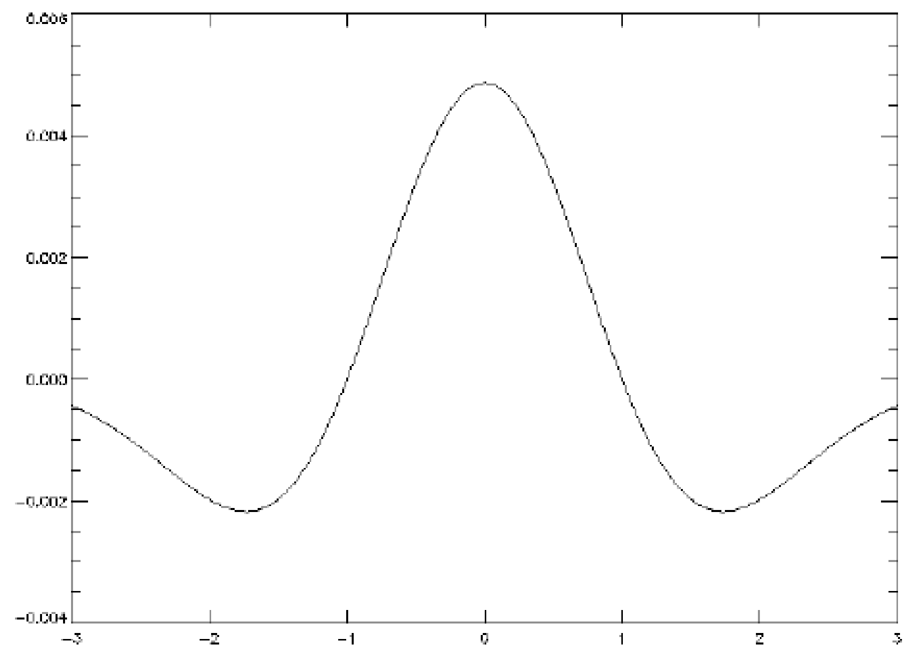
The genus of a surface G is

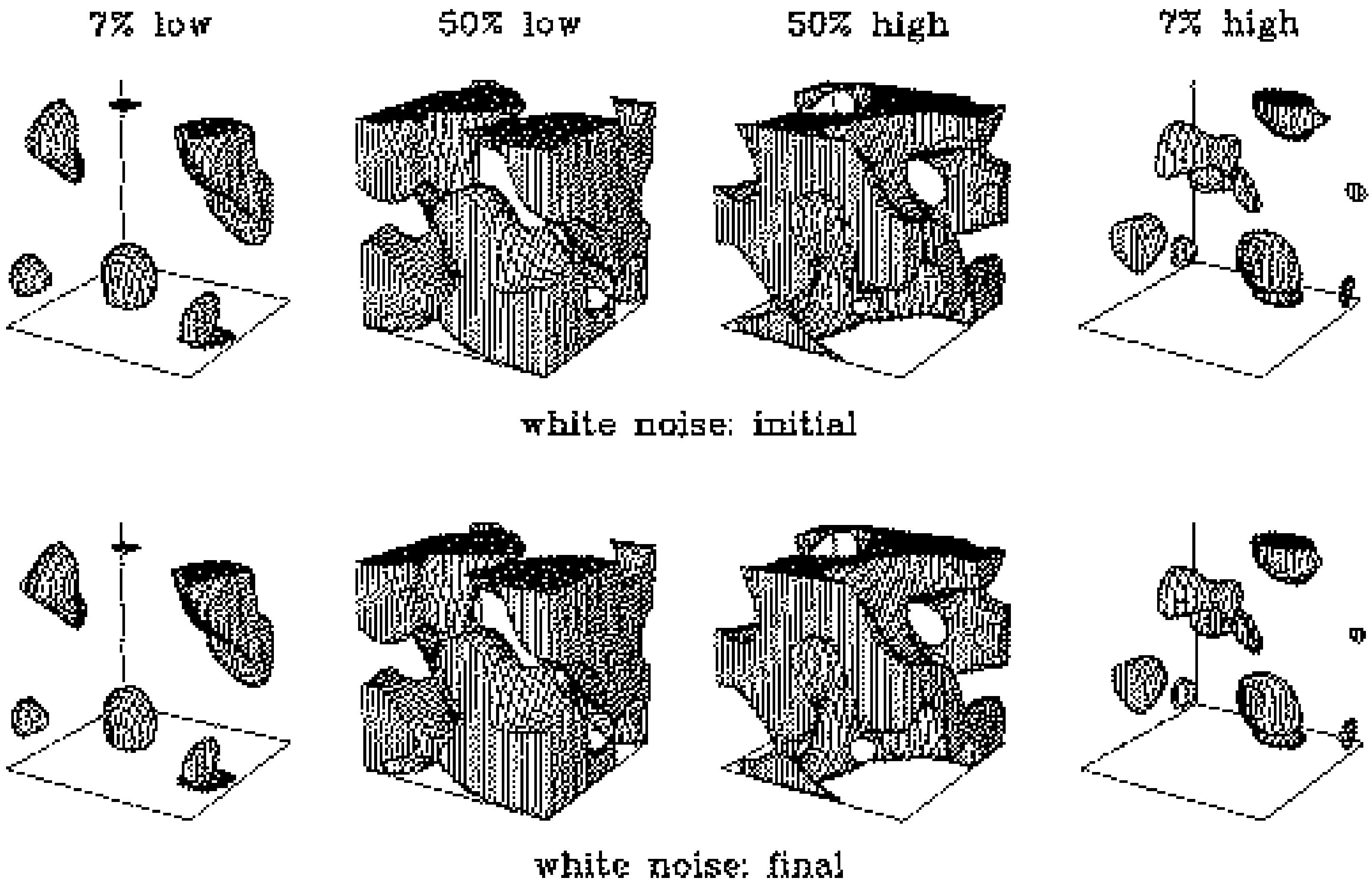
**G(T) = (number of holes) - (number of isolated regions) + 1**

- Convolve the data by a Gaussian
  - Threshold all values under a threshold level T
  - G(T) = (number of holes) - (number of isolated regions) + 1

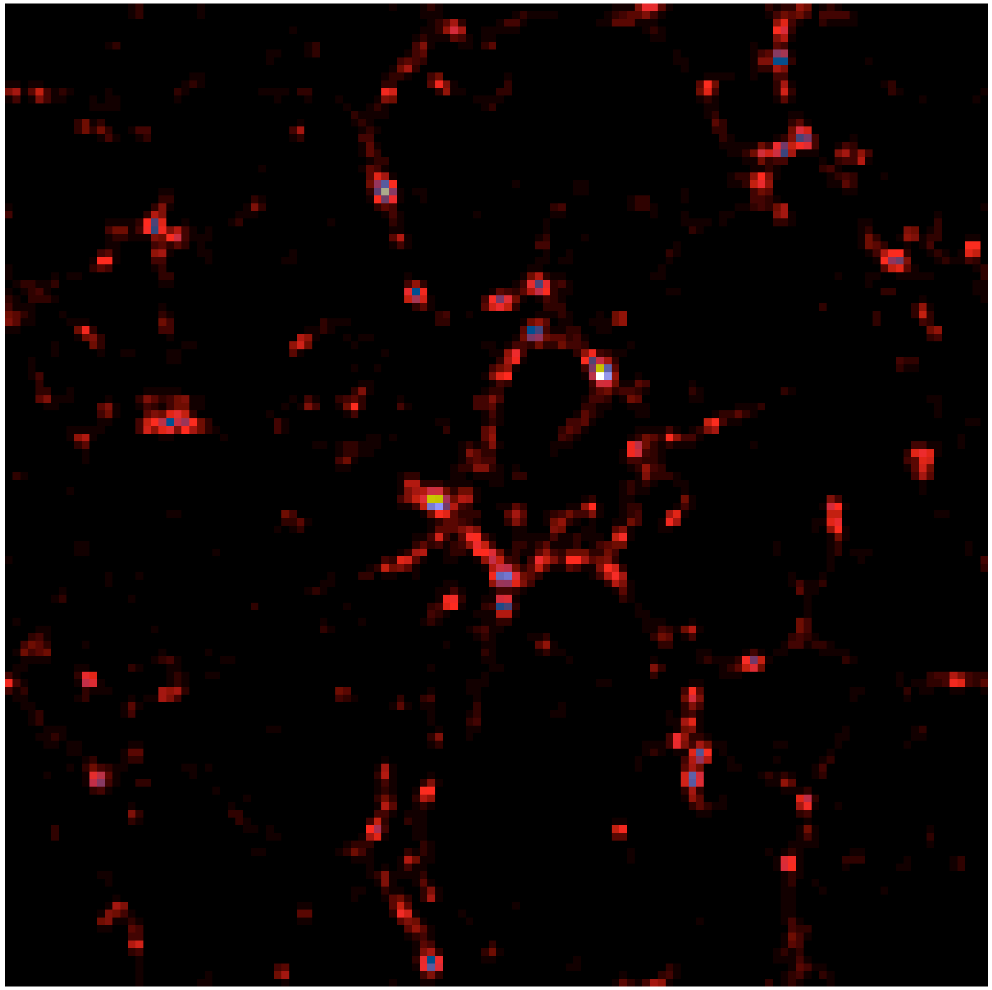
For a Gaussian field, the genus curve is:

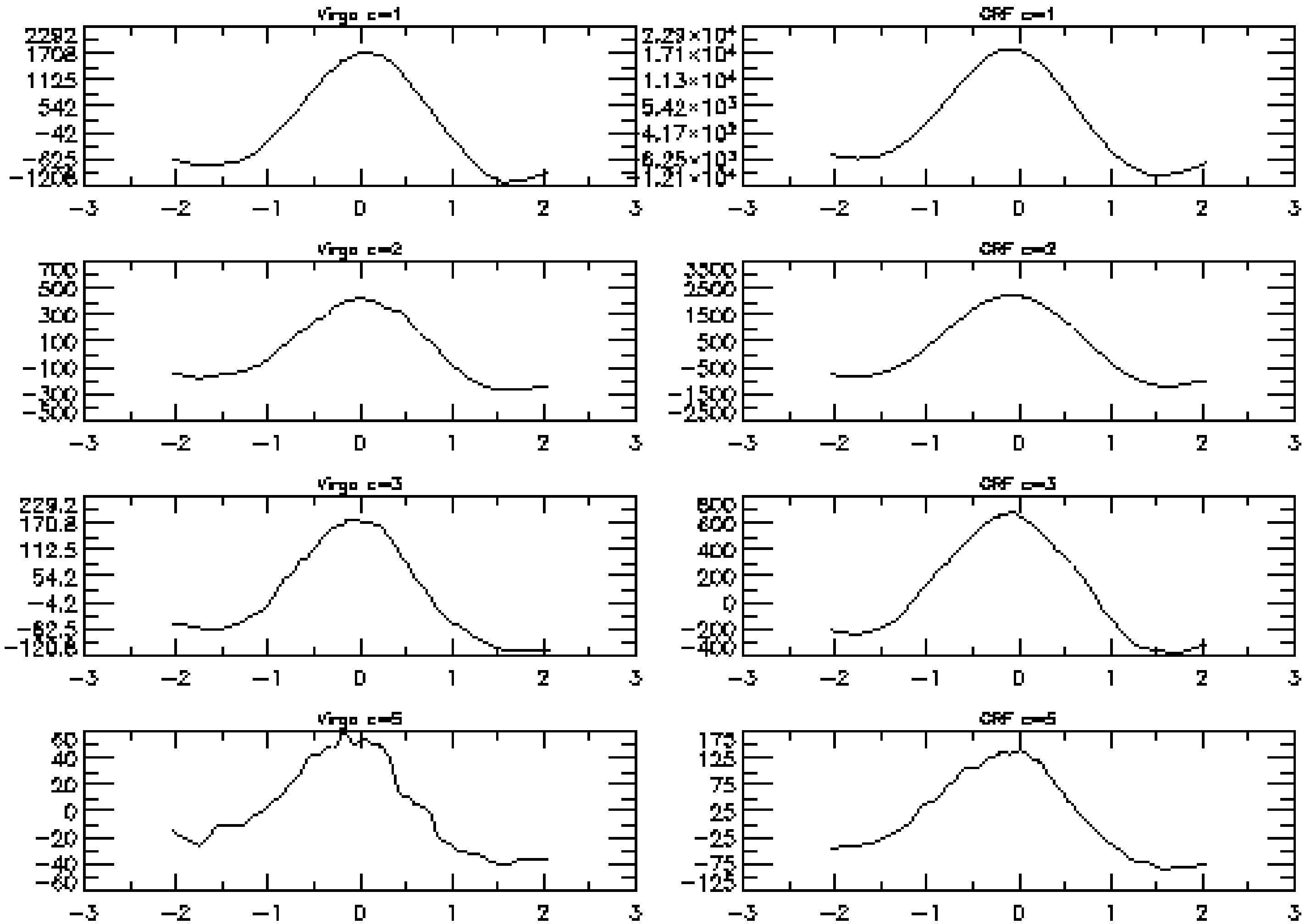
$$g(\nu) = N(1-\nu^2)\exp(-\frac{\nu^2}{2})$$



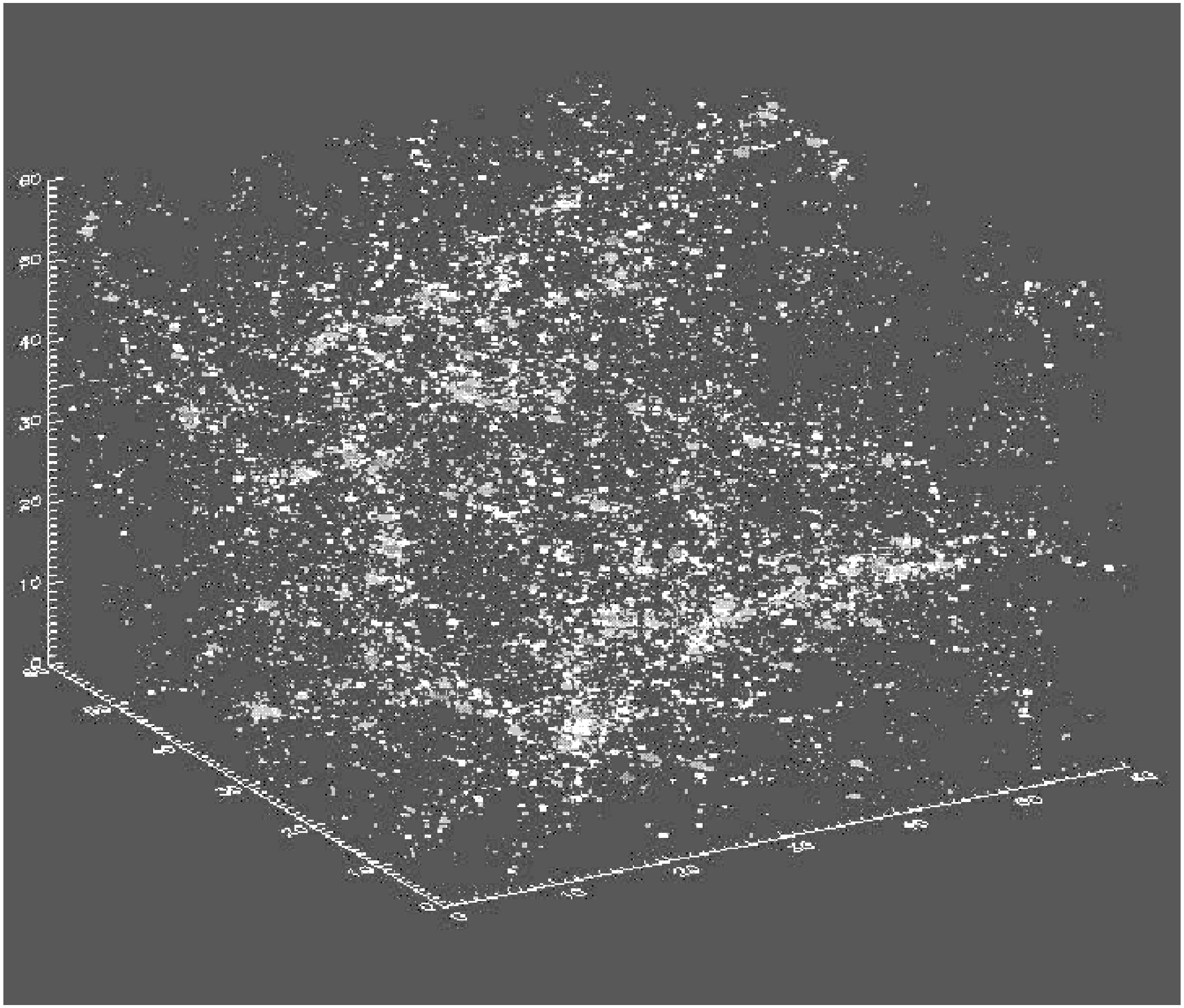

7% low
50% low
50% high
7% high
white noise: initial
white noise: final

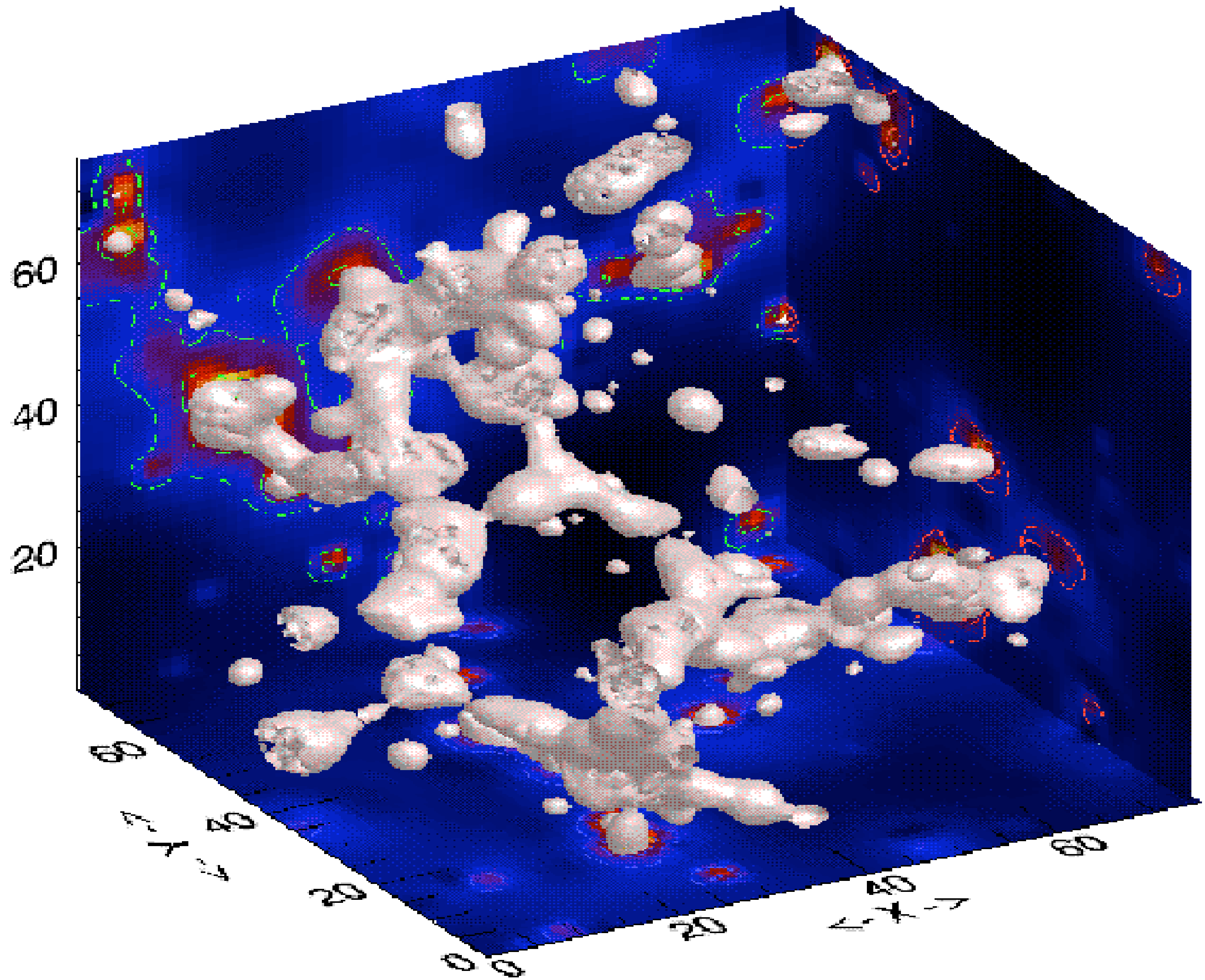
[illegible]

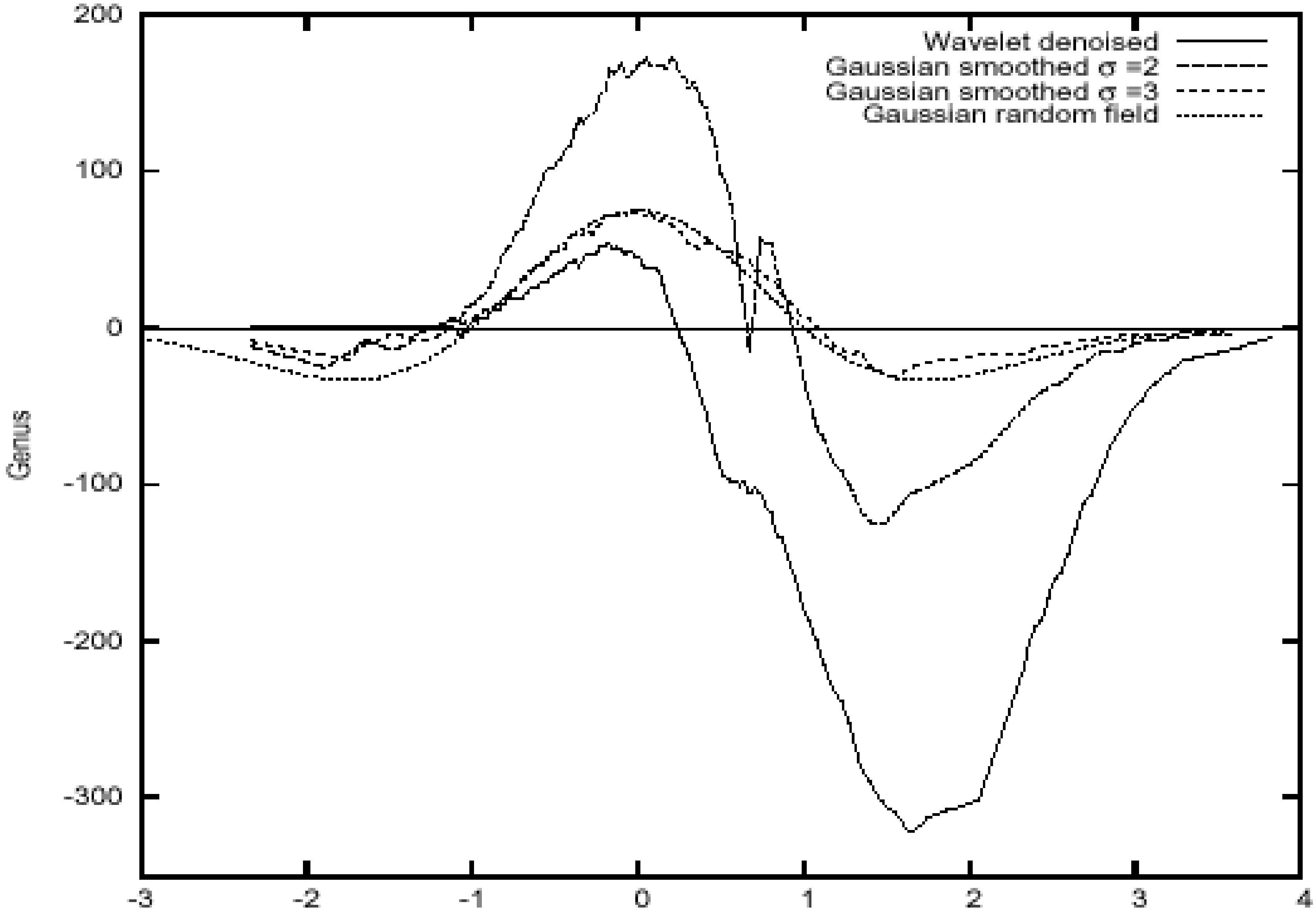




Virgo c=1
2292
1708
1125
542
−42
−625
−1208
CRF c=1
2.29×10^4
1.71×10^4
1.13×10^4
5.42×10^3
4.17×10^3
6.25×10^3
1.21×10^4
Virgo c=2
700
500
300
100
−100
−300
−500
CRF c=2
3500
2500
1500
500
−500
−1500
−2500
Virgo c=3
229.2
170.8
112.5
54.2
−4.2
−62.5
−120.8
CRF c=3
800
600
400
200
0
−200
−400
Virgo c=5
60
40
20
0
−20
−40
−60
CRF c=5
175
125
75
25
−25
−75
−125
−3
−2
−1
0
1
2
3


60
50
40
30
20
10
0

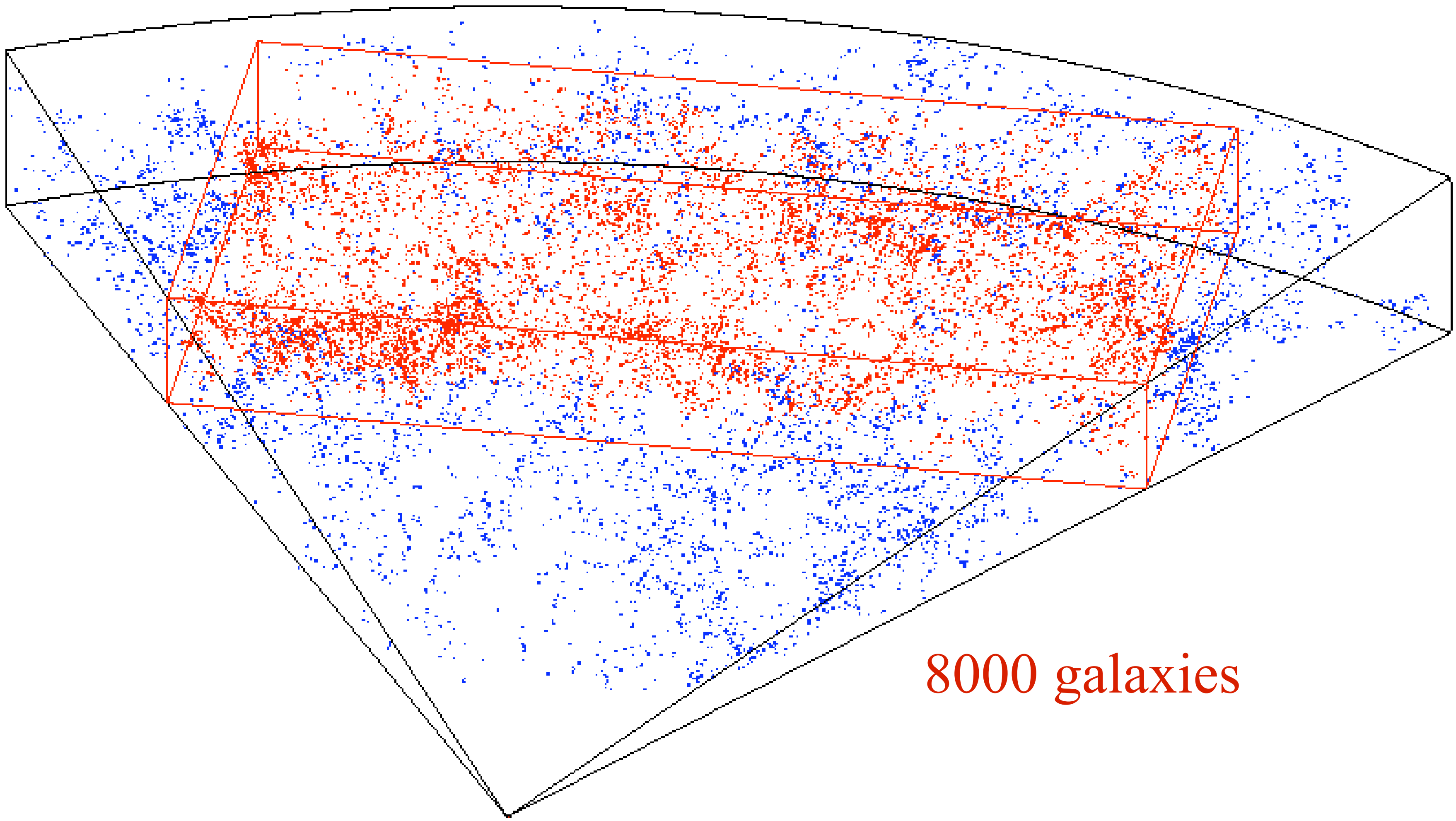

60
40
20
60
40
20
0
0
20
40
60



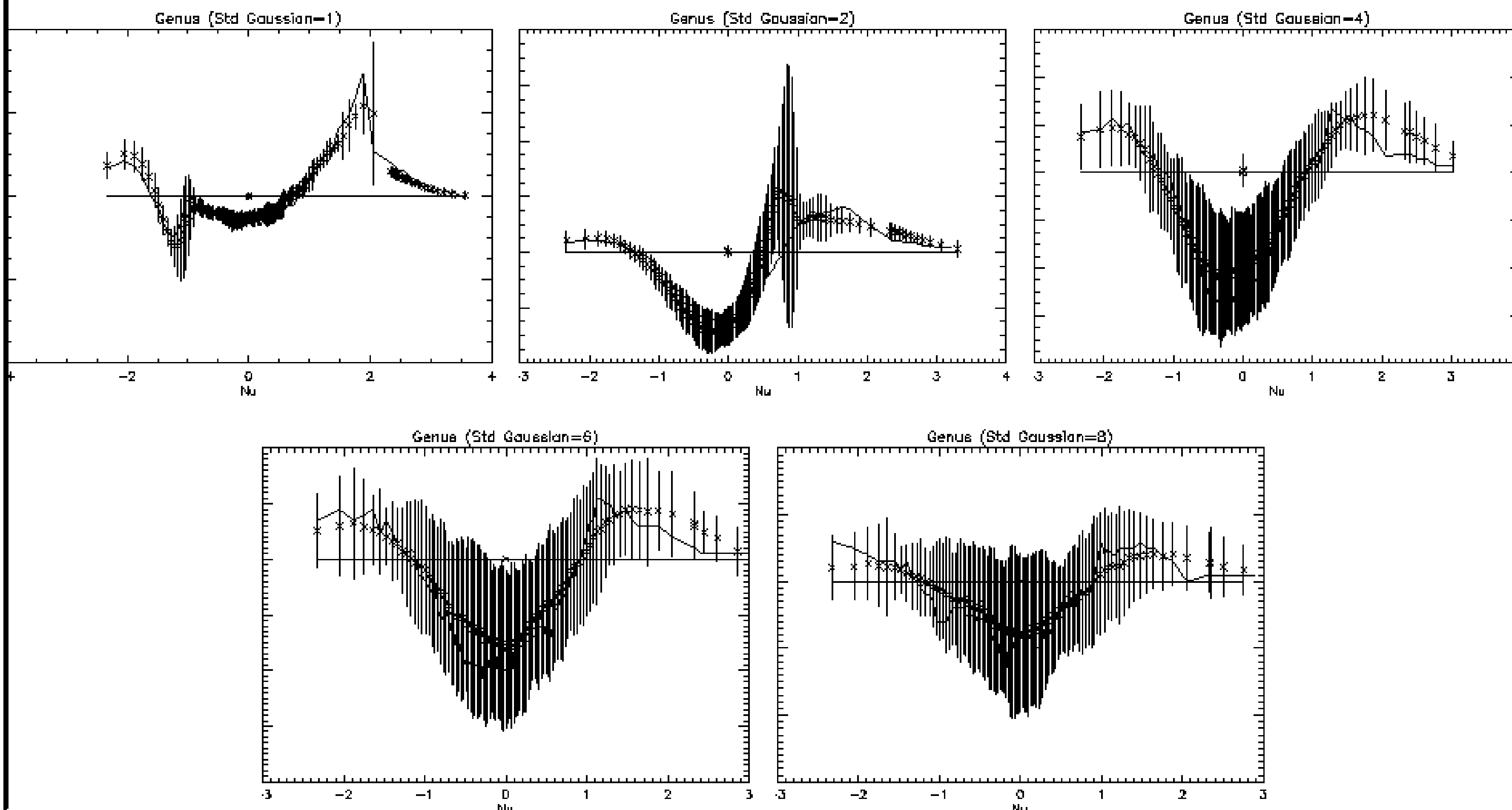

V. Martinez, J.-L. Starck, E. Saar, D.L. Donoho, P. de la Cruz, S. Paredes and S. Reynolds, "Morphology of the Galaxy Distribution from Wavelet Denoising", ApJ, 634, pp 744--755, 2005.

The genus curve of this adaptive reconstructed density field is much more informative because it is unique and does not depend of the particular choice of the filter radius. Additionally, the genus curves of Gaussian-smoothed density fields mimic those of Gaussian random fields, describing thus more the properties of the filter than the real morphology of the density distribution.

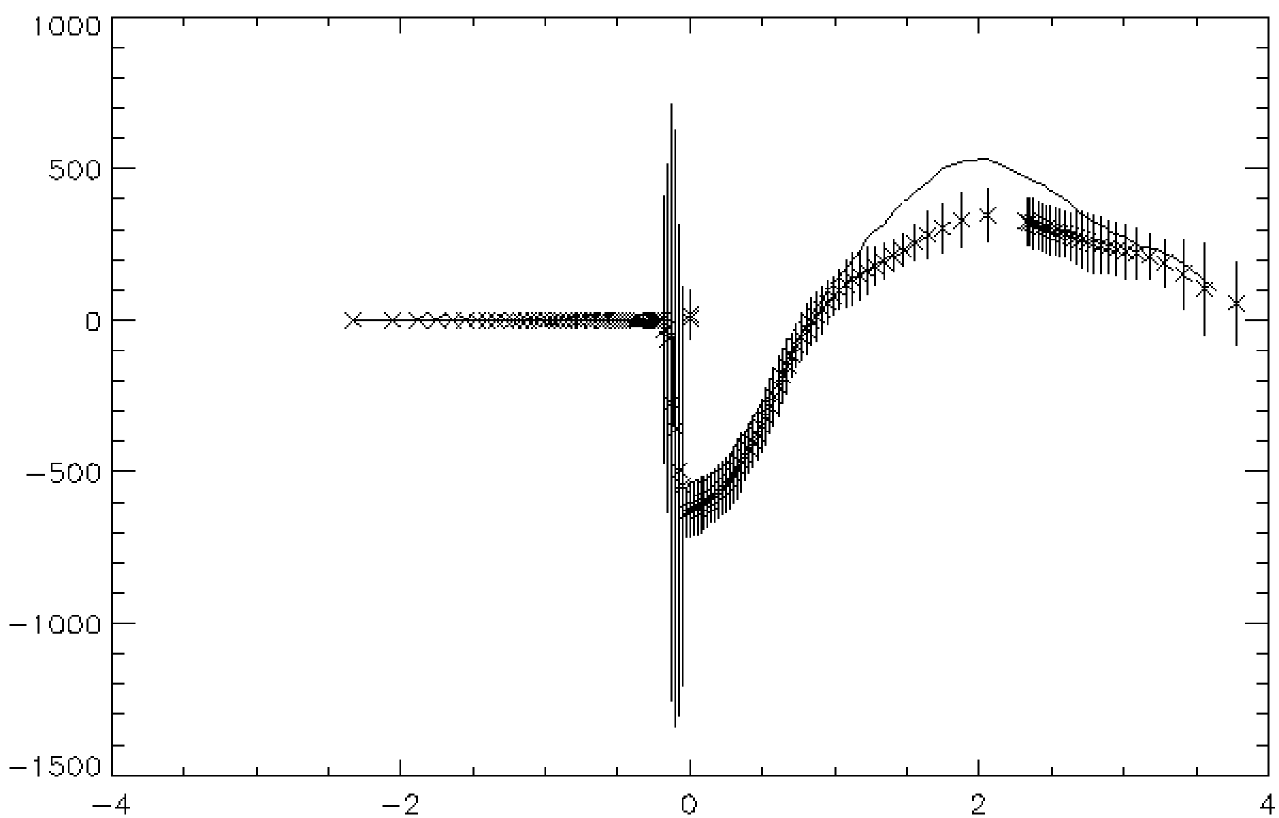
# 2dfGRS northern slice



# Genus: 2DF and Lambda-CDM simulations



# Wavelet Denoising+Genus: 2DF and Lambda-CDM Sim.



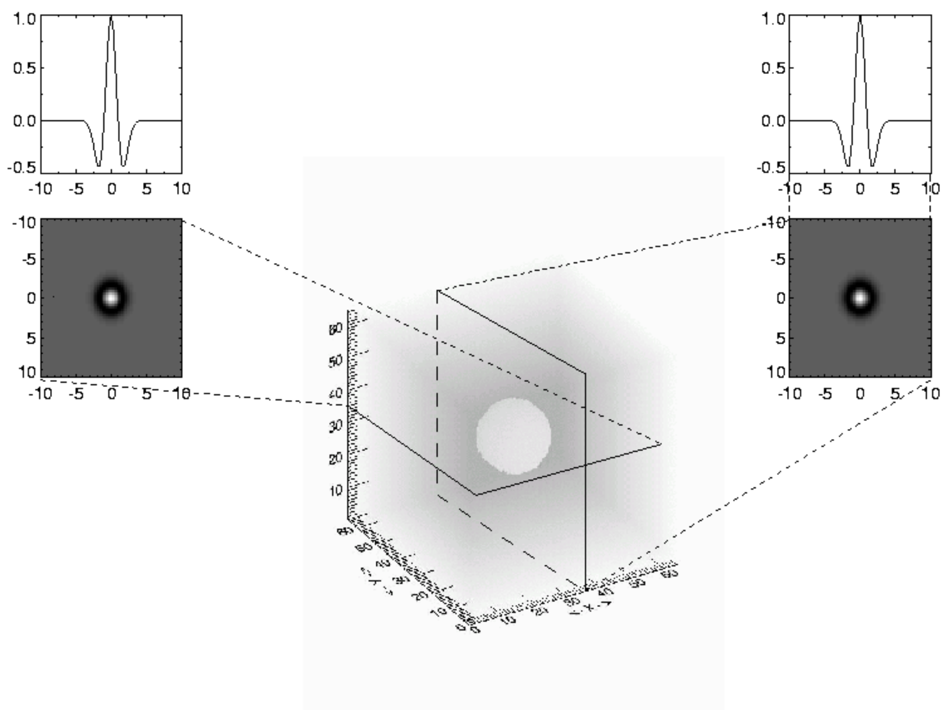
# *3D MULTISCALE TRANSFORMS*

1) **3D WAVELET TRANSFORM: Isotropic Structures**
2) **3D RIDGELET TRANSFORM: Sheet like Structures**
3) **3D BEAMLET TRANSFORM: Filaments**

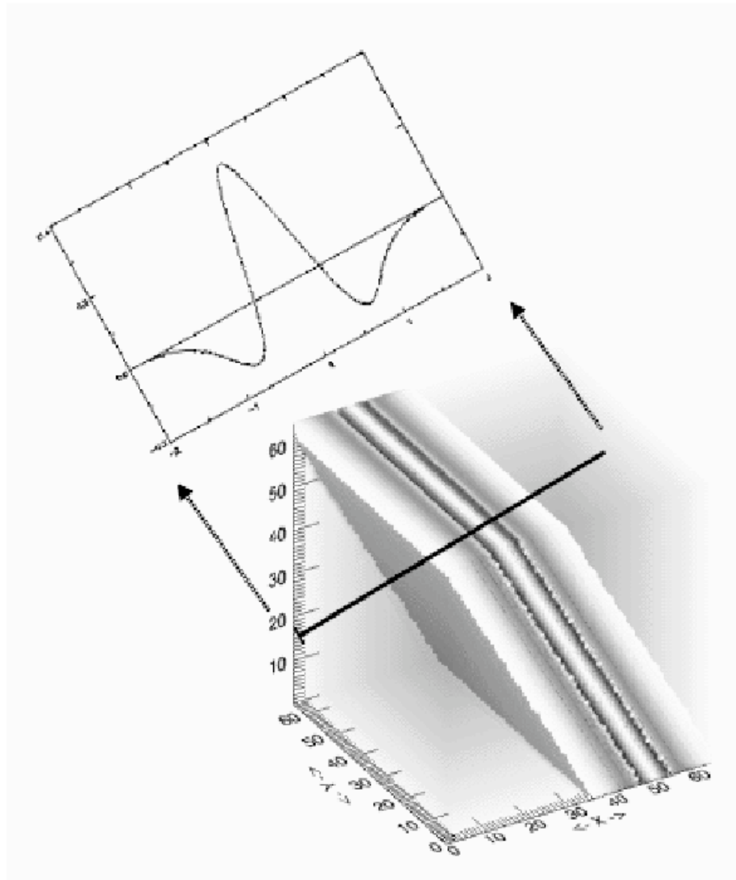
⇒ **Statistical information extraction from all transforms**

***Analysis of the spatial distribution of galaxies by multiscale methods***, Eurasip Journal on Applied Signal Processing, 15, pp 2455-2469, 2005.

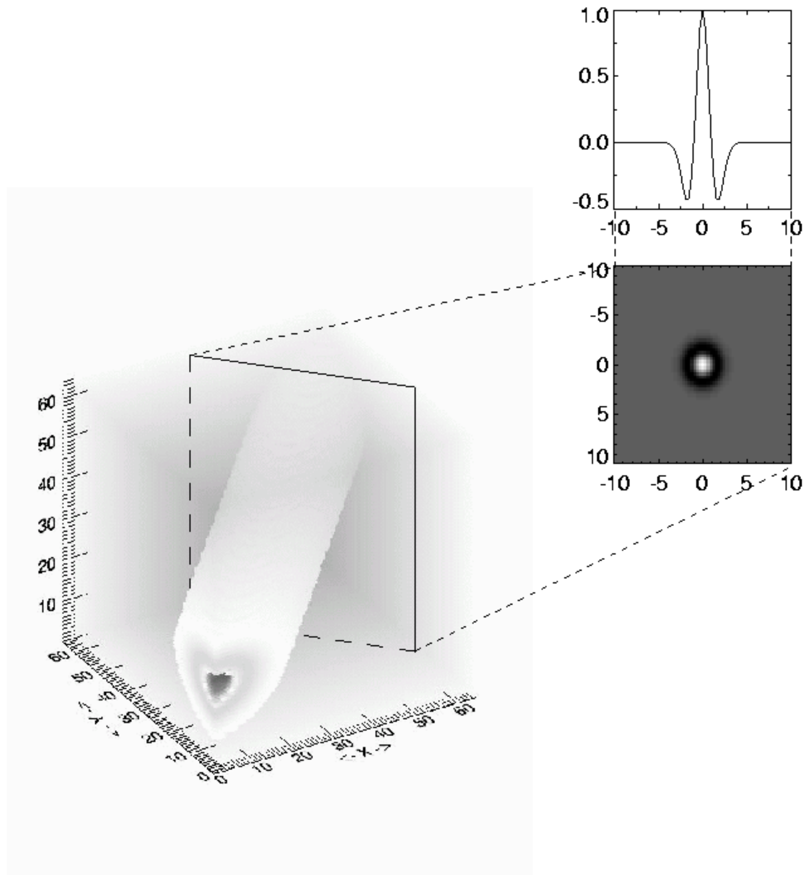
# 3D Wavelet Function

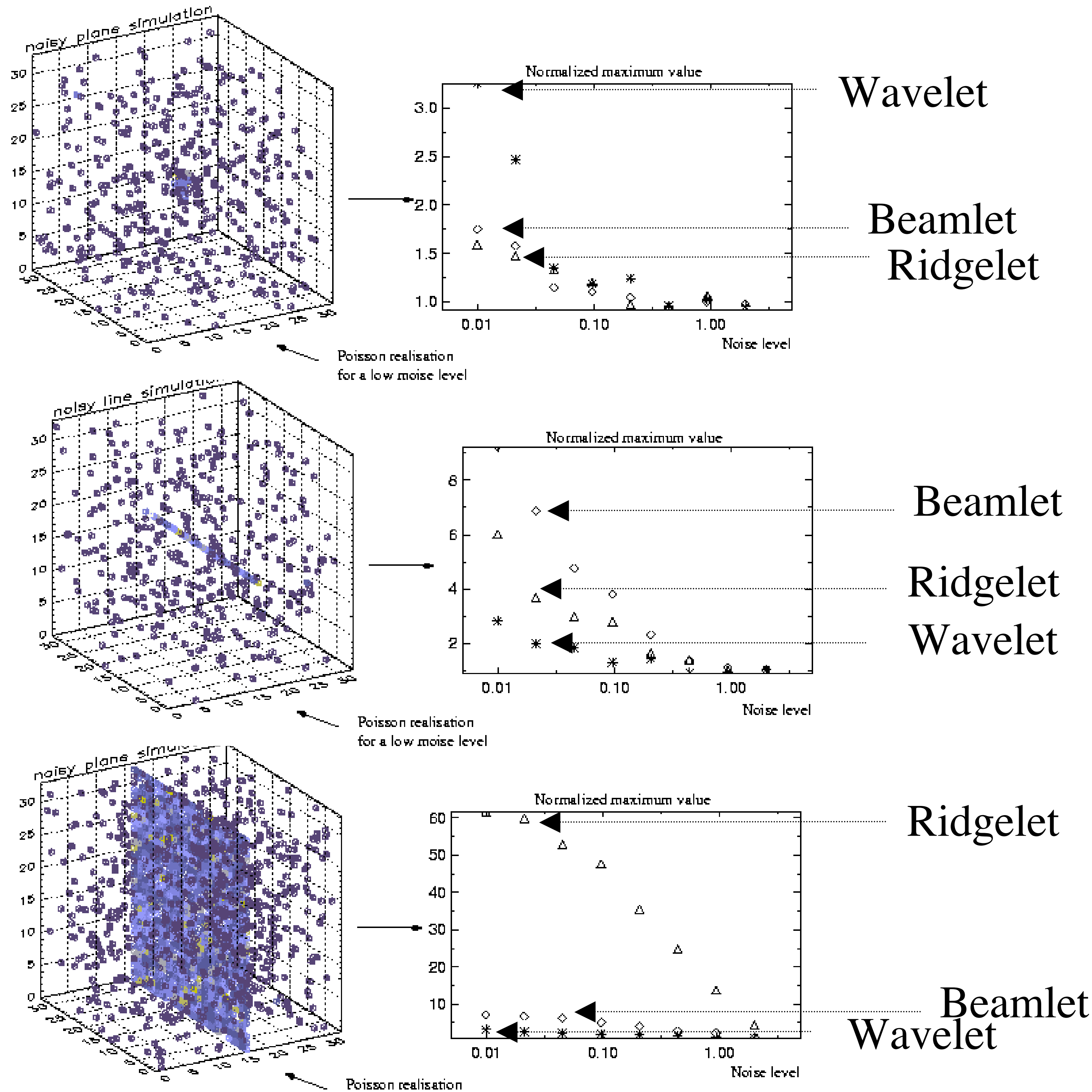


# 3D Ridgelet Function

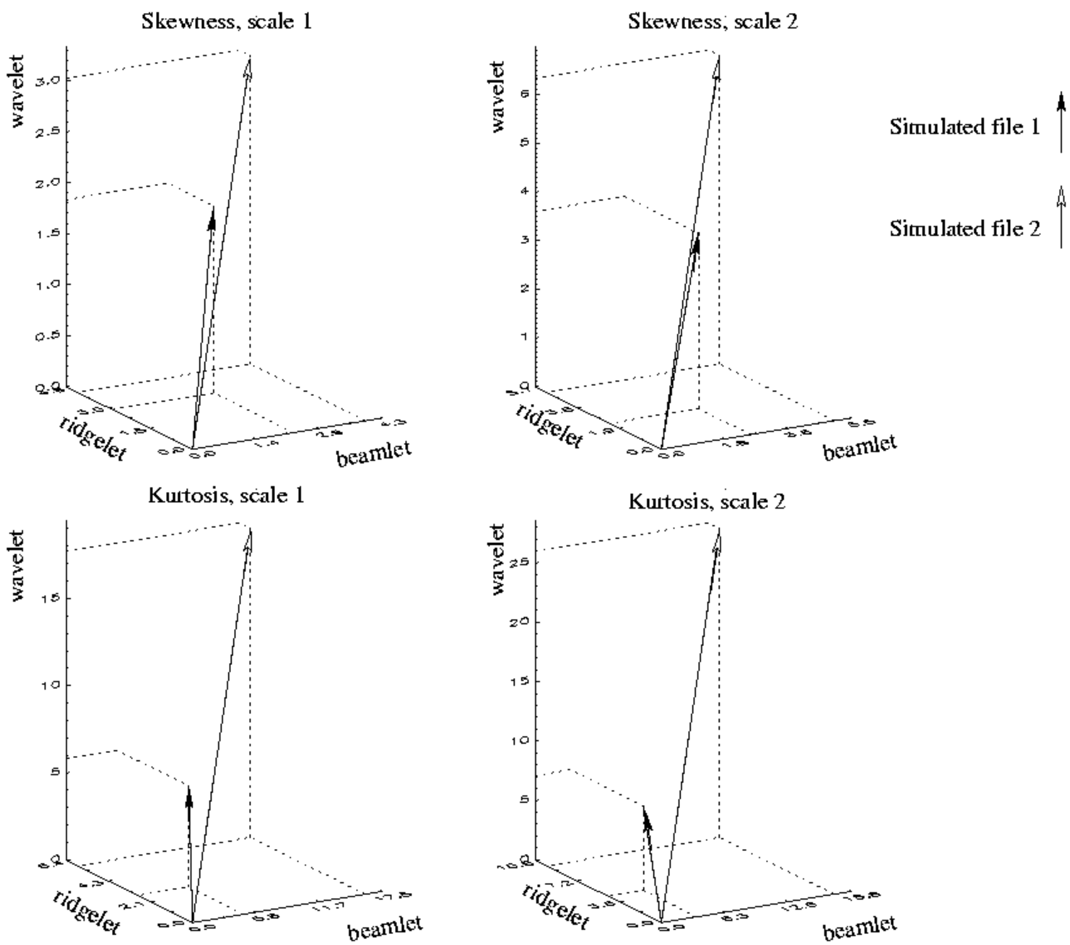


# 3D Beamlet Function




noisy plane simulation
Poisson realisation
for a low moise level
Normalized maximum value
Noise level
0.01
0.10
1.00
1.0
1.5
2.0
2.5
3.0
Wavelet
Beamlet
Ridgelet
noisy line simulation
Poisson realisation
for a low moise level
Normalized maximum value
Noise level
0.01
0.10
1.00
2
4
6
8
Beamlet
Ridgelet
Wavelet
noisy plane simulation
Poisson realisation
Normalized maximum value
Noise level
0.01
0.10
1.00
10
20
30
40
50
60
Ridgelet
Beamlet
Wavelet

# Skewness and Kurtosis

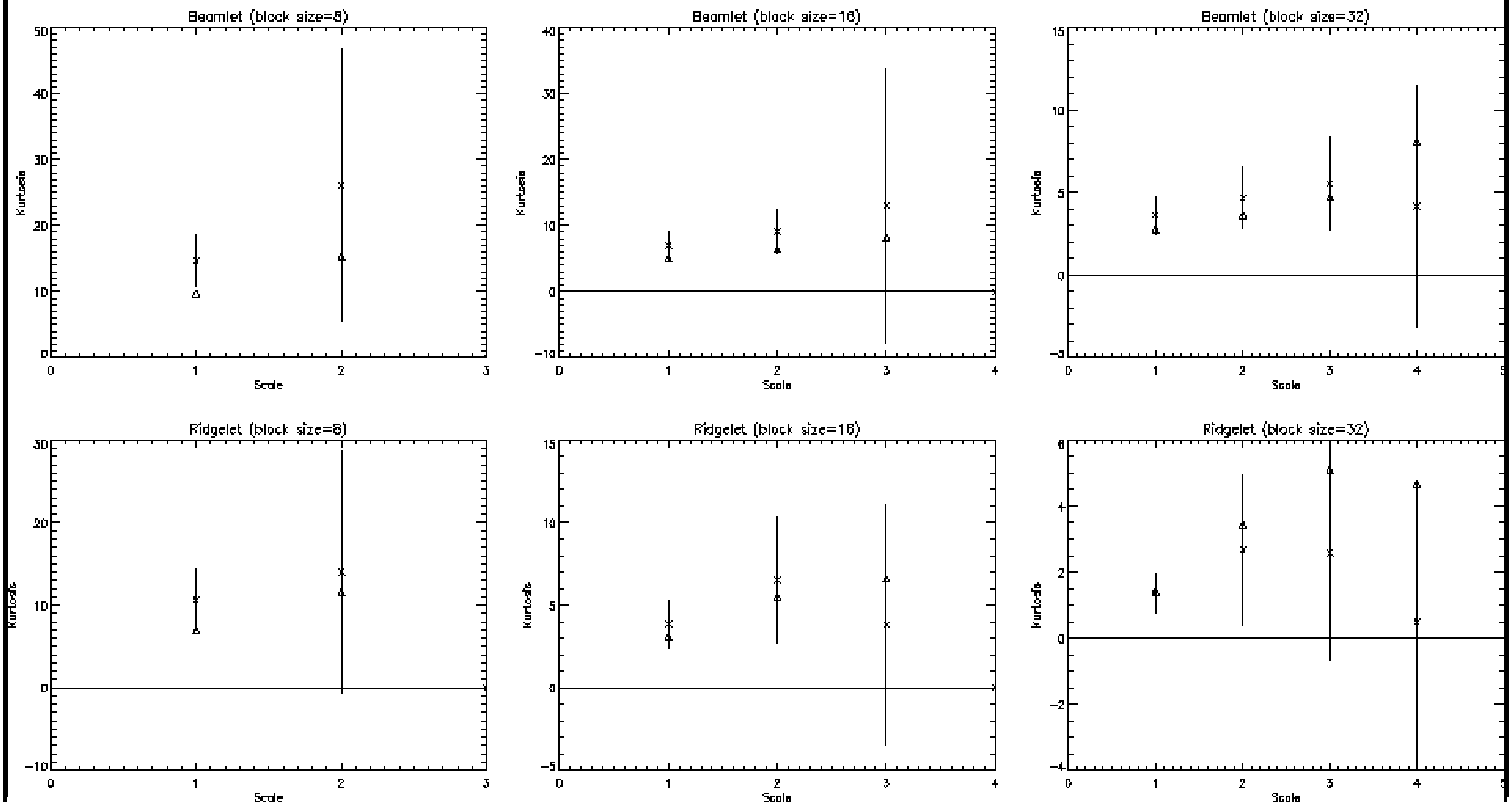


## MGA and the 2DF

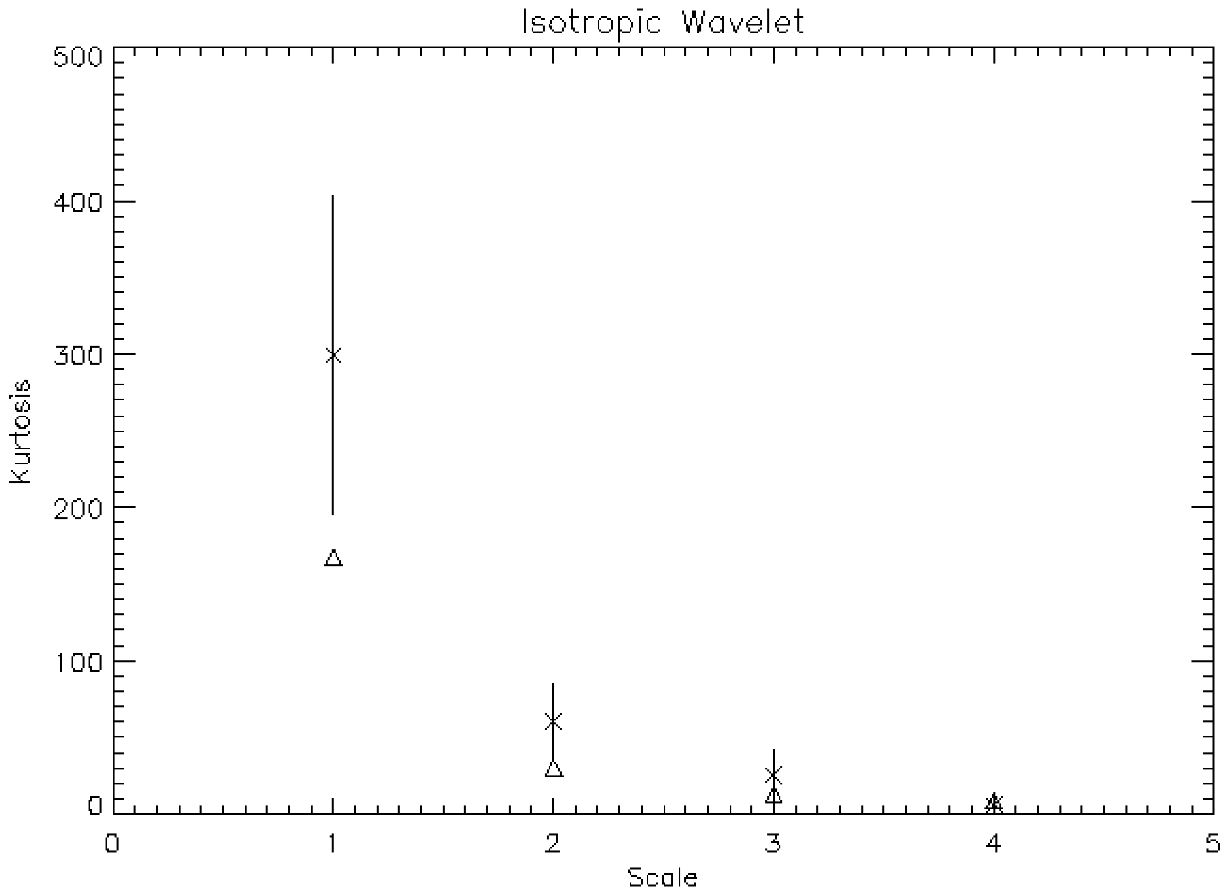
We have considered 7 transforms:

1. 3D Isotropic Wavelet Transfrom with 4 dyadic scales.
2. 3D Ridgelet Transform using a block size of 8 Mpc and two scales. Here the scale is related to the width of the ridgelet function, its length being fixed by the block size.
3. 3D Ridgelet Transform using a block size of 16 Mpc and three scales.
4. 3D Ridgelet Transform using a block size of 32 Mpc and three scales.
5. 3D Beamlet Transform using a block size of 8 Mpc and two scales. Here the scale is related to the width of the beamlet function, its length being fixed by the block size.
6. 3D Beamlet Transform using a block size of 16 Mpc and three scales.
7. 3D Beamlet Transform using a block size of 32 Mpc and three scales.

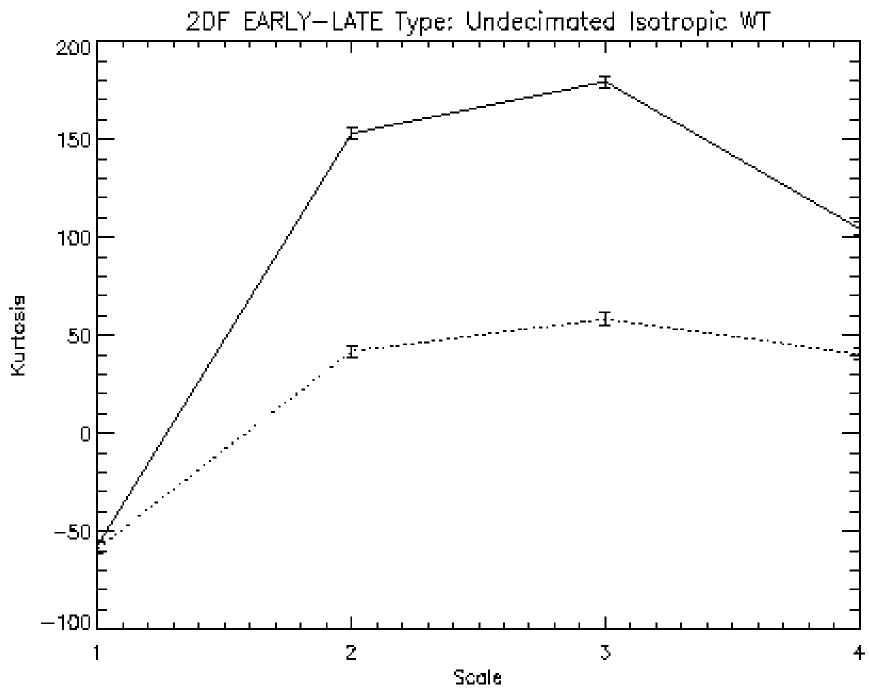
# Beamlet Kurtosis and Ridgelet Kurtosis.



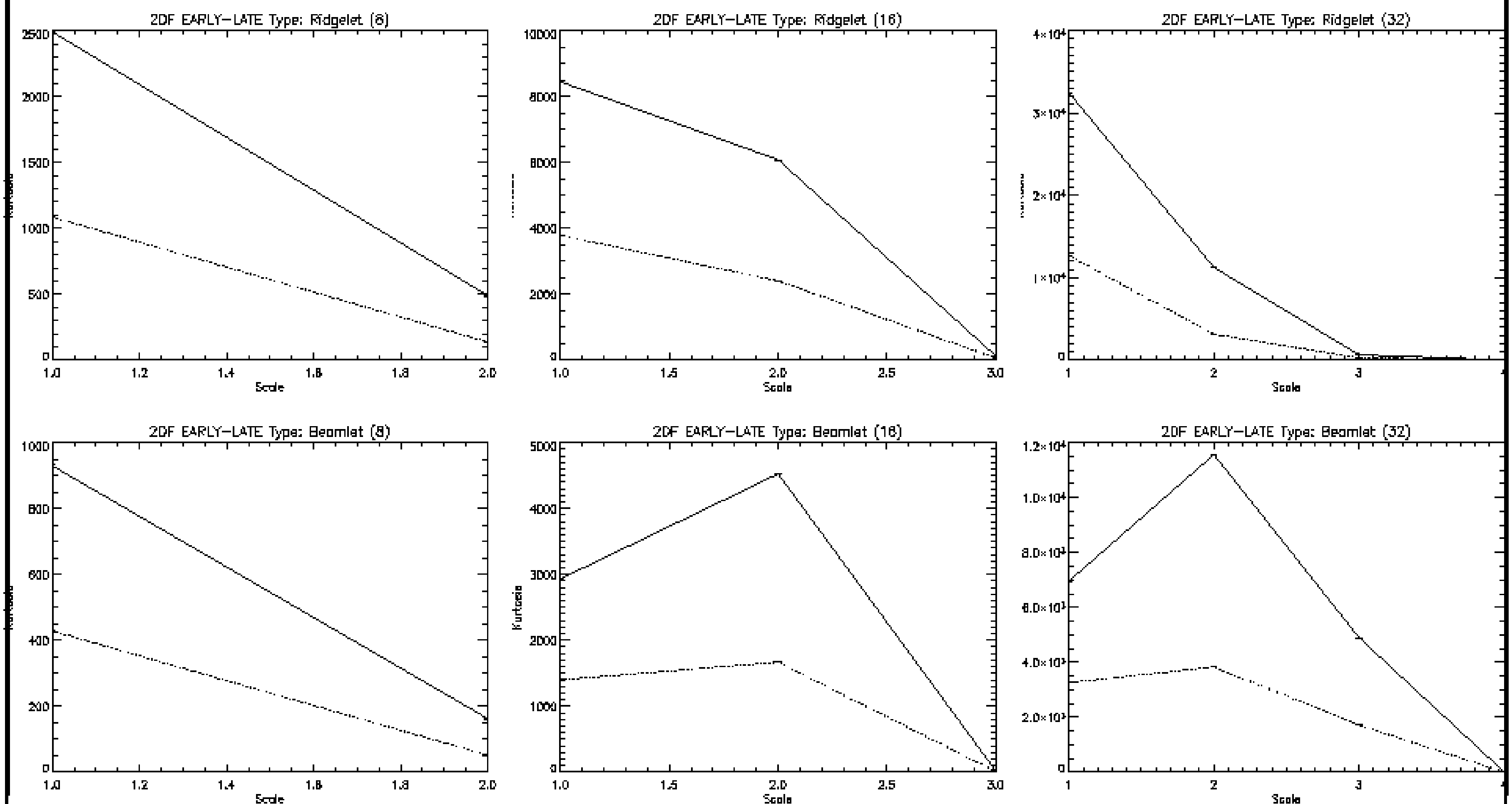
# Wavelet Kurtosis.



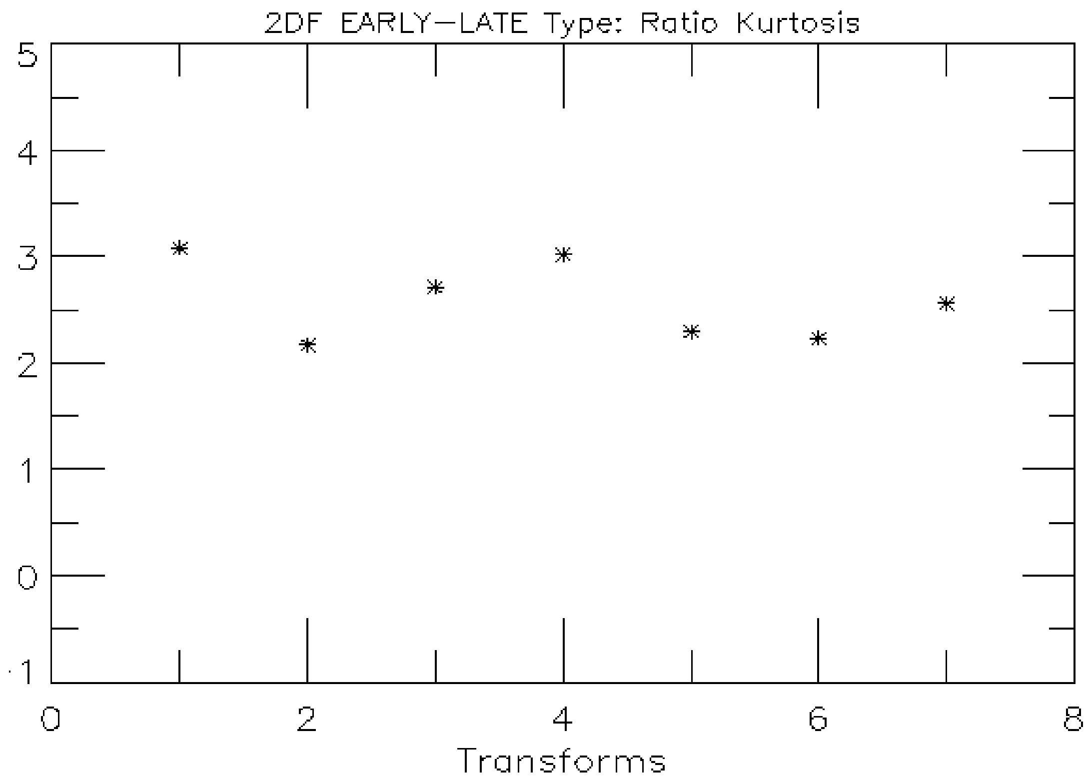
# Isotropic WT Kurtosis: ETG (solid line) and LTG (dashed line).



# Ridgelet and Beamlet Kurtosis: ETG (solid line) and the LTG (dashed).



**Kurtosis ratio:** $R_K(t) = \frac{K_{max}^{(ETG)}(t)}{K_{max}^{(LTG)}(t)}$



# Conclusions

- Quantitative descriptors -being reliable, robust, unbiased, and physically interpretable- are needed to extract cosmological information from the data ==> MGA approach seems very promising.

- Using MGA on 2DF Data:

1) The mock catalogs are NOT compatible with the data.

2) We do not see any tendency toward homogeneity up to the scale of 32 Mpc.

3) Early type galaxies are more clustered than late type galaxies in filaments, walls, and clusters in a similar way.

- Morphological Diversity can be used for many other applications such the separation of components in mono or multi-channel data ==> MCA and multi-channel MCA.

# If you want to know more...

*Second edition available in september 2006*