Source Separation based on Morphological Diversity

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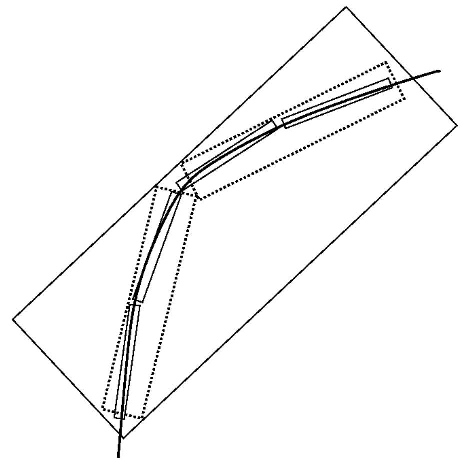
Morphological Component Analysis: *MCA allows us to separate features in an image which present different morphological aspects. MCA is based on fast transform/reconstruction operators.*

TRANSFORMS

. DCT
. Orthogonal WT: Mallat, 1989.
. Bi-orthogonal WT: Daubechies, Cohen, … 1992
. Lifting Scheme: Swelden, 1996 (JPEG 2000 Norm).

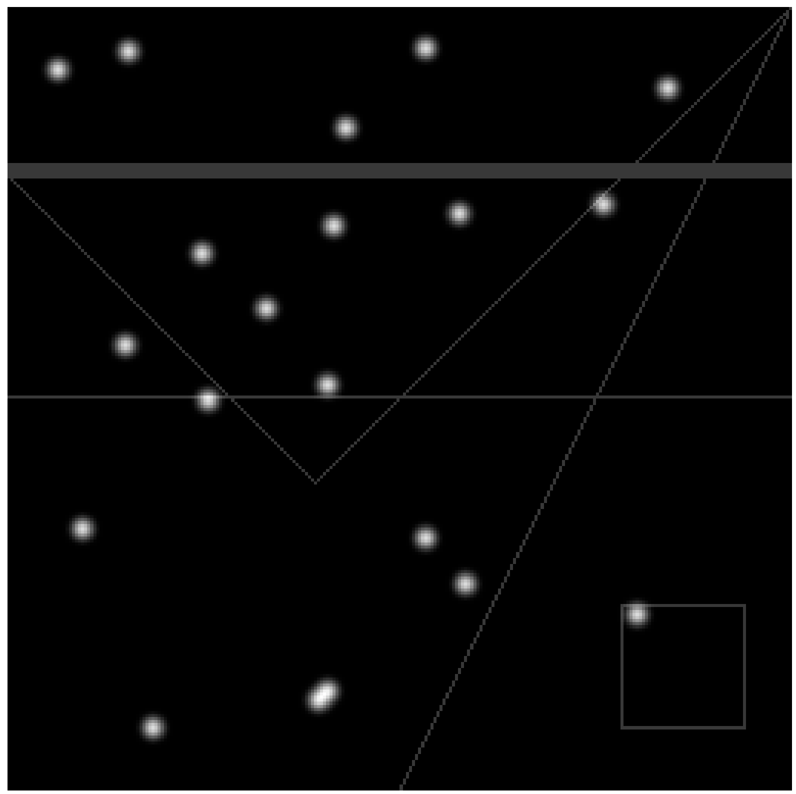
REDUNDANT TRANSFORMS

- Local DCT ([overlapping] blocks + DCT)
- Undecimated Wavelet Transform
- Isotropic Undecimated Wavelet Transform
- Ridgelet Transform
- **Curvelet Transform**

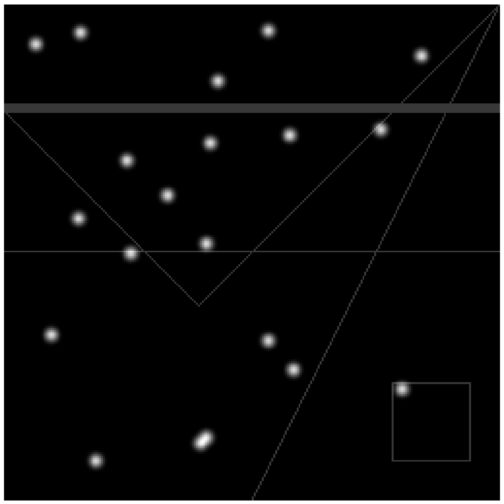


A difficult issue

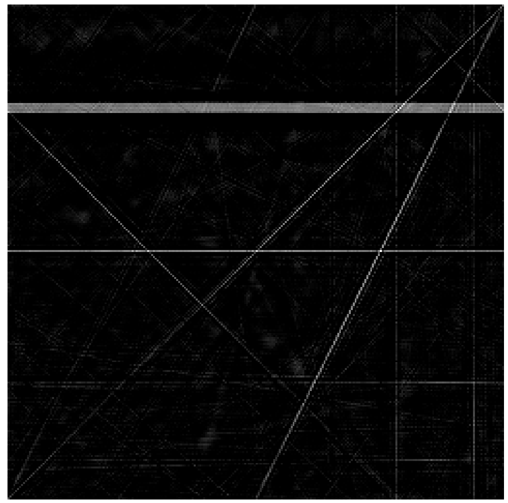
Is there any representation that well represents the following image ?



Going further

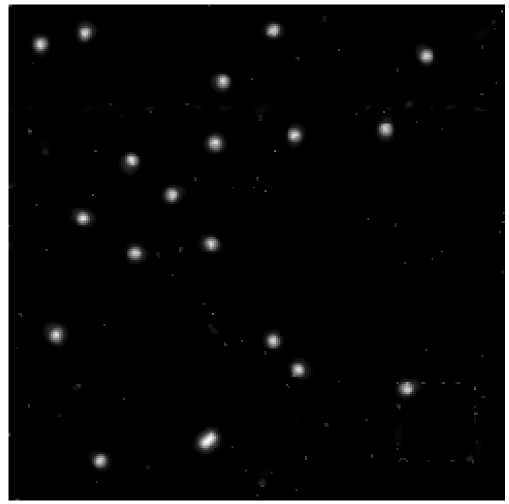


=



Lines

+



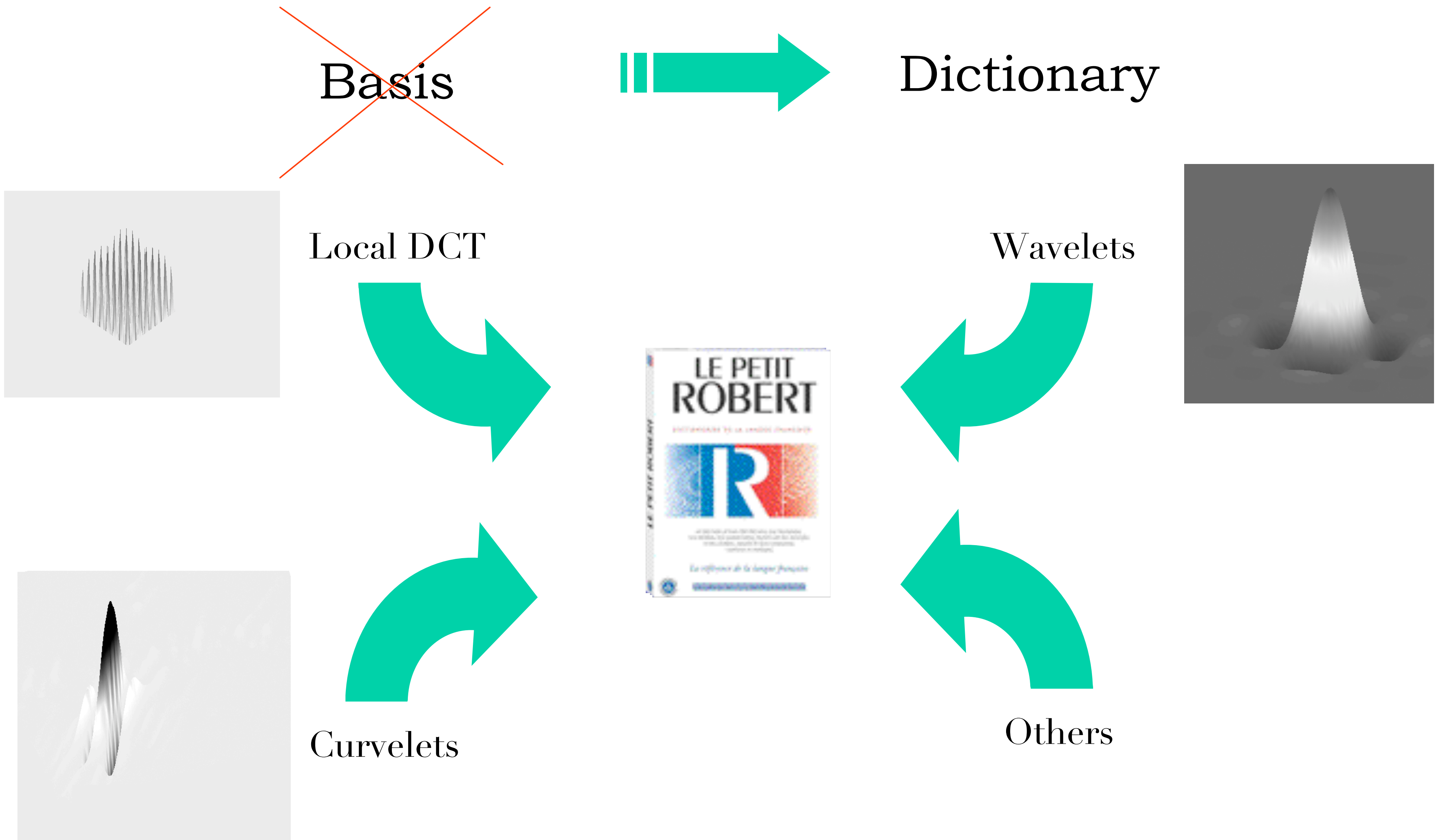
Gaussians

Curvelets

Wavelets

REDUNDANT REPRESENTATIONS

How to choose a representation ?



Sparse Representation in a Redundant Dictionary

Given a signal s, we assume that it is the result of a sparse linear combination of atoms from a known dictionary D.

A dictionary D is defined as a collection of waveforms $(\phi_\gamma)_{\gamma\in\Gamma}$, and the goal is to obtain a representation of a signal s with a linear combination of a small number of basis such that:

$$s = \sum_\gamma \alpha_\gamma \phi_\gamma$$

Or an approximate decomposition:

$$s = \sum_\gamma \alpha_\gamma \phi_\gamma + R$$

Formally, the sparsest coefficients are obtained by solving the optimization problem:

$$\text{(P0)} \quad \text{Minimize} \quad \|\alpha\|_0 \quad \text{subject to} \quad s = \phi\alpha$$

It has been proposed (*to relax and*) to replace the l_0 norm by the l_1 norm (Chen, 1995):

$$\text{(P1)} \quad \text{Minimize} \quad \|\alpha\|_1 \quad \text{subject to} \quad s = \phi\alpha$$

It can be seen as a kind of convexification of (P0).

It has been shown (Donoho and Huo, 1999) that for certain dictionary, it there exists a highly sparse solution to (P0), then it is identical to the solution of (P1).

We consider now that the dictionary is built of a set of L dictionaries related to multiscale transforms, such wavelets, ridgelet, or curvelets.

Considering L transforms, and α_k the coefficients relative to the kth transform:

$$\phi = [\phi_1, \ldots, \phi_L], \quad \alpha = \{\alpha_1, \ldots, \alpha_L\}, \quad s = \phi\alpha = \sum_{k=1}^{L} \phi_k \alpha_k$$

Noting $T_1, \ldots T_L$ the L transform operators, we have:

$$\alpha_k = T_k s_k, \qquad s_k = T_k^{-1} \alpha_k, \qquad s = \sum_{k=1}^{L} s_k$$

A solution α is obtained by minimizing a functional of the form:

$$J(\alpha) = \left\| s - \sum_{k=1}^{L} T_k^{-1} \alpha_k \right\|_2^2 + \|\alpha\|_p$$

Different Problem Formulation

$$J(s_1,\ldots,s_L) = \left\| s - \sum_{k=1}^{L} s_k \right\|_2^2 + \lambda \sum_{k=1}^{L} \| T_k s_k \|_p$$

.We do not need to keep all transforms in memory.

. There are less unknown (because we use non orthogonal transforms).

.We can easily add some constraints on a given component

Morphological Component Analysis (MCA)

"Redundant Multiscale Transforms and their Application for Morphological Component Analysis", Advances in Imaging and Electron Physics, 132, 2004.

$$J(s_1, \ldots, s_L) = \left\| s - \sum_{k=1}^{L} s_k \right\|_2^2 + \lambda \sum_{k=1}^{L} \| T_k s_k \|_p + \sum_{k=1}^{L} \gamma_k C_k(s_k)$$

$C_k(s_k)$ = constraint on the component s_k

Compare to a standard matching or basis pursuit:

- We do not need to keep all transforms in memory.
- There are less unknown (because we use non orthogonal transforms).
- We can easily add some constraints on a given component

The MCA Algorithm

The MCA algorithm relies on an iterative scheme: at each iteration, MCA picks in alternately in each basis the most significant coefficients of a residual term:

. Initialize all s_k to zero

. Iterate t=1,...,Niter

- Iterate k=1,..,L

Update the kth part of the current solution by fixing all other parts and minimizing:

$$J(s_k) = \left\| s - \sum_{i=1, i \neq k}^{L} s_i - s_k \right\|_2^2 + \lambda_t \left\| T_k s_k \right\|_1$$

Which is obtained by a simple soft/hard thresholding of : $s_r = s - \sum_{i=1, i \neq k}^{L}$

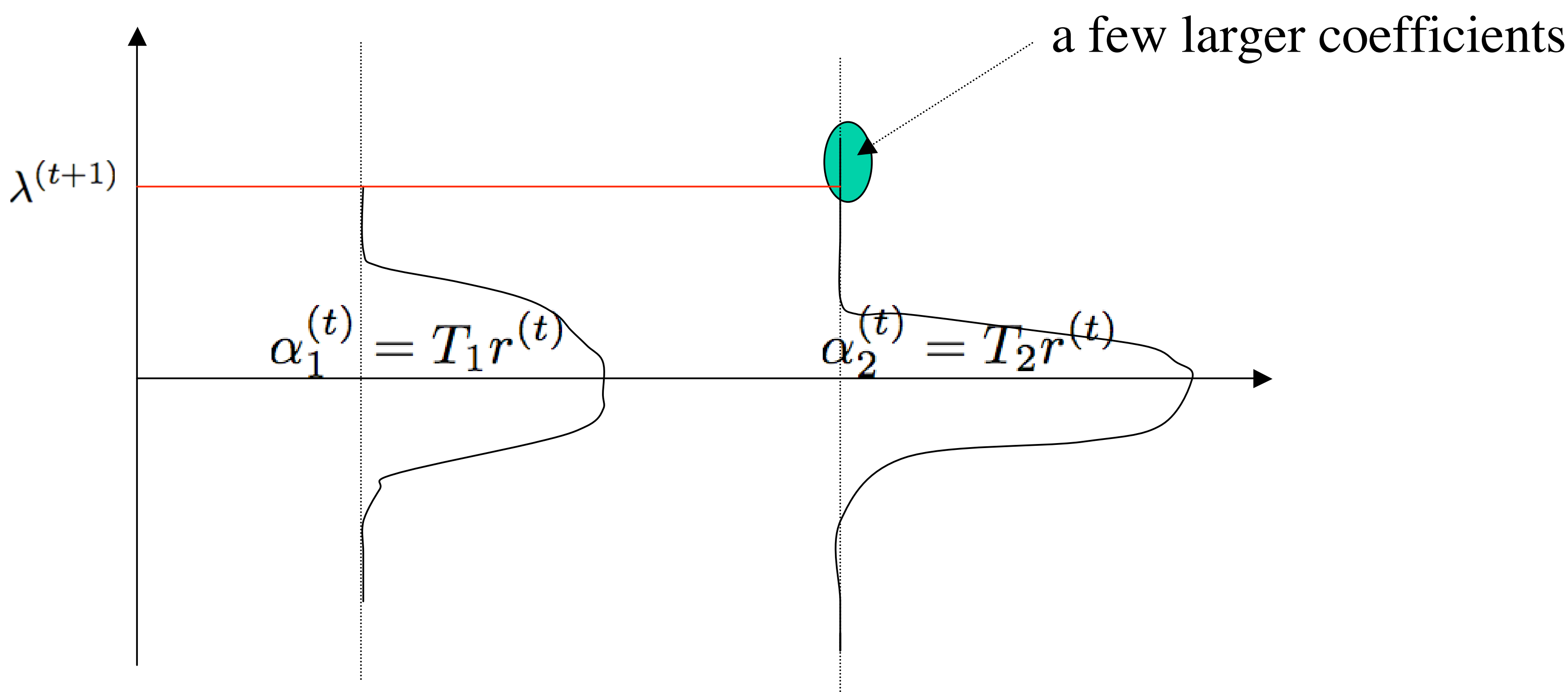
- Decrease λ_t

How to optimally tune the thresholds ?

- The thresholds play a key role as they manage the way coefficients are selected and thus determine the sparsity of the decomposition.

- As K transforms per iteration are necessary :
  the least number of iterations, the faster the decomposition.

$$r^{(t)} = s - s_1^{(t)} - s_2^{(t)}$$


a few larger coefficients
$\lambda^{(t+1)}$
$\alpha_1^{(t)} = T_1 r^{(t)}$
$\alpha_2^{(t)} = T_2 r^{(t)}$

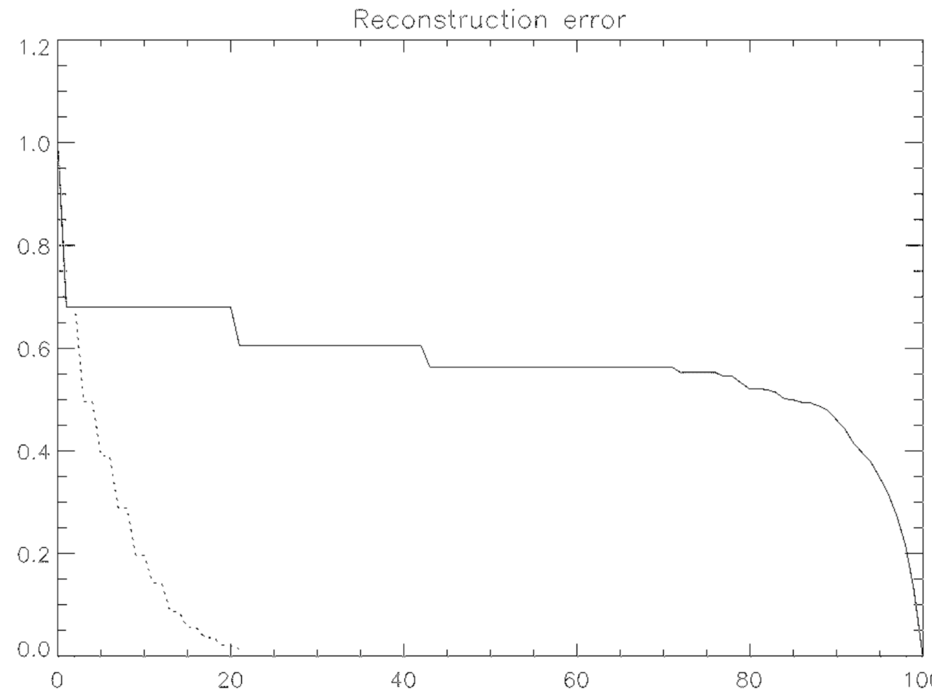
In practice : an empirical approach: The « MOM » strategy

In practice, we would like to use an adaptative tuning strategy.
For a union of 2 orthogonal bases, the threshold is selected such that:

$$\min\{||r^{(k)}\mathbf{\Phi}_1||_\infty, ||r^{(k)}\mathbf{\Phi}_2||_\infty\} < \lambda < \max\{||r^{(k)}\mathbf{\Phi}_1||_\infty, ||r^{(k)}\mathbf{\Phi}_2||_\infty\}$$

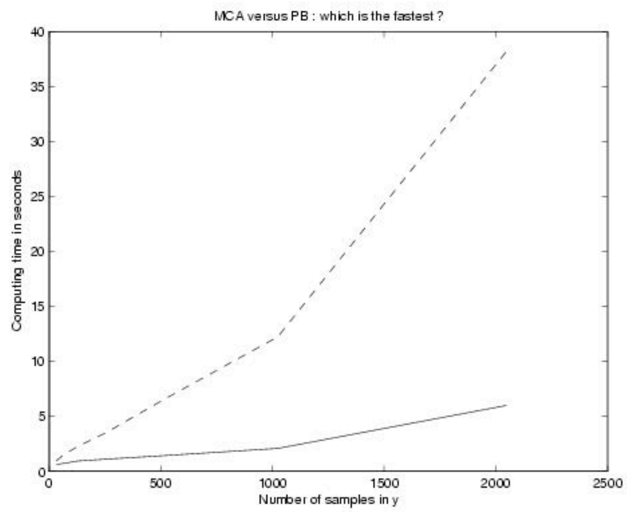
That's why this strategy is called « Min Of Max » (MOM)

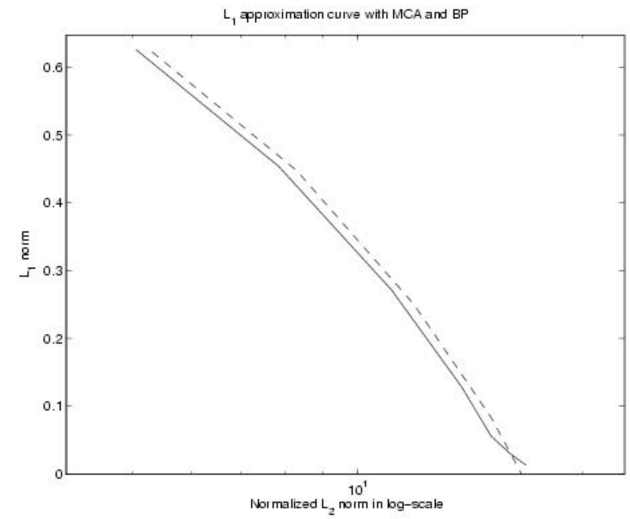
Mom in action

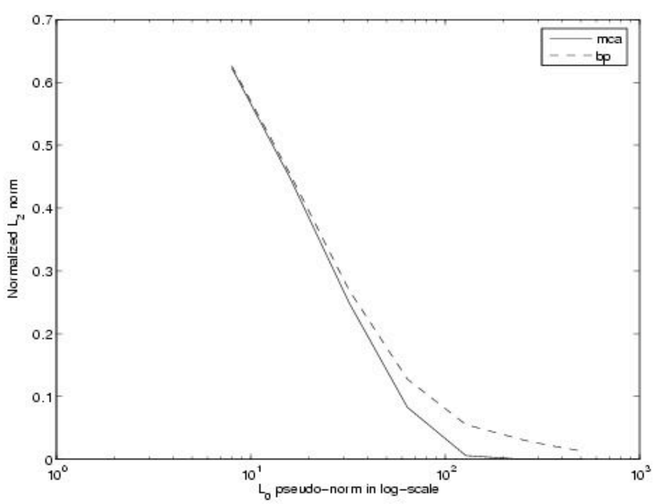


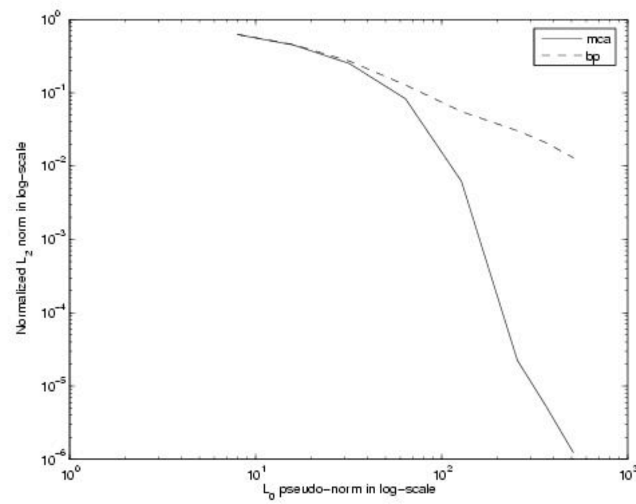
$$\Phi = \text{Curvelets} + \text{Global DCT}$$

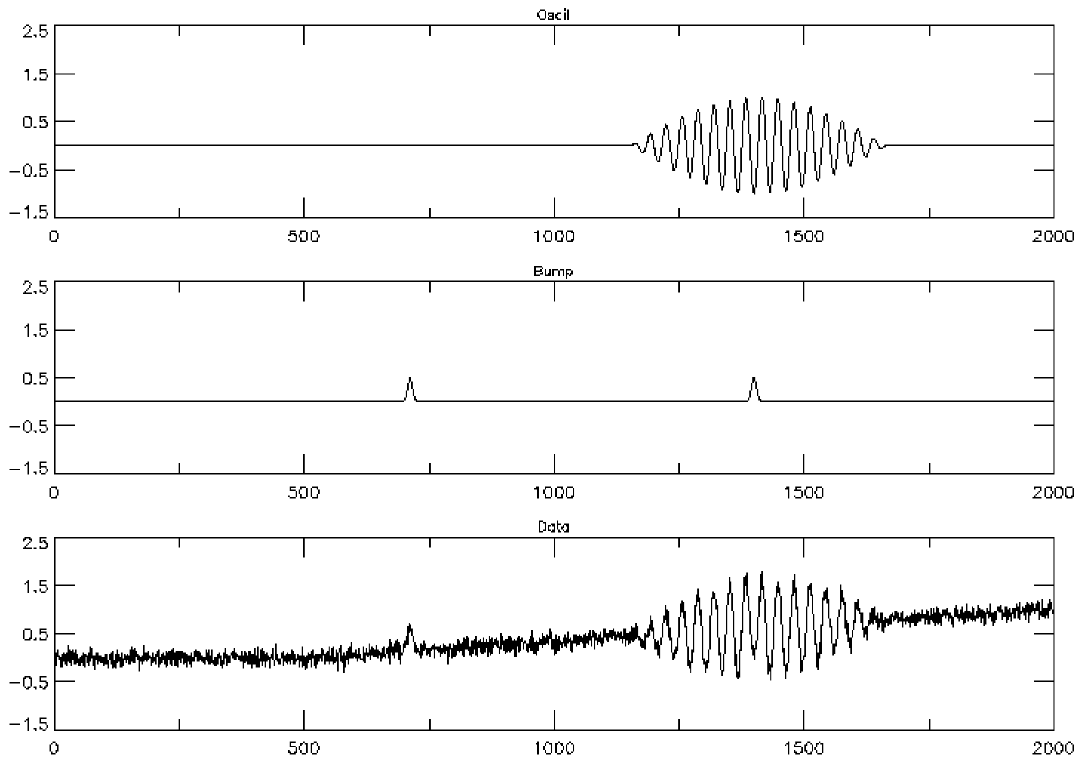
MCA versus Basis Pursuit



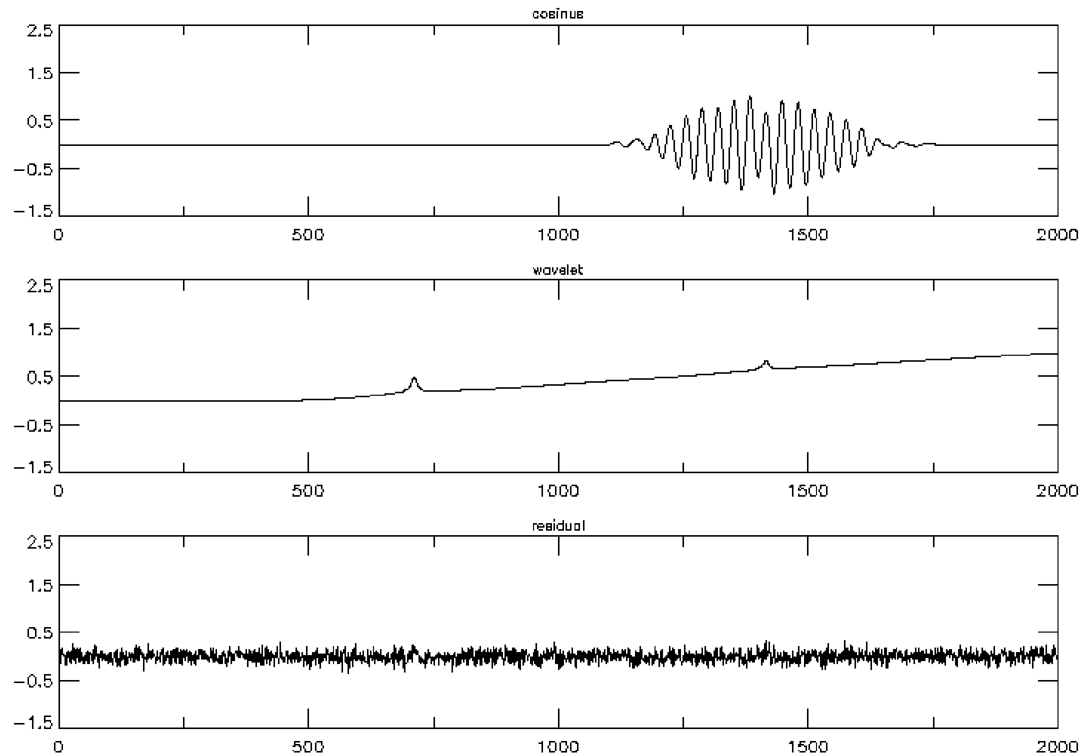




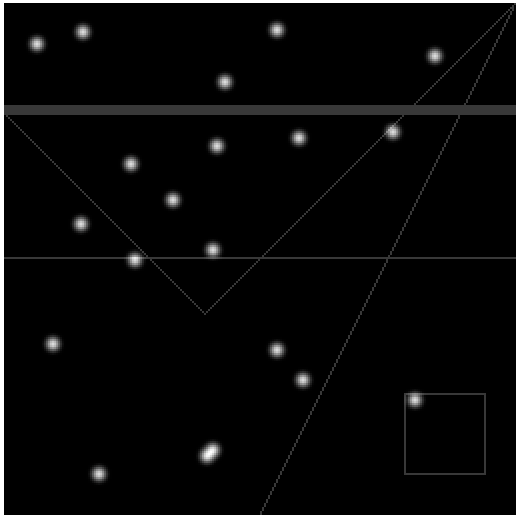




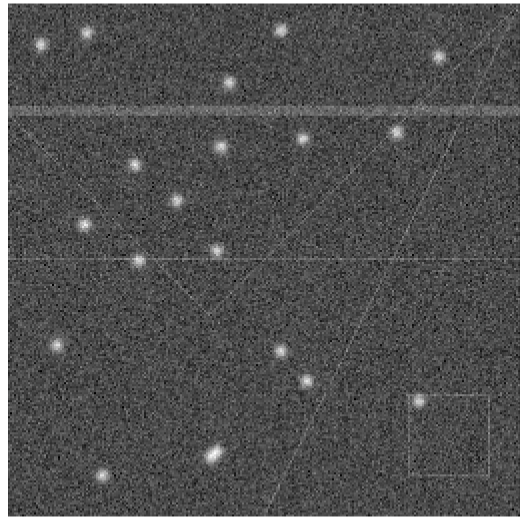
From top to bottom, oscillating component, component with bumps, and simulated data



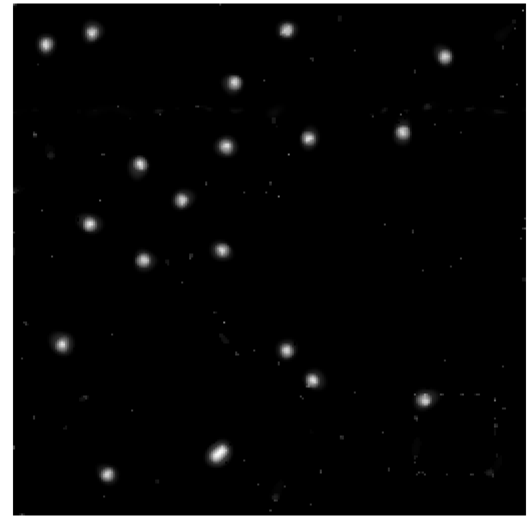
From top to bottom, reconstructed oscillating component, reconstructed component with bumps, and residual.



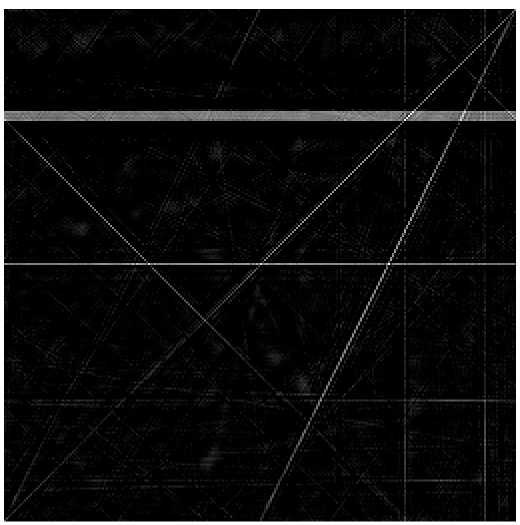
a) Simulated image (Gaussians+lines)



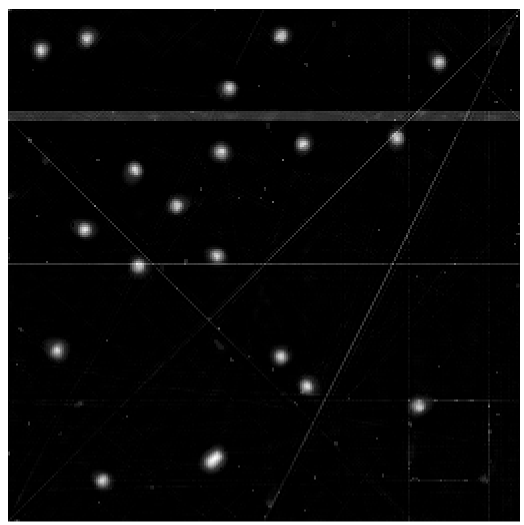
b) Simulated image + noise



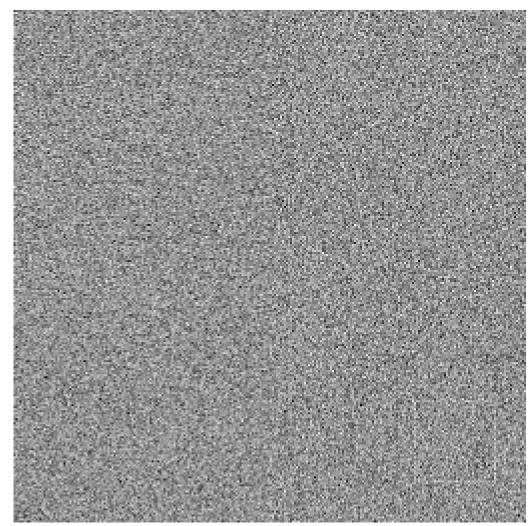
c) A trous algorithm



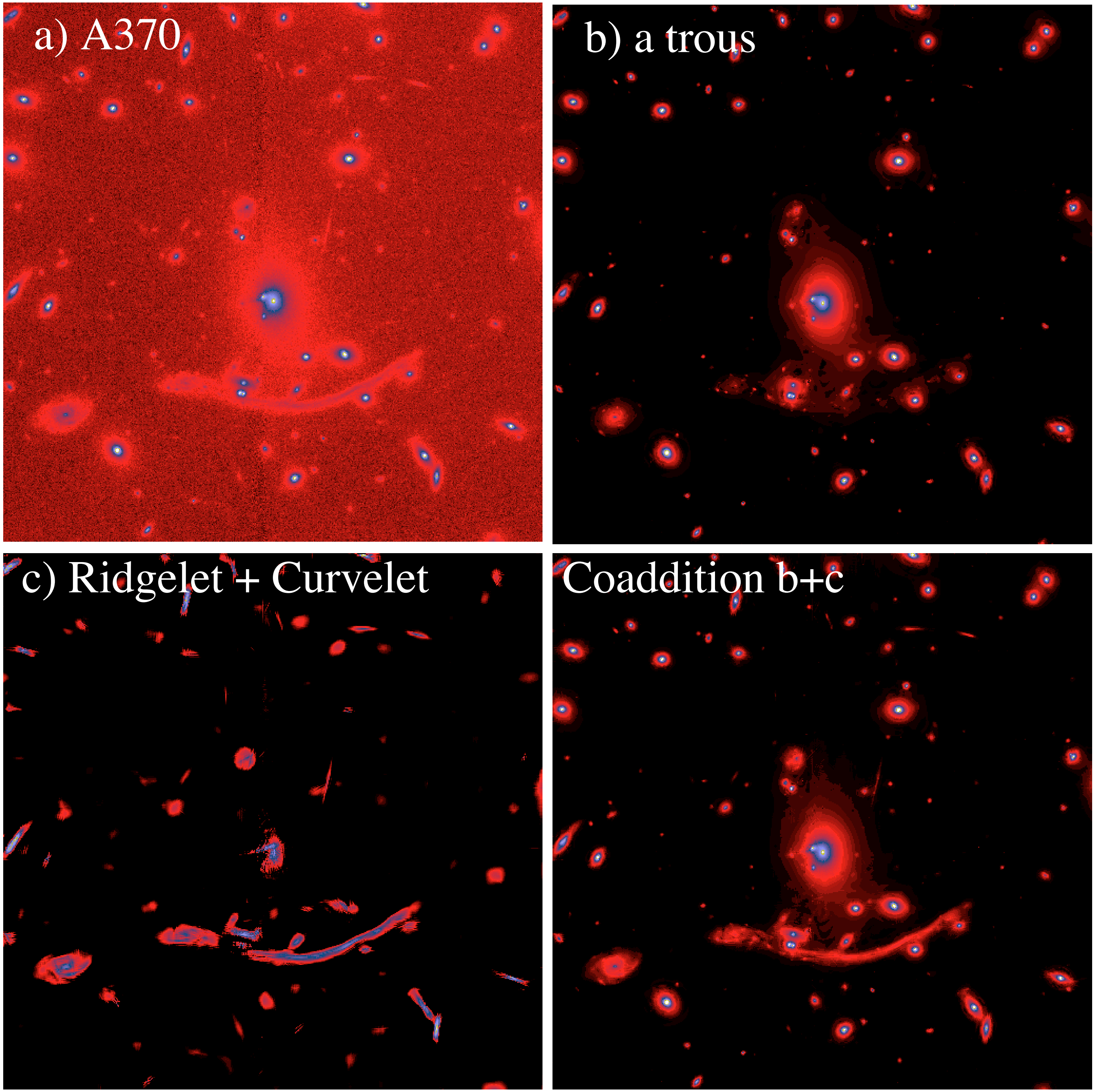
d) Curvelet transform

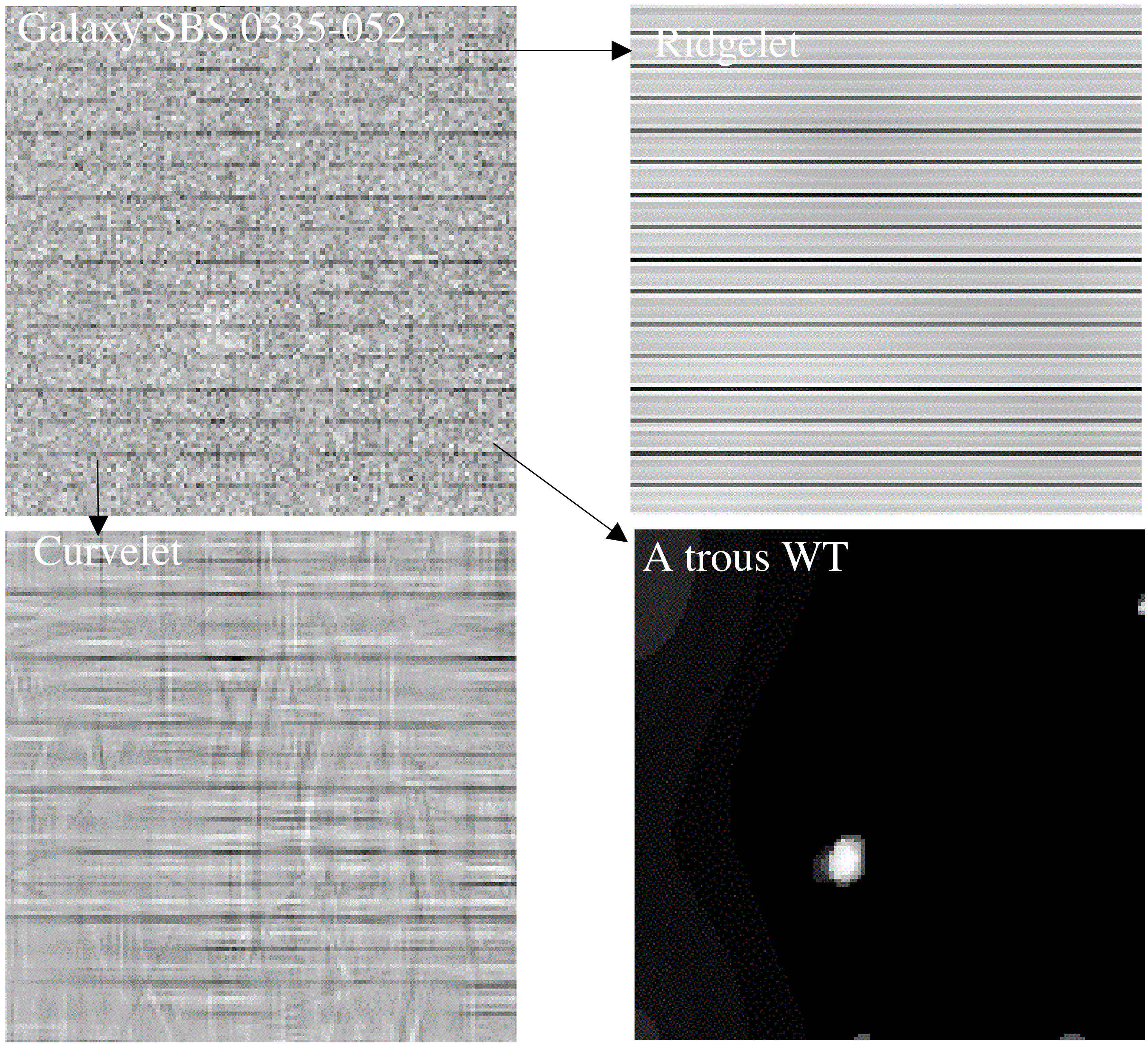


e) coaddition c+d

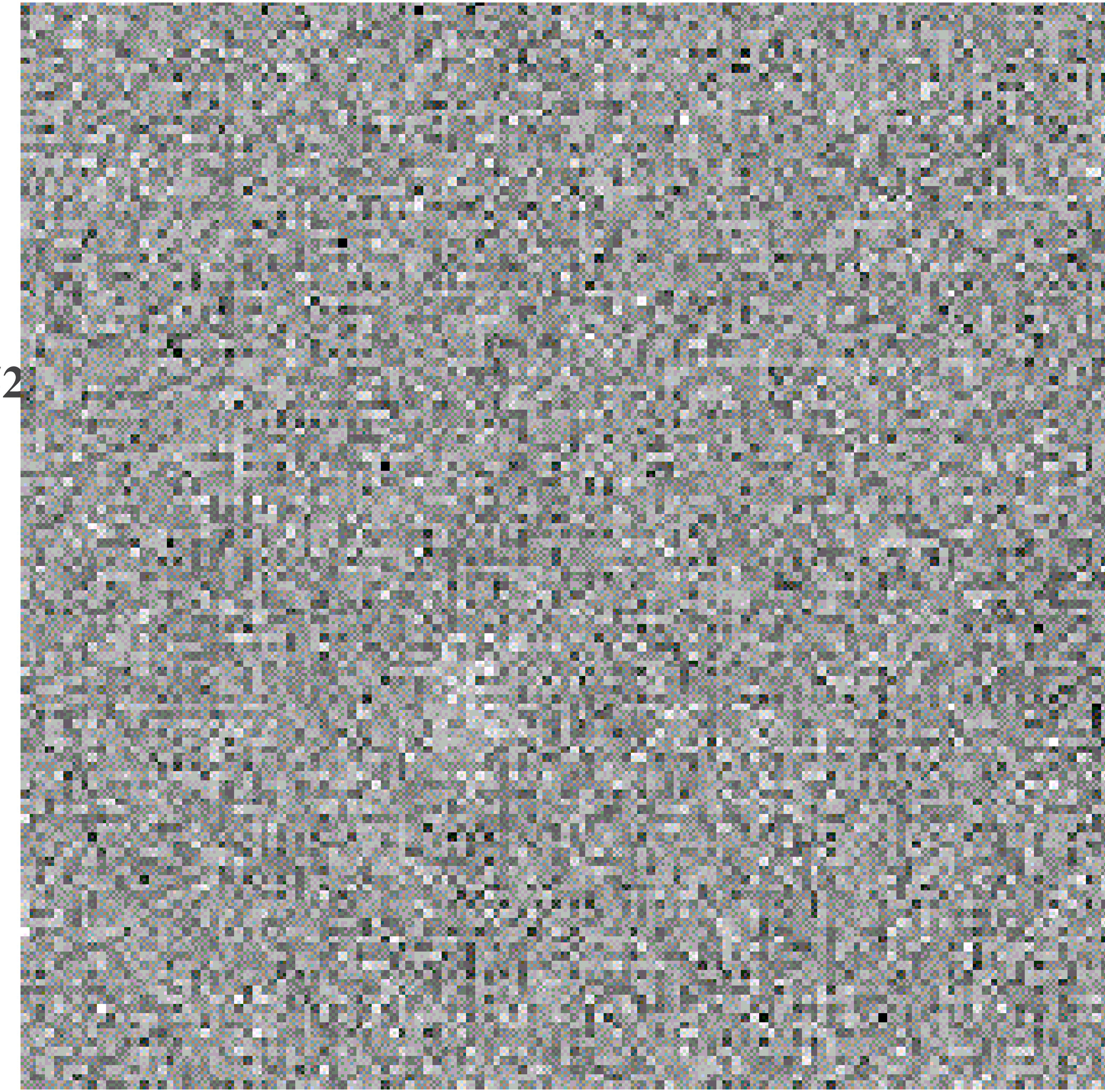


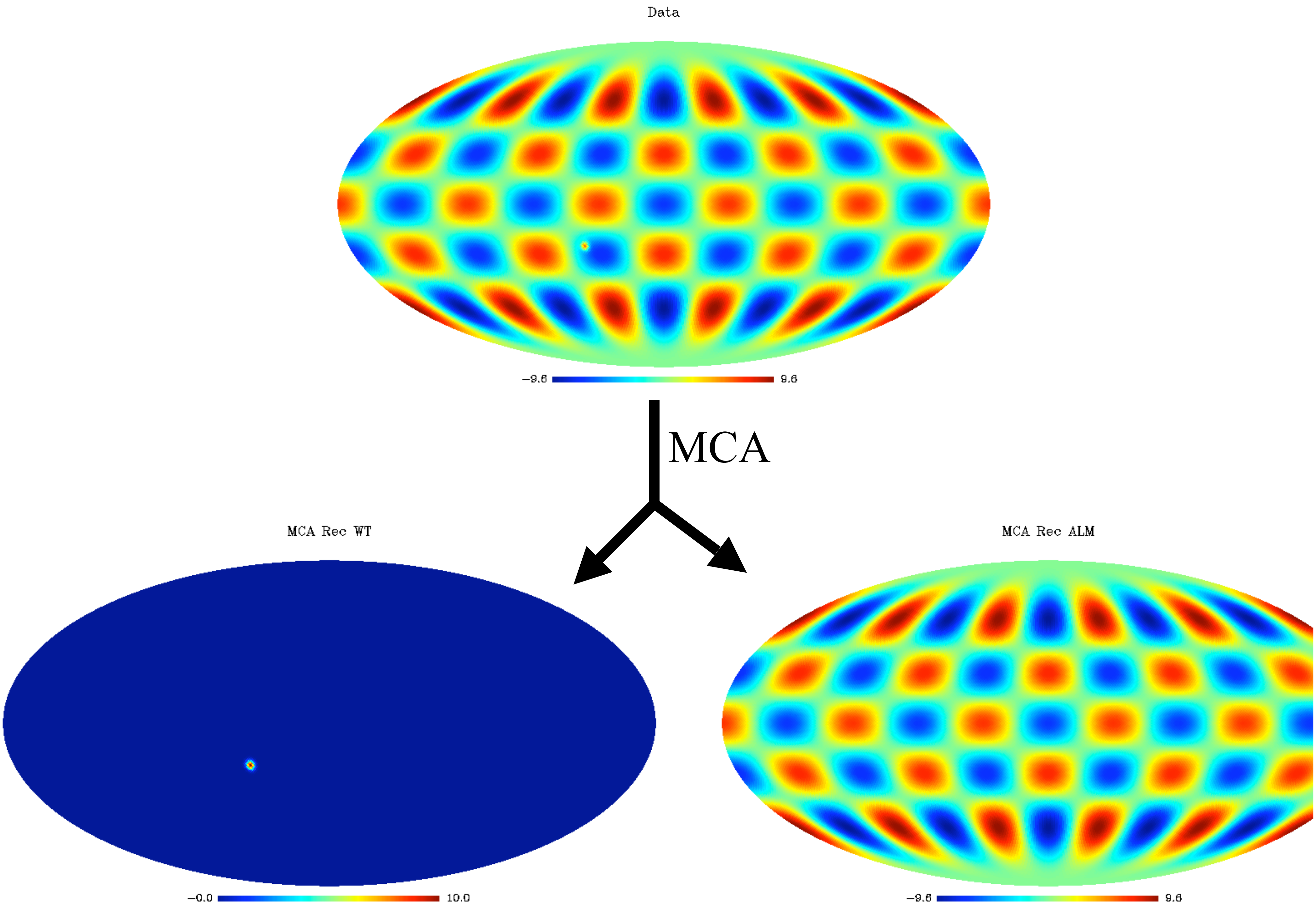
f) residual = e-b


a) A370
b) a trous
c) Ridgelet + Curvelet
Coaddition b+c


Galaxy SBS 0335-052
Ridgelet
Curvelet
A trous WT

Galaxy SBS 0335-052
10 micron
GEMINI-OSCIR




Data
−9.6
9.6
MCA
MCA Rec WT
−0.0
10.0
MCA Rec ALM
−9.6
9.6

Interpolation of Missing Data

$$J(s_1,\ldots,s_L) = \left\| M\left(s - \sum_{k=1}^{L} s_k\right) \right\|_2^2 + \lambda \sum_{k=1}^{L} \left\| T_k s_k \right\|_p$$

Where M is the mask: M(i,j) = 0 ==> missing data

M(i,j) = 1 ==> good data

If the data are composed of a piecewise smooth component + texture

$$J(X_t, X_n) = \left\| M(X - X_t - X_n) \right\|_2^2 + \lambda\left(\left\| \mathbf{C} X_n \right\|_1 + \left\| \mathbf{D} X_t \right\|_1 \right) + \gamma\, \mathrm{TV}(X_n)$$

M. Elad, J.-L. Starck, D.L. Donoho, P. Querre, "Simultaneous Cartoon and Texture Image Inpainting using Morphological Component Analysis (MCA)", ACHA, Vol. 19, pp. 340-358, November 2005.

M.J. Fadili, J.-L. Starck, "Sparse Representations and Bayesian Image Inpainting" , SPARS'05, Vol. I, Rennes, France, Nov., 2005.

. Initialize all s_k to zero

. Iterate j=1,...,Niter

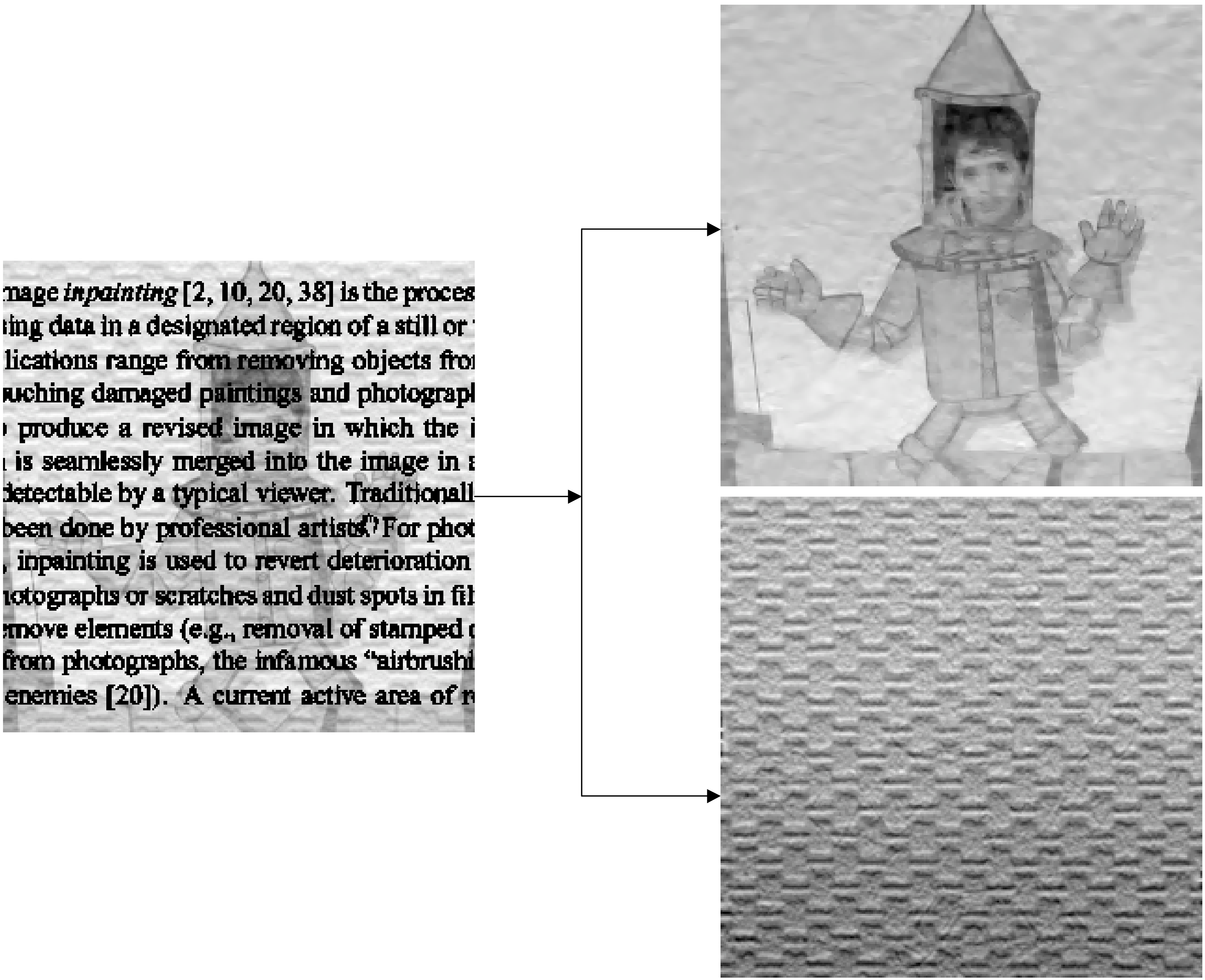
- Iterate k=1,..,L

- Update the kth part of the current solution by fixing all other parts and minimizing:

$$J(s_k) = \left\| M(s - \sum_{i=1, i \neq k}^{L} s_i - s_k) \right\|_2^2 + \lambda \| T_k s_k \|_1$$

Which is obtained by a simple soft thresholding of :

$$s_r = M(s - \sum_{i=1, i \neq k}^{L} s_i)$$

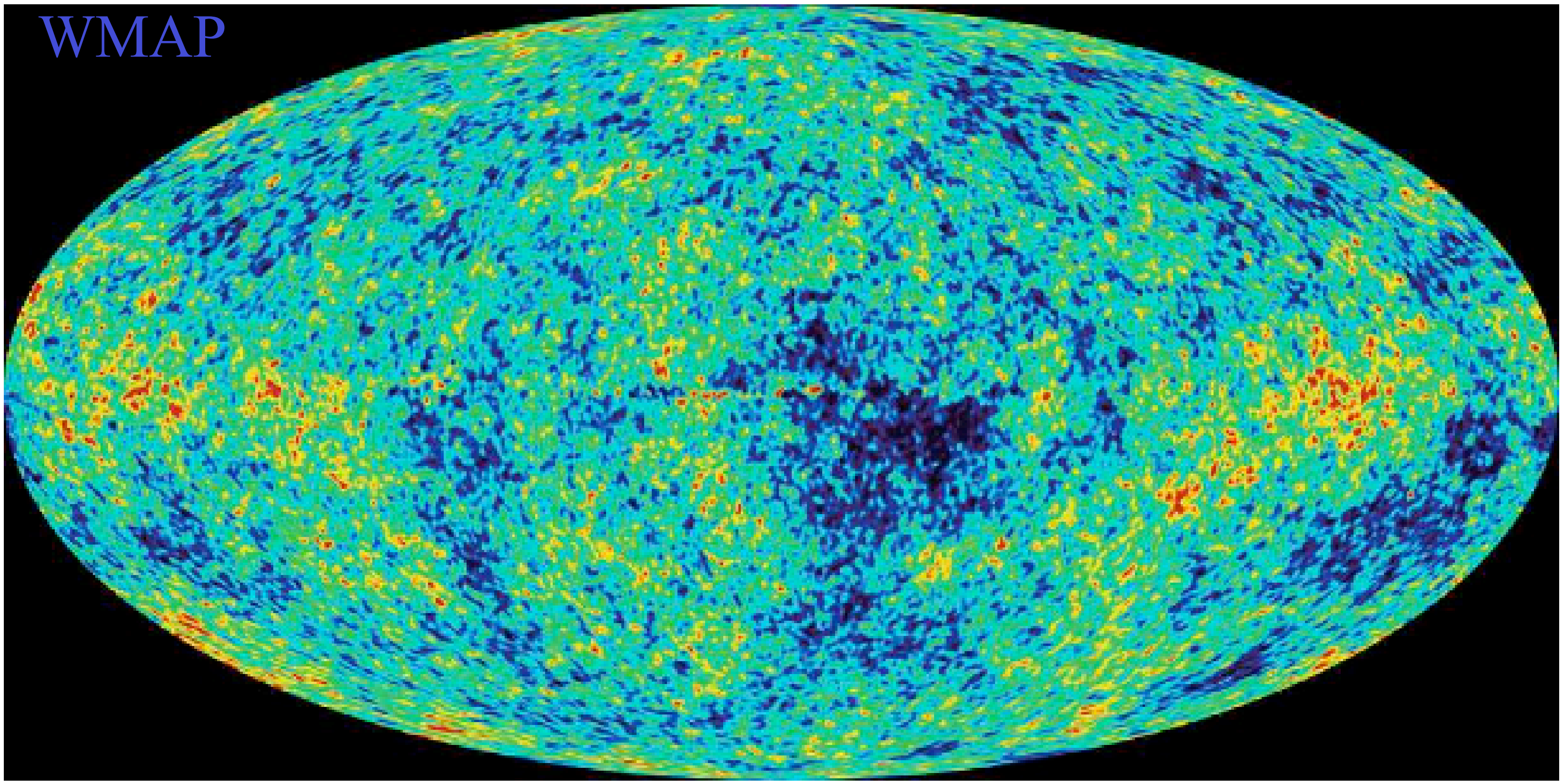

nage inpainting [2, 10, 20, 38] is the proces
ing data in a designated region of a still or
lications range from removing objects fro
uching damaged paintings and photograp
produce a revised image in which the
is seamlessly merged into the image in
detectable by a typical viewer. Traditionall
been done by professional artists. For phot
inpainting is used to revert deterioration
otographs or scratches and dust spots in fil
move elements (e.g., removal of stamped
from photographs, the infamous "airbrushi
enemies [20]). A current active area of

20%

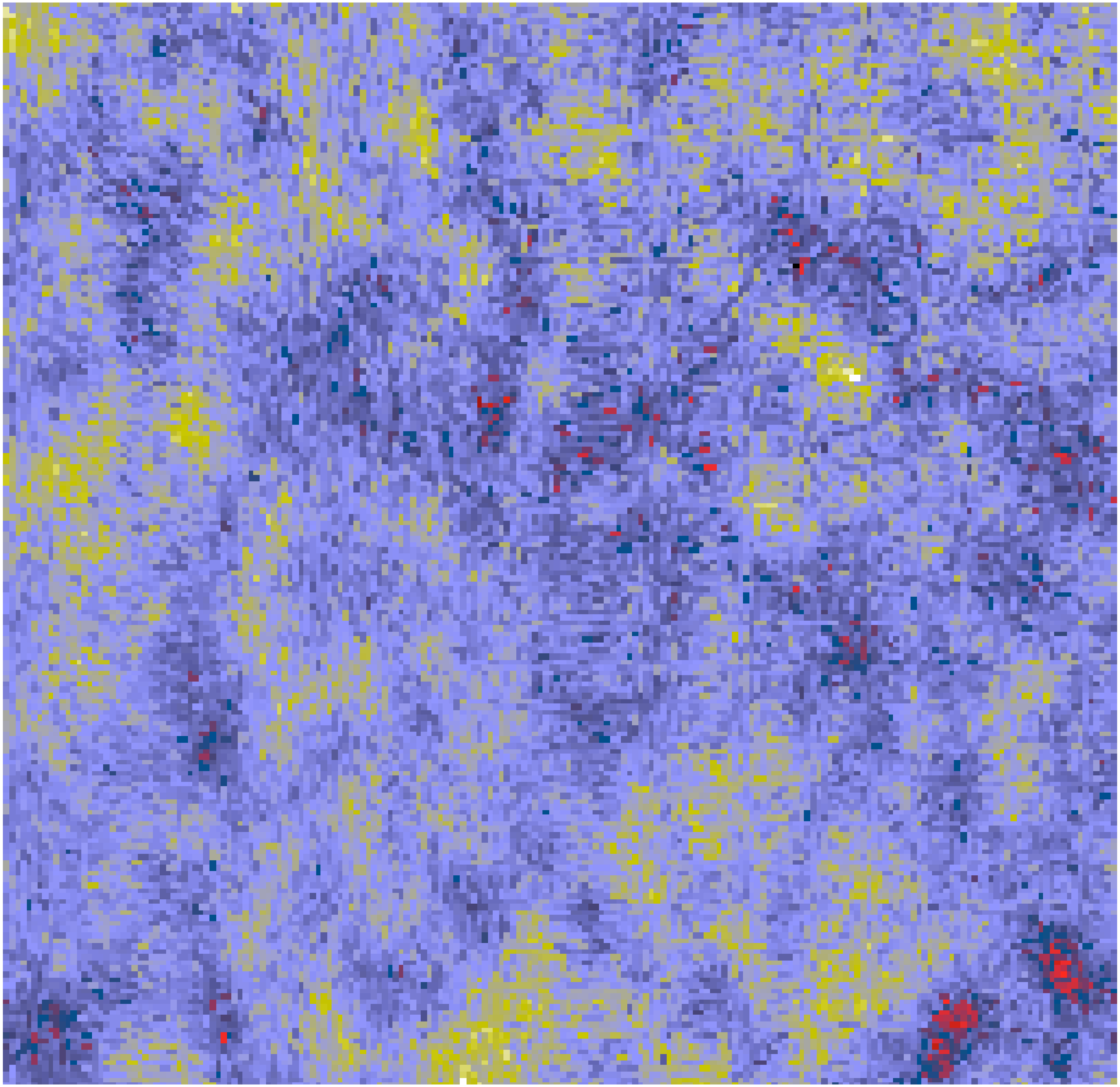
50%

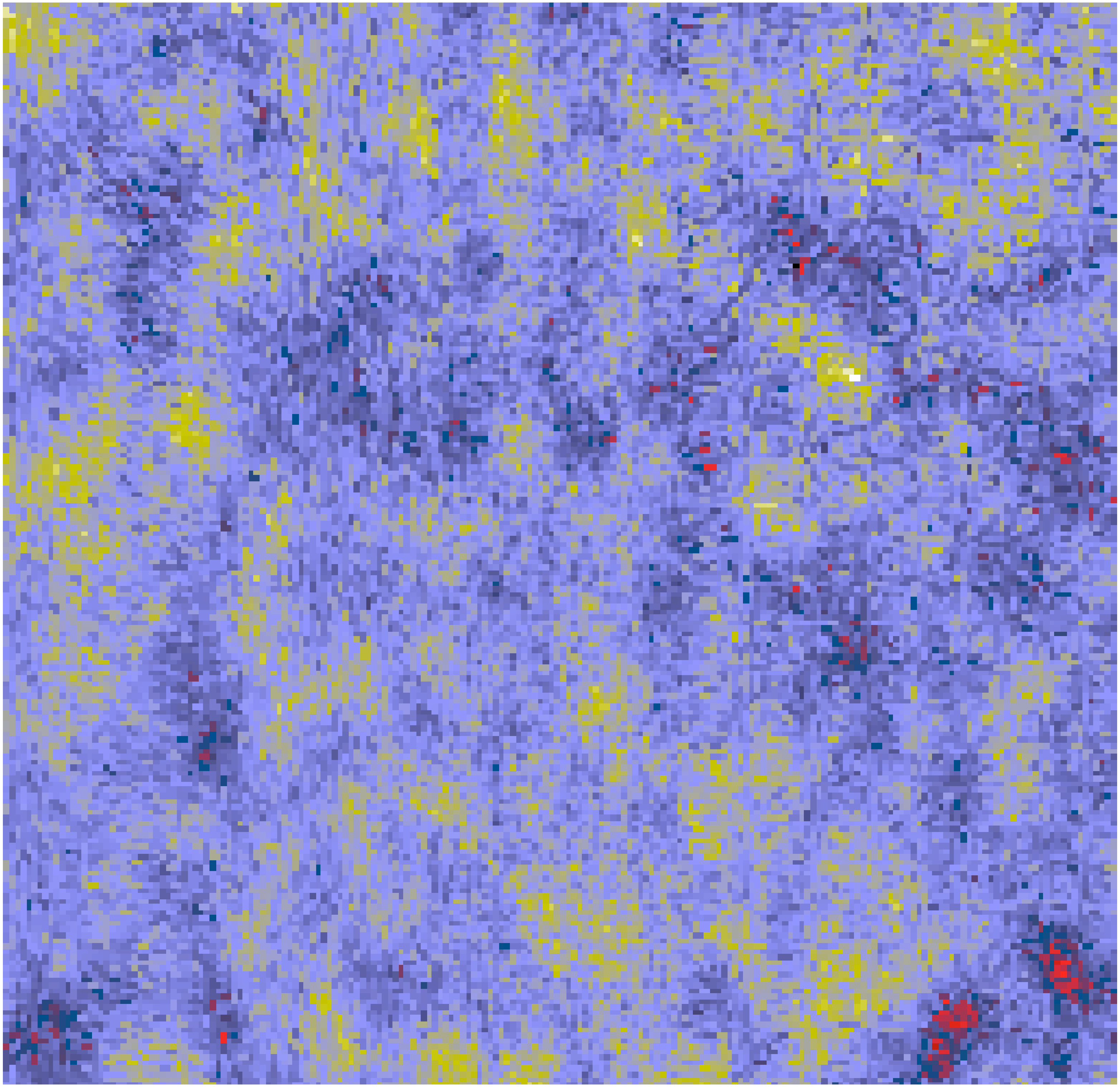
80%

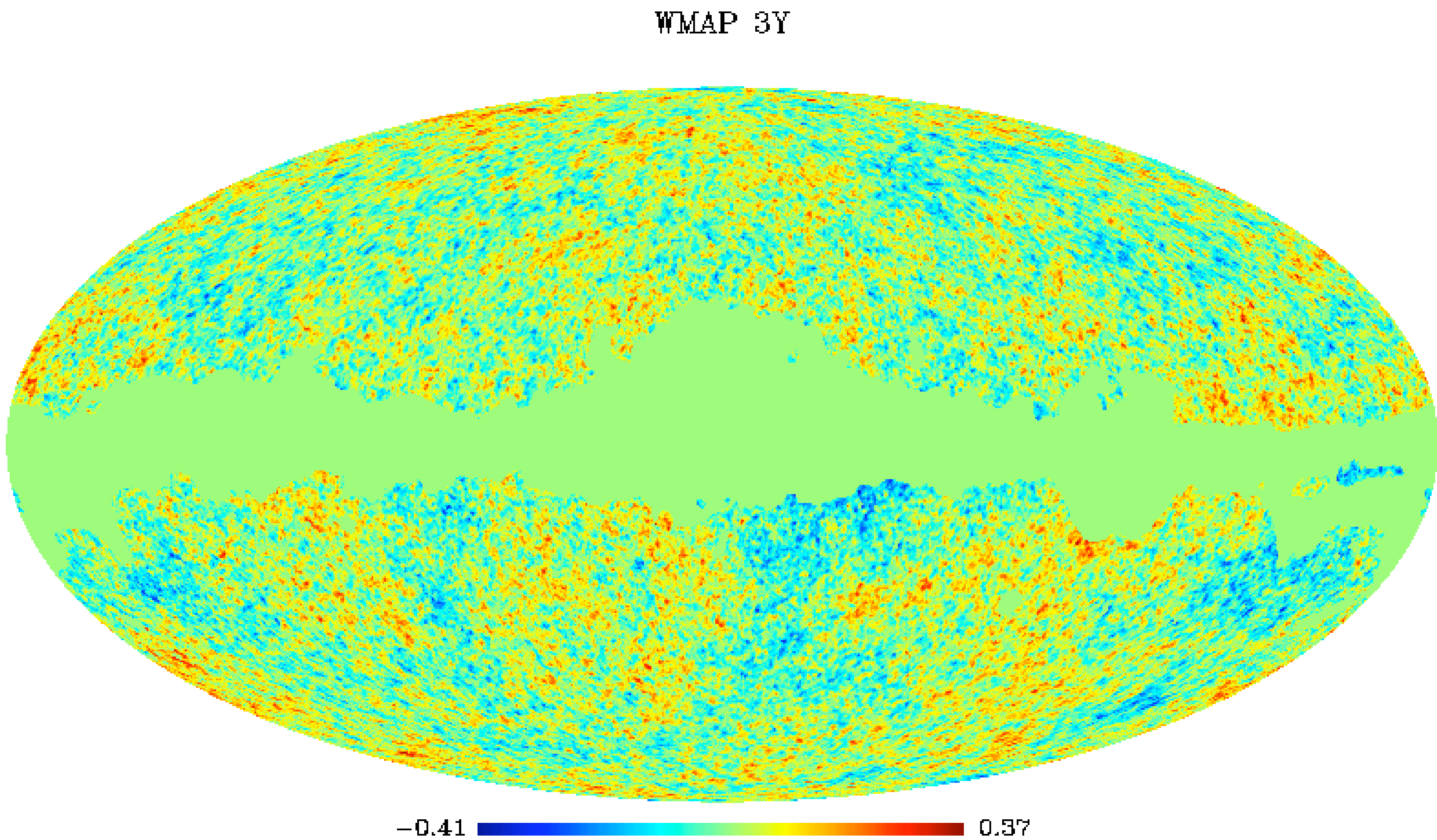
Application in Cosmology



The cosmic Microwave Background is a relic radiation (with a temperature equals to 2.726 Kelvin) emitted 13 billion years ago when the Universe was about 370000 years old.



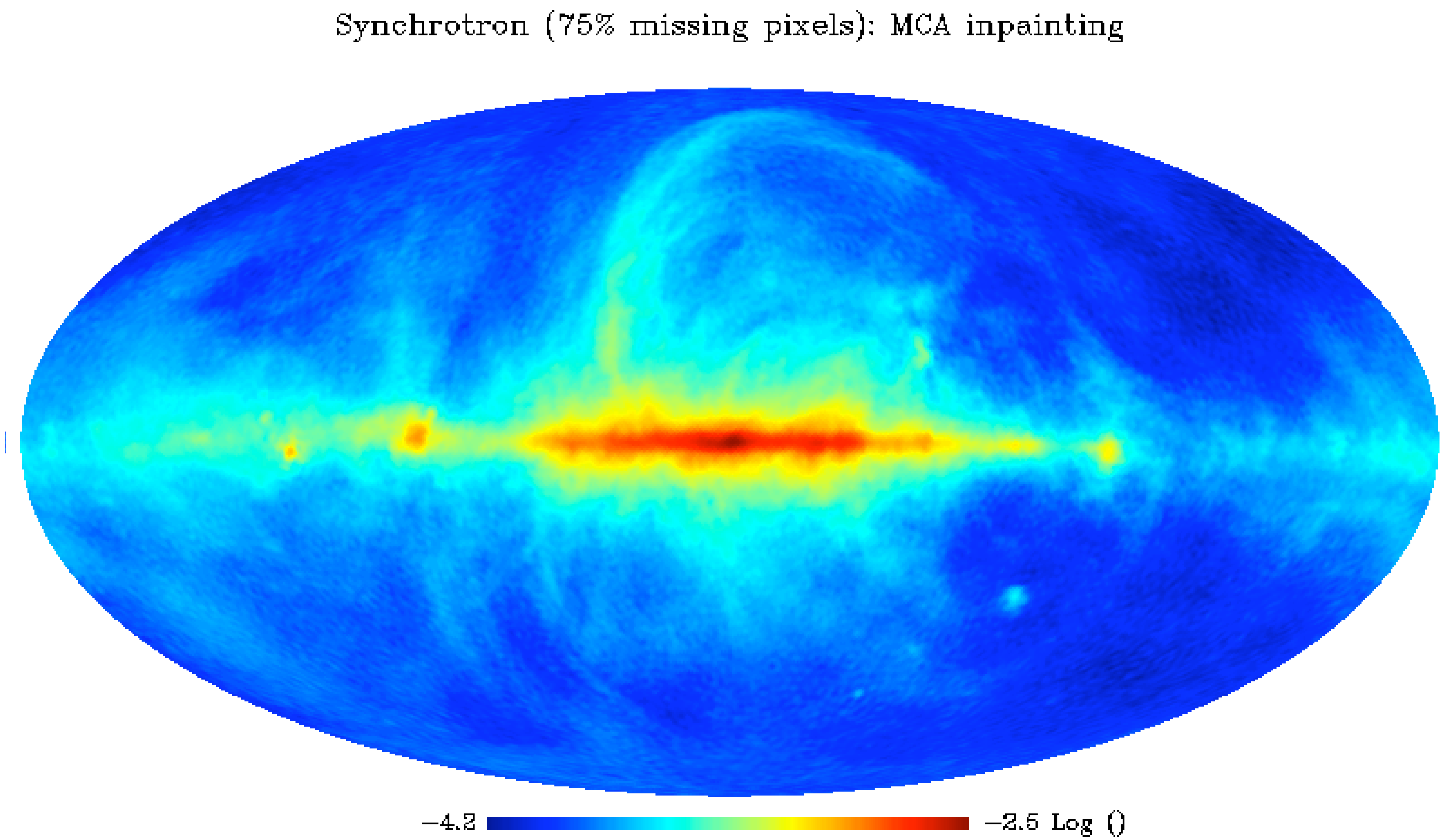


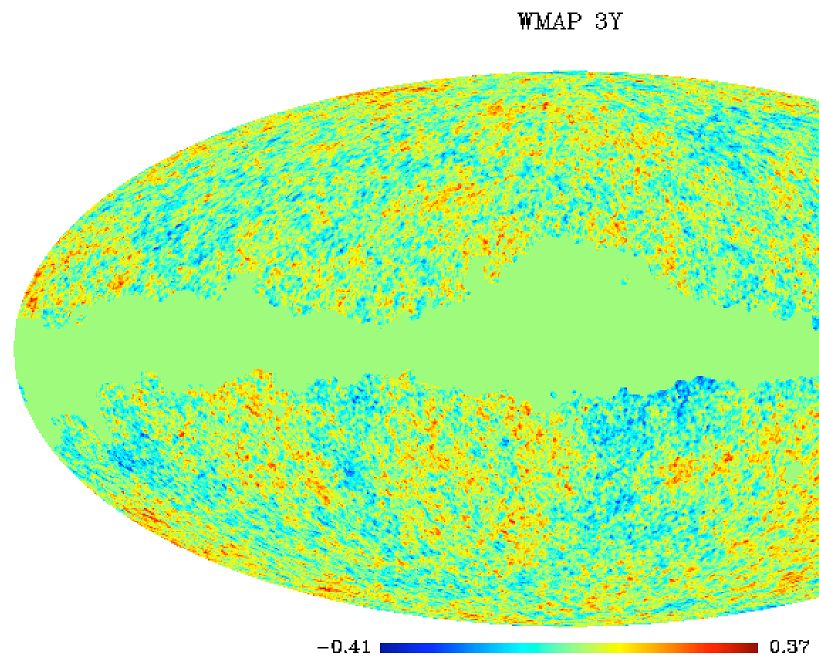


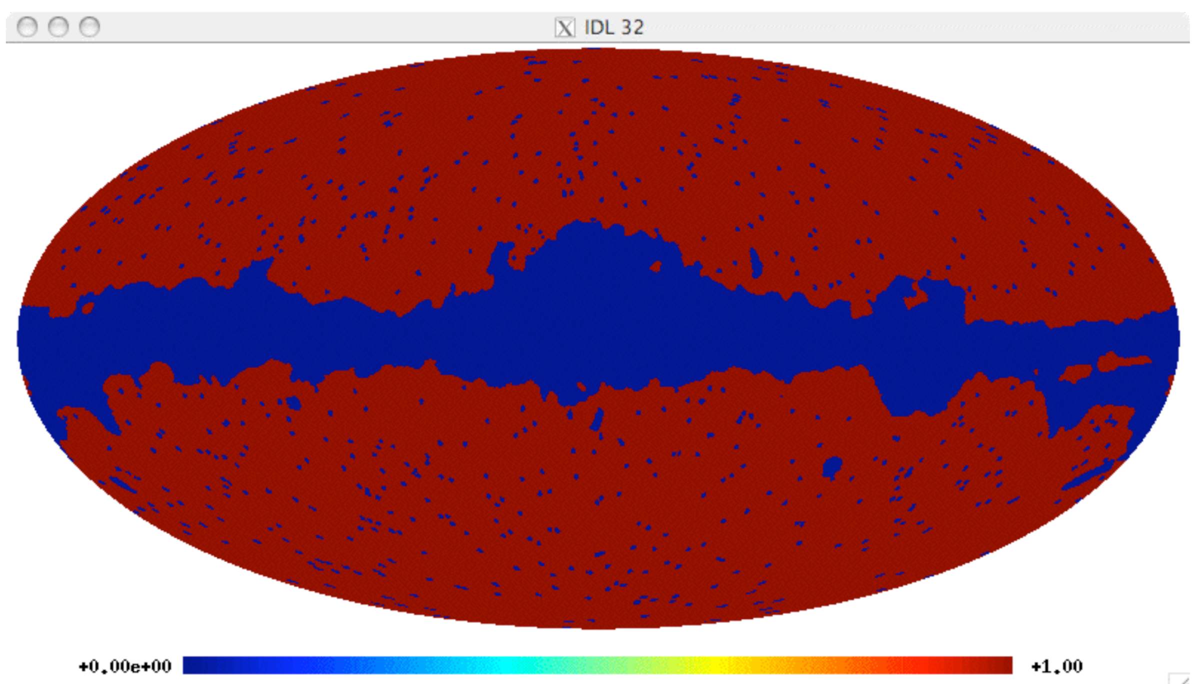
=> we need an *inpainting* method to fill properly the gaps.

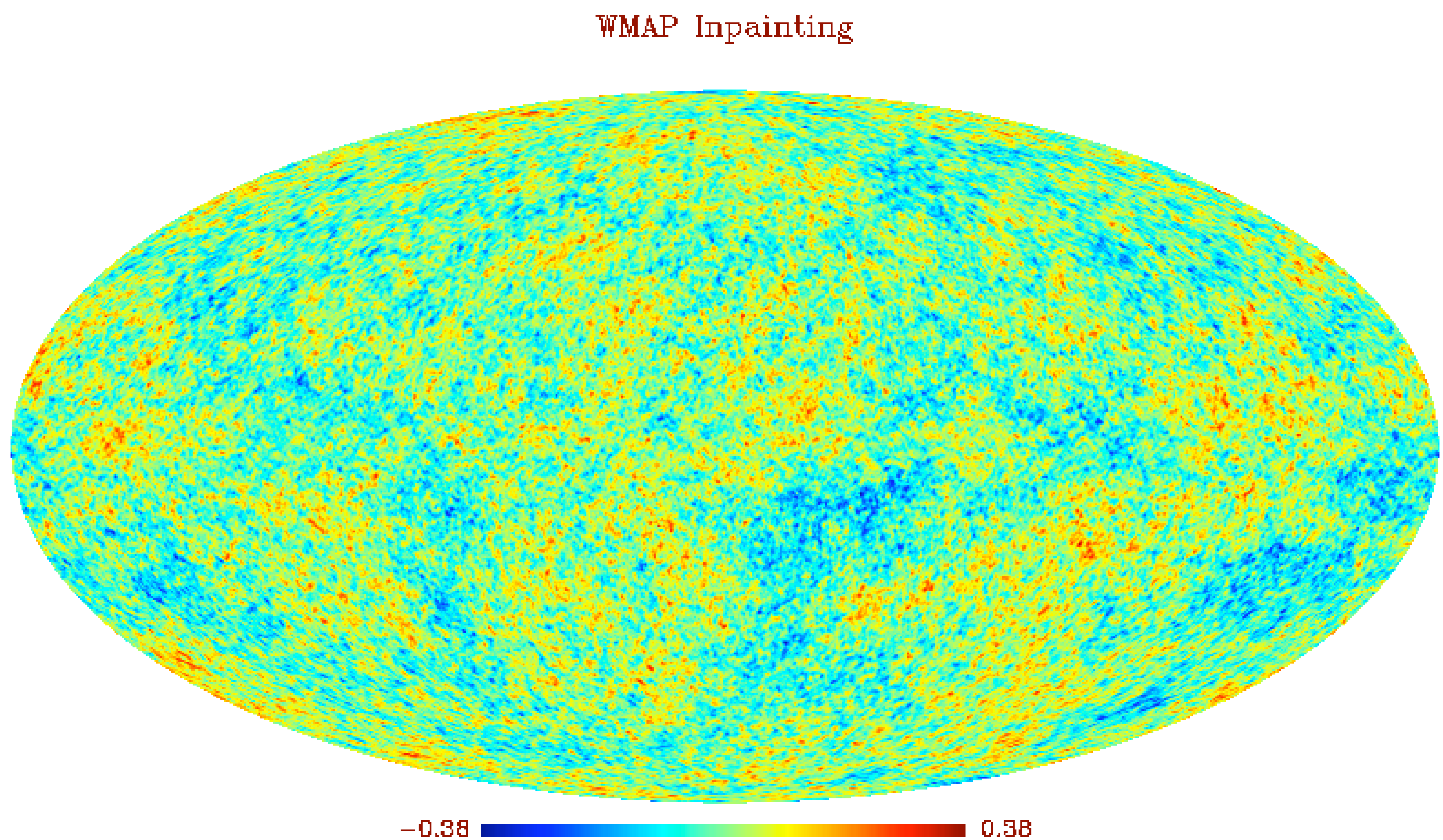
$$J(X) = \| Y - MX \| \|^2 + \lambda \|\mathcal{T} X\|_1$$

Simultaneous Cartoon and Texture Image Inpainting using Morphological Component Analysis (MCA), Applied and Computational Harmonic Analysis, Vol. 19, pp. 340-358, November 2005.


Synchrotron (75% missing pixels): MCA inpainting
−4.2
−2.5 Log ()


WMAP 3Y
−0.41
0.37


IDL 32
+0.00e+00
+1.00


WMAP Inpainting
−0.38
0.38

Multichannel MCA (MMCA)

$$X = AS \quad \text{or} \quad X_i = \sum_{k=1}^{K} a_{i,k} s_k, \;\; \exists T_k \text{ such that } \alpha_k = T_k s_k \text{ is sparse}$$

According to the MCA paradigm, each source is morphologically different from the others. Each source s_k is then well sparse in a specific basis Φ_k. Thus MMCA aims at solving the following minimization problem:

$$\min_{A,s_1,\ldots,s_k} = \sum_{l=1}^{m} \left\| X_l - \sum_{k=1}^{K} A_{k,l} s_k \right\|_2^2 + \lambda \sum_{k=1}^{K_i} \left\| T_k s_k \right\|_p$$

Both the source matrix S and the mixing matrix A are estimated alternately for fixed values of λ_k from a Maximum A Posteriori.

Defining a multichannel residual D_k: $\mathbf{D}_k = \mathbf{X} - \sum_{k' \neq k} a^{k'} s_{k'}$

the parameters are **alternately** estimated such that :

$$J(s_k) = \left\| D_k - s_k \right\|_2^2 + \lambda_n \left\| T_k s_k \right\|_p$$

The MMCA Algorithm

. Initialize all s_k to zero

. Iterate t=1,...,Niter

- Iterate k=1,..,L

  Update the kth part of the current solution by fixing all other parts and minimizing:

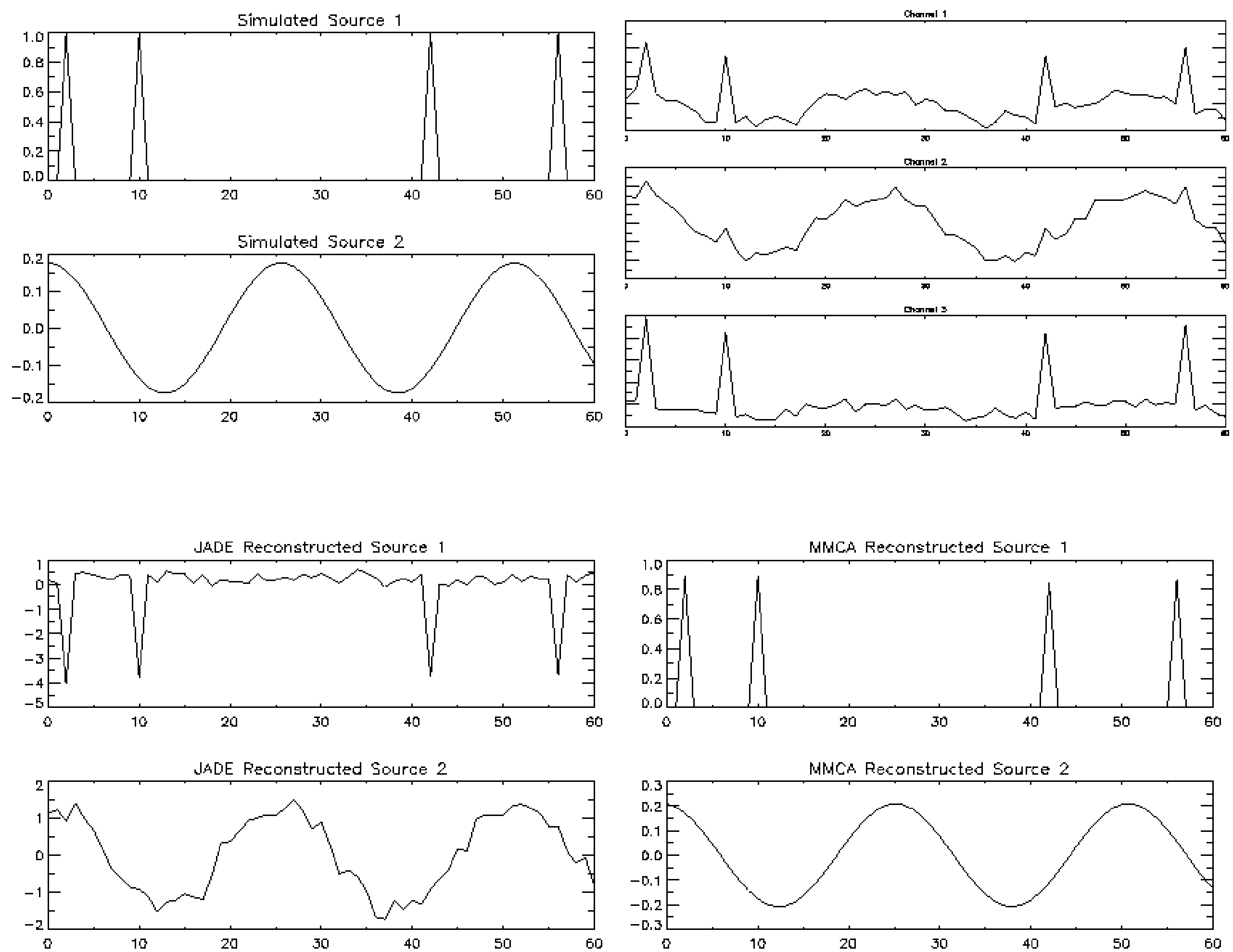
  $$J(s_k) = \left\| D_k - s_k \right\|_2^2 + \lambda_t \left\| T_k s_k \right\|_1 \quad \text{with} \quad D_k = a^{k^T} (X - \sum_{i=1, i \neq k}^{L} a^i s_i$$

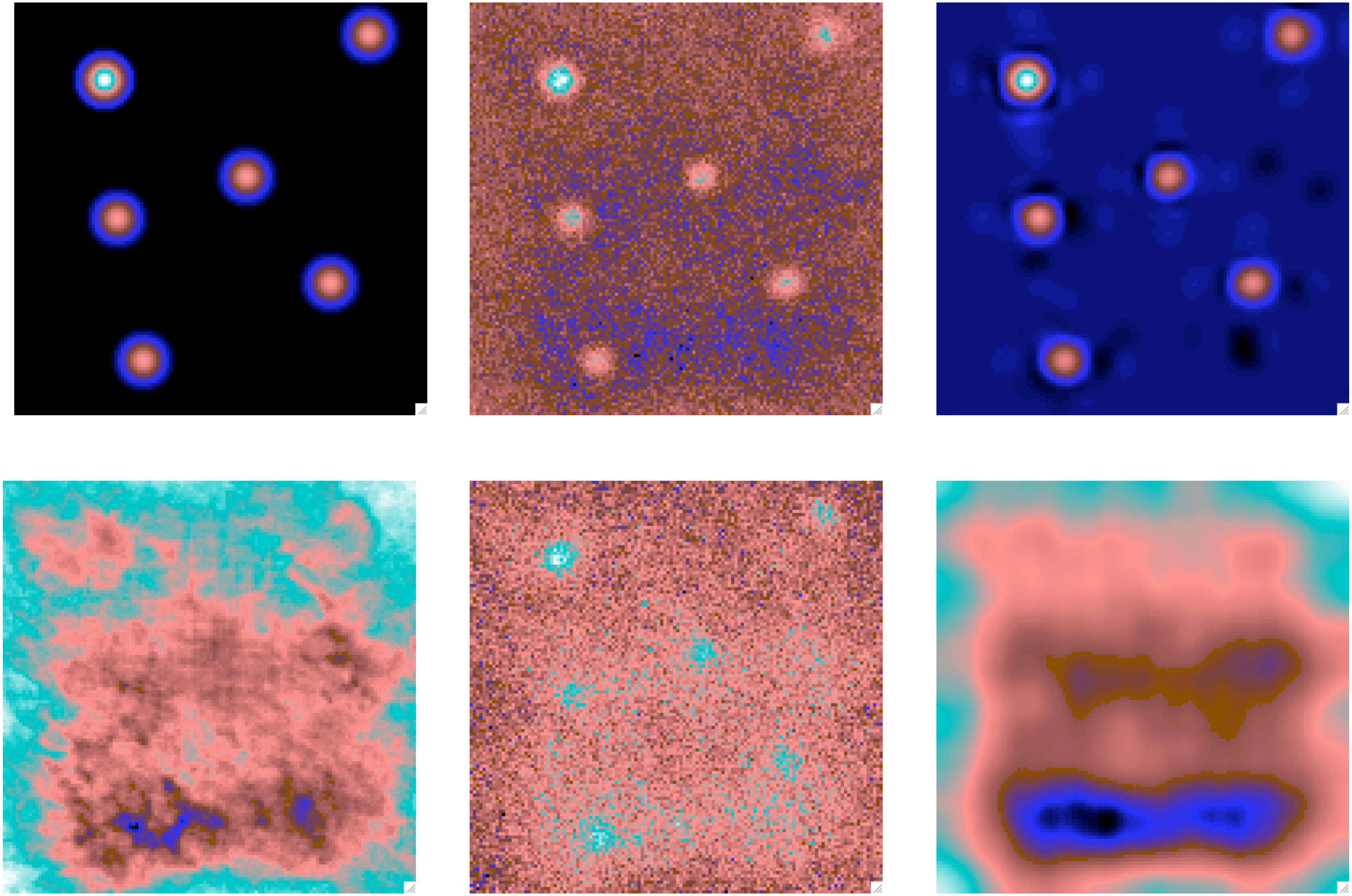
  which is obtained by a simple hard/soft thresholding of D_k

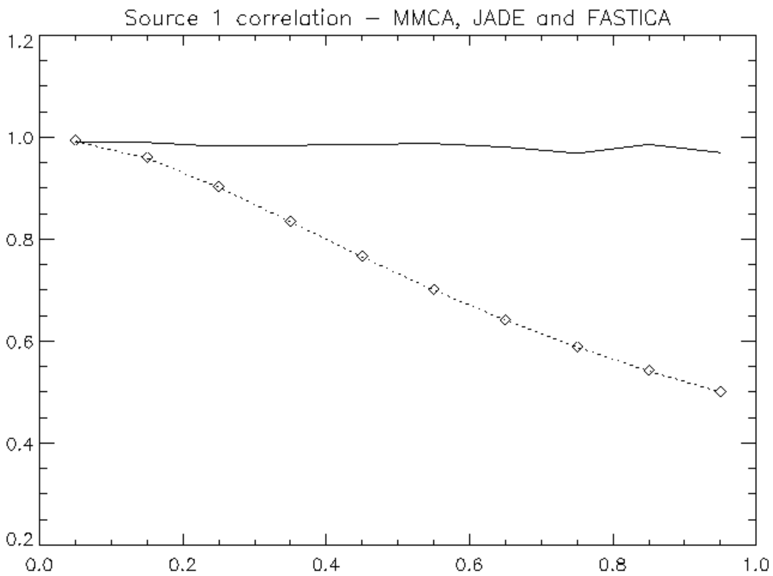
- estimation of a^k assuming all s_l and $a^l_{l \neq k}$ fixed

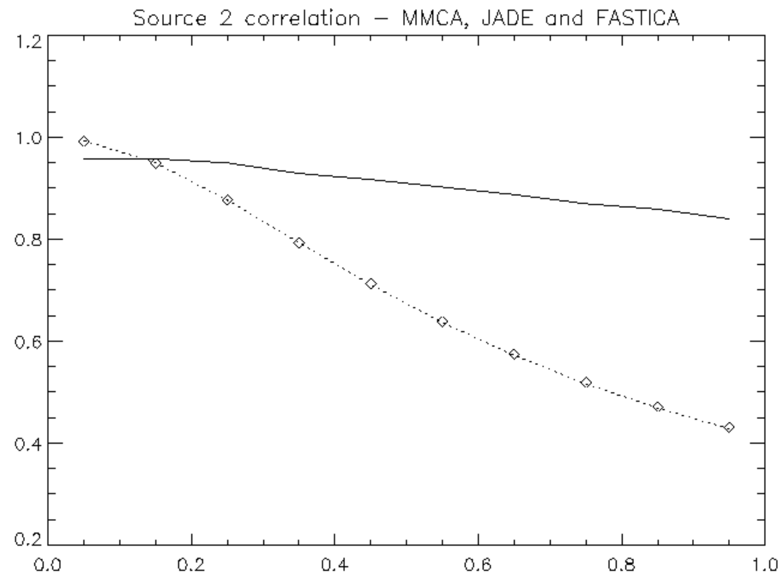
  $$a^k = \frac{1}{s_k s_k^T} D_k s_k^T$$

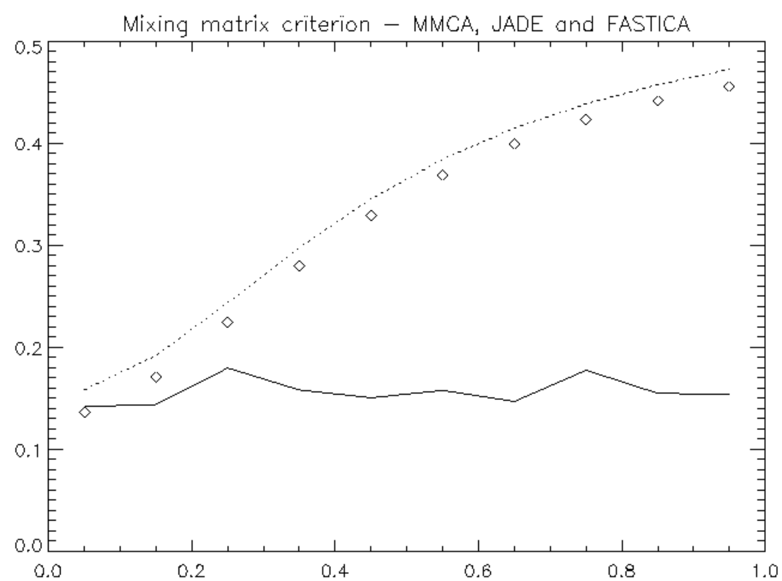
- Decrease λ_t


Simulated Source 1
Simulated Source 2
Channel 1
Channel 2
Channel 3
JADE Reconstructed Source 1
MMCA Reconstructed Source 1
JADE Reconstructed Source 2
MMCA Reconstructed Source 2




Source 1 correlation – MMCA, JADE and FASTICA
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0


Source 2 correlation – MMCA, JADE and FASTICA
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0


Mixing matrix criterion – MMCA, JADE and FASTICA
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Generalized MCA (GMCA)

Source: $S = [s_1, \ldots, s_n]$ Data: $X = [x_1, \ldots, x_m] = AS$

We now assume that the sources are linear combinations of morphological components :

$$s_i = \sum_{k=1}^{K} c_{i,k} \quad \text{such that} \quad \alpha_{i,k} = T_{i,k} c_{i,k} \text{ sparse}$$

$$\Longrightarrow \qquad X_l = \sum_{i=1}^{n} A_{i,l} s_i = \sum_{i=1}^{n} A_{i,l} \sum_{k=1}^{K} c_{i,k}$$

$$\phi = [[\phi_{1,1}, \ldots, \phi_{1,K}], \ldots, [\phi_{n,1}, \ldots, \phi_{n,K}]], \quad \alpha = S\phi^t = [[\alpha_{1,1}, \ldots, \alpha_{1,K}], \ldots, [\alpha_{n,1}, \ldots, \alpha_{n,K}]]$$

GMCA aims at solving the following minimization:

$$\min_{A, c_{1,1}, \ldots, c_{1,K}, \ldots, c_{n,1}, \ldots, c_{n,K}} = \sum_{l=1}^{m} \left\| X_l - \sum_{i=1}^{n} A_{i,l} \sum_{k=1}^{K} c_{i,k} \right\|_2^2 + \lambda \sum_{i=1}^{n} \sum_{k=1}^{K} \left\| T_{i,k} c_{i,k} \right\|_p$$

The GMCA Algorithm

. Initialize all c_k to zero, $\lambda_1 = \max(\alpha), \delta = \max(\alpha)/\text{Niter}$

. Iterate t=1,...,Niter

- Iterate i=1,..,NbrSource

  Defining a multichannel residual $\mathbf{D_i}$: $D_i = X - \sum_{i' \neq i} a^{i'} s_{i'}$

  Iterate k=1,..,K_k

  - Least square estimate of $c_{i,k}$: $l_{i,k} = \frac{1}{a^{i^T} a^i} a^{i^T} (D_i - a^i \sum_{k' \neq k} c_{i,k'})$

  - Minimize: $J(\tilde{l}_{i,k}) = \left\| l_{i,k} - \tilde{l}_{i,k} \right\|_2^2 + \lambda_t \left\| T_{i,k} \tilde{l}_{i,k} \right\|_1$

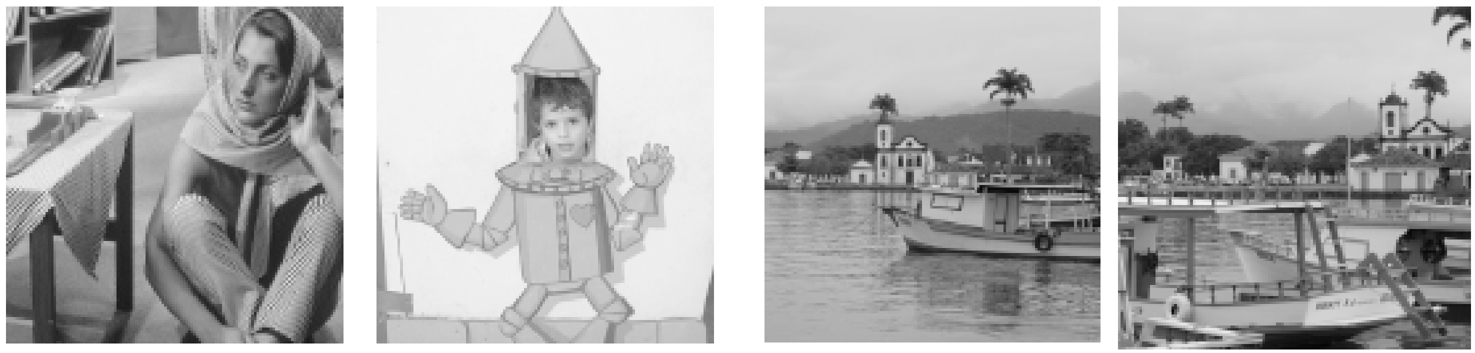
    which is obtained by a simple hard/soft thresholding of $l_{i,k}$

  $s_k = \sum_i l_{k,i}$

- $S = [s_1, \ldots, s_K]^t$
- Estimation of the matrix A: $A = XS^t (SS^t)^{-1}$
- Decrease $\lambda_{t+1} = \lambda_t - \delta$

A first result (1)

Original Sources



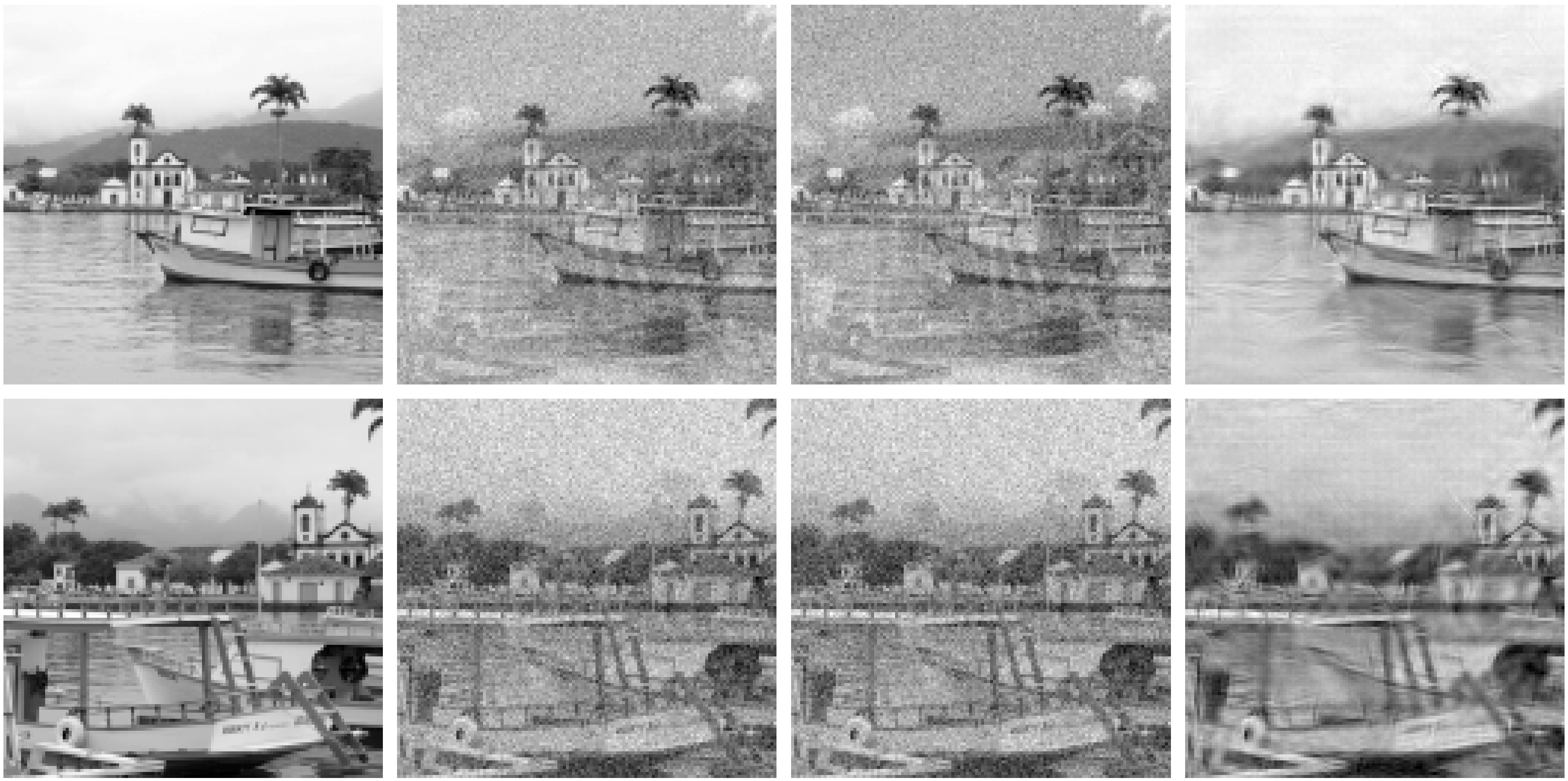
Mixtures

Noiseless experiment, 4 random mixtures, 4 sources

A first result (2)

2 mixtures SNR = 10.4dB

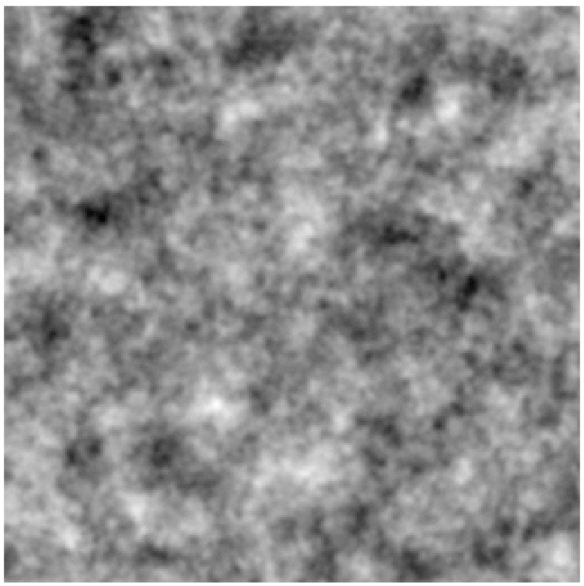
$\Phi =$ Curvelets + DCT



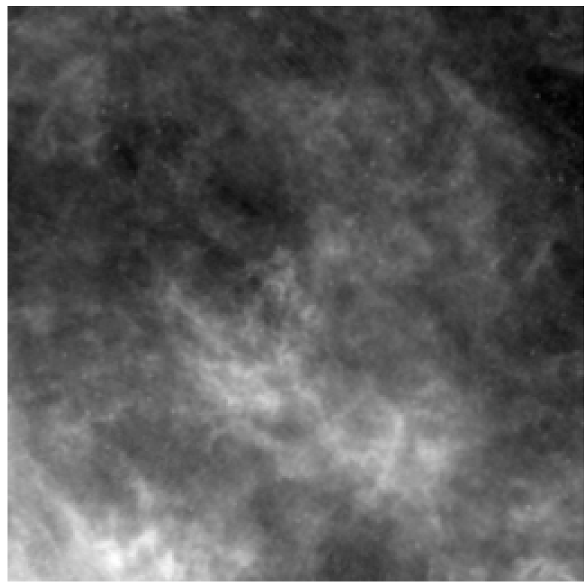
Sources Mixtures JADE

The source images: 300x300 pixels corresponding to a field of 12,5x12,5 degres.

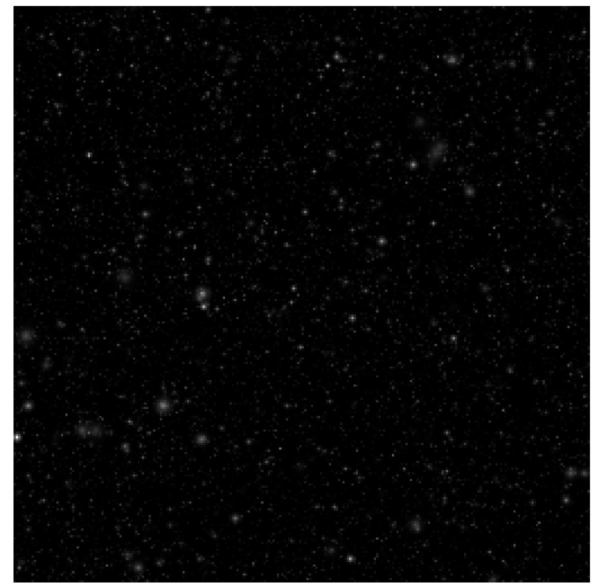
CMB



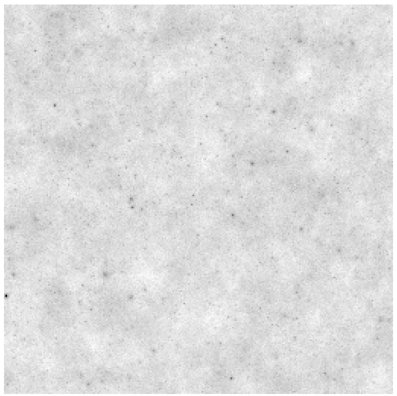
DUST



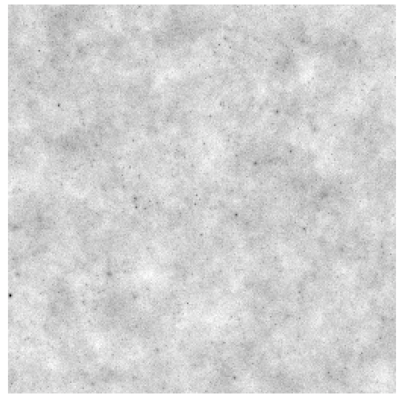
SZ



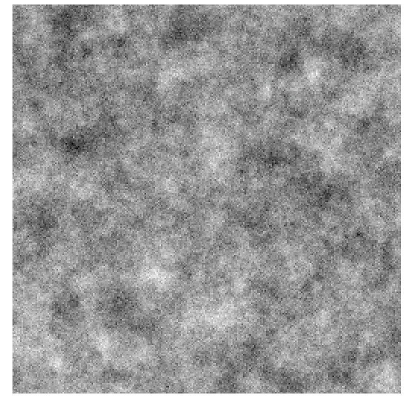
The six simulated HFI Channels
(100, 143, 217, 353, 545 and 857 GHz)



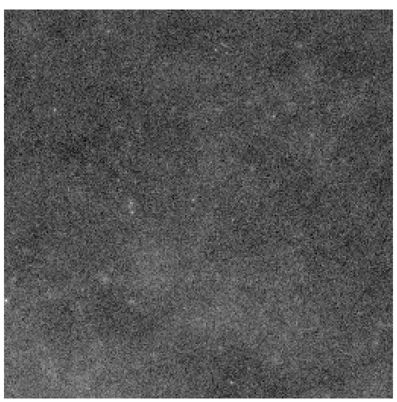
3.6 dB



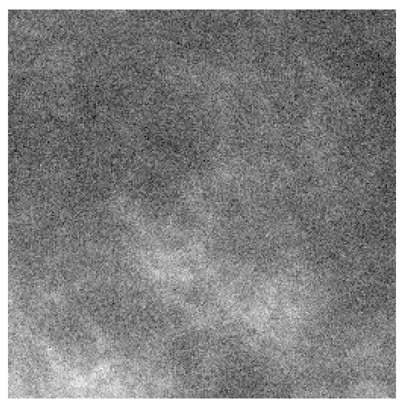
4.3 dB



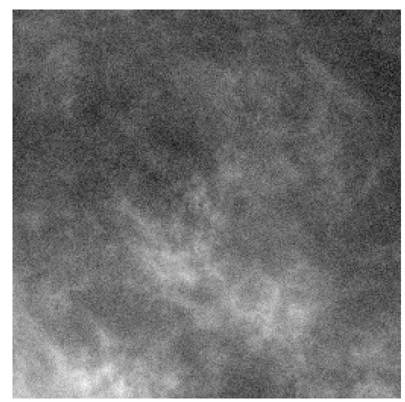
1.4 dB



-3.7 dB

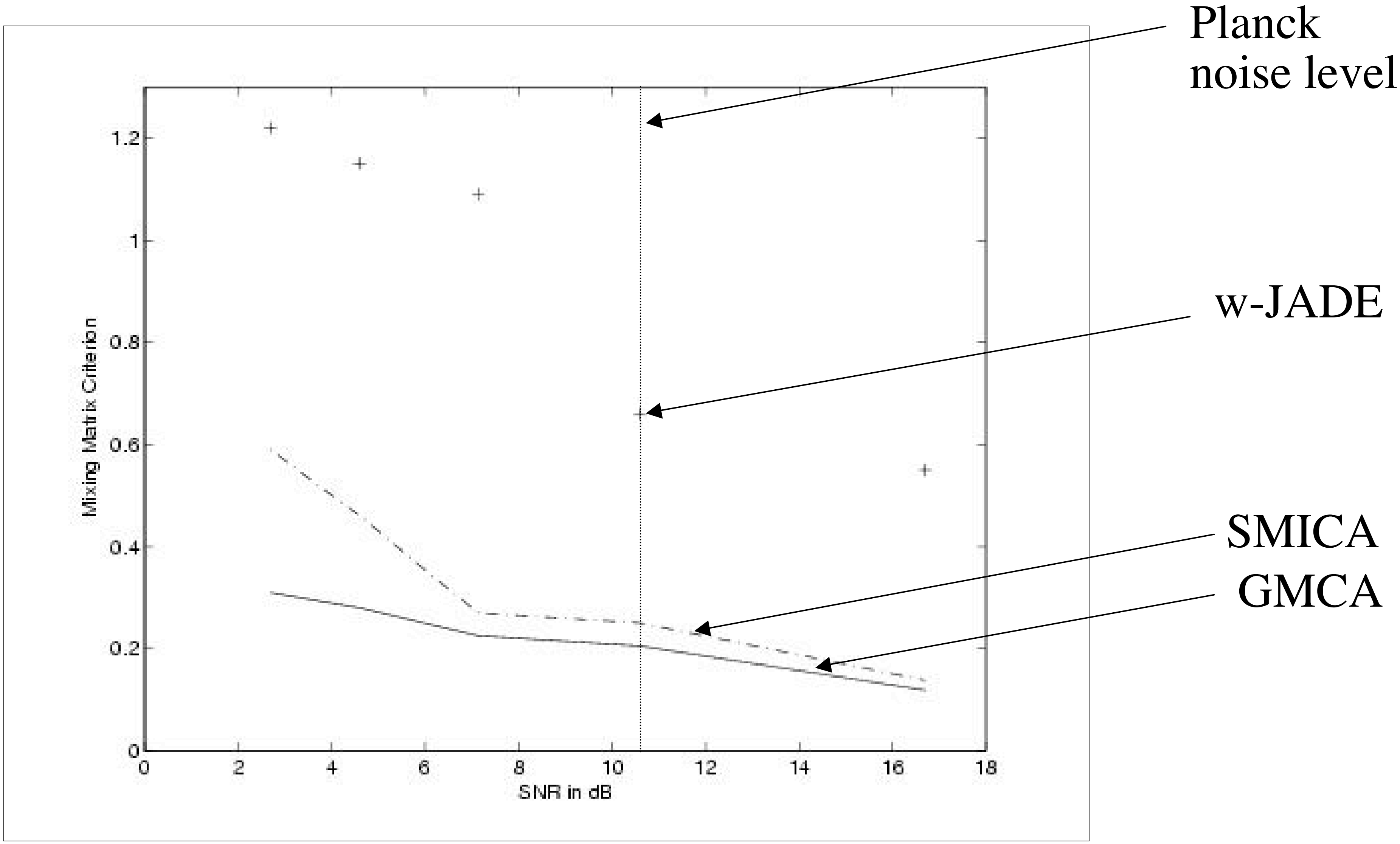


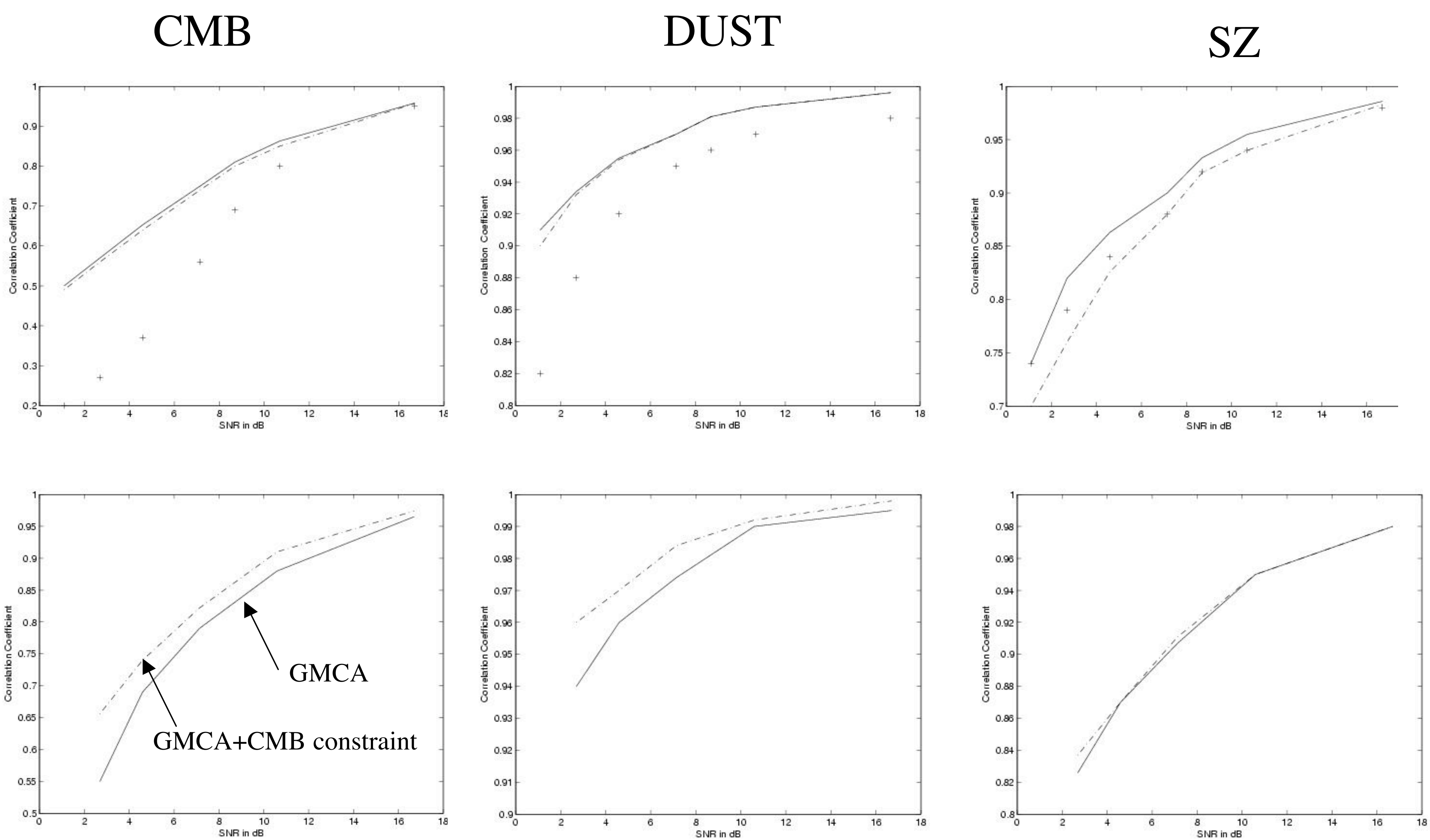
1.25 dB



9.35 dB

Mixing Matrix Estimation Error




CMB
DUST
SZ
Correlation Coefficient
SNR in dB
GMCA
GMCA+CMB constraint

Conclusions

MCA method can be useful in different applications such texture separation or inpainting.

.**Redundant Multiscale Transforms and their Application for Morphological Component Analysis,** *Advances in Imaging and Electron Physics, 132, 2004*.

. **Image Decomposition Via the Combination of Sparse Representation and a Variational Approach**, *IEEE Transaction on Image Processing*, *14, 10, pp 1570--1582, 2005.*

. **Simultaneous Cartoon and Texture Image Inpainting using Morphological Component Analysis (MCA),** ACHA, *19, pp. 340-358, 2005.*

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The MMCA algorithm brings a very strong and robust component separation as long as the MMCA hypothesis is verified (sources are sparsified in different bases) i.e. for morphologically diverse sources.

. **Morphological Diversity and Source Separation",** *IEEE Trans. on Signal Processing letters, in press.*

GMCA is more general, and can be applied for many applications:

$$s_i = \sum_{k=1}^{K_i} c_{i,k} \quad \text{and} \quad X_l = \sum_{i=1}^{n} A_{i,l} s_i = \sum_{i=1}^{n} A_{i,l} \sum_{k=1}^{K_i} c_{i,k}$$

More MCA experiments available at http://jstarck.free.fr/mca.html and Jalal Fadili's web page (http://www.greyc.ensicaen.fr/~jfadili).