Analysis of Vector Inflation Models Using Dynamical Systems

José Fernando Rodríguez Ruiz[1]

Yeinzon Rodríguez García[1,2]

[1] School of Physics - Industrial University of Santander - Bucaramanga - Colombia

[2] University of Antonio Narinño - Bogota D.C. - Colombia

Contents

Introduction

- Large number of vector fields are present in the standard model and its extensions.
- It is natural to consider vector field models of inflation.
- Anomalies found in the Planck data could be explained with the addition of vector fields.

Introduction

The main purpose is to analyze possible vector field inflation models using the techniques of dynamical systems.
The following conditions must be fulfilled:

- Inflation must last long enough to solve the classical problems of cosmology.
- Inflation must be finite.
- The anisotropic expansion must agree with the observational data.

Scalar field inflation

- The methodology will be illustrated by the dynamics of the scalar field whose lagrangian density is:

$$\mathcal{L}_\varphi = \varphi_{,\mu}\varphi^{,\mu} - V(\varphi). \tag{1}$$

- The energy density and pressure associated to the scalar field are:

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{2}$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{3}$$

Scalar field inflation

- The field equations are:

$$3m_p^2 H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{4}$$

$$\dot{H} = -\left[\frac{3}{2}H^2 + \frac{1}{2m_p^2}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)\right] = -\frac{\rho + p}{2m_p^2}, \tag{5}$$

$$\ddot{\varphi} = -3H\dot{\varphi} - V_{,\varphi}, \tag{6}$$

where

$$m_p = \frac{1}{\sqrt{8\pi G}},$$

is the reduced Planck mass and G is the gravitational constant.

Dynamics of a scalar field

- We define the following dimensionless variables:

$$x = \frac{\dot{\varphi}}{\sqrt{2} m_p H}, \quad y = \frac{\sqrt{\rho_{\rm V}}}{m_p H} = \frac{\sqrt{V}}{m_p H}. \tag{7}$$

- Autonomous dynamical system:

$$H' = -H x^2, \tag{8}$$

$$x' = -3x - \Lambda y^2 + x^3, \tag{9}$$

$$y' = \Lambda x y + x^2 y, \tag{10}$$

where a prime denotes derivation with respect to the number of e-folds and the parameter Λ is:

$$\Lambda = \frac{m_p}{\sqrt{2}} \frac{V_{,\varphi}}{V}. \tag{11}$$

- The Friedmann equation becomes a constraint:

$$3 = x^2 + y^2. \tag{12}$$

Dynamics of scalar field

- The definition of the variables allows us to decouple the Hubble parameter.
- Using the Friedmann equation the dynamics is reduced to the following single equation:

$$x' = g(x) = -3x - 3\Lambda + \Lambda x^2 + x^3. \tag{13}$$

- $x_c = -\Lambda$ is an equilibrium point.
- The condition for the existence of inflation is that the potential dominates i.e. y dominates. From the Friedmann equation we obtain the following condition $\Lambda^2 < 3$.

Stability

- To analyse the stability, the equation (13) is linearized:

$$\begin{aligned} x' &= g'(x_c)x, \\ &= (-3 + \Lambda^2)x. \end{aligned} \tag{14}$$

- When $-3 + \Lambda^2 < 0$ the critical point is stable. Therefore inflation is stable.

Slow-roll inflation

- The following are the conditions for slow-roll inflation:

$$\epsilon = -\frac{\dot{H}}{H} \ll 1, \quad |\eta| = |-\frac{\ddot{H}}{2H\dot{H}} + \epsilon| = |2\epsilon - \frac{1}{2H}\frac{\dot{\epsilon}}{\epsilon}|,$$
$$= |2\epsilon - \frac{1}{2}\frac{\epsilon'}{\epsilon}| \ll 1. \quad (15)$$

- The slow-roll parameters evaluated at the critical point are:

$$\epsilon = \Lambda^2, \eta = \Lambda^2. \quad (16)$$

Inflation is attractive

Slow-roll inflation requires $\Lambda^2 \ll 1 < 3 \Longrightarrow$ Asymptotic stability.

Evolution of Λ

- The evolution of the system approaching to the critical point must be faster than the evolution of the critical point itself.

$$\left|\frac{1}{x_c}\frac{dx_c}{dN}\right| < \left|\frac{1}{x}\frac{dx}{dN}\right| = |m_\beta|. \tag{17}$$

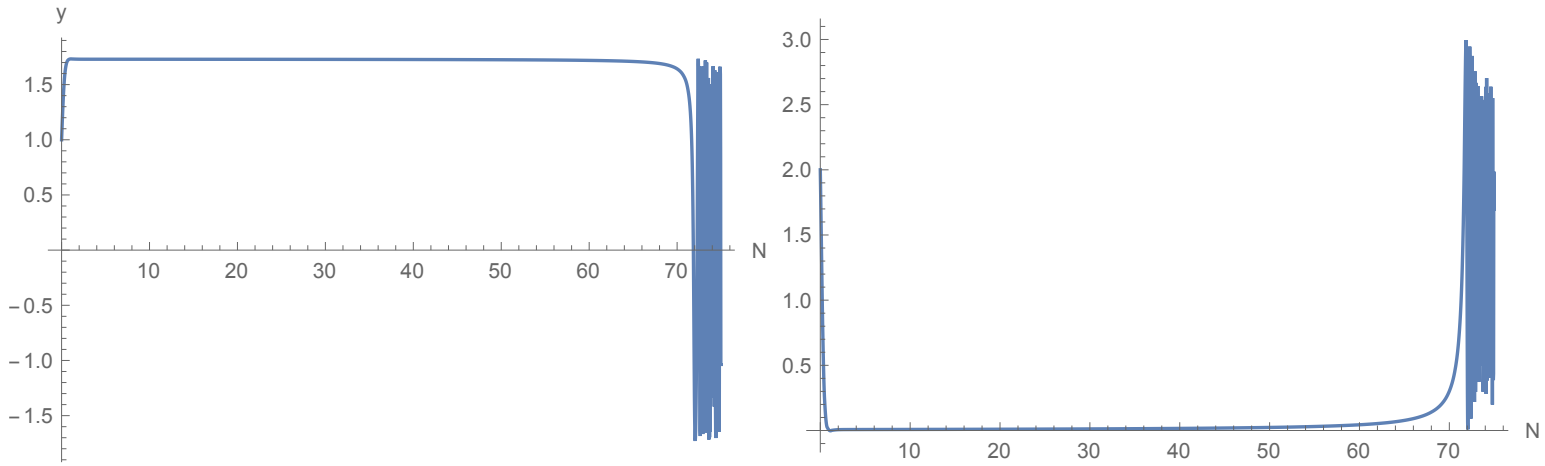
- The last condition is equivalent to:

$$|2\epsilon - \eta| < |-3 + \epsilon|. \tag{18}$$

- The conditions of slow-roll guarantee the condition (18), so during slow-inflation the parameter Λ evolves slowly.

Particular case $V = \frac{1}{2}m^2\varphi^2$

- For the case $V = \frac{1}{2}m^2\varphi^2$, the condition for existence of inflation is $2/3 < (\varphi/m_p)^2$.

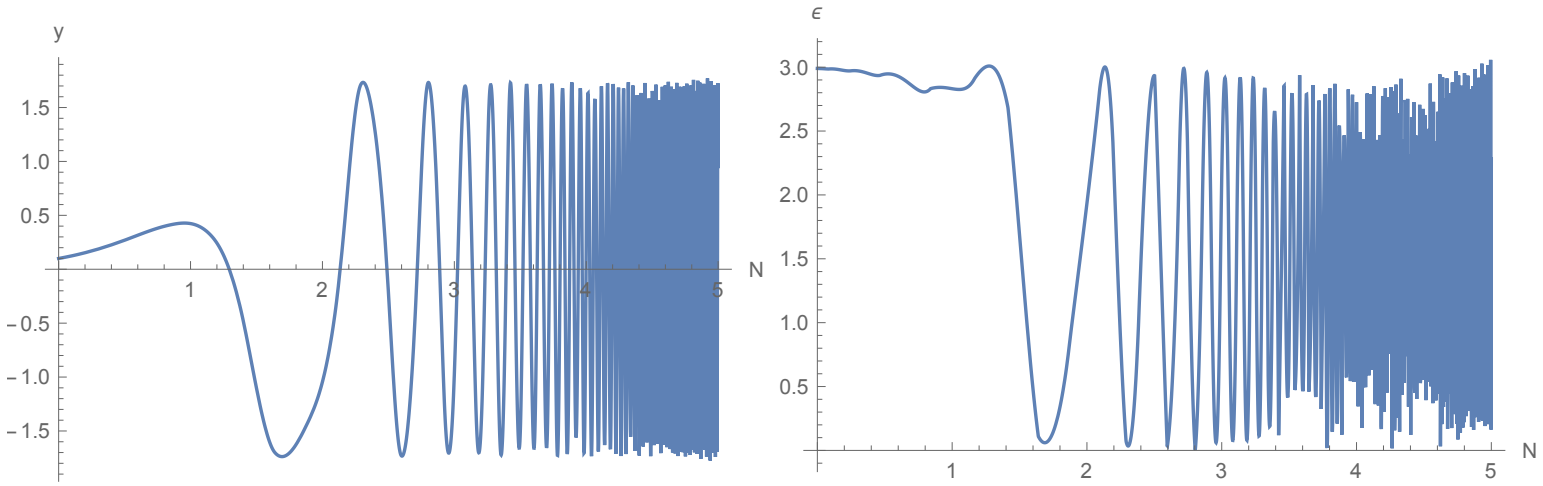


(a) y vs number of e-folds N.

(b) ϵ vs number of e-folds N.

Figure: Behaviour of the system when the condition for the existence of inflation is satisfied.

Particular case $V = \frac{1}{2}m\varphi^2$



(a) y vs number of e-folds N.

(b) ϵ vs number of e-folds N.

Figure: Behaviour of the system when the condition for the existence of inflation is not satisfied.

Geometry of spacetime

- We assume a homogeneous but anisotropic spacetime. We choose Bianchi Type I metric expressed in the following way:

$$g_{tt} = 1, \quad g_{xx} = g_{yy} = -e^{2(\alpha+\sigma)}, \quad g_{zz} = -e^{2\alpha-4\sigma}. \tag{19}$$

- Global Hubble parameter:

$$H := \dot{\alpha}. \tag{20}$$

- Cosmic shear:

$$\Sigma := \frac{H_{\mathrm{xy}} - H}{H} = \frac{\dot{\sigma}}{\dot{\alpha}}. \tag{21}$$

- Current value of the cosmic shear Σ_0[1]:

$$-0.012 < \Sigma_0 < 0.012. \tag{22}$$

[1] L. Campanelli, P. Cea, G. Fogli, and A. Marrone, Testing the Isotropy of the Universe with Type Ia Supernovae, Phys. Rev. **D83**, 103503 (2011)

U(1) field vector Model

- The lagrangian density of the abelian vector field is [Cembranos, 2012a]:

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(A_\mu A^\mu), \tag{23}$$

  where $V = \lambda(-A_\mu A^\mu)^n$.
- The vector field is homogeneous and the spatial components can be lined up in the z-direction $A_\mu = (A_t, 0, 0, A_z)$.
- The reduced lagrangian is:

$$L_r = e^{3\alpha} \left[m_p^2 \left(-3\dot{\alpha}^2 + 3\dot{\sigma}^2\right) - \frac{1}{2} g^{zz} \dot{A}_z^2 - \lambda \left(-g^{zz} A_z^2\right)^n \right]. \tag{24}$$

Field equations

$$3m_p^2\left(H^2+\dot{\sigma}^2\right)=-\frac{1}{2}g^{33}\dot{A}^2+\lambda\left(-g^{33}A^2\right)^n, \tag{25}$$

$$\dot{H}=-3H^2-\frac{1}{3m_p^2}\left[\frac{1}{2}g^{33}\dot{A}_z^2+\lambda(n-3)\left(-g^{33}A_z^2\right)^n\right], \tag{26}$$

$$\ddot{\sigma}=-3H\dot{\sigma}-\frac{1}{3m_p^2}\left[g^{33}\dot{A}_z^2+2n\lambda\left(-g^{33}A_z^2\right)^n\right], \tag{27}$$

$$\frac{dV}{d(A^2)}A_0=0 \tag{28}$$

$$\ddot{A}_z=-\dot{A}_z\left(4\dot{\sigma}+H\right)-2n\lambda\left(-g^{33}\right)^{n-1}A_z^{2n-1}. \tag{29}$$

Dynamical system

- We define the following dimensionless variables:

$$\Sigma \equiv \frac{\dot{\sigma}}{H}, \quad x \equiv \frac{\sqrt{-\frac{1}{2}g^{zz}\dot{A}_z^2}}{\sqrt{3}m_p H}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}m_p H}. \tag{30}$$

- The field equations are equivalent to the following dynamical system together with a constraint:

$$\Sigma' = x^2 - 2ny^2 - 3\Sigma + \epsilon\Sigma, \tag{31}$$

$$x' = -2x - 2x\Sigma - 2n\Gamma y^2 + \epsilon x, \tag{32}$$

$$y' = ny(2\Sigma - 1) + 2n\Gamma xy + \epsilon y, \tag{33}$$

$$1 = \Sigma^2 + x^2 + y^2, \tag{34}$$

where Γ is:

$$\Gamma = m_p V_{,A}/V.$$

Possibility of inflation

- The slow-roll parameter ϵ in terms of the new variables is:

$$\epsilon = 2x^2 + ny^2 + 3\Sigma^2. \tag{35}$$

- The Friedmann equation implies $\epsilon > 1$ for $n \geq 1$.
- This model is ruled out as an inflation model.

Possibility of inflation

- The slow-roll parameter ϵ in terms of the new variables is:

$$\epsilon = 2x^2 + ny^2 + 3\Sigma^2. \tag{35}$$

- The Friedmann equation implies $\epsilon > 1$ for $n \geq 1$.
- This model is ruled out as an inflation model.

SU(2) vector field model

- The lagrangian density corresponding to the vector field is [Cembranos, 2012b]:

$$\mathcal{L}_m = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\xi}{2} (A^{\mu\ a}_{;\mu})^2 - V(M_{ab} A^a_\mu A^{b\mu}), \tag{36}$$

where

$$F^a_{\mu\nu} = A^a_{\nu,\mu} - A^a_{\mu,\nu} + g c^a_{\ bc} A^b_\mu A^b_\nu.$$

Field equations

$$\xi\left[3\ddot{A}_0^a A_0^a + 21\dot{A}_0^a A_0^a H + 9(A_0^a)^2\dot{H} + 3(\dot{A}_0^a)^2 + 9(A_0^a)^2 H\right] - 6m_p(\dot{H} + H^2) = - 9m_p(H^2 - \dot{\sigma}^2) - 3V - V' g^{ij}{}_{,\alpha} M_{ab} A_i^a A_j^b - \frac{1}{2} g^{ij}\dot{A}_i^a \dot{A}_j^a - g\dot{A}_i^a c_{abc} A_0^b A_j^c g^{ij} - c_{abc}c_{ade}\left(\frac{1}{2}g^2 A_0^b A_i^c g^{ij} A_j^e + \frac{1}{4}A_i^b A_j^c g^{im} A_m^d g^{jn} A_n^e\right) + \frac{3}{2}\xi(\dot{A}_0^a + 3A_0^a H). \quad (37)$$

Field equations

$$6m_p^2(\ddot{\sigma} + 3\dot{\sigma}H) = -\frac{1}{2}g^{ij}{}_{,\sigma}\dot{A}_i^a\dot{A}_j^a - g\dot{A}_i^a c_{abc}A_0^b A_j^c g^{ij}$$
$$- c_{abc}c_{ade}\left(\frac{1}{2}g^2 A_0^b A_i^c g^{ij}{}_{,\sigma}A_j^e + \frac{1}{4}A_i^b A_j^c (g^{im}g^{jn})_{,\sigma}A_m^d A_n^e\right)$$
$$- V' g^{ij}{}_{,\sigma}M_{ab}A_i^a A_j^b. \quad (38)$$

Field equations

Equation for the vector field:

- For the time component:

$$\xi(\ddot{A}_0^b + 3\dot{A}_0^b H + 3A_0^b \dot{H}) + g\dot{A}_i^a c_{abc} A_0^b g^{ij} A_j^c + g^2 c_{abc} c_{ade} A_0^b A_i^c A_0^d A_j^e g^{ij} + 2VM_{ab} A_0^a = 0. \quad (39)$$

- For the spatial components:

$$g^{lj}\ddot{A}_j^a + gc_{abc}\dot{A}_0^b A_j^c g^{lj} + \dot{g}^{lj}\dot{A}_j^a + gc_{abc}A_0^b \dot{A}_j^c g^{lj} gc_{abc}A_0^b A_j^c \dot{g}^{lj} + 3\dot{\alpha}(g^{lj}\dot{A}_j^a + gc_{abc}A_0^b A_j^c g^{lj}) g\dot{A}_j^c c_{abc}A_0^b g^{ij} + g^2 c_{abc}c_{cde}A_0^b A_0^d A_j^e g^{lj} + g^2 c_{ade}c_{dbc}A_n^e g^{jn} A_j^c A_i^b g^{il} - 2V' M_{ab} A_l^b = 0. \quad (40)$$

Particular case

- We study the particular case $A^a_\mu = A^a_0 \delta^0_\mu$.
- Similar to a scalar field but it is not the same:

$$\varepsilon = T^\mu{}_\mu = V + \frac{\xi}{2}(\dot{A}^a_0 + 3A^a_0 H)^2, \tag{41}$$

$$p = \frac{\varepsilon - T^\mu{}_\mu}{3} = -V - \frac{3}{2}(\dot{A}^a_0 + 3A^a_0 H)^2 - \xi A^a_0(\dot{A}^a_0 + 3A^a_0 H). \tag{42}$$

Reduced lagrangian

- The equations in this particular case can be deduced from the following lagrangian:

$$L_r = e^{3\alpha}\left[-3m_p^2\left(\dot{\alpha}^2 - \dot{\sigma}^2\right) + \frac{\xi}{2}(\dot{A}_0^a + 3A_0^a\dot{\alpha})^2 - V(M_{ab}A_0^aA_0^b)\right]. \quad (43)$$

Change of coordinates

- When M_{ab} is diagonal, the lagrangian (43) is invariant under "rotations" in the (A_0^1, A_0^2, A_0^3) subspace.
- Conserved quantities $\Longrightarrow$ the system is constrained in a "plane".
- Change to polar cylindrical coordinates

$$L_r = e^{3\alpha}\left[-3m_p^2\left(\dot{\alpha}^2 - \dot{\sigma}^2\right) + \frac{\xi}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + 6\dot{r}r\dot{\alpha} + 9r^2\dot{\alpha}^2) - V(r^2)\right]. \quad (44)$$

Change of coordinates

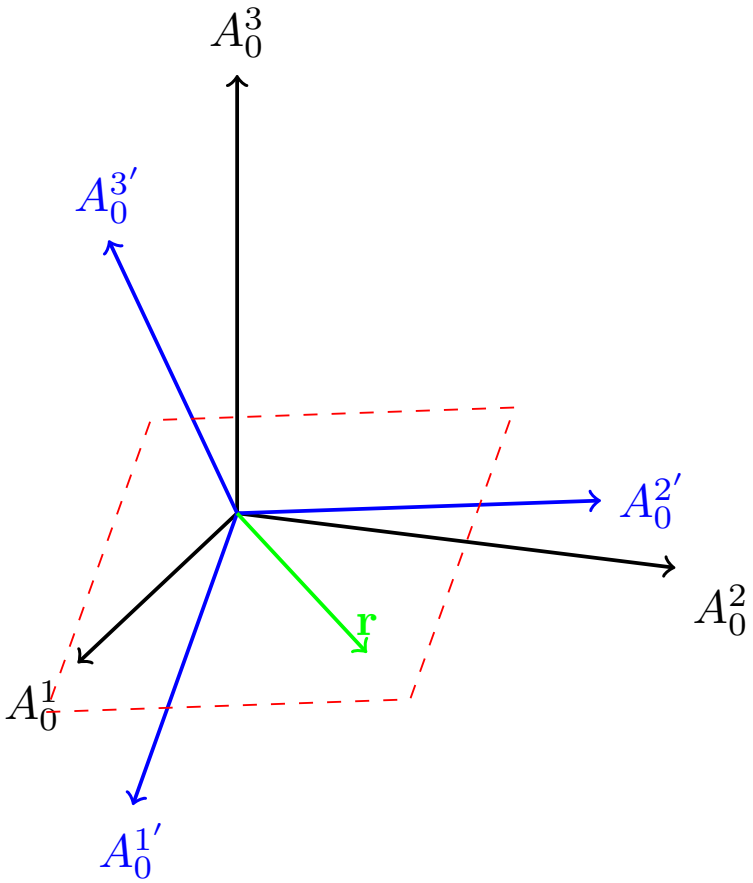


Figure: The green vector represents $\mathbf{r} = (A_0^1, A_0^2, A_0^3)$. The red dashed line represents the plane where the system is constrained. The prime coordinate system is chosen so that the $A_0^{3'}$ is perpedincular to this plane.

Dynamical variables

- We define the following variables to formulate an autonomous dynamical system:

$$\Sigma \equiv \frac{\dot{\sigma}}{H}, \quad x \equiv \frac{\sqrt{\xi}\dot{r}}{\sqrt{2}\sqrt{3}m_p H}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}m_p H}, \quad w \equiv \frac{\sqrt{3}\sqrt{\xi}r}{\sqrt{2}m_p},$$

$$\Theta \equiv \frac{\dot{\theta}}{H}.$$

and the parameter

$$\Lambda = \frac{m_p V_{,r}}{\sqrt{\xi}V}.$$

- The Friedmann equation is:

$$\Sigma^2 + \frac{2\Theta^2 w^2}{3\xi} + w^2 + 2wx + x^2 + y^2 = 1. \tag{45}$$

- The slow parameter is: $\epsilon = 3\Sigma^2 + \sqrt{\frac{3}{2}}\Lambda w y^2$.

Dynamical system

- Autonomous dynamical system:

$$\Sigma' = \Sigma\left(\epsilon - 3\right), \tag{46}$$

$$\begin{aligned} x' = & -\frac{\Theta^2 w^3}{\xi} - \frac{3w^3}{2} - 3w^2 x + \sqrt{\frac{3}{2}}\Lambda w^2 y^2 \\ & + \frac{2\Theta^2 w}{\xi} + \frac{3\Sigma^2 w}{2} - \frac{3wx^2}{2} - \frac{3wy^2}{2} + \frac{3w}{2} \\ & + x\epsilon - 3x - \sqrt{\frac{3}{2}}\Lambda y^2, \end{aligned} \tag{47}$$

$$y' = y\left(\sqrt{\frac{3}{2}}\Lambda x + \epsilon\right), \tag{48}$$

$$w' = 3x, \tag{49}$$

$$\Theta' = \Theta\left(\epsilon - 3 - \frac{6x}{w}\right). \tag{50}$$

Behaviour around the critical points

- We find a critical point $X_c = (0, 0, 0, 1, 0)$.
- ϵ evaluated at the critical point is equal to 0. Therefore this point corresponds to inflation.
- To analyze the stability we perform a change of coordinates so that the critical point is located at the origin: $\tilde{w} = w - 1$
- We linearize the equations around the origin:

$$\begin{pmatrix} \Sigma' \\ x' \\ y' \\ \Theta' \\ \tilde{w}' \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Sigma \\ x \\ y \\ \Theta \\ \tilde{w} \end{pmatrix} \tag{51}$$

Behaviour around the critical points

- The eigenvalues of the corresponding matrix are negative except one which is 0.

$$\text{eig} = (-3, -3, -3, -3, 0)$$

- The stable manifold is $y = 0$.
- This implies that if the potential is equal to 0 we obtain an infinite inflationary period.
- What happens when $V \neq 0$?

Local center manifold

Theorem (Local center manifold theorem)

Suppose that the dynamical system $\mathbf{X} = \mathbf{f}(\mathbf{X})$ *has a critical point in the origin and that* $D\mathbf{f}(0)$ *has c eigenvalues with zero real parts and s eigenvalues with negative real parts. The system then can be written in the form*

$$\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y}) \tag{52}$$

$$\dot{\mathbf{y}} = P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y}), \tag{53}$$

where C *is a square matrix having c eigenvalues with 0 real part and* P *is a square matrix having s eigenvalues with negative real part. Furthermore there exists a* $\delta > 0$ *and function* $\mathbf{h}$ *that defines the local center manifold and satisfies*

$$D\mathbf{h}(\mathbf{x})[C\mathbf{x} + F(\mathbf{x}, \mathbf{h}(\mathbf{x}))] - P\mathbf{h}(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = 0. \tag{54}$$

The flow on the center manifold for $|\mathbf{x}| < \delta$ *is given by:*

$$\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \tag{55}$$

Local center manifold

- The center manifold theorem allows us to decouple the equations making the analysis easier.
- The local center manifold can be approximated by:

$$\Sigma = 0, \tag{56}$$

$$x = \mathcal{O}(y^4), \tag{57}$$

$$\tilde{w} = \frac{1}{2}y^2 + \mathcal{O}(y^4), \tag{58}$$

$$\Theta = 0. \tag{59}$$

- The behaviour in the center manifold is governed by the following equation:

$$y' = \sqrt{\frac{3}{2}}\Lambda y^3 + \mathcal{O}(y^5). \tag{60}$$

Saddle point

- When the parameter $\Lambda > 0$ the critical point is a saddle point.
- For a potential of the form λr^{2n}, $\Lambda > 0$.

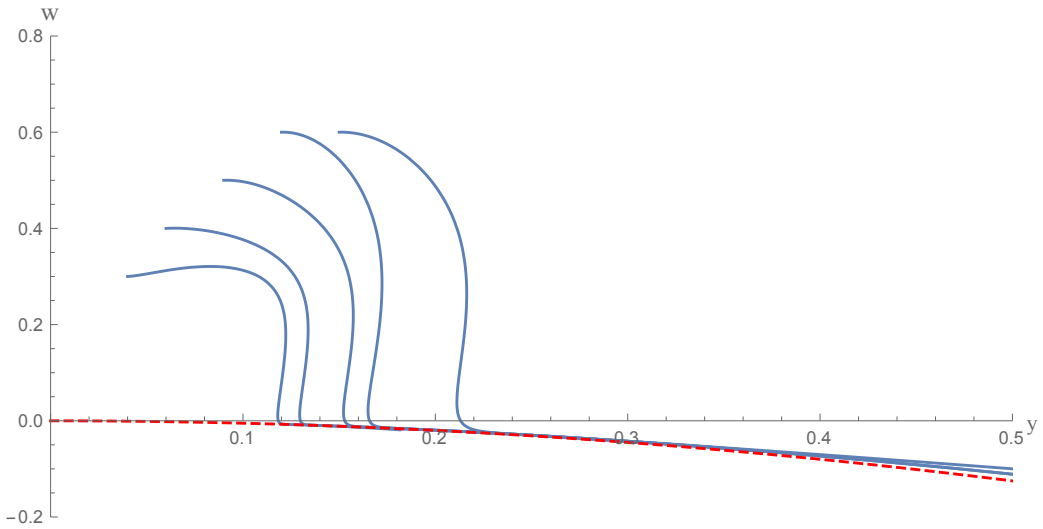


Figure: Numerical plot of w vs y. The dashed red line represents the center manifold.

Slow-roll inflation

- We need inflation lasts at least 60 e-folds.
- When the system is in the local center manifold, the slow-roll parameter ϵ is:

$$\epsilon = \sqrt{\frac{3}{2}}\Lambda y^2 + \mathcal{O}(y^4). \tag{61}$$

- $\epsilon = 1$ is the end of inflation which equivalent to $y_{\max} = \sqrt{\frac{2}{3}}$.
- Maximum speed $v_{\max} = y_{\max}/N = \sqrt{\frac{2}{3}}/N$.
- If the speed of y in the center manifold is lower than this maximum, we can obtain an inflationary period long enough as required.

Numerical solution

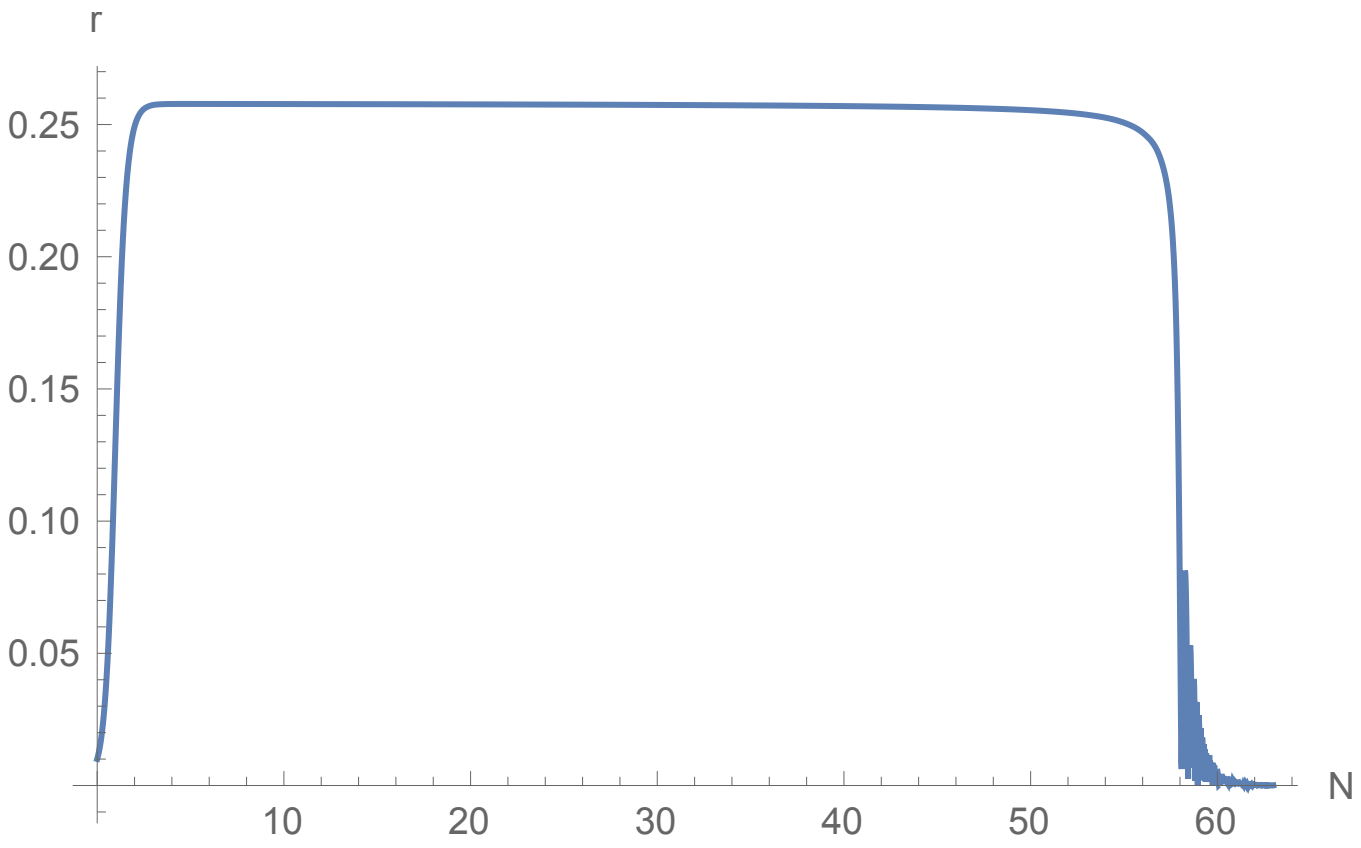


Figure: r is in units of the Planck mass m_p

Numerical solution

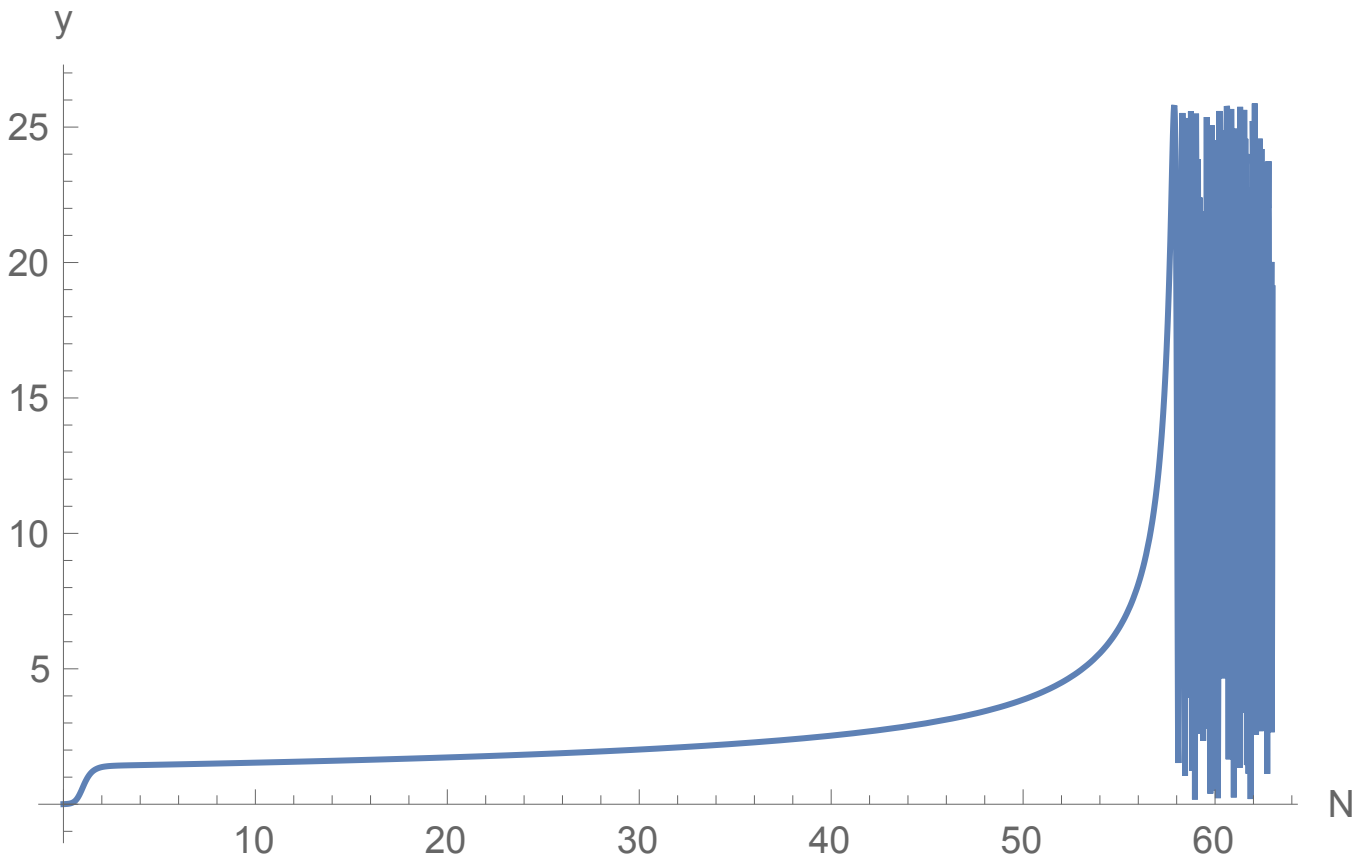


Figure: y vs number of e-folds N.

Numerical solution

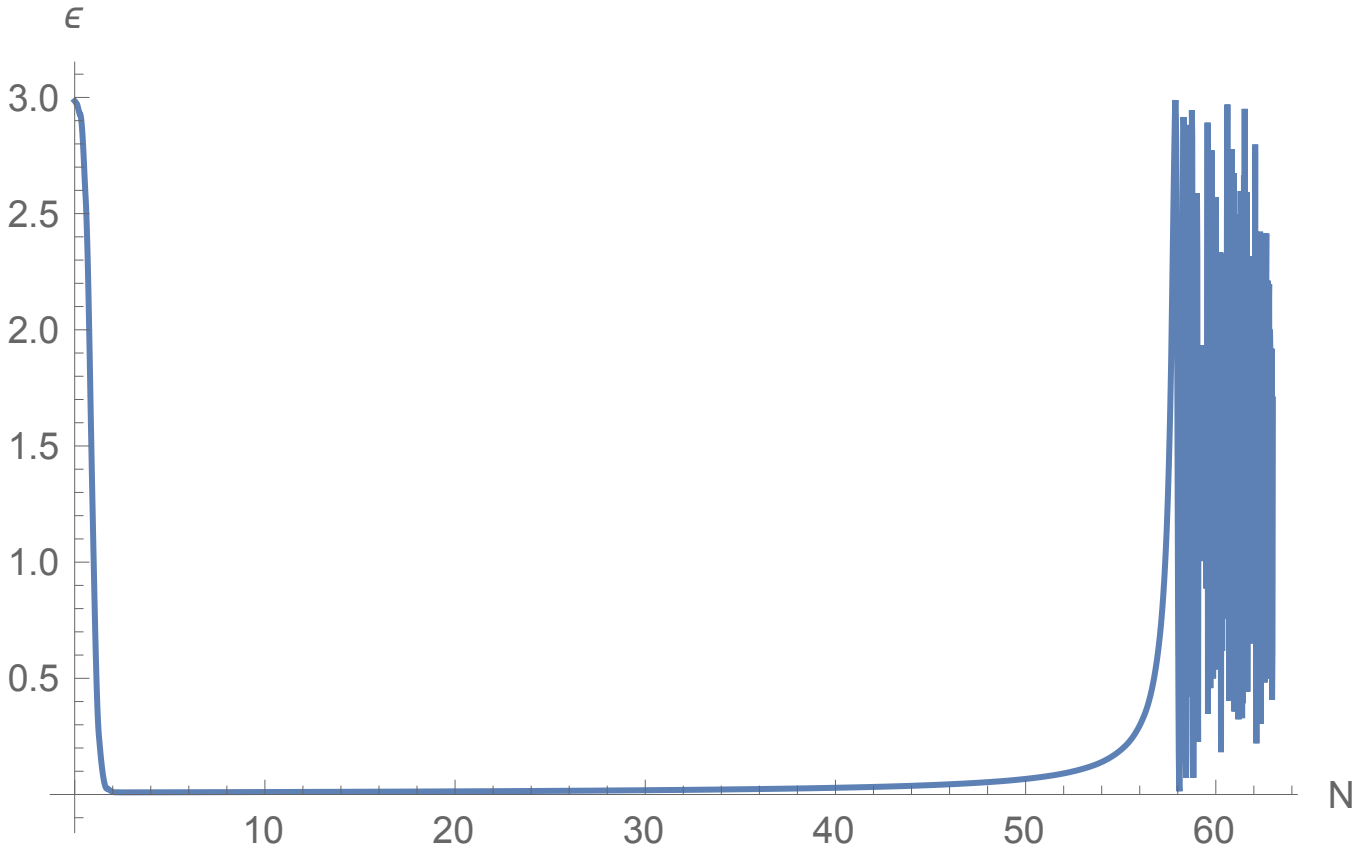


Figure: ϵ vs number of e-folds N.

Conclusions

- We find that for a potential $V = \lambda r^{2n}$, it is possible to obtain a finite inflationary period.
- The condition for the existence of inflation is $V \ll 3m_p^2 H^2$. This means that the potential must be negligible compared to $3m_p^2 H^2$ in contrast to scalar field inflation.
- $V < 2m_p^2 H^2 N^{-2/3}$ is required so that inflation lasts at least N e-folds.

What's next?

- Regardin the SU(2) field model, to analyse the case where both the temporal and spatial components are present.
- To analyse the following model

$$\mathcal{L} = \frac{1}{2}\varphi_{,\mu}\varphi^{,\mu} - V(\varphi) - \frac{1}{4}f(\varphi)F_{\mu\nu}F^{\mu\nu} - h(\varphi)\frac{1}{4}F_{\mu\nu}\frac{1}{2}\frac{\epsilon^{\mu\nu\sigma\tau}}{\sqrt{-g}}F_{\sigma\tau}, \quad (62)$$

and check if $h(\varphi) = f(\varphi) \propto e^{(-1\pm 3)\alpha}$ which correspond to flat perturbation spectrum[2] is attractive.

[2] K. Dimopoulos & M. Karciauskas, *Parity Violating Statistical Anisotropy*, JHEP **1206**, 040 (2012)