Equations of motion of compact binaries at the fourth post-Newtonian order

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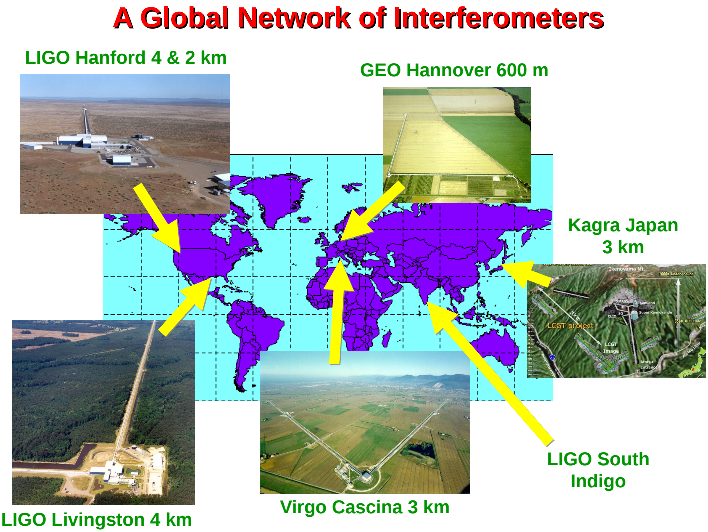
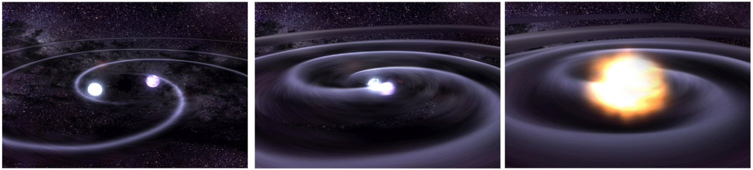
Outline

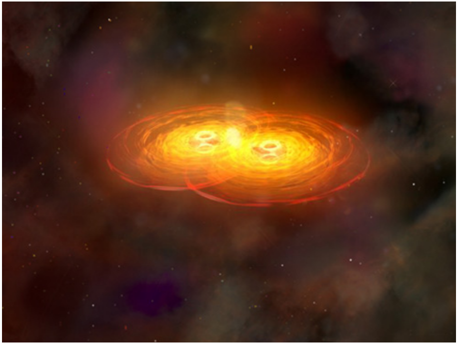
Motivations



Coalescing compact binary systems

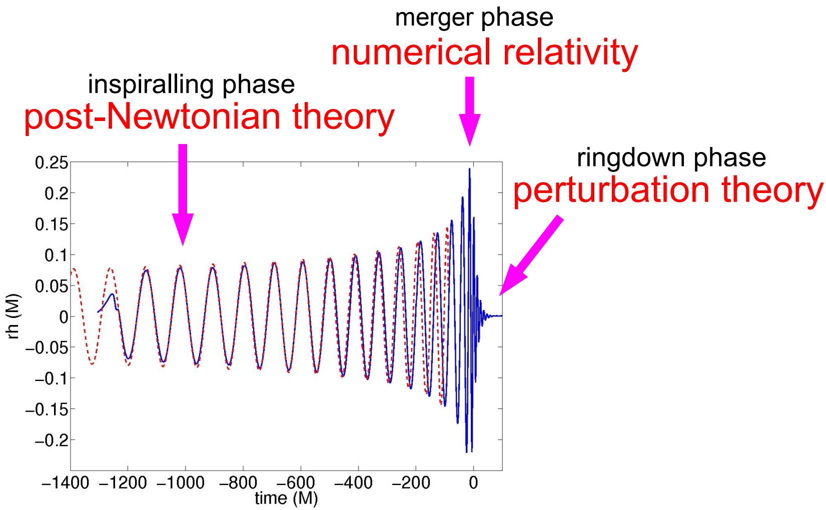


NS-NS merger



BH-BH merger

Coalescing compact binary systems



Principle of the Fokker action

- Starting from the action

$$S_{\text{tot}}\left[g_{\mu\nu}, \mathbf{y}_B(t), \mathbf{v}_B(t)\right] = S_{\text{grav}}\left[g_{\mu\nu}\right] + S_{\text{mat}}\left[(g_{\mu\nu})_B, \mathbf{y}_B(t), \mathbf{v}_B(t)\right]$$

- we solve the Einstein equation $\frac{\delta S_{\text{tot}}}{\delta g_{\mu\nu}} = 0 \to \overline{g}_{\mu\nu}\left[\mathbf{y}_A(t), \mathbf{v}_A(t), \cdots\right]$
- and construct the Fokker action

$$S_{\text{Fokker}}\left[\mathbf{y}_B(t), \mathbf{v}_B(t), \cdots\right] = S_{\text{tot}}\left[\overline{g}_{\mu\nu}\left(\mathbf{y}_A(t), \mathbf{v}_A(t), \cdots\right), \mathbf{y}_B(t), \mathbf{v}_B(t)\right]$$

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- The dynamics for the particles is unchanged

$$\begin{aligned} \frac{\delta S_{\text{Fokker}}}{\delta y_A} &= \underbrace{\left.\frac{\delta S_{\text{tot}}}{\delta g_{\mu\nu}}\right|_{g=\overline{g}}}_{=0} \cdot \frac{\delta g_{\mu\nu}}{\delta y_A} + \left.\frac{\delta S_{\text{tot}}}{\delta y_A}\right|_{g=\overline{g}} \\ &= \left.\frac{\delta S_{\text{tot}}}{\delta y_A}\right|_{g=\overline{g}} \end{aligned}$$

Our Fokker action

$$S_{\text{grav}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left[g^{\mu\nu}\left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right) - \underbrace{\frac{1}{2}g_{\mu\nu}\Gamma^{\mu}\Gamma^{\nu}}_{\text{gauge fixing term}}\right],$$

$$S_{\text{mat}} = -\sum_A m_A c^2 \int \mathrm{d}t\sqrt{-\left(g_{\mu\nu}\right)_A \frac{v_A^{\mu}v_A^{\nu}}{c^2}}\,.$$

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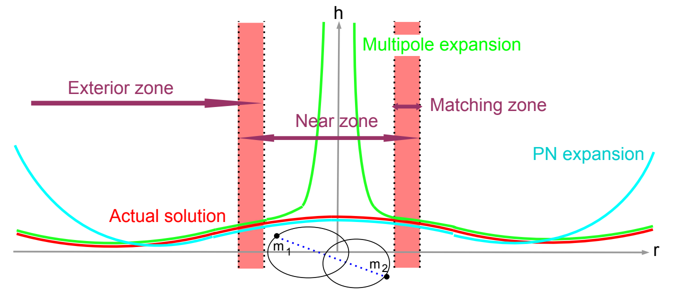
Relaxed Einstein equations

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}\left[h, \partial h, \partial^2 h\right]$$

- with $h^{\mu\nu} = \sqrt{|g|}g^{\mu\nu} - \eta^{\mu\nu}$ the metric perturbation variable.
- We don't impose the harmonicity condition $\partial_\nu h^{\mu\nu} = 0$.
- $\Lambda^{\mu\nu}$ encodes the non-linearities, with supplementary harmonicity terms containing $H^\mu = \partial_\nu h^{\mu\nu}$.

Near zone / Wave zone



- **Near zone** : Post-Newtonian expansion $h = \overline{h}$,
- **Wave zone** : Multipole expansion $h = \mathcal{M}(h)$,
- **Matching zone** : $\overline{h} = \mathcal{M}(h) \quad \Longrightarrow \quad \mathcal{M}\left(\overline{h}\right) = \overline{\mathcal{M}(h)}$.

$$S_g = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_F + \underset{B=0}{\mathrm{FP}} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \mathcal{M}\left(\mathcal{L}_F\right)$$

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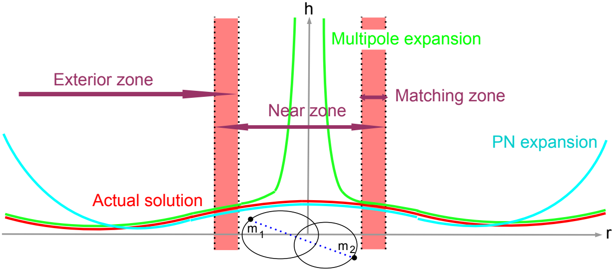

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$$S_g = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_F + \underbrace{\mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \mathcal{M}\left(\mathcal{L}_F\right)}_{\mathcal{O}(5.5PN)}$$

Post-Newtonian counting in a Fokker action

Thanks to the property of the Fokker action, cancellations between gravitational and matter terms in the action occur.

- To get the Lagrangian at nPN *i.e.* $\mathcal{O}\left(\frac{1}{c^{2n}}\right)$, we only need to know the metric at roughly half the order we would have expected :

$$\left(h^{00ii}, h^{0i}, h^{ij}\right) = \mathcal{O}\left(\frac{1}{c^{n+2}}\right).$$

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$$\left(h^{00ii},\, h^{0i},\, h^{ij}\right) = \mathcal{O}\left(\frac{1}{c^{n+2}}\right).$$

- **For 4 PN :** $\left(h^{00ii},\, h^{0i},\, h^{ij}\right) = \mathcal{O}\left(\frac{1}{c^{6}}, \frac{1}{c^{5}}, \frac{1}{c^{6}}\right)$

Tail effects at 4PN

- At 4PN we have to insert some tail effects,

$$\overline{h}^{\mu\nu} = \overline{h}^{\mu\nu}_{\text{part}} - \frac{2G}{c^4} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \partial_L \left\{ \frac{\mathcal{A}_L^{\mu\nu}(t - r/c) - \mathcal{A}_L^{\mu\nu}(t + r/c)}{r} \right\}$$

- When inserted into the Fokker action it gives in the following contribution

$$S_{\text{tail}} = \frac{G^2(m_1 + m_2)}{5c^8} \underset{\frac{2s_0}{c}}{\text{Pf}} \int\int \frac{\mathrm{d}t\, \mathrm{d}t'}{|t - t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

  - The two constant of integration are linked through $s_0 = r_0\, \mathrm{e}^{-\alpha}$.

Different regularization schemes

IR Singularity of the PN expansion at infinity : $\boldsymbol{r_0}$

Tail effects : $\boldsymbol{s_0}$

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- α will be determined by comparison with self-force results.

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UV Singularity at the location of the point particles

- Dimensional regularization,

  1. We calculate the Lagrangian in $d = 3 + \varepsilon$ dimensions.
  2. We expand the results when $\varepsilon \to 0$: appearance of a pole $1/\varepsilon$.
  3. We eliminate the pole through a redefinition of the variables.

- **The physical result should not depend on $\boldsymbol{\varepsilon}$.**

The equations of motion at 4PN

The generalized Lagrangian

$$L_{4\text{PN}} = \frac{Gm_1m_2}{r_{12}} + \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + L_{1\text{pn}} + L_{2\text{pn}} + L_{3\text{pn}} + L_{4\text{pn}}[y_A(t), v_A(t), a_A(t), \partial a_A(t), \cdots]$$

The equations of motion

$$a^i_{1,4\text{PN}} = -\frac{Gm_2}{r_{12}^2}n^i_{12} + a^i_{1,1\text{pn}} + a^i_{1,2\text{pn}} + a^i_{1,3\text{pn}} + a^i_{1,4\text{pn}}[\alpha]$$

- Previous results at 4PN were obtained with the Hamiltonian formalism (Jaranowski, Schaffer 2013 and Damour, Jaranowski, Schaffer 2014) and partially with EFT (Foffa, Sturani 2012).

Binding energy for circular orbits

- The constant α is determined by comparison of the binding energy for circular orbits with another method, such as self-force calculations:

$$E(x;\nu) = -\frac{\mu c^2 x}{2}\left[1 - \left(\frac{3}{4}+\frac{\nu}{12}\right)x + \left(-\frac{27}{8}+\frac{19\nu}{8}-\frac{\nu^2}{24}\right)x^2 + \left(-\frac{675}{64}+\left(\frac{34445}{576}-\frac{205\pi^2}{96}\right)\nu - \frac{155\nu^2}{96}-\frac{35\nu^3}{5184}\right)x^3 + \left(-\frac{3969}{128}+\left(\frac{9037\pi^2}{1536}-\frac{123671}{5760}+\frac{448}{15}\left(2\gamma+\ln(16x)\right)\right)\nu - \left(\frac{3157\pi^2}{576}-\frac{198449}{3456}\right)\nu^2+\frac{301\nu^3}{1728}+\frac{77\nu^4}{31104}\right)x^4\right]$$

with $x = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{2/3}$ and $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ the symmetric mass ratio.

Consistency checks

We have checked that

- the IR regularization is in agreement with the tail part : **no** r_0,
- the result does not depend on the UV regularization : **no pole** $1/\varepsilon$,
- the equations of motion are manifestly **Lorentz invariant**,
- in the test mass limit we recover the **Schwarzschild geodesics**,
- we recover the **conserved energy for circular orbits**.

Summary

- We obtained the equations of motion at 4PN from a Fokker Lagrangian, in harmonic coordinates.
- We recover all the physical results that we expected.

Summary

- We obtained the equations of motion at 4PN from a Fokker Lagrangian, in harmonic coordinates.
- We recover all the physical results that we expected.
- We are now systematically computing the conserved quantities.
- The important goal is now to compute the gravitational radiation field at 4PN.