Key Consequences of Theoretical Self-Consistency in GR

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A. Constraints on space-time transformations due to contracted-tensor covariance

A1. The theorem that the contraction of an upper index with a lower index of any tensor itself transforms as a tensor whose rank is less by two—e.g., that the contraction T^{μ}_{μ} of the mixed tensor T^{μ}_{ν} is a *scalar*—is obviously *an indispensable linchpin of Einstein's GR*.

A2. But this theorem holds *only* at space-time points where $\bar{x}^\alpha(x^\mu)$, the space-time transformation that is involved, satisfies, in conjunction with its inverse $x^\nu(\bar{x}^\alpha)$, the relation,

$$(\partial\bar{x}^\alpha/\partial x^\mu)(\partial x^\nu/\partial\bar{x}^\alpha) = \delta^\nu_\mu. \tag{1}$$

Although Eq. (1) of course follows from the chain rule of the calculus, *that doesn't of itself* imply that Eq. (1) is *always* true.

A3. In fact, if any component of the Jacobian matrix $\partial\bar{x}^\alpha/\partial x^\mu$ happens to be nonfinite at some space-time point, or if any component of the matrix-inverse thereof happens to be nonfinite there, then at that space-time point the left-hand side of Eq. (1) is ill-defined as a finite real number, whereas the right-hand side of Eq. (1) remains well-defined as a finite real number.

A4. Therefore at any such point Eq. (1) *is self-inconsistent*. Thus the space-time transformation $\bar{x}^\alpha(x^\mu)$ *cannot be regarded as physical in Einstein's GR at such a point*; indeed, in classical theoretical physics *nonfinite entities don't even make sense*.

A5. Because of Einstein's Principle of Equivalence, space-time transformations are *fundamental* to GR. Therefore *the physical need to bar infinities* from these transformations' Jacobian matrices and the inverses thereof impacts the *entirety* of GR.

B. Constraints on metric tensors due to contracted-tensor covariance

B1. According to Einstein's Principle of Equivalence, any metric tensor is locally *a matrix congruence transform* of the Minkowski metric tensor with a Jacobian matrix of some space-time transformation.

B2. Therefore, in light of the results of the previous section, a metric tensor can be physical *only* at space-time points where it and its inverse have exclusively *finite* components and as well have signatures that are equal to the $(+,-,-,-)$ signature of the Minkowski metric tensor.

B3. However, there actually exist metric-tensor *solutions of the Einstein field equation* which on a subset of space-time *flout* these requirements to be physical, just as there exist solutions of the Maxwell and Schrödinger equations which likewise flout conditions that are required for those solutions to be physical: such unphysical solutions of *the latter two* field-theoretic equations *are discarded*.

B4. To gain familiarity with the factors that foster occurrences of unphysical solutions of field-theory equations, and to as well gain familiarity with the proper way to handle such occurrences, we begin with a class of very simple unphysical solutions of the source-free Maxwell equations.

C. Unphysical static uniform-field solutions of source-free electromagnetism

C1. The source-free Maxwell equations, namely,

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -(1/c)\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = (1/c)\dot{\mathbf{E}},$$

clearly are satisfied by *all* static uniform $\mathbf{E}$ and $\mathbf{B}$ fields.

C2. However *unless* those static uniform-field solutions *completely vanish*, their resulting *electromagnetic field energy*, namely $(1/2)\int d^3\mathbf{r}\,(|\mathbf{E}|^2 + |\mathbf{B}|^2)$, *diverges*. Therefore those solutions are *unphysical*, and indeed *are shunned* in source-free electromagnetic field theory.

C3. The divergent field energies of those unphysical static uniform Maxwell-equation field solutions are strikingly reminiscent of the *divergent wave-function normalizations* which occur for a class of *unphysical wave-function solutions of Schrödinger equations*.

D. Unphysical non-normalizable Schrödinger-equation solutions

D1. The stationary-state Schrödinger equation for the simple harmonic oscillator,

$$(1/2)[-(\hbar^2/m)(d^2/dx^2) + m\omega^2 x^2]\psi_{E_{\rm osc}}(x) = E_{\rm osc}\psi_{E_{\rm osc}}(x),$$

has for each nonnegative value of $E_{\rm osc}$ two linearly-independent solutions (parabolic cylinder functions). When $x \to +\infty$ or $x \to -\infty$, all linear combinations of those two solutions are either strongly unbounded or else strongly approach zero.

D2. But it is *only* when $E_{\rm osc}$ takes on one of the *discrete* values $[n + (1/2)]\hbar\omega$, $n = 0, 1, 2, \ldots$, that there *exists* a linear combination of the two solutions which *isn't* strongly unbounded *under at least one of the two circumstances* $x \to +\infty$ and $x \to -\infty$.

D3. *All the remaining nonnegative values of* $E_{\rm osc}$ are therefore associated to solutions of the stationary-state harmonic oscillator Schrödinger equation *that are not normalizable and hence are unphysical*. All non-normalizable, unphysical Schrödinger-equation solutions *are discarded without further ado*.

D4. The *discrete* negative energy spectrum of the hydrogen atom is *likewise* associated with the massive *discarding* of non-normalizable, unphysical Schrödinger-equation solutions. But we shall now see that non-normalizability *isn't the only unphysical boundary condition* which Schrödinger-equation solutions can manifest.

E. Unphysical rotationally non-periodic Schrödinger-equation solutions

E1. The stationary-state Schrödinger equation for the simple rotator of moment of inertia I is,

$$-(1/2)(\hbar^2/I)(d^2/d\theta^2)\psi_{E_{\rm rot}}(\theta) = E_{\rm rot}\psi_{E_{\rm rot}}(\theta).$$

For each nonnegative value of $E_{\rm rot}$ this equation has the two linearly-independent solutions,

$$\psi^{\pm}_{E_{\rm rot}}(\theta) = C^{\pm}_{E_{\rm rot}} \exp\left[\pm i(E_{\rm rot}(2I/\hbar^2))^{\frac{1}{2}}\theta\right].$$

E2. These Schrödinger-equation solutions lack the physically-required *rotational periodicity of* 2π *in* θ unless $E_{\rm rot}$ assumes one of the *discrete* values $(n\hbar)^2/(2I)$, $n = 0, 1, 2, \dots$. The Schrödinger-equation solutions for the *remaining* nonnegative values of $E_{\rm rot}$ are rotationally non-periodic and hence *unphysical*; they are *discarded without further ado*.

E3. The last three sections have made it apparent that *unphysical solutions* of field-theory equations (1) *must be discarded* and (2) *reflect unphysical boundary conditions*—for example field behavior at large distances that precludes normalization or finite energy, or field non-periodicity in variables in which the physics is periodic. With these two guidelines firmly in mind, we now cogitate on the unphysical points present in the Einstein field equation's empty-space Schwarzschild metric-tensor solution.

F. Are Schwarzschild-solution unphysical points really located in empty space?

F1. The empty-space Schwarzschild solution is usually combined with the Newtonian *positive point-mass* idealization as its *source*. But is this Newtonian idealization self-consistent in a *relativistic* gravitational theory where an object's *negative* internal gravitational energy *diminishes* its effective mass?

F2. Let's *check* the relativistic self-consistency of this Newtonian idealization by attempting to produce such a positive point mass by progressively *reducing the separation* d between *two* such idealized point masses which each have positive mass $M_>/2$. The effective mass M of this system is of course given by,

$$Mc^2 = M_> c^2 - G(M_>/2)^2/d.$$

When $d \to \infty$, $M \to M_>$. But when $d \to 0$ for *fixed* $M_>$, $M \to -\infty$!

F3. The *optimal cure* for this is to *choose* $M_>$ *at each value of* d *so as to maximize* M. The result thereof is,

$$M_{\max}(d) = (c^2/G)d. \text{ [This } \textit{maximum value} \text{ of } M \text{ at } d \text{ occurs for the } \textit{choice } M_>(d) = 2(c^2/G)d.] \quad (2)$$

The $M_{\max}(d)$ result of Eq. (2) shows conclusively that as $d \to 0$ *a positive point mass can't be produced*.

F4. In addition, Eq. (2) draws our attention to an inherent self-gravitational limit on a system's effective mass that is proportional to *its largest linear dimension*, with the constant of proportionality being of order (c^2/G). This implies that a system of effective mass M is compelled to have its largest linear dimension *be of order* $(G/c^2)M$ *or greater*.

F5. The spherically-symmetric empty-space Schwarzschild metric-tensor solution with a source of effective mass M has *unphysical points* which lie on a spherical shell whose radius is of order $(G/c^2)M$—namely *the very same order* as the *smallest possible radius* of its spherically-symmetric *source* of effective mass M.

F6. It is therefore highly plausible that the *unphysical points* of the spherically-symmetric *empty-space* Schwarzschild solution *always lie within its source*, which is *not* empty space, and therefore is a region *where the empty-space Schwarzschild solution doesn't even apply*.

F7. That this is *indeed* the case is *confirmed* by the fact that in spherically-symmetric "standard" coordinates the self-gravitationally shrinking dust cloud of effective mass M treated by Oppenheimer and Snyder *never* (quite) shrinks to the radius $2(G/c^2)M$, which is *precisely* the radius of *the shell of unphysical points* of the Schwarzschild solution that *also* has a source of effective mass M and is expressed in those *same* spherically-symmetric "standard" coordinates.

F8. We thus see that the *unphysical boundary conditions* which *permit* the *presence of the unphysical points* of the spherically-symmetric *empty-space* Schwarzschild solution are *always* at loggerheads with *the smallest physically-possible radius* of its spherically-symmetric *nonempty-space source* (which has effective mass M and therefore radius of order $(G/c^2)M$ or greater).

F9. For those who are wondering about the *singularities* in the Oppenheimer-Snyder solution in spherically-symmetric "comoving" coordinates, it is to be noted that *the space-time transformation* of the Oppenheimer-Snyder solution from "standard" coordinates, *where that solution is well-behaved*, to "comoving" coordinates *is unphysical*. Moreover, the definition of "comoving" *time* requires the clocks *of an infinite number of observers*, and therefore *isn't physically observable*. The *related* property of "comoving" metrics that $g_{00} = 1$ is *incompatible* with the requirement that in the static weak-field limit $(g_{00} - 1)/2$ becomes the Newtonian gravitational potential ϕ, as well as with the requirement that in the static limit $(g_{00})^{-\frac{1}{2}}$ is the gravitational time dilation factor. Therefore "results" presented in terms of "comoving" coordinates *fail to have direct physical interpretation.*