Asymptotic Safety and Quantum Gravity

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Based on

“A flow equation for $f(R)$ gravity and some of its exact solutions,”
arXiv:1507.00968 [hep-th], with Roberto Percacci and Gian Paolo Vacca

and work in preparation.

1 Introduction

A way to quantum gravity $\cdots$ personal view

4 years ago

Einstein theory is non-renormalizable but is only a low-energy effective theory! If one considers quantum gravity, e.g. string theory, higher-order terms alway appear! ⇒ This might lead to possible UV completion because of the following reasons

– In 4D, quadratic (higher derivative) theory is known to be renormalizable but non-unitary! (Stelle)

Exciting developments in 3D gravity: unitary higher-derivative gravity (Bergshoeff-Hohm-Townsend)

$$S = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \Big[\sigma R - 2\Lambda_0 + \alpha R^2 + \beta R_{\mu\nu}^2 + \frac{1}{2\mu} \mathcal{L}_{LCS} \Big],$$

3D Einstein theory $\cdots$ no dynamical degree of freedom

With higher derivatives, gravity gets dynamics!

Not only that, we have the possibility of renormalizability!!

For the first time, we have the possibility of having both unitary and renormalizable gravity theory!, though in three dimensions.

A complete classification of all unitary and stable theories is done including all possible coupling and maximally symmetric vacua, by looking at the pole residues from the action (off-shell analysis). (NO)

Unfortunately, the renormalizability fails precisely for unitary theories They are just incompatible with each other. (Muneyuki and NO)

In this situation, the only possible way to make sense of the quantum effects in gravity seems to be the asymptotic safety.

What is the asymptotic safety?
A non-renormalizable theory ⇒ effectively renormalizable by a rearrangement of the perturbation series or addition of higher derivative terms.

To be more precise

Terms of finite order in the perturbation series then contain what appear to be unphysical singularities. Such unphysical singularities may be avoided if the couplings approach a fixed point in the ultraviolet energy. This is the asymptotic safety (Weinberg).

Note: The asymptotic safety is a wider notion than the renormalizability (includes renormalizable theories).

∵) Any theory $\cdots$ always have a fixed point at the origin.
The asymptotic safety
⇒ the couplings lie on the attractive surface to the fixed point(s)
⇒ All couplings with negative dimensionality (in powers of mass) should vanish (because the couplings are driven away from the origin)
⇒ The non-renormalizable interactions should vanish
⇒ Theory is renormalizable in the usual sense (asymptotic safety includes these theories)

Rationale:
We do not stick to perturbative unitarity. It only matters if we can compute effective action without any instability and other problems.

2 Wilsonian method for renormalization group

Wilsonian RG:
Effective action describing physical phenomenon at a momentum scale k = the result of integrating out all fluctuations of the fields with momenta larger than k.
k: the lower limit of the functional integration (the infrared cutoff). The

dependence of the effective action on k gives the Wilsonian RG flow.

One can obtain one-loop beta functions from this functional equation.

We apply this method to our theory on arbitrary background in arbitrary dimensions.
The beta functions were computed maximally symmetric spaces.
$\Rightarrow$ Non-trivial fixed points were found for Λ and G. (NO)

The complete beta functions are obtained for arbitrary dimensions on arbitrary backgrounds with Roberto Percacci.

The action

$$S = \int d^D x \sqrt{-g} \Big[\frac{1}{\kappa^2}(\sigma R - 2\Lambda) + \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E + \frac{1}{\xi} R^2 + \tau \Box R \Big],$$

and derive the beta functions.
We find the beta functions of the dimensionless couplings in 4 dims.
Fixed points: The fixed point value of the cosmological constant is gauge-invariant! (The precise value may not be important.)

We also find nontrivial fixed points in other dimensions including 3 dims.

3 $f(R)$ Gravity

In order to facilitate the program, one has to truncate the theory, e.g. derivative expansion, polynomial expansion etc.

Still there is accumulating evidence that there are always nontrivial fixed points. $\Rightarrow$ **Asymptotic safety program may be the right direction.**

Problems with earlier attempts: Gauge-dependence, parametrization-dependence

It is interesting to consider actions of the general form

$$S = \int d^d x \sqrt{-g} f(R),$$

in d **dimensions, and derive FRGE for the function** $f(R)$**!**

Two different parametrizations of the metric fluctuation:

linear split: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \cdots$ **most often used**

exponential split: $g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu, \cdots$ **new parametrization**

The latter has the advantage that the result is gauge-independent.

The background space is d-dimensional Einstein space with

$$\bar{R}_{\mu\nu} = \frac{\bar{R}}{d} g_{\mu\nu}, \qquad \bar{R} = \textbf{const.}$$

For the exponential split, the quadratic terms are given by

$$\begin{aligned} I^{(2)} &= \frac{1}{2} f''(\bar{R}) \Big[\Box h - \nabla_\mu \nabla_\nu h^{\mu\nu} + \bar{R}_{\mu\nu} h^{\mu\nu} \Big]^2 + \frac{1}{2} f'(\bar{R}) \Big[\frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + (\nabla_\mu h^{\mu\nu})^2 \\ &\quad + h(\nabla_\mu \nabla_\nu h^{\mu\nu} - \bar{R}_{\mu\nu} h^{\mu\nu}) + \bar{R}_{\mu\alpha} h^{\mu\nu} h^\alpha_\nu - \frac{1}{2} h \Box h + \bar{R}_{\mu\alpha\nu\beta} h^{\mu\nu} h^{\alpha\beta} \Big] \\ &\quad + \frac{1}{8} f(\bar{R})(h^2 - 2h^2_{\mu\nu}) + \frac{1}{2} \Big[\frac{1}{2} f(\bar{R}) \bar{g}^{\mu\nu} - f'(\bar{R}) \bar{R}^{\mu\nu} \Big] h^2_{\mu\nu}, \end{aligned}$$

up to terms which do not contribute to our results.

York decomposition: $h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{d} \bar{g}_{\mu\nu} \nabla^2 \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h,$

where

$$\nabla_\mu h^{TT}_{\mu\nu} = \bar{g}^{\mu\nu} h^{TT}_{\mu\nu} = \nabla_\mu \xi^\mu = 0.$$

$$\begin{aligned} \Rightarrow I^{(2)}_{exp} &= -\frac{1}{4} f'(\bar{R}) h^{TT}_{\mu\nu} \Big(-\nabla^2 + \frac{2}{d(d-1)} \bar{R} \Big) h^{TT\,\mu\nu} + h \Big(\frac{1}{8} f(\bar{R}) - \frac{1}{4d} f'(\bar{R}) \bar{R} \Big) h \\ &\quad + \frac{d-1}{4d} s \Big[\frac{2(d-1)}{d} f''(\bar{R}) \Big(-\nabla^2 - \frac{\bar{R}}{d-1} \Big) + \frac{d-2}{d} f'(\bar{R}) \Big] \Big(-\nabla^2 - \frac{\bar{R}}{d-1} \Big) s \end{aligned}$$

on the sphere $\bar{R}_{\mu\rho\nu\sigma} = \frac{\bar{R}}{d(d-1)}(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})$**,**

$s = h - \bar{\nabla}^2\sigma$ **is the gauge-invariant variable**

Remarkably all terms containing ξ_μ cancel out!

Gauge fixing: $S_{GF} = \frac{Z}{2\alpha}\int d^dx\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu$

with

$$F_\mu = \bar{\nabla}_\rho h^\rho{}_\mu - \frac{\beta+1}{d}\bar{\nabla}_\mu h \ .$$

$$\Rightarrow S_{GF} = \frac{Z}{2\alpha}\int dx\sqrt{\bar{g}}\left[\xi_\mu\left(-\bar{\nabla}^2 - \frac{\bar{R}}{d}\right)^2\xi^\mu + \frac{(d-1-\beta)^2}{d^2}\chi(-\bar{\nabla}^2)\left(-\bar{\nabla}^2 - \frac{\bar{R}}{d-1-\beta}\right)^2\chi\right].$$

$\xi,\ \chi = \frac{[-(d-1)\bar{\nabla}^2-\bar{R}]\sigma+\beta h}{-(d-1-\beta)\bar{\nabla}^2-\bar{R}}$**: gauge degrees of freedom**

h^{TT}**,** s**: the physical degrees of freedom.**

The ghost action for this gauge fixing:

$$S_{gh} = \int dx\sqrt{\bar{g}}\bar{C}^\mu\left(\delta_\mu^\nu\bar{\nabla}^2 + \left(1 - 2\frac{\beta+1}{d}\right)\bar{\nabla}_\mu\bar{\nabla}^\nu + \bar{R}_\mu{}^\nu\right)C_\nu$$

Contribution from the gauge fixing term of the gauge variant fields (ξ, χ)

$$\mathrm{Det}\Big(-\bar\nabla^2-\frac{1}{d}\bar R\Big)^{-1}\mathrm{Det}(-\bar\nabla^2)^{-1/2}\mathrm{Det}\Big(-\bar\nabla^2-\frac{1}{d-1-\beta}\bar R\Big)^{-1}$$

The ghost determinants

$$\mathrm{Det}\Big(-\bar\nabla^2-\frac{1}{d}\bar R\Big)\mathrm{Det}\Big(-\bar\nabla^2-\frac{1}{d-1-\beta}\bar R\Big)$$

Finally the Jacobian (in York decomposition)

$$\mathrm{Det}\Big(-\bar\nabla^2-\frac{1}{d}\bar R\Big)^{1/2}\mathrm{Det}(-\bar\nabla^2)^{1/2}\mathrm{Det}\Big(-\bar\nabla^2-\frac{\bar R}{d-1}\Big)^{1/2}$$

The gauge-dependent factors cancel out, and the result is gauge independent!!

4 Functional renormalization group equation

By the standard procedure, we get the functional renormalization group equation (FRGE)

$$\dot{\Gamma}_k = \frac{1}{2}\mathbf{Tr}_{(2)}\left[\frac{\dot{f}'(\bar{R})R_k(\Box) + f'(\bar{R})\dot{R}_k(\Box)}{f'(\bar{R})\left(P_k(\Box) - E_{(2)} + \frac{2}{d(d-1)}\bar{R}\right)}\right] + \frac{1}{2}\mathbf{Tr}_{(0)}\left[\frac{\dot{f}''(\bar{R})R_k(\Box) + f''(\bar{R})\dot{R}_k(\Box)}{f''(\bar{R})\left(P_k(\Box) - E_{(0)} - \frac{1}{d-1}\bar{R}\right) + \frac{d-2}{2(d-1)}f'(\bar{R})}\right] - \frac{1}{2}\mathbf{Tr}_{(1)}\left[\frac{\dot{R}_k(\Box)}{P_k(\Box) - E_{(1)} - \frac{1}{d}\bar{R}}\right],$$

where the dot denote the logarithmic derivative with respect to the scale k, $E_{(s)}$ $(s = 0, 1, 2)$ are terms linear in the scalar curvature, and

$$P_k(\Box) = \Box + R_k(\Box), \qquad \Box = -\bar{\nabla}^2 + E_{(s)},$$

with the cutoff function $R_k(\Box)$. The subscripts to the traces represent contributions from different spin sectors.

Heat kernel expansion: $\mathbf{Tr}_{(s)}[e^{-\sigma\Box}] = \frac{1}{(4\pi\sigma)^{d/2}} \int_{S^d} d^dx\sqrt{\bar{g}} \sum_{n\geq 0} b_{2n}^{(s)} \sigma^n \bar{R}^n,$

$$\Rightarrow \mathbf{Tr}_{(s)}[W(\Box)] = \frac{1}{(4\pi)^{d/2}} \int_{S^d} d^dx\sqrt{\bar{g}} \sum_{n\geq 0} b_{2n}^{(s)} Q_{d/2-n}[W] \bar{R}^n,$$

where

$$Q_m[W] = \frac{1}{\Gamma(m)} \int_0^\infty dz z^{m-1} W[z].$$

Choose $R_k(z) = (k^2 - z)\theta(k^2 - z) = k^2(1-y)\theta(1-y)$**,** $\Rightarrow \dot{P}_k = \dot{R}_k = 2k^2\theta(k^2 - z)$

$$E_{(2)} = -\alpha\bar{R}, \quad E_{(0)} = -\beta\bar{R}, \quad E_{(1)} = -\gamma\bar{R},$$

and define

$$r \equiv \bar{R}k^{-2}, \qquad \varphi(r) = k^{-d} f(\bar{R}),$$

Our main result in 4 dims.

$$32\pi^2(\dot{\varphi} - 2r\varphi' + 4\varphi) = \frac{c_1(\dot{\varphi}' - 2r\varphi'') + c_2\varphi'}{\varphi'[6 + (6\alpha + 1)r]} + \frac{c_3(\dot{\varphi}'' - 2r\varphi''') + c_4\varphi''}{2\{\varphi''[3 + (3\beta - 1)r] + \varphi'\}} - \frac{c_5}{4 + (4\gamma - 1)r},$$

where

$$c_1 = 5 + 5\Big(3\alpha - \frac{1}{2}\Big)r + \Big(15\alpha^2 - 5\alpha - \frac{1}{72}\Big)r^2 + \Big(5\alpha^3 - \frac{5}{2}\alpha^2 - \frac{\alpha}{72} + \frac{311}{9072}\Big)r^3,$$
$$c_2 = 40 + 15(6\alpha - 1)r + \Big(60\alpha^2 - 20\alpha - \frac{1}{18}\Big)r^2 + \Big(10\alpha^3 - 5\alpha^2 - \frac{\alpha}{36} + \frac{311}{4536}\Big)r^3,$$
$$c_3 = 1 + \Big(3\beta + \frac{1}{2}\Big)r + \Big(3\beta^2 + \beta - \frac{511}{360}\Big)r^2 + \Big(\beta^3 + \frac{1}{2}\beta^2 - \frac{511}{360}\beta + \frac{3817}{9072}\Big)r^3,$$
$$c_4 = 2\Big[3 + (6\beta + 1)r + \Big(3\beta^2 + \beta - \frac{511}{360}\Big)r^2\Big],$$
$$c_5 = 12 + 2(12\gamma + 1)r + \Big(12\gamma^2 + 2\gamma - \frac{607}{180}\Big)r^2.$$

If we include the contribution of the constant mode of trace h, we have an additional term to the r.h.s.

$$\frac{8}{3}\frac{r^2}{16 + 2\varphi - r\varphi'}$$

We can also compute the traces by summing directly the corresponding functions of the eigenvalues of the Laplacian on the sphere as

$$Tr_{(s)}[W(-\bar{\nabla}^2 + E_{(s)})] = \sum_l M_l(d,s)W(\lambda_l(d,s) + E_{(s)})$$

$\lambda_l, E_{(s)}$: **the eigenvalues and the corresponding multiplicities.**

$\Rightarrow$ **The structure of the flow equations is the same.**

5 Scaling solutions in 4D

Properties of differential equations obtained from $\dot{\varphi} = 0 \Rightarrow$ fixed points.

$$\varphi(r) = \sum_{m=0}^{N} g_m r^m, \qquad \dot{\varphi}(r) = \sum_{m=0}^{N} \beta_{g_m} r^m,$$

g_m: **the k-dependent running couplings**
$\beta_{g_m} = \partial_t g_m$: **their beta functions.**

Substituting and expanding the result in powers of r up to order N yields a system of $N+1$ algebraic equations for the beta functions. Setting these to zero, we can find the fixed points.

The FRGE tells us that the large-r behavior of φ is

$$\varphi \sim a_2 r^2 + a_1 r + a_0 + a_{-1}/r + \cdots \quad \textbf{at most quadratic!}$$

Exact solutions:
We first treat these parameters as unknowns to solve for.
The simplest possible solutions are of the form

$$\varphi(r) = g_0 + g_1 r + g_2 r^2 \; . \cdots \quad \textbf{Similar to Starobinsky model!}$$

We obtains a system of six equations for the six unknowns g_0, g_1, g_2, α, β and γ. This system has a number of solutions in Table 1.

$10^3\alpha$	$10^3\beta$	$10^3\gamma$	$10^3\tilde{g}_{0*}$	$10^3\tilde{g}_{1*}$	$10^3\tilde{g}_{2*}$	θ
−593	−73.5	−177	7.28	−8.42	1.71	3.78
−616	−70.7	−154	7.42	−8.64	1.74	3.75
−564	−80.3	−168	6.82	−8.77	1.83	3.70
−543	−87.4	−126	6.31	−9.47	2.06	3.43
−420	−100.5	−3.19	4.90	−10.2	2.83	2.93
−173	−2.98	244	4.53	−8.34	2.70	2.18
−146	−64973	250	2.90	−10.7	0.0006	2.58
−109	−22267	307	2.90	−10.4	0.0045	2.45
109	−3564	526	2.84	−7.83	0.094	C
377	−1305	794	2.57	−4.37	0.214	> 4

Table 1: Exact quadratic solutions

We also study the system for vanishing endmorphism $\alpha = \beta = \gamma = 0$. The non-Gaussian fixed points (NGFP) are summarized in Table 2. The critical exponents for $N = 8$ are $4, 2.21, -2.51, -5.21, -7.50, -9.53, -11.46, -12.9, -15.1$.

Numerical analysis for fixed α, β, γ:

When we solve for the differential equations of the fixed point solutions, which are third order in general, the zero's of the third order coefficients rc_3 give the singularities.

N	g_0^*	g_1^*	g_2^*	$10^5 g_3^*$	$10^6 g_4^*$	$10^7 g_5^*$	$10^7 g_6^*$	$10^8 g_7^*$	$10^9 g_8^*$
1	0.00290	−0.00770							
2	0.00467	−0.00492	0.00148						
3	0.00466	−0.00472	0.00143	−1.78					
4	0.00478	−0.00473	0.00136	−1.39	−4.17				
5	0.00479	−0.00466	0.00134	−1.79	−4.34	−7.09			
6	0.00481	−0.00465	0.00132	−1.79	−4.83	−7.72	−1.53		
7	0.00482	−0.00464	0.00132	−1.85	−4.93	−8.81	−1.73	−3.60	
8	0.00482	−0.00463	0.00131	−1.86	−5.02	−9.11	−2.00	−4.23	−9.51

Table 2: The NGFPs for $\alpha = \beta = \gamma = 0$

The normal form of the flow equation has a singularity at $r = 0$ and further fixed singularities depending on β.

The isolated solutions are expected to occur when the number of fixed singularities matches the order of the equation.
$\Rightarrow$ work in progress

We can also study beta functions in other dimensions. Our results confirm that there are always nontrivial UV fixed point functions.

6 Discussions

We have constructed a novel functional renormalization group equation for gravity which encodes the gravitational degrees of freedom in terms of general function $f(R)$ of the scalar curvature.

We use a new parametrization of exponential type of the metric which avoids unphysical singularity.

The advantage of this parametrization is that it gives gauge-independent result.

We have shown that there are ultraviolet fixed points essential for Asymptotic Safety for the function $f(R)$.

We have studied if this approach may be used to determine possible UV completion of gravitational theory and the result contains exact solutions similar to Starobinsky model $(R + R^2)$, consistent with the current observation on inflation.

Possible future directions:
extending the analysis to more general theory, background-independence etc.