On the uniqueness of Einstein-Hilbert kinetic term (in massive (multi-)gravity)

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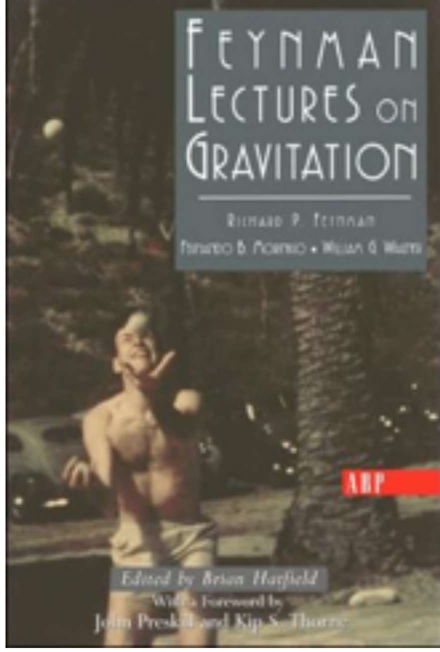
Based on: de Rham, Matas, Tolley,
``New Kinetic Terms for Massive Gravity and Multi-gravity:
A No-Go in Vielbein Form,'' 1505.00831
``New Kinetic Interactions for Massive Gravity?,''
1311.6485
de Rham, Matas, Ondo and Tolley,
``Interactions of Charged Spin-2 Fields,''1410.5422

Einstein approach to GR

- Equivalence Principle as guiding principle
- Spacetime Geometry is fundamental
- Diffeomorphism (General Coordinate) invariance is fundamental
- Spacetime Curvature encodes strength of gravity

Field theory approach to GR

Gupta, Feynman, Weinberg, Deser, Boulware, Wald …



- Gravity is a force like EM propagated by a massless spin-2 particle
- GR (with a cosmological constant) is the unique Lorentz invariant low energy effective theory of a single massless spin 2 particle coupled to matter
- Diffeomorphism invariance is a derived concept
- Equivalence Principle is a derived concept (Weinberg ``Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass~1964)
- Form of action is derived by principles of LEEFT

Sketch of proof

Spin 2 field is encoded in a 10 component symmetric tensor

$$h_{\mu\nu}$$

But physical degrees of freedom of a massless spin 2 field are
d.o.f. = 2

We need to subtract 8 = 2 x 4

This is achieved by introducing 4 local symmetries

Every symmetry removes one component since 1 is pure gauge and the other is fixed by associated first class constraint (Lagrangian counting)

Sketch of proof

Lorentz invariance demands that the 4 symmetries form a vector (there are only 2 possible distinct scalar symmetries) and so we are led to the **unique** possibility

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

We can call this linear Diff symmetry but its really just 4 U(1) symmetries, its sometimes called **spin 2 gauge invariance**

Quadratic action

Demanding that the action is **local** and starts at lowest order in derivatives (two), we are led to a unique quadratic action which respects linear diffs

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$S = \int d^4x \frac{M_P^2}{8} h^{\mu\nu} \Box (h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) + \ldots$$

Where … are terms which vanish in de Donder/harmonic gauge. It has an elegant representation with the Levi-Civita symbols …..

$$S \propto \int d^4x \epsilon^{ABCD} \epsilon^{abcd} \eta_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

Nonlinear theory

In order to construct the nonlinear theory we must have a **nonlinear completion** of the linear Diff symmetry to ensure that nonlinearly the degrees of freedom are

$$10 - 2 \times 4 = 2$$

So the relevant question, and what all the proofs in effect rely on is, what are the nonlinear extensions of the symmetry which are consistent (i.e. form a group)

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Nonlinear theory

The nonlinear symmetry should preserve Lorentz invariance so

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

becomes schematically

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + h_\mu^\alpha h_\nu^\beta (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha) + h^n (\partial h) \xi + h^m \partial \xi$$

$$+\text{higher derivatives}$$

but the form of the transformation is **strongly constrained** by the requirement that it forms a group

Unique result

Most complete proof Wald 1986

There are **only two** nonlinear extensions of the linear Diff symmetry, (assumption over number of derivatives)

1. Linear Diff -> Linear Diff

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

2. Linear Diff -> Full Diffeomorphism

$$h_{\mu\nu} \to h_{\mu\nu} + \xi^\omega \partial_\omega h_{\mu\nu} + g_{\mu\omega} \partial_\nu \xi^\omega + g_{\omega\nu} \partial_\mu \xi^\omega$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Metric emerges as derived concept

Case 1: Coupling to matter

1. Linear Diff -> Linear Diff

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

The coupling to matter must respect this symmetry, e.g. if we consider

$$\int d^4x \frac{1}{2} h_{\mu\nu}(x) J^{\mu\nu}(x)$$

then we must have

performing transformation:

$$\int d^4x \partial_\mu \xi_\nu J^{\mu\nu} \longrightarrow \partial_\mu J^{\mu\nu}(x) = 0$$

Case 1: Coupling to matter

$$\int d^4x \frac{1}{2} h_{\mu\nu}(x) J^{\mu\nu}(x)$$ then we must have

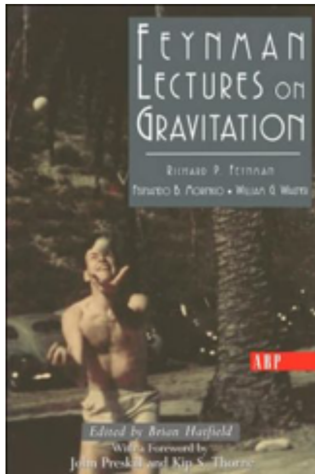
$$\partial_\mu J^{\mu\nu}(x) = 0$$

The problem is that this must hold as an IDENTITY!!

We cannot couple h to the stress energy of matter which is conserved in the absence of the coupling because as soon as we add the interaction, the equations of motion for matter are modified in such a way that the stress energy is no longer conserved

$$J^{\mu\nu} \neq T^{\mu\nu}$$

e.g. Feynman goes through expample of a point particle in his book …



Case 1:Non-gravitational spin 2 theory

$$\partial_\mu J^{\mu\nu}(x) = 0$$

An interacting theory does exist in case 1, by taking J to be identically conserved

Example: `Galileon combinations'

$$J^{\mu\nu} = \epsilon^{\mu abc}\epsilon^{\nu ABC} A_{aA} A'_{bB} A''_{cC}$$

where each entry is either

$$A_{aA} = \partial_a \partial_A \pi \text{ or } \eta_{aA}$$

Precisely these terms arise in the Decoupling Limit of Massive Gravity de Rham, Gabadadze 2010

Case 2: Coupling to matter

2. Linear Diff -> Full Diffeomorphism

$$h_{\mu\nu} \to h_{\mu\nu} + \xi^\omega \partial_\omega h_{\mu\nu} + g_{\mu\omega}\partial_\nu \xi^\omega + g_{\omega\nu}\partial_\mu \xi^\omega$$

The coupling to matter must respect this symmetry, but this is now easy, we just couple matter covariantly to

$$g_{\mu\nu}$$

any such coupling is perturbatively equivalent to

$$\int d^4x\, h_{\mu\nu} T^{\mu\nu}$$

and so is a theory of gravity!

Kinetic Terms

Case 1: Non-Gravitational Spin 2.

Since nonlinear symmetry is linear Diff, existing kinetic term is leading term at two derivative order (however there is a second term ….)

$$S = \int d^4x \frac{M_P^2}{8} h^{\mu\nu} \Box (h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) + \dots$$

Case 2: Gravitational Spin 2

Since nonlinear symmetry is nonlinear Diff, kinetic term must be leading two derivative diffeomorphism invariant operator

$$S = \int d^4x \frac{M_P^2}{2} \sqrt{-g} R$$

HENCE GR!!!!

Basic Question

What happens if we repeat this arguments starting with the assumption of a **massive spin 2 field?**

i.e. suppose that the graviton is massive, are we inevitably led to the Einstein-Hilbert action (plus mass term)?

One argument says no

In a Massive theory of Gravity Diffeomorphism invariance is completely broken. Thus **superficially** it appears that everything that makes GR nice is completely lost

For instance, already at 2 derivative order we can imagine an infinite number of possible kinetic terms which are schematically

$$S = \int d^4x - \frac{M_P^2}{2}\left(\partial h \partial h + \cdots \sum \alpha_n h^{n-2} \partial h \partial h\right)$$

Fortunately this is wrong

If we really allowed for such a completely general form, then we would be at risk that all 10 components of metric are dynamical

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \dots$$

Even if we ensure that $h_{0\mu}$ is not dynamical, we are at risk that the 6 remaining spatial components are dynamical

h_{ij} which is one two many

6 = 5 + Ostrogradski ghost

A toy example, Proca theory

For a massive spin 1 field, we break gauge invariance, so we may think that we can allow non-gauge invariant kinetic terms of the form

$$S = \frac{1}{2}F_{\mu\nu}^2 + \alpha(\partial_\mu A^\mu)^2$$

However this would lead to 4 propagating degrees of freedom, instead of 2s+1 = 3

The key point is that A^0 must remain non-dynamical to impose a second class constraint

A toy example, Proca theory

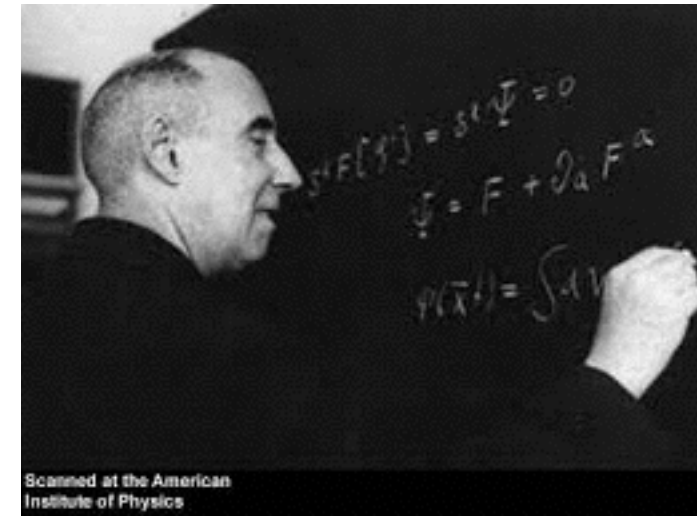
In passing from massless to massive theory
what happens is:

A^0 goes from a Lagrange multiplier of a first class constraint
(which generates a symmetry)

to a Lagrange multiplier of a second class constraint

this fixes the lowest order Lagrangian

Stuckelberg picture



All of this is much easier to understand in the Stuckelberg picture in which reintroduce gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$S = \frac{1}{4} F_{\mu\nu}^2 + \alpha(\Box\chi + \partial_\mu A^\mu)^2$$

Massive theory is now gauge invariant

$$A_\mu \to A_\mu + \partial_\mu \xi \,,\; \chi \to \chi - \xi$$

But is now clearly higher derivative for χ

Therefore number of degrees of freedom are χ

2 A_μ + 1 χ + 1 Ostrogradski

Now to massive spin 2

The general principle is the same in the spin 2 case

Although the massive theory breaks the 4 nonlinear gauge symmetries, we still need that at least one second class constraint to ensure 5 degrees of freedom

Equivalently, if we Stuckelberg back the symmetries of the massless theory then we must demand that the Stuckelberg fields do not admit Ostrogradski instabilities

However, how we do this depends on whether we are looking at non-gravitational (SPIN 2 MESONS) or gravitational spin 2 fields (GRAVITONS)

Case 1. Non-gravitational massive spin 2

In this case we should Stuckelberg the linear Diff symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remarkably there is a unique extension to the kinetic term already at two derivative level which is cubic

Hinterbichler 2013
Folkerts, Pritzel, Wintergerst 2011

$$S_{(3)} = \int d^4x \epsilon^{ABCD} \epsilon^{abcd} h_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

Thus for Case 1 theories, linearized E-H kinetic term, i.e. Fierz-Pauli kinetic term is not unique!!!

Note this is NOT a limit of a Lovelock term as seen by counting derivatives

Case 2. Gravitational massive spin 2

In this case we should Stuckelberg the nonlinear Diff symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \xi^\omega \partial_\omega h_{\mu\nu} + g_{\mu\omega}\partial_\nu \xi^\omega + g_{\omega\nu}\partial_\mu \xi^\omega$$

This is done explicitly by replacing h with a tensor

$$h_{\mu\nu} = g_{\mu\nu} - \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} \qquad \phi^a = x^a + \frac{A^a}{m M_P} + \frac{\partial^a \pi}{m^2 M_P}$$

In this case we are led (after much calculation) to a unique kinetic term in four dimensions (up to total deriavatives), i.e. Einstein-Hilbert kinetic term

$$S = \int d^4x \frac{M_P^2}{2} \sqrt{-g} R$$

Case 2. Gravitational massive spin 2

de Rham, Matas, Tolley,
``New Kinetic Interactions for Massive Gravity?,''
1311.6485

I'm leaving out all the details of the proof which is complicated but what it means is there is no `graviton' analoge of the spin-2 meson kinetic term ….

$$S_{(3)} = \int d^4x \epsilon^{ABCD} \epsilon^{abcd} h_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

$$S = \int d^4x \frac{M_P^2}{2} \sqrt{-g} R$$

Thus **all of the key features of Einstein gravity** emerge equally from the assumption that the graviton is massive even though Diffeomorphism invariance is strictly broken

Coupled with the uniqueness of the mass terms this means the theory of a massive spin 2 particle is unique!

de Rham, Gabadadze, Tolley (2010)

This is remarkable!

Extensions I

de Rham, Matas, Tolley,
``New Kinetic Terms for Massive Gravity and Multi-gravity:
A No-Go in Vielbein Form,'' 1505.00831

This result extends to all bigravity and multi-gravity theory

Hassan, Rosen 2011, Hinterbichler, Rosen 2012

E.g. the unique kinetic term in metric language for a single massless and a single massive spin 2 field is a direct sum of 2 E-H kinetic terms (up to field redefinitions)

$$S = \int d^4x \frac{M_P^2}{2} \sqrt{-g} R[g] + \frac{M_f^2}{2} \sqrt{-f} R[f]$$

If this were not the case, then it would be possible to take a decoupling limit in which the f metric fluctuations decouple and generate a new kinetic term for the metric g which we have already rules out   c.f. Luc Blanchet talk

Extensions II

de Rham, Matas, Tolley,
``New Kinetic Terms for Massive Gravity and Multi-gravity:
A No-Go in Vielbein Form,'' 1505.00831

This result extends to the Einstein-Cartan (first order formalism)

For example, in bigravity we have 2 vierbeins and 2 spin connections, but we respect only a single copy of Diffs and a single copy of local Lorentz invariance

Thus superficially the following looks ok

$$S = \int e \wedge e \wedge R[\omega] + f \wedge f \wedge R[\Omega] + \alpha(\omega - \Omega) \wedge (\omega - \Omega) \wedge e \wedge e$$

or

$$S = \int e \wedge e \wedge R[\omega] + f \wedge f \wedge R[\Omega] + e \wedge f \wedge R[\omega] + \ldots$$

Extensions II

$$S = \int e \wedge e \wedge R[\omega] + f \wedge f \wedge R[\Omega] + e \wedge f \wedge R[\omega] + \dots$$

However in this case, the massless theory would have the symmetry

$$Diff \times Lorentz \times Diff \times Lorentz$$

which is broken to

$$(Diff)_{Diagonal} \times (Lorentz)_{\rm Diagonal}$$

We thus must introduce 4 Diff stuckelberg fields and 6 local Lorentz stuckelberg fields

Extensions II

$$S = \int e \wedge e \wedge R[\omega] + f \wedge f \wedge R[\Omega] + \alpha(\omega - \Omega) \wedge (\omega - \Omega) \wedge e \wedge e$$

We thus must introduce 4 Diffeomorphism Stueckelberg fields and 6 local Lorentz Stueckelberg fields

$$f_\mu^a \to \partial_\mu \phi^A f_A^{a'} \Lambda^a{}_{a'} \qquad \Lambda \eta \Lambda^T = \eta$$

Demanding that these have no Ostrogradski ghosts fixes the form of the kinetic term to the sum of two separate E-H kinetic terms

Kaluza-Klein theory

The most famous example of this is Kaluza-Klein theory

EH Kinetic term in 5 dimensions = `Sum of'
EH Kinetic terms in 4 dimensions

$$S = \int d^5 x \, e \wedge e \wedge R^{(5)} = \int dy \int d^4 x \, \left(e \wedge e \wedge R^{(4)} + \dots \right)$$

Discretize through DECONSTRUCTION

c.f. Claudia de Rham talk

In mass eigenstate basis these are not diagonal but are field redefinable to diagonaliable - thus KK theory is the prototypical example of a theory of gravitional massive spin 2 particles

Charged Spin 2

A significant consequence of these results is the following:

It is impossible to write down a consistent effective field theory of a single charged spin 2 particle coupled to gravity, i.e. a theory in which the number of degrees of freedom is only

2 + 5 (particle)+5 (antiparticle)

de Rham, Matas, Ondo and Tolley,
``Interactions of Charged Spin-2 Fields,''1410.5422

Charged Spin 2

This was a surprise to us: but the reason is very simple

A charged spin 2 field is described at the linearized level by a complex tensor

$$H_{\mu\nu} \neq H^*_{\mu\nu}$$

we want to couple this to gravity, so we will have in effect 3 tensors

$$g_{\mu\nu} \; H_{\mu\nu} \; H^*_{\mu\nu}$$

Charged Spin 2

The kinetic term is determined by the symmetries that arise in the massless limit

There are two possibilities:

Case 1: One nonlinear Diff (g) and a complex linear Diff

$$H_{\mu\nu} \ H^*_{\mu\nu}$$

Case 2: Three nonlinear Diffs acting in some combination of

$$g_{\mu\nu} \ H_{\mu\nu} \ H^*_{\mu\nu}$$

Charged Spin 2

Case 1: One nonlinear Diff (g) and a complex linear Diff

$$H_{\mu\nu} \; H^*_{\mu\nu}$$

This just corresponds to covariantizing Fierz-Pauli Lagrangian for $H_{\mu\nu}$

However it is an old result that this covariantization does not preserve the correct number of degrees of freedom as soon as g is not an Einstein space

Buchdahl 1958
Aragone and Deser 1980

Charged Spin 2

de Rham, Matas, Ondo and Tolley,
``Interactions of Charged Spin-2 Fields,''1410.5422

Case 2: Three nonlinear Diffs acting in some combination of

$$g_{\mu\nu} \ H_{\mu\nu} \ H^*_{\mu\nu}$$

This is equivalent to considering a tri-gravity theory, and asking that the trigravity Lagrangian admits a global U(1) symmetry that ultimately may be gauged.

However from out uniqueness statements, the unique tri-gravity kinetic term is

$$e_1 \wedge e_1 \wedge R[\omega_1] + e_2 \wedge e_2 \wedge R[\omega_2] + e_3 \wedge e_3 \wedge R[\omega_3]$$

which admits no global U(1) symmetry

Implications

Of course this does not mean that charged spin 2 fields do not exist, rather it means

1. That there is a built in cutoff at/above which the theory must be UV completed by new degrees of freedom
2. Or new degrees of freedom arise already at a lower scale and must be included into the EFT (however no known case of finite number of d.o.f.)

E.g. the spin 2 may be completed by a tower of Kaluza-Klein states or it may be a composite, not fundamental, excitation in some otherwise partially UV complete theory like QCD

Coupling to Electromagnetism

Porrati and Rahman 2008

In fact already in the absence of gravity, the theory of a single charged spin 2 theory has a build in cutoff

Consider a charged spin 2 coupled to EM with a Pauli-term (magnetic moment) $D_\mu = \partial_\mu - iqA_\mu$

$$S = \int \mathrm{d}^4x \left(\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\mu'\nu'\rho'}{}_{\sigma} \; H^*_{\mu\mu'} D_\nu D_{\nu'} H_{\rho\rho'} - m^2 \left([H^* H] - [H^*][H] \right) - \frac{1}{4} F^2_{\mu\nu} + iq(2g-1) H^*_{\mu\nu} F^{\nu\rho} H_\rho{}^{\mu} \right).$$

Introduce Stuckelberg fields

$$H_{\mu\nu} = h_{\mu\nu} + D_{(\mu} \left(\frac{1}{m} B_{\nu)} + \frac{1}{2m^2} D_{\nu)} \pi \right)$$

Cutoff for charged spin 2

Taking the decoupling limit

$$q \to 0, \quad m \to 0, \quad \Lambda_{q,n} \equiv \frac{m}{q^{1/n}} \text{ fixed.}$$

$$S_{\rm kin} = \int \mathrm{d}^4x \left(h^*_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} - \frac{1}{4}|G_{\mu\nu}|^2 - \frac{3}{4}|\partial\pi|^2 - \frac{1}{4}F^2_{\mu\nu} \right)$$

$$\mathcal{L}_{\Lambda_{q,4}} = (2g-1)\frac{i}{\Lambda^4_{q,4}}\partial_\mu\partial_\nu\pi^* F^{\nu\rho}\partial_\rho\partial^\mu\pi$$

This gives a ghost! or cutoff at scale

$$\Lambda_{q,4} = \frac{m}{q^{1/4}}$$

Federbush

Federbush 1961

However if we make the special choice first of gyromagnetic ratio $g = \frac{1}{2}$

$$S_{\rm kin} = \int \mathrm{d}^4x \, \left(h^*_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{1}{4} |G_{\mu\nu}|^2 - \frac{3}{4} |\partial \pi|^2 - \frac{1}{4} F^2_{\mu\nu} \right)$$

$$\mathcal{L}_{\Lambda_{q,3}} = -\frac{i}{\Lambda^3_{q,3}} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'}{}_{\sigma} \partial_\mu \partial_{\mu'} \pi^* F_{\nu\rho} G_{\nu'\rho'} + c.c.$$

No ghost, i.e. no Ostrogradski instability - strong coupling scale at $\Lambda_{q,4} = \frac{m}{q^{1/3}}$

EFT understanding

Already in the absence of gravity, theory of charged spin-2 field has cutoff of either

$$\Lambda_{q,4} = \frac{m}{q^{1/4}} \qquad g \neq \frac{1}{2} \qquad\qquad \Lambda_{q,3} = \frac{m}{q^{1/3}} \qquad g = \frac{1}{2}$$

Thus when we add gravity, we can happily live with an Ostrogradski ghost whose mass is above these scales!

Specific UV completions will indicate precisely how LEFT is resolved at or before cutoff but this is model dependent

Conclusions

- There are two types of interacting spin-2 fields, non-gravitational (spin 2 mesons) and gravitational (gravitons)
- In the case of gravitational, for any number of gravitons the kinetic terms must be a direct sum of Einstein-Hilbert kinetic terms, thus Einstein gravity always arises in some limit of a theory of a interacting massive spin 2 field that couples to matter
- For charged spin 2 fields there are **two built in cutoffs**, one dependent on m and q and one dependent on m and M_Planck. Nevertheless such theories make sense as LEEFTs
- NOTE WE CAN LIVE WITH NON-STANDARD KINETIC TERMS PROVIDED THAT THEY ARE SUPRESSED BY THE CUTOFF