1. Initial boundary operator corresponding to Drazin invertible operators

Let $X$, and $E$ be a complex Banach spaces. Denoted $\mathcal{C}(X)$ the set of all closed linear operators from $X$ into $X$. The identity operator on a Banach space $E$ is denoted by $I_E$.

We consider the following boundary value problem for unknown $x \in \mathcal{D}(A)$ by the system

$$\begin{cases}
(A - \lambda I)x = f \\
\Gamma x = \varphi
\end{cases}$$

where $f \in X$, $\varphi \in E$ and $\lambda \in \mathbb{C}$.

1.1. Definition. An operator $\Gamma : X \to E$ is said to be an initial boundary operator for a Drazin invertible operator (resp. right Drazin invertible operator) $T$ corresponding to its Drazin inverse $T^D$ (resp. to its right Drazin inverse $S \in B(X)$) if:

(i) $\Gamma T^D = 0$ on $\mathcal{D}(T^D)$ (resp. $\Gamma S = 0$ on $X$);
(ii) There exists an operator $\Pi : E \to X$ such that $\Pi I = I_E$ and $R(\Pi) = \mathcal{N}(T^m)$, with $m = a(T) = d(T)$ (resp. $R(\Pi) = \mathcal{N}(T^{m+1})$, with $d(T) = m$).

1.2. Definition. An operator $\Gamma : X \to E$ is said to be an initial boundary operator for a left Drazin invertible operator $T$ corresponding to its left Drazin inverse $S \in B(X)$ if:

(i) $\Gamma T = 0$ on $\mathcal{D}(T)$;
(ii) There exists an operator $\Pi : E \to X$ such that $\Pi I = I_E$ and $R(\Pi) = \mathcal{N}(S) \cap R(T^m)$, with $m = a(T)$.

2. Main results

The following results are given to establish the existence and uniqueness of the solution for the boundary value problem $(P)$.

It well known that there is a helpful explicit formula for the Drazin inverse $A^D$ of a closed operator $A$ in terms of the spectral projection $P$ of $A$ at 0:

$$A^D = (A + \lambda I)^{-1}(I - P)$$

for any $\lambda \neq 0$. We also observe that $P = I_X - AA^D$. If $A = A_1 \oplus A_2$ is the decomposition of a Drazin invertible operator $A \in \mathcal{C}(X)$ described in the preceding section, then

$$A^D = A_1^D \oplus 0.$$ 

So we can assert that there exists $\epsilon > 0$ such that $\mu I_X - A^D$ is invertible for $|\mu| < \epsilon$. Now, in the case where $A$ is Drazin invertible, the problem $(P)$ is well-posed and its unique solution is explicitly obtained.

2.1. Theorem. Let $A \in \mathcal{C}(X)$ be Drazin invertible operator with Drazin inverse $A^D \in B(X)$. Then there exists $\epsilon > 0$ such that $(I_X - \lambda A^D)$ is invertible for $|\lambda| < \epsilon$ and the boundary value problem $(P)$ has a unique solution given by

$$x_\lambda^e = A^D(I_X - \lambda A^D)^{-1} f + (I_X - \lambda A^D)^{-1} \Pi \varphi$$

for every $f \in R(A^m)$, with $a(A) = d(A) = m$.

2.2. Theorems

1. If $A$ be left Drazin inverse of the operator $T \in \mathcal{C}(X)$ with $a(T) = m < \infty$ and $I_X - \lambda T$ is invertible where $\lambda \neq 0$, then the boundary value problem $(P)$ has unique solution given by:

$$x_\lambda^e = T(I_X - \lambda T)^{-1} f + (I_X - \lambda T)^{-1} \Pi \varphi$$

for $f \in R(T^m)$.

2. If $A \in \mathcal{C}(X)$ is right Drazin invertible with right Drazin inverse $R$ such that $d(A) = m < \infty$ and $(I_X - \lambda R)$ is invertible where $\lambda \neq 0$, then the boundary value problem $(P)$ has unique solution:

$$x_\lambda^e = R(I_X - \lambda R)^{-1} f + (I_X - \lambda R)^{-1} \Pi \varphi.$$ 

for $f \in R(A^m)$.

3. Example

We consider second order Cauchy problem

$$\begin{cases}
\frac{du}{dx} = \lambda u + f(x) \\
u(0) = u_0
\end{cases}$$

where $\lambda \in \mathbb{C}$. $UCB(\Omega)$ denote the family of all bounded, uniformly continuous valued complex functions on an interval $\Omega$. Let $UCB^k(\Omega)$ denote the set of all $k$ times differentiable functions in $UCB(\Omega)$ whose derivatives belongs to $UCB(\Omega)$. Let $X = UCB(\mathbb{R})$ equipped with the uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. We consider the operator $A = \frac{d}{dx}$ on $X$ with the domain $\mathcal{D}(A) = UCB^1(\mathbb{R})$.

The null space $\mathcal{N}(A)$ of the operator $A$ is the set of all constant functions on $\mathbb{R}$ (any such function belongs to $UCB(\mathbb{R})$).

In [2], P.L. Butzer and J.J. Koliha showed that $A = \frac{d}{dx}$ is Drazin invertible with $a(A) = d(A) = 1$, and it’s Drazin inverse $A^D$ is given by:

$$A^D f(x) = (I_X - P) h(x) - (Qh)(x), \quad \text{for } f \in X$$

where

$$\begin{align*}
P f & = \lim_{\xi \to \infty} \frac{1}{\xi} \int_{-\xi}^{\xi} f(t) dt, \quad \text{for } \xi > 0, \\
h(x) & = \int_{-\infty}^{\infty} (f(t) - Pf)dt, \\
Qh & = \lim_{|x| \to \infty} \frac{h(x)}{x},
\end{align*}$$

whenever the (finite) limit exists for $h : \mathbb{R} \to \mathbb{C}$. See [2] for more details.

Let $E = \mathbb{R}$, we define the initial boundary operator $\Gamma$ by $\Gamma(x) = u_0$ and the maps $\Pi$ by $(\Pi u_0)(x) = u_0$. Then $\Pi I = 1$, $\Gamma A^D f(x) = 0$ on $X$ and $R(\Pi) = \mathcal{N}(A)$. Now, due to Theorem 1, we have,

Theorem. There exists $\epsilon > 0$ such that $(I_X - \lambda A^D)$ is invertible for $|\lambda| < \epsilon$ and the boundary value problem (3) has unique solution given by

$$a(x) = (I_X - \lambda A^D)^{-1}(A^D f + \Pi u_0)(x).$$

References