Linear boundary value problems described by Drazin invertible operators

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1.Initial boundary operator corresponding to Drazin invertible operators

Let X, and E be a complex Banach spaces. Denoted $\mathcal{C}(X)$ the set of all closed linear operators from X into X. The identity operator on a Banach space E is denoted by I_E.

We consider the following boundary value problem for unknown $x \in \mathcal{D}(A)$ by the system

$$(\mathcal{P}) \begin{cases} (A-\lambda I)x = f \\ \Gamma x = \varphi \end{cases} \qquad (1)$$

where $f \in X$, $\varphi \in E$ and $\lambda \in \mathbb{C}$.

1.1. Definition. An operator $\Gamma : X \to E$ is said to be an initial boundary operator for a Drazin invertible operator (resp. right Drazin invertible operator) T corresponding to its Drazin inverse T^D (resp. to its right Drazin inverse $S \in \mathcal{B}(X)$) if,

(i) $\Gamma T^D = 0$ on $\mathcal{D}(T^D)$ (resp. $\Gamma S = 0$ on X);

(ii) There exists an operator $\Pi : E \to X$ such that $\Gamma\Pi = I_E$ and $\mathcal{R}(\Pi) = \mathcal{N}(T^m)$, with $m = a(T) = d(T)$ (resp. $\mathcal{R}(\Pi) = \mathcal{N}(T^{m+1})$, with $d(T) = m$).

1.2. Definition. An operator $\Gamma : X \to E$ is said to be an initial boundary operator for a left Drazin invertible operator T corresponding to its left Drazin inverse $S \in \mathcal{B}(X)$ if

(i) $\Gamma T = 0$ on $\mathcal{D}(T)$;

(ii) There exists an operator $\Pi : E \to X$ such that $\Gamma\Pi = I_E$ and $\mathcal{R}(\Pi) = \mathcal{N}(S) \cap \mathcal{R}(T^m)$, with $m = a(T)$.

2. Main results

The following results are given to establish the existence and uniqueness of the solution for the boundary value problem $(\mathcal{P})$.

It well known that there is a useful explicit formula for the Drazin inverse A^D of a closed operator A in terms of the spectral projection P of A at 0:

$$A^D = (A + \xi P)^{-1}(I_X - P) \quad \text{for any } \xi \neq 0. \qquad (2)$$

We also observe that $P = I_X - AA^D$. If $A = A_1 \oplus A_2$ is the decomposition of a Drazin invertible operator $A \in \mathcal{C}(X)$ described in the preceding section, then

$$A^D = A_1^{-1} \oplus 0.$$

So we can assert that there exists $\epsilon > 0$ such that $\mu I_X - A^D$ is invertible operator for $|\mu| < \epsilon$. Now, in the case where A is Drazin invertible, the problem $(\mathcal{P})$ is well-posed and its unique solution is explicitly obtained.

2.1. Theorem. Let $A \in \mathcal{C}(X)$ be Drazin invertible operator with Drazin inverse $A^D \in \mathcal{B}(X)$. Then there exists $\epsilon > 0$ such that $(I_X - \lambda A^D)$ is invertible for $|\lambda^{-1}| < \epsilon$ and the boundary value problem $(\mathcal{P})$ has a unique solution given by

$$x_\lambda^{f,\varphi} = A^D(I_X - \lambda A^D)^{-1}f + (I_X - \lambda A^D)^{-1}\Pi\varphi$$

for every $f \in \mathcal{R}(A^m)$, with $a(A) = d(A) = m$.

2.2. Theorems

1. If A be left Drazin inverse of the operator $T \in \mathcal{C}(X)$ with $a(T) = m < \infty$ and $I_X - \lambda T$ is invertible where $\lambda \neq 0$, then the boundary value problem $(\mathcal{P})$ has unique solution given by:

$$x_\lambda^{f,\varphi} = T(I_X - \lambda T)^{-1}f + (I_X - \lambda T)^{-1}\Pi\varphi$$

for $f \in \mathcal{R}(T^m)$.

2. If $A \in \mathcal{C}(X)$ is right Drazin invertible with right Drazin inverse R such that $d(A) = m < \infty$ and $(I_X - \lambda R)$ is invertible where $\lambda \neq 0$, then the boundary value problem $(\mathcal{P})$ has unique solution:

$$x_\lambda^{f,\varphi} = R(I_X - \lambda R)^{-1}f + (I_X - \lambda R)^{-1}\Pi\varphi.$$

for $f \in \mathcal{R}(A^m)$.

3. Example

We consider second order Cauchy problem

$$\begin{cases} \frac{d^2u(x)}{dx^2} = \lambda u(x) + f(x) \\ u(0) = u_0 \end{cases} \qquad (3)$$

where $\lambda \in \mathbb{C}$. $UCB(\Omega)$ denote the family of all bounded, uniformly continuous complex valued functions on an interval Ω. Let $UCB^k(\Omega)$ denote the set of all k times differentiable functions in $UCB(\Omega)$ whose derivatives belongs to $UCB(\Omega)$. Let $X = UCB(\mathbb{R})$ equipped with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$. We consider the operator $A = \frac{d^2}{dx^2}$ on X with the domain

$$\mathcal{D}(A) = UCB^2(\mathbb{R}).$$

The null space $\mathcal{N}(A)$ of the operator A is the set of all constant functions on $\mathbb{R}$ (any such function belongs to $UCB(\mathbb{R})$).

In [2], P.L. Butzer and J.J.Koliha showed that $A = \frac{d^2}{dx^2}$ is Drazin invertible with $a(A) = d(A) = 1$, and it's Drazin inverse A^D is given by :

$$A^D f(x) = (I_X - P)h(x) - (Qh)(x), \quad \textit{for} \;\; f \in X$$

where

$$Pf = \lim_{\zeta\to\infty} \frac{1}{2\xi}\int_{-\xi}^{\xi} f(t)dt, \quad \text{for} \;\; \xi > 0,$$

$$h(x) = \int_0^x \int_0^s (f(t) - Pf)dtds,$$

and

$$Qh = \lim_{|x|\to\infty} \frac{h(x)}{x},$$

whenever the (finite) limit exists for $h : \mathbb{R} \to \mathbb{C}$. See [2] for more details.

Let $E = \mathbb{R}$, we define the initial boundary operator Γ by $\Gamma u(x) = u_0$ and the maps Π by $(\Pi u_0)(x) = u_0$. Then $\Gamma\Pi = 1$, $\Gamma A^D f(x) = 0$ on X and $\mathcal{R}(\Pi) = \mathcal{N}(A)$. Now, due to Theorem 1, we have,

Theorem. There exists $\epsilon > 0$ such that $(I_X - \lambda A^D)$ is invertible for $|\lambda^{-1}| < \epsilon$ and the boundary value problem (3) has unique solution given by

$$u(x) = (I_X - \lambda A^D)^{-1}(A^D f + \Pi u_0)(x).$$

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