Methods for cosmological simulations

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Many collaborators

Outline

Basic concepts ⟷ Advanced topics

Equations ⟷ Applications

Details ⟷ Overview

Outline

1. What we do we want to solve?
2. Gravity: Solvers & Co.
3. A quick detour: Integrators
4. Is gravity enough?
5. Simulation types $< - >$ Initial conditions.

Our aim

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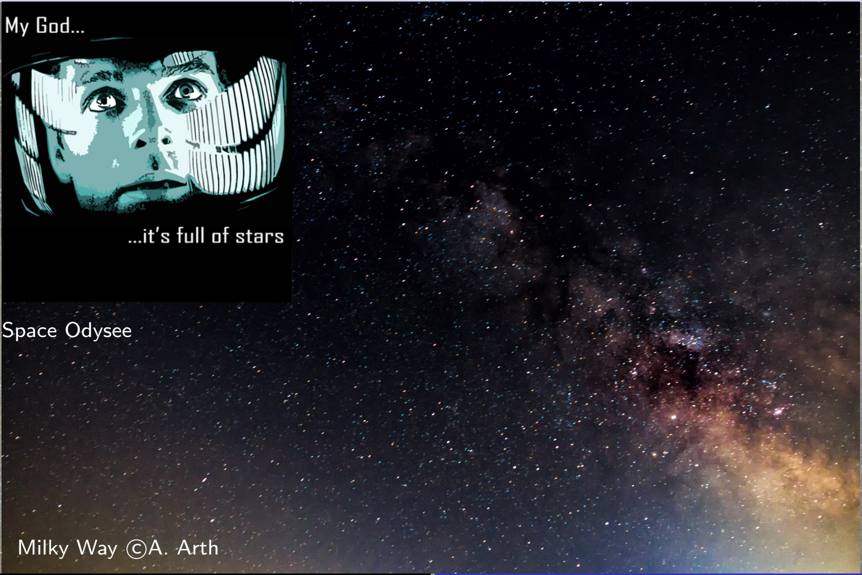

Space Odysee

Milky Way ©A. Arth

Our aim

Space Odysee

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Not only, but even full of galaxies!

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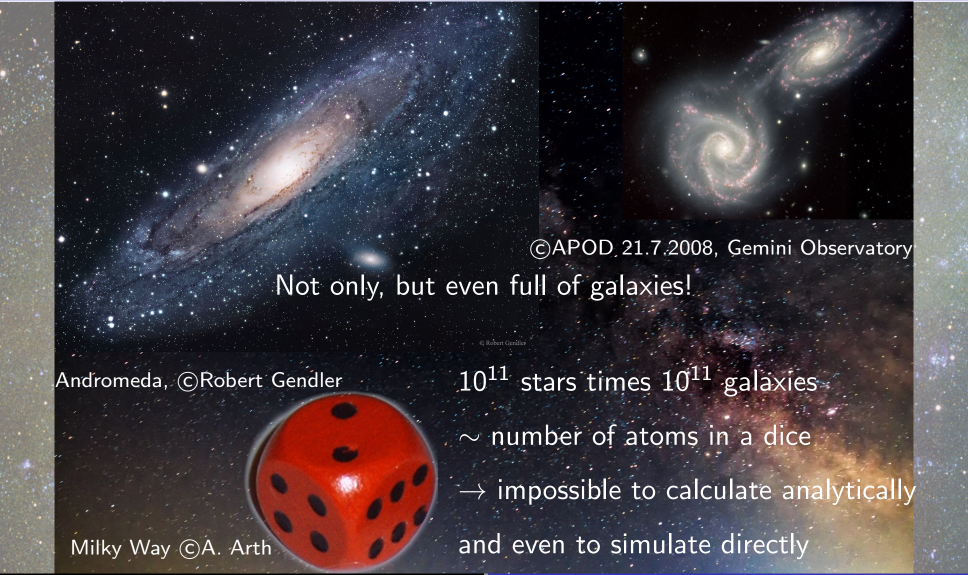
Our aim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Not only, but even full of galaxies!
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Andromeda, ©Robert Gendler
10^{11} stars times 10^{11} galaxies
$\sim$ number of atoms in a dice
$\rightarrow$ impossible to calculate analytically
and even to simulate directly
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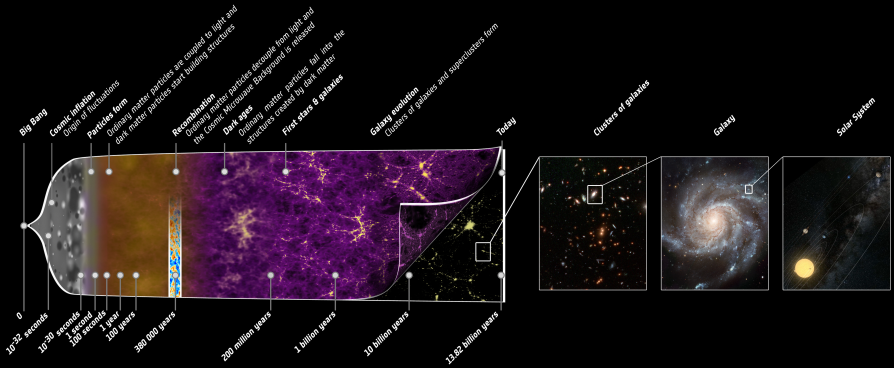
Huge range of time and spatial scales



Outline

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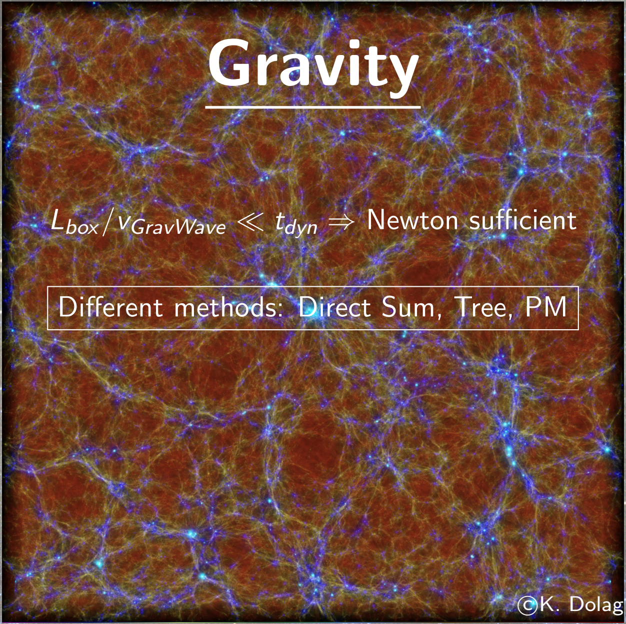
Gravity

$L_{box}/v_{GravWave} \ll t_{dyn} \Rightarrow$ Newton sufficient

Different methods: Direct Sum, Tree, PM



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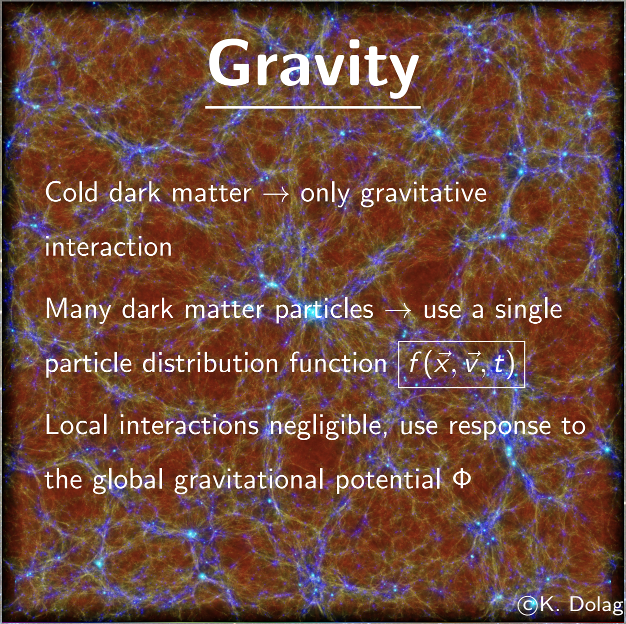

Gravity

Cold dark matter → only gravitative interaction

Many dark matter particles → use a single particle distribution function $\boxed{f(\vec{x}, \vec{v}, t)}$

Local interactions negligible, use response to the global gravitational potential Φ

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Gravity

Vlasov eq.:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}} \cdot \vec{v} + \frac{\partial f}{\partial \vec{v}} \cdot \left(-\frac{\partial \Phi}{\partial \vec{x}} \right) = 0}$$

Poisson eq.: $\boxed{\Delta \Phi = 4\pi G \int f \, d\vec{v}}$

Stars: two body relaxation time $\gg t_H \rightarrow$

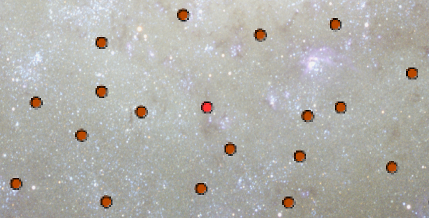
also collisionless system like DM

$\Rightarrow$ Hardly analytically solvable

Basic equations

Use N test "particles" to discretize the medium and write down Newton's law $\forall i$ (basically Monte Carlo method):

$$\boxed{\ddot{\vec{x}}_i = -\nabla_i \Phi\left(\vec{x}_i\right) \quad \& \quad \Phi\left(\vec{x}\right) = -G \sum_{j=1}^{N} \frac{m_j}{\left(\left(\vec{x} - \vec{x}_j\right)^2\right)}}$$



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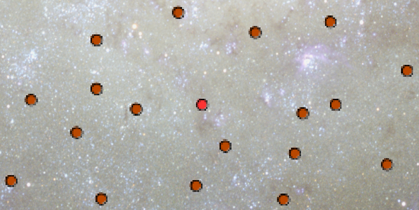
Introducing softening ϵ



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Introducing softening ϵ

Problem: $3N$ coupled non-linear 2nd order differential equations very time consuming to solve

Reasons for softening

* Consistent with Plummer potential plugged into Lagrangian
* $\Rightarrow$ Adhere to global rather than local potential

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- Prevent large angle scatterings and bound particle pairs
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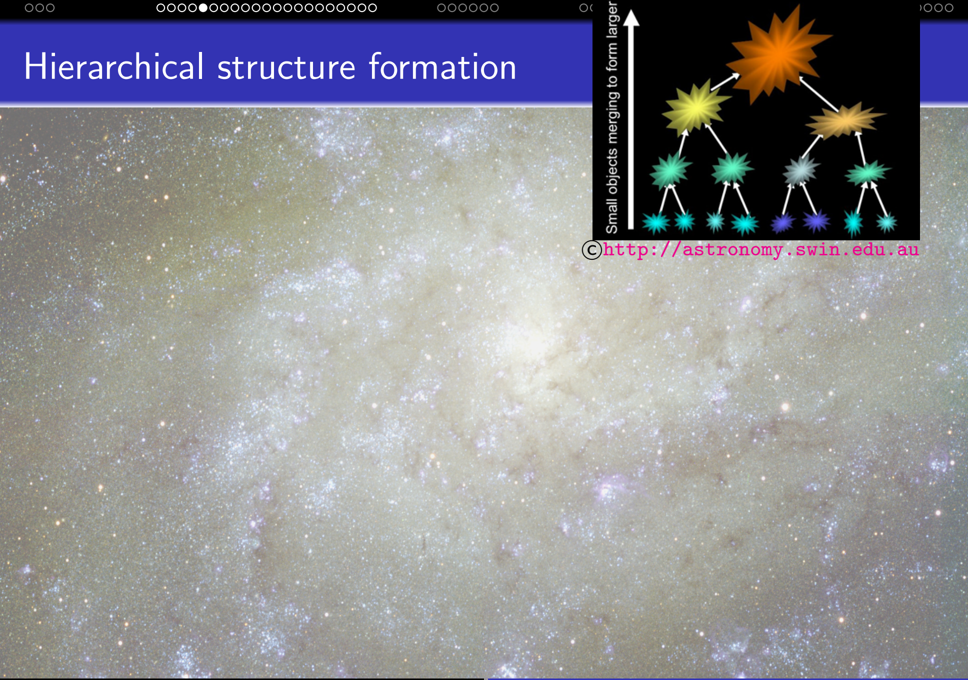
- ∗ Ensure that two-body relaxation time is sufficiently large
- ∗ Allow integration with a low-order integrator

Hierarchical structure formation



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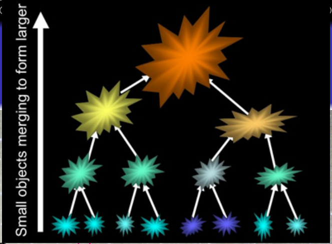
Hierarchical structure formation



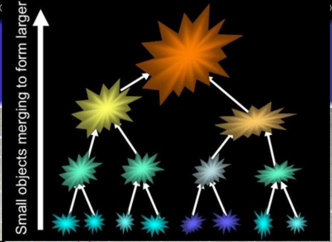
Small particle mass

Internal structure

Small scale physics

Small objects

Hierarchical structure formation



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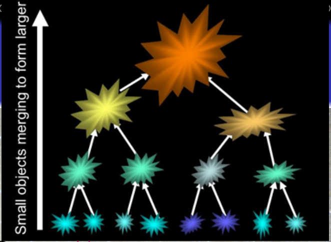
Small particle mass	Large volume
Internal structure	Representative statistics
Small scale physics	Rare objects
Small objects	Max simulation time ↔ large density modes

Hierarchical structure formation



Small particle mass	Large volume
Internal structure	Representative statistics
Small scale physics	Rare objects
Small objects	Max simulation time $\leftrightarrow$ large density modes

Essentially want to produce a large mass range of haloes

Solving Poisson's equation

Introduce a Green's function:

$$\Phi\left(\vec{x}\right) = \int g\left(\vec{x} - \vec{x}'\right) \rho\left(\vec{x}\right) d\vec{x}'$$

which in Fourier space comes down to a simple multiplication

$$\hat{\Phi}\left(\vec{k}\right) = \hat{g}\left(\vec{k}\right) \cdot \hat{\rho}\left(\vec{k}\right)$$

Example for vacuum boundary conditions:

$$g\left(\vec{x}\right) = -\frac{G}{|(\vec{x})|}$$

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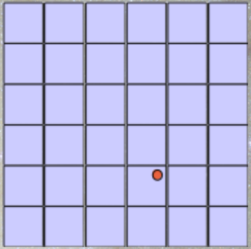
Steps to solution:

* Forward Fourier transformation of density
* Multiplication with Green's function
* Backwards Fourier transformation to obtain potential

First numerical approach: Particle Mesh



Four basic steps:

(1) Density assignment to cells

(2) Computation of potential

(3) Determination of force field

(4) Assignment of forces to particles

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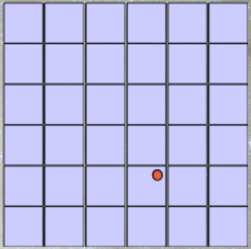

Four basic steps:

(1) Density assignment to cells

~~(2) Computation of potential~~ Potential from Green's function

(3) Determination of force field

(4) Assignment of forces to particles

(1) Density assignment

Put a mesh over the simulation domain, give each particle a shape $S(\vec{x})$ and assign the overlap mass fraction to each cell (cell coordinates x_m and particles x_i). Overlap function:

$W(\vec{x}_m - \vec{x}_i) = \int \Pi\left(\frac{\vec{x}' - \vec{x}_m}{h}\right) \cdot S(\vec{x}' - \vec{x}_i)\ d\vec{x}'$ (convolution) with

$$\Pi(x) = \begin{cases} 1 & |x| < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

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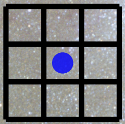
Then density: $\rho(\vec{x}_m) = \frac{1}{h^3} \sum_{i=1}^{N} m_i W(\vec{x}_m - \vec{x}_i)$

Shape functions

NGP

Delta function

1 cell



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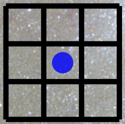
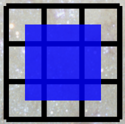


CIC

Constant shape

$2^3 = 8$ cells



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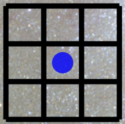

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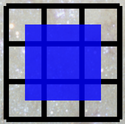
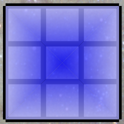


TSC

Triangular shape

$3^3 = 27$ cell

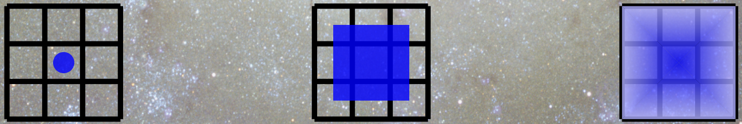


(3,4) Determination of force field & assignment to particles

In general: $\boxed{\vec{f} = -\nabla\Phi}$

Approximate with a discretization scheme, e.g. finite difference

Interpolate with **same** overlap function to get back to particle picture: $\boxed{F(\vec{x}_i) = \sum_m W(\vec{x}_m - \vec{x}_i)\, f_m}$



PM: Pros and Cons

- Fast
- Straight forward
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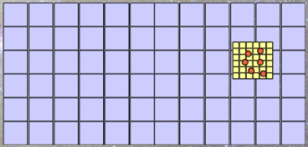
- Force resolution limited to mesh $\rightarrow$ missing adaptivity for large dynamic range
- Force errors anisotropic on the scale of cell size

Modifications: P^3M

* Supplement particle mesh with direct summation (details in a moment)
* Short range (scale of mesh cells)
* Larger dynamic range
* Slow with clustered of particles
* Straight forward to use with additional force term

Modifications: AP^{3}M

* Additional mesh refinement on clustered regions
* Avoid clustering slow down
* Complex because of interaction region
* Arbitrary to some degree in mesh placement
* Typically 2 initial fixed mesh layers
* Used e.g. for zoom simulations



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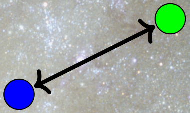
Direct sum / Tree based force calculation

- Calculate $\vec{F}_G$ between each particle pair i, j
- Exploit symmetry
- Scaling still $\mathcal{O}\left(N^2\right)$

$\Rightarrow$ Very expensive for large N



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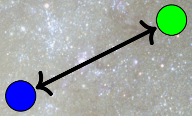

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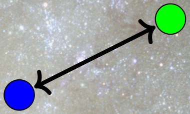
- Idea: Group distant particles together and consider them a bound blob (multipole expansion)

$\rightarrow$ Better scaling $\mathcal{O}\left(N \log N\right)$

Direct sum / Tree based force calculation



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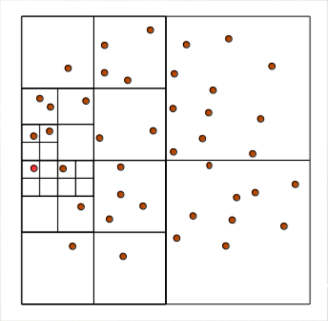

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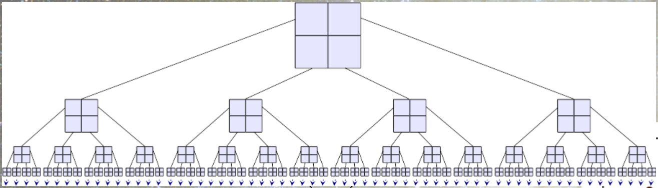

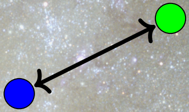

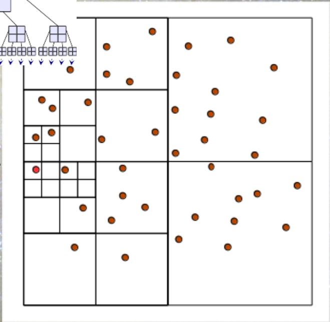

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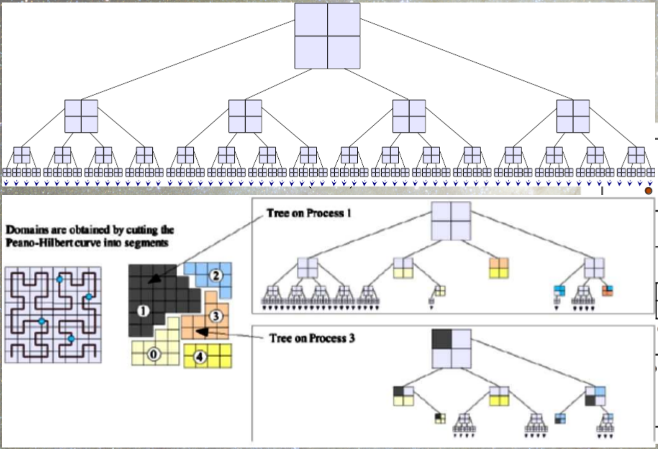

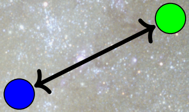

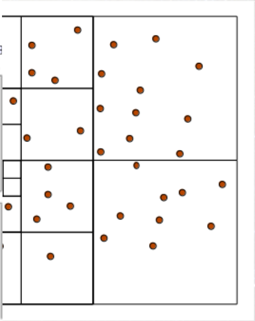

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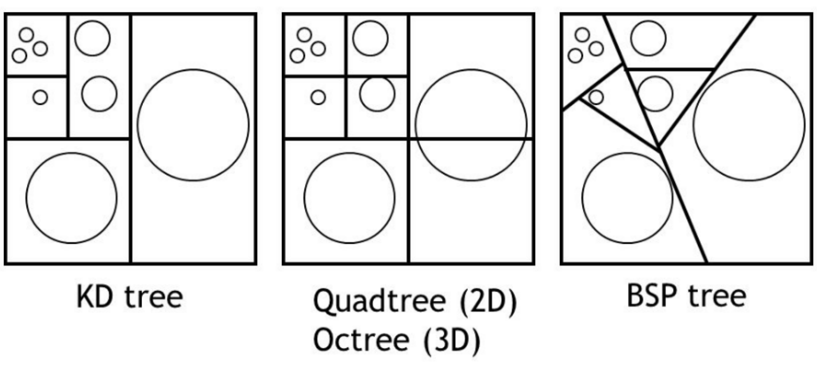
Different Tree types



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Multipole expansion of the potential

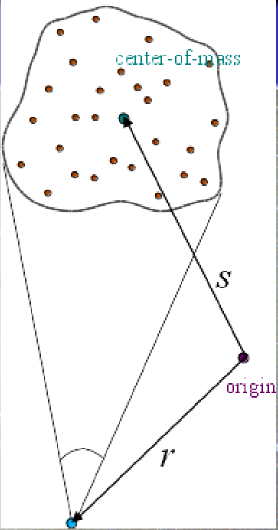
$$\Phi(\vec{x}) = -G\sum_i \frac{m_i}{|\vec{x}-\vec{x}_i|}$$

$$|\vec{x}-\vec{x}_i| = |(\vec{x}-\vec{s}) - (\vec{x}_i-\vec{s})|$$

$$=: |\vec{y} - (\vec{x}_i-\vec{s})| \text{ with } \vec{y} \gg (\vec{x}_i-\vec{s})$$

Dipole vanishes under $\sum_i$

$\Rightarrow$ monopole, quadrupole



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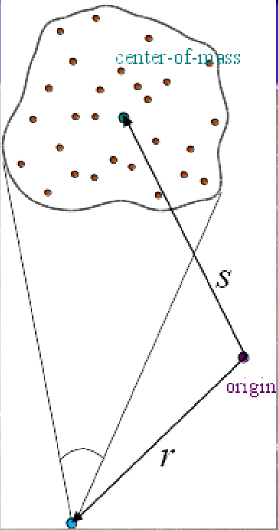

$\Phi(\vec{x}) = -G \sum_i \frac{m_i}{|\vec{x}-\vec{x}_i|}$

$|\vec{x} - \vec{x}_i| = |(\vec{x} - \vec{s}) - (\vec{x}_i - \vec{s})|$

$=: |\vec{y} - (\vec{x}_i - \vec{s})|$ with $\vec{y} \gg (\vec{x}_i - \vec{s})$

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Barnes & Hut 1986: Use cell if

(cell size > (distance particle $\leftrightarrow$ cell center) / opening angle)

Improvement: Use s and r

Resulting potential

$$\Rightarrow \Phi\left(\vec{x}\right) = \cdots = -G\left[\frac{M}{|\vec{y}|} + \frac{1}{2}\frac{\vec{y}^{\,T}\mathbf{Q}\vec{y}}{|\vec{y}|^5}\right]$$

* No intrinsic restrictions for dynamic range since adaptive
* Accuracy depends on the opening criterion and can be adjusted to a desired level
* Speed depends only weakly on clustering
* Flexible, different optimal tree structures depending on geometry

Merging approaches yet again: TreePM

* Split particles potential in Fourier space: long-range PM and short-range tree part:
* Poisson eq.: $\hat{\Phi}\left(\vec{k}\right) = -\frac{4\pi G}{\vec{k}^2 \rho(\vec{k})}$
* $\hat{\Phi}_{Long}\left(\vec{k}\right) \propto \exp\left(-A \cdot \vec{k}^2\right)$

  $\Rightarrow$ CIC, FFT, Overlap, FFT, Solver, Interpolate back to particles
* $\hat{\Phi}_{Short}\left(\vec{k}\right) \propto 1 - \exp\left(-A \cdot \vec{k}^2\right)$

  $\Rightarrow \Phi\left(\vec{x}\right) = -\frac{Gm}{r} \mathrm{erf}\left(\frac{|\vec{x}|}{2\sqrt{A}}\right)$ & Tree

Final GADGET approach

Three stages of solvers

(1) PM or APM for long range

(2) Tree for mid range

(3) Direct sum for short range

Final GADGET approach

Three stages of solvers

(1) PM or APM for long range

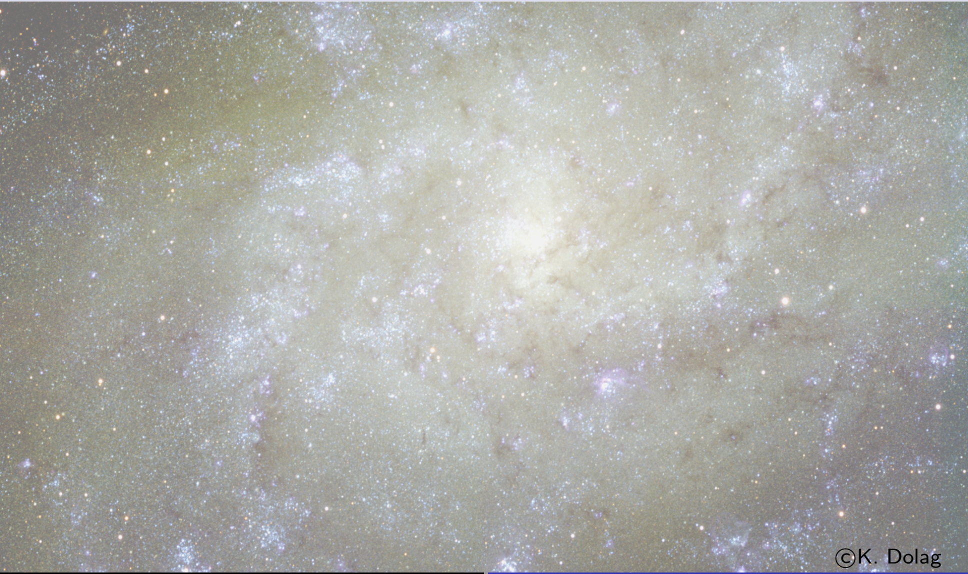
(2) Tree for mid range

(3) Direct sum for short range

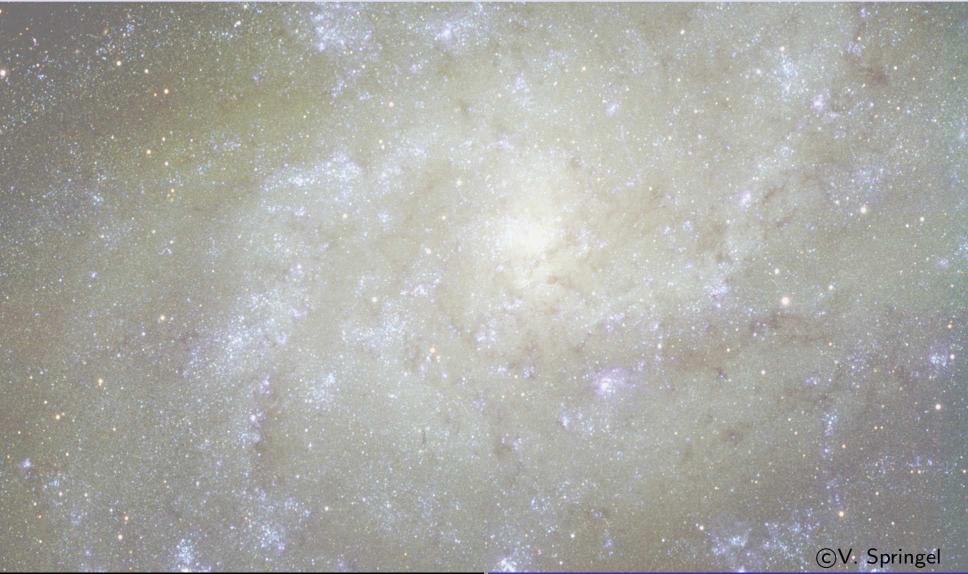
$\Rightarrow$ Trade-of between Accuracy and Computation Time

$\Rightarrow$ Complex to implement

Structure formation simulation



Millenium Simulation



Outline

1. What we do we want to solve?
2. Gravity: Solvers & Co.
3. A quick detour: Integrators
4. Is gravity enough?
5. Simulation types $< - >$ Initial conditions.

Numerical integration

Consider an ODE like $\dot{\vec{x}} = f(\vec{x})$. Many ways to solve this. For example:

* Explicit Euler: $\vec{x}_{n+1} = \vec{x}_n + f(\vec{x}_n)\,\Delta t$ (simple, straight forward, 1st order accurate)

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* Implicit Euler: $\vec{x}_{n+1} = \vec{x}_n + f(\vec{x}_{n+1})\,\Delta t$ (stable, complicated since implicit)
* Implicit Mid-Point: $\vec{x}_{n+1} = \vec{x}_n + f\left(\frac{\vec{x}_n + \vec{x}_{n+1}}{2}\right)\Delta t$ (2nd order accurate, symplectic, implicit)

Numerical integration

Consider an ODE like $\dot{\vec{x}} = f(\vec{x})$. Many ways to solve this. For example:

* Runge-Kutta (e.g. 4th order accurate):

$$\vec{k}_1 = f(\vec{x}_n, t_n)$$
$$\vec{k}_2 = f\left(\vec{x}_n + \vec{k}_1 \Delta t/2, t_n + \Delta t/2\right)$$
$$\vec{k}_3 = f\left(\vec{x}_n + \vec{k}_2 \Delta t/2, t_n + \Delta t/2\right)$$
$$\vec{k}_4 = f\left(\vec{x}_n + \vec{k}_3 \Delta t/2, t_n + \Delta t\right)$$
$$\vec{x}_{n+1} = \vec{x}_n + \frac{1}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right)\Delta t$$

Numerical integration

Consider an ODE like $\dot{\vec{x}} = f(\vec{x})$. Many ways to solve this. For example:

* Runge-Kutta (e.g. 4th order accurate):

$$\vec{k}_1 = f(\vec{x}_n, t_n)$$
$$\vec{k}_2 = f\left(\vec{x}_n + \vec{k}_1 \Delta t/2, t_n + \Delta t/2\right)$$
$$\vec{k}_3 = f\left(\vec{x}_n + \vec{k}_2 \Delta t/2, t_n + \Delta t/2\right)$$
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$$\vec{x}_{n+1} = \vec{x}_n + \frac{1}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right)\Delta t$$

* Leapfrog (2nd order accurate, explicit and symplectic!)

Leap frog

We typically deal with a 2nd order ODE: $\ddot{\vec{x}} = f\left(\vec{x}\right)$

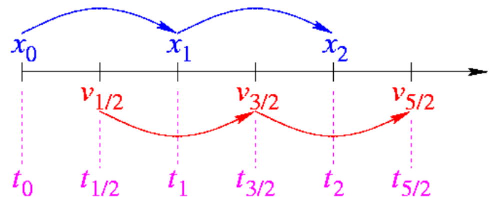
Drift-Kick-Drift

$$\vec{x}_{n+1/2} = \vec{x}_n + \vec{v}_n \Delta t/2$$

$$\vec{v}_{n+1} = \vec{v}_n + f\left(\vec{x}_{n+1/2}\right)\Delta t$$

$$\vec{x}_{n+1} = \vec{x}_{n+1/2} + \vec{v}_{n+1}\Delta t/2$$

Leap frog



We typically deal with a 2nd

Drift-Kick-Drift

$$\vec{x}_{n+1/2} = \vec{x}_n + \vec{v}_n \Delta t/2$$

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$$\vec{x}_{n+1} = \vec{x}_{n+1/2} + \vec{v}_{n+1} \Delta t/2$$

Kick-Drift-Kick

$$\vec{v}_{n+1/2} = \vec{v}_n + f\left(\vec{x}_n\right)\Delta t/2$$

$$\vec{x}_{n+1} = \vec{x}_n + \vec{v}_{n+1/2} \Delta t/2$$

$$\vec{v}_{n+1} = \vec{v}_{n+1/2} + f\left(\vec{x}_{n+1}\right)\Delta t/2$$

For deeper investigation see tutorial today!

Symplectic Integrators

Formally: Preserve Hamiltonian structure of the system by formulating each integration step as a canonical transformation.

$\Rightarrow$ Time evolution operator applied to the Hamiltonian.

Symplectic Integrators

Formally: Preserve Hamiltonian structure of the system by formulating each integration step as a canonical transformation.

$\Rightarrow$ Time evolution operator applied to the Hamiltonian.

Idea operator splitting: $H = H_{kin} + H_{pot} \left(+H_{num\ err}\right)$

Then drift and kick operators:

$$D\left(\Delta t\right) := \exp\left(\int_t^{t+\Delta t} dt\ H_{kin}\right)$$

$$K\left(\Delta t\right) := \exp\left(\int_t^{t+\Delta t} dt\ H_{pot}\right)$$

$\Rightarrow D\left(\Delta t/2\right) K\left(\Delta t\right) D\left(\Delta t/2\right)$ and $K\left(\Delta t/2\right) D\left(\Delta t\right) K\left(\Delta t/2\right)$

Timesteps

* Accuracy Vs Computational cost
* Courant-Friedrichs-Levy criterion for hydro codes (see next section):

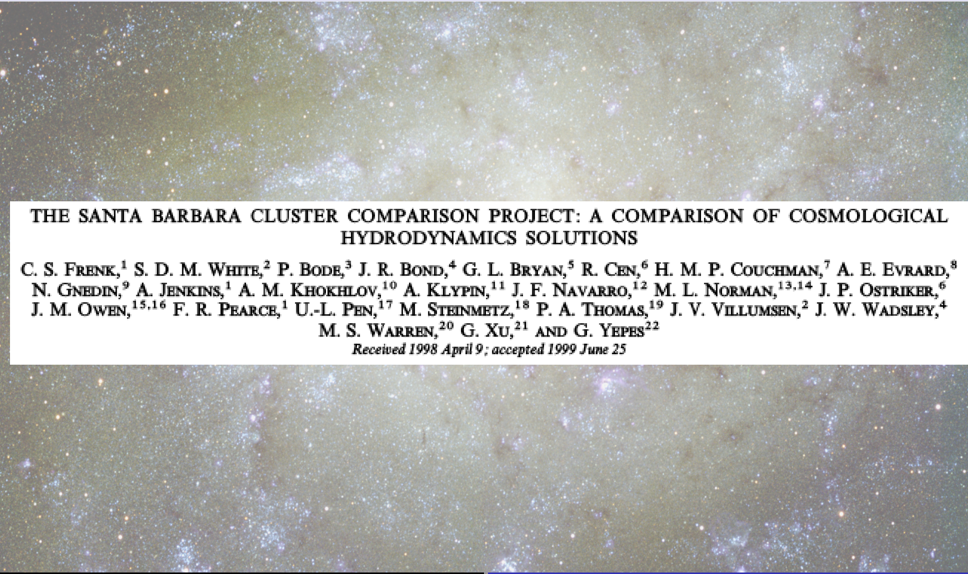
  $\Delta t = C_{CFL} \cdot \frac{l_{res}}{c_s}$ with $C_{CFL} \sim 0.1 - 0.3$

Timesteps

* Accuracy Vs Computational cost
* Courant-Friedrichs-Levy criterion for hydro codes (see next section):

  $\Delta t = C_{CFL} \cdot \frac{l_{res}}{c_s}$ with $C_{CFL} \sim 0.1 - 0.3$
* Idea: Individual timesteps
  - Accuracy where required
  - Complex: Interactions of active with inactive particles
  - Additional drifts required

DM only Code Comparison



THE SANTA BARBARA CLUSTER COMPARISON PROJECT: A COMPARISON OF COSMOLOGICAL HYDRODYNAMICS SOLUTIONS

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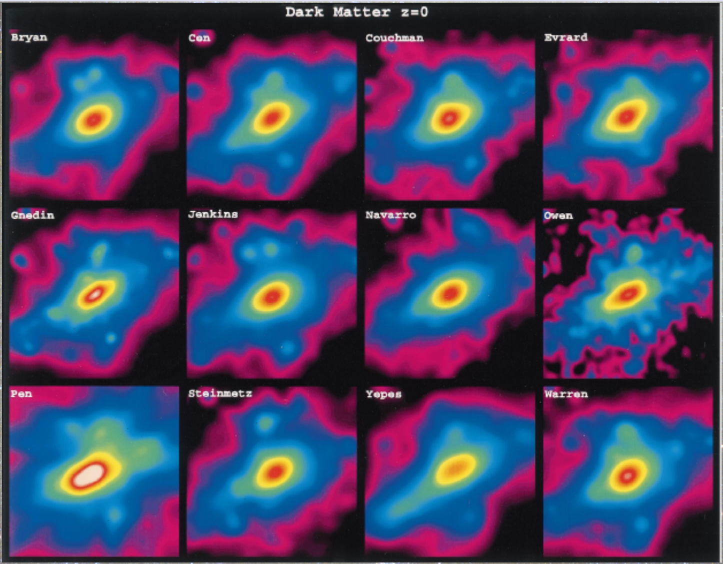

DM only Code Comparison

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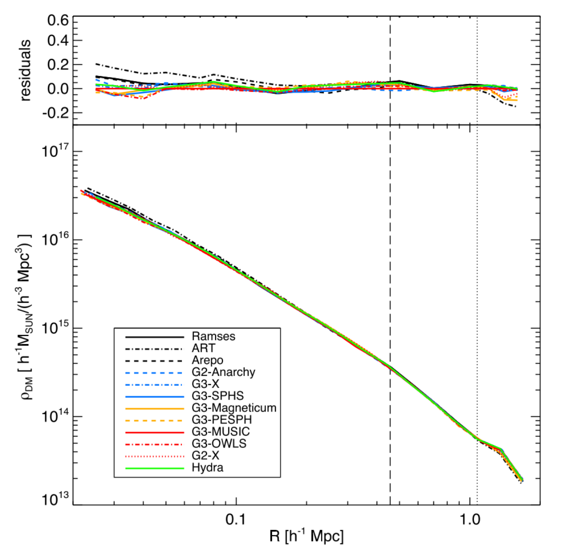
nIFTy galaxy cluster simulations – I. Dark matter and non-radiative models

Federico Sembolini,[1,2,3★] Gustavo Yepes,[1,2] Frazer R. Pearce,[4] Alexander Knebe,[1,2] Scott T. Kay,[5] Chris Power,[6] Weiguang Cui,[6] Alexander M. Beck,[7,8,9] Stefano Borgani,[10,11,12] Claudio Dalla Vecchia,[13,14] Romeel Davé,[15,16,17] Pascal Jahan Elahi,[18] Sean February,[19] Shuiyao Huang,[20] Alex Hobbs,[21] Neal Katz,[20] Erwin Lau,[22,23] Ian G. McCarthy,[24] Guiseppe Murante,[10] Daisuke Nagai,[22,23,25] Kaylea Nelson,[23,25] Richard D. A. Newton,[5,6] Valentin Perret,[26] Ewald Puchwein,[27] Justin I. Read,[28] Alexandro Saro,[7,29] Joop Schaye,[30] Romain Teyssier[26] and Robert J. Thacker[31]

DM only Code Comparison



Outline

1. What we do we want to solve?
2. Gravity: Solvers & Co.
3. A quick detour: Integrators
4. **Is gravity enough?**
5. Simulation types $< - >$ Initial conditions.

Is gravity enough?

Of course not!

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Of course not!

* Baryonic matter can collide, dissipate energy, clump, . . .
* Idea: Mainly H, He $\Rightarrow$ Hydrodynamics

Is gravity enough?

Of course not!

* Baryonic matter can collide, dissipate energy, clump, ...
* Idea: Mainly H, He $\Rightarrow$ Hydrodynamics
* Requirement: Mean free path λ_e small enough:
  - $\lambda_e \approx 22.5 \left(\frac{T_e}{10^8 K}\right)^2 \left(\frac{n_e}{10^{-3} cm^{-3}}\right)^{-1} kpc$ (Spitzer 1956)
  - Influence of magnetic fields (see tomorrow)

    Typical scales: $r_{g,e} = \frac{m_e c v}{eB}$ & $\frac{|\vec{B}|}{\nabla \cdot \vec{B}}$
  - $kpc \Rightarrow km$ scale

Reminder: Basics of Hydrodynamics

Euler: $\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}\Phi$

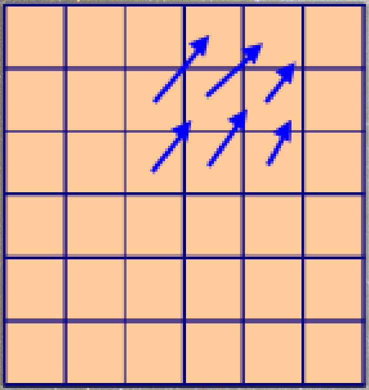
Continuity: $\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$

1st law t-d: $\frac{du}{dt} = -\frac{p}{\rho} \vec{\nabla} \cdot \vec{v} - \frac{\Lambda(u,\rho)}{\rho}$

Eq of state: $p = (\gamma - 1)\,\rho u$ (adiabatic $\gamma = \frac{5}{3}$)

Different Methods

Eulerian



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Different Methods

Eulerian

Discretize volume

Grid cells = volume elements

Solve fluxes, capture shocks natively

Not Galilean invariant

Mixing implicitly at cell level

Low numerical viscosity

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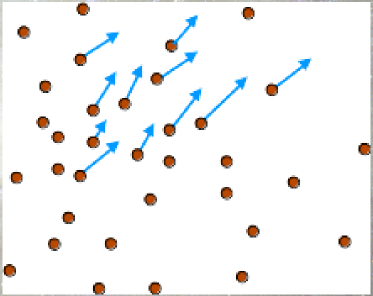

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Different Methods

Eulerian

Discretize volume

Grid cells = volume elements

Solve fluxes, capture shocks natively

Not Galilean invariant

Mixing implicitly at cell level

Low numerical viscosity

Lagrangian

Discretize mass

Particles = mass elements

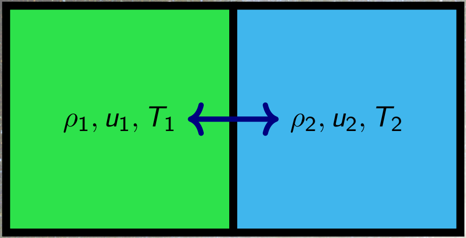
Inherent adaptivity through particle movement

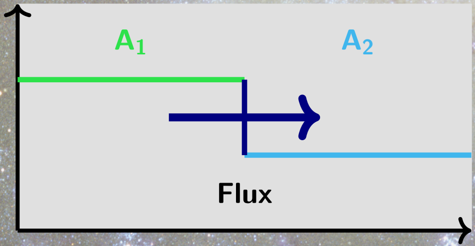
Galilean invariant

Mixing suppressed at particle level

Artificial viscosity, conductivity

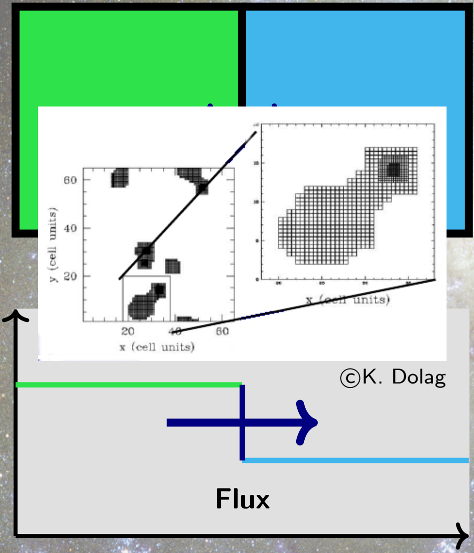
Eulerian in a nutshell





* Godunov method: solve fluxes through cell faces
* 1st order accurate scheme: Riemann problem
* Exact vs approximate Riemann solvers
* Typically finite volume scheme

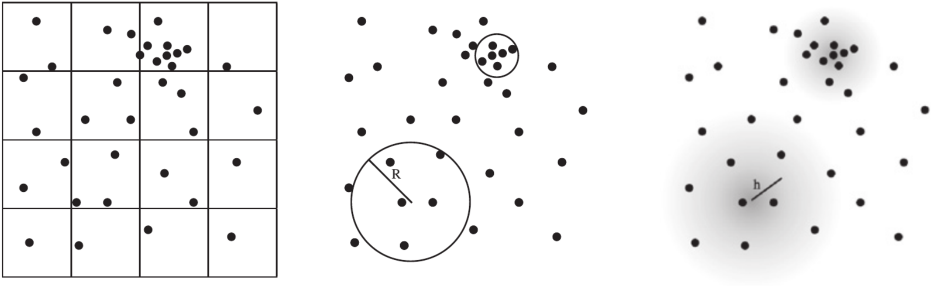
Eulerian in a nutshell



- Slope limitiers: positive, smooth solutions
- Typically 2nd order scheme
- Adaptive Mesh Refinement: Adaptivity but memory expensive
- Self Gravity difficult

SPH in a nutshell

Sample mass instead of volume → "Particles" instead of cells



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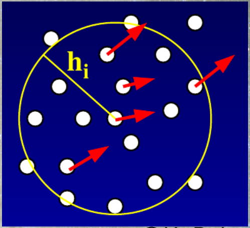
Fundamental quantity: $\boxed{\rho\left(\vec{x}\right) = \sum_{j}^{N_{ngb}} m_j W\left(\left|\vec{x} - \vec{x}_j\right|, h\right)}$

Kernel theory

$$\rho(\vec{x}) = \sum_{j}^{\boxed{N_{ngb}}} m_j \boxed{W\left(\left|\vec{x} - \vec{x}_j\right|, h\right)}$$



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* Remember Overlap function earlier
* Positive
* Monotonically decreasing
* Radial symmetry
* Central plateau
* Normalised
* Finite

Kernel theory

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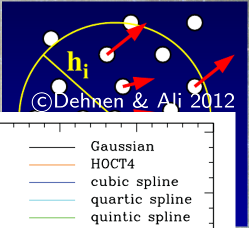

- Remember Overlap function e
- Positive
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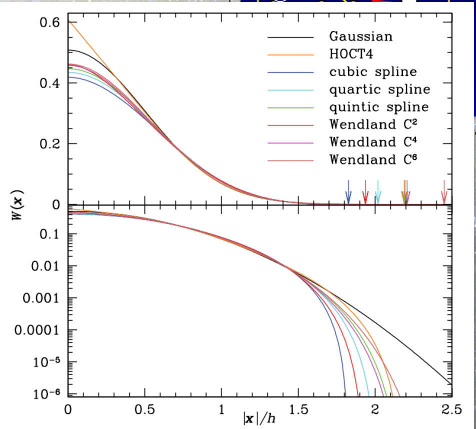

Discretization in general

$$\rho_i\left(\vec{x}\right) = \sum_{j}^{N_{ngb}} m_j W\left(\left|\vec{x} - \vec{x}_j\right|, h_i\right)$$

$$h_i = \eta\left(\frac{m_i}{\rho_i}\right)^{1/d}$$

Discretization in general

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$$h_i = \eta \left(\frac{m_i}{\rho_i}\right)^{1/d}$$

$$A_i \approx \sum_j^{N_{ngb}} mj \frac{A_j}{\rho_j} W_{ij}(h_i)$$

$$\mathcal{D}A_i \approx \sum_j^{N_{ngb}} mj \frac{A_j}{\rho_j} \mathcal{D}W_{ij}(h_i)$$

for a differential operator $\mathcal{D}$

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for a differential operator $\mathcal{D}$

Many modifications possible! E.g. subtracting error terms:

$$\vec{\nabla}A_i \approx \left\langle\vec{\nabla}A_i\right\rangle - A_i\left\langle\vec{\nabla}1\right\rangle = \sum_j^{N_{ngb}} \frac{mj}{\rho_j}\left(A_j - A_i\right)W_{ij}(h_i)$$

Equation of motion in SPH

- Can be derived directly from fluid Lagrangian:

  $$L = \frac{1}{2}\sum_i m_i \dot{\vec{x}}_i^2 - \sum_i m_i u_i$$

- $$\frac{d\vec{v}_i}{dt} = -\sum_j m_j \left(\frac{p_i}{\Omega_i \rho_i^2} \vec{\nabla}_i W_{ij}(h_i) + \frac{p_j}{\Omega_j \rho_j^2} \vec{\nabla}_i W_{ij}(h_j) \right)$$

  with variable smoothing lengthes h:

  $$\Omega_i = 1 - \frac{\partial h_i}{\partial \rho_i} \sum_j^{N_{ngb}} m_j \frac{\partial W_{ij}(h_i)}{\partial h_i}$$

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  with variable smoothing lengthes h:

$$\Omega_i = 1 - \frac{\partial h_i}{\partial \rho_i} \sum_j^{N_{ngb}} m_j \frac{\partial W_{ij}(h_i)}{\partial h_i}$$

* Equation of state: $p_i = (\gamma - 1)\, \rho_i u_i$
* Classical description in terms of density and "entropy":

$$A(S) = \frac{P}{\rho^\gamma} = (\gamma - 1) \frac{u}{\rho^{\gamma-1}}$$

Artificial Viscosity (Beck, Arth et al. 2016)

Ideal Euler eq. $\rightarrow$ no dissipative terms $\rightarrow$ problems at discontinuities e.g. shocks

Remove post-shock oscillations & noise, smooth velocity field

Energy conserving

Artificial Viscosity (Beck, Arth et al. 2016)

Ideal Euler eq. $\rightarrow$ no dissipative terms $\rightarrow$ problems at discontinuities e.g. shocks

Remove post-shock oscillations & noise, smooth velocity field

Energy conserving

$$\left.\frac{d\mathbf{v}_i}{dt}\right|_{\mathrm{visc}} = \frac{1}{2}\sum_j \frac{m_j}{\rho_{ij}} \left(\mathbf{v}_j - \mathbf{v}_i\right) \alpha_{ij}^{v} f_{ij}^{\mathrm{shear}} v_{ij}^{\mathrm{sig,v}} \overline{F}_{ij}$$

$$\left.\frac{du_i}{dt}\right|_{\mathrm{visc}} = -\frac{1}{2}\sum_j \frac{m_j}{\rho_{ij}} \left(\mathbf{v}_j - \mathbf{v}_i\right)^2 \alpha_{ij}^{v} f_{ij}^{\mathrm{shear}} v_{ij}^{\mathrm{sig,v}} \overline{F}_{ij}$$

Shear flow limiter $f_i^{\mathrm{shear}} = \frac{|\nabla\cdot\mathbf{v}|_i}{|\nabla\cdot\mathbf{v}|_i + |\nabla\times\mathbf{v}|_i + \sigma_i}$

Kernel gradient $\vec{\nabla}_i W_{ij}\left(h_i\right) = F_{ij}\hat{r}_{ij}$

Artificial Conductivity (Beck, Arth et al. 2016)

SPH does not mix energy at particle level

Discontinuities in internal energy u

Required in density-entropy formalism, less e.g. in pressure-entropy

Artificial Conductivity (Beck, Arth et al. 2016)

SPH does not mix energy at particle level

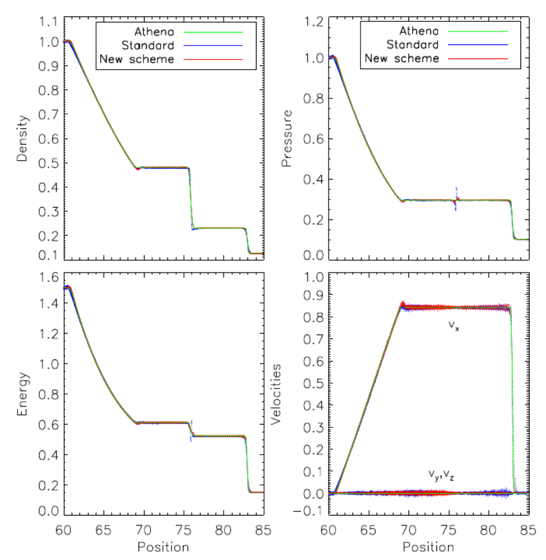
Discontinuities in internal energy u

Required in density-entropy formalism, less e.g. in pressure-entropy

$$\left.\frac{du_i}{dt}\right|_{\mathrm{cond}} = \sum_j \frac{m_j}{\rho_{ij}}(u_j - u_i)\alpha^c_{ij} v^{\mathrm{sig,c}}_{ij} \overline{F}_{ij}$$

Coefficient $\alpha^{\mathrm{c}}_i = \frac{h_i}{3}\frac{|\nabla u|_i}{|u_i|}$

Modern SPH (Beck, Arth et al. 2016)



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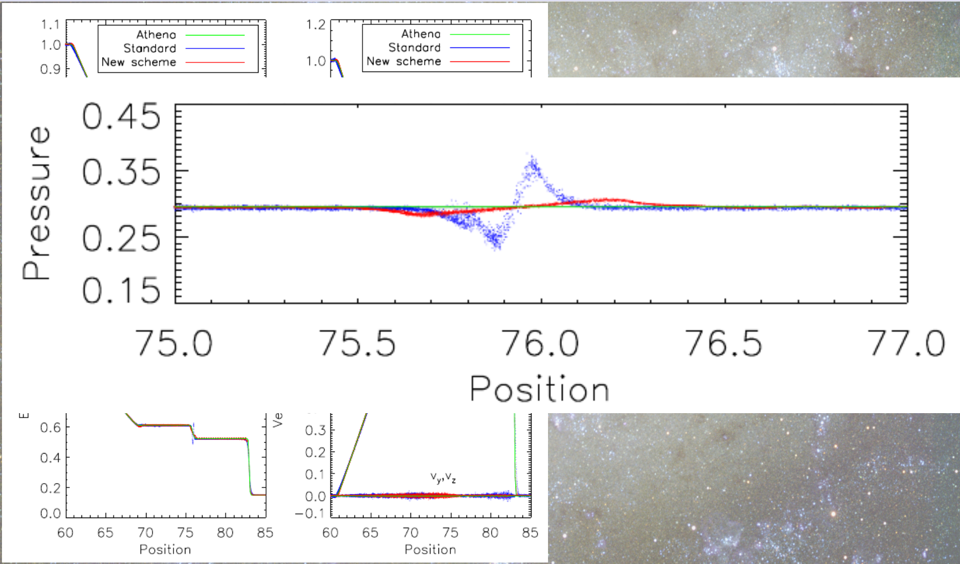

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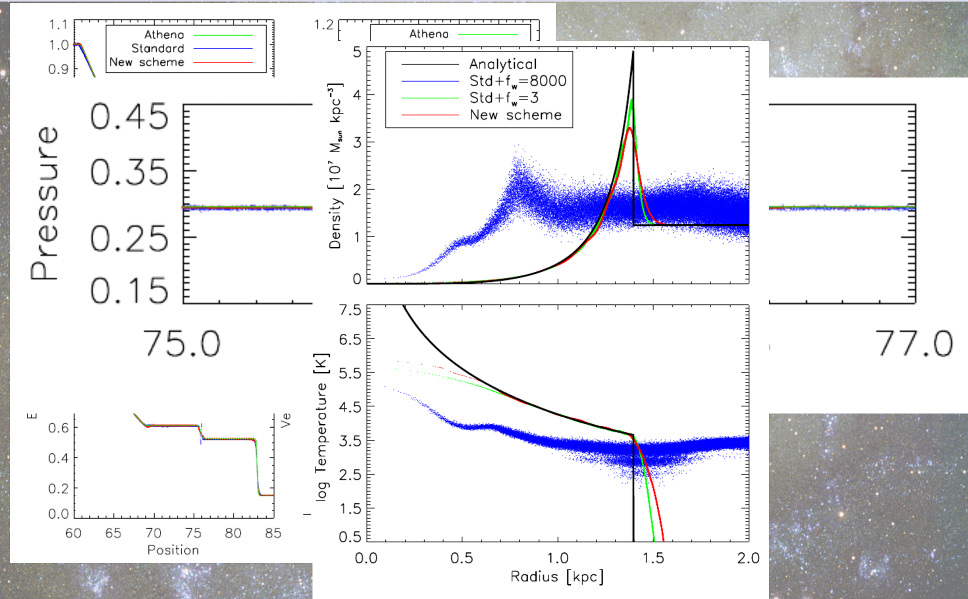

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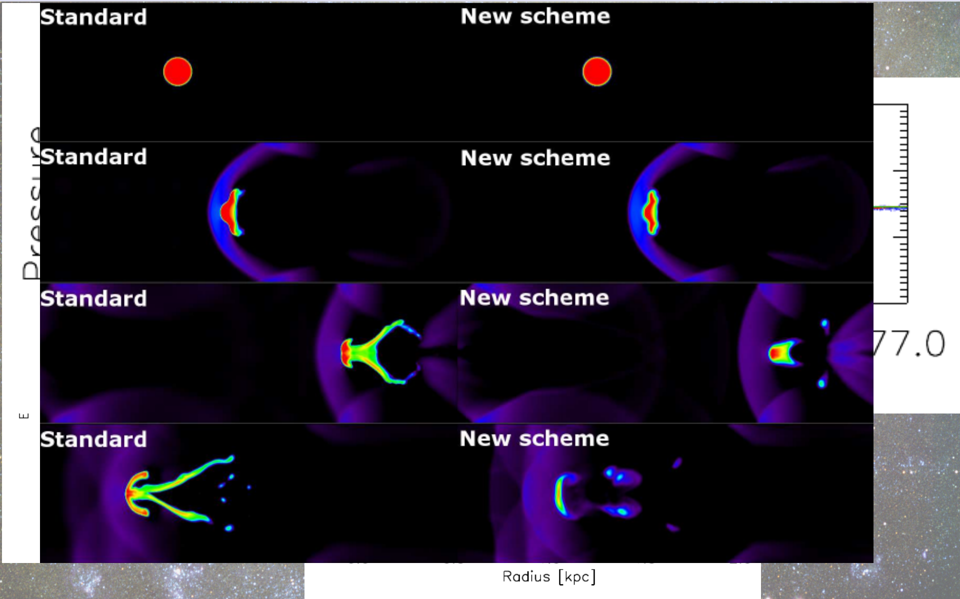

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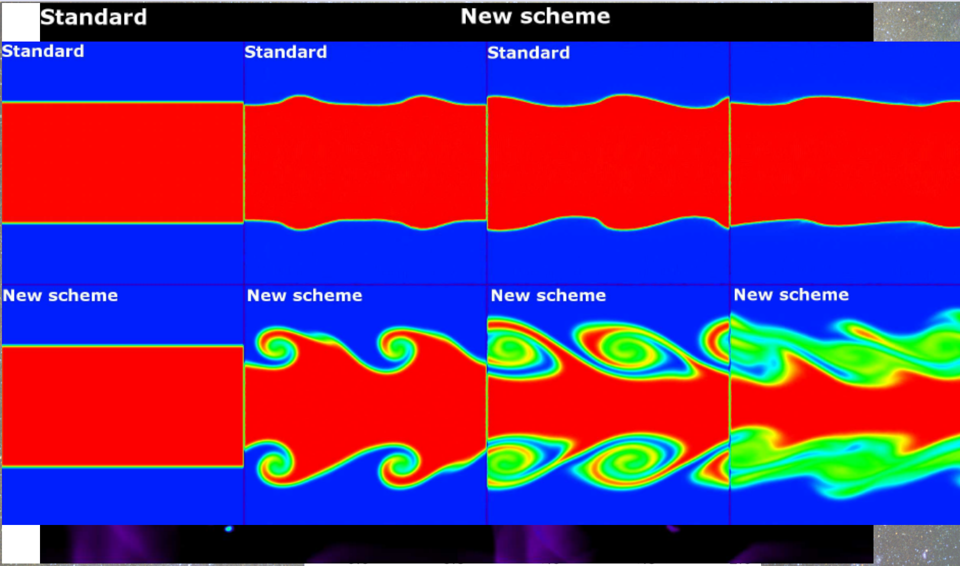
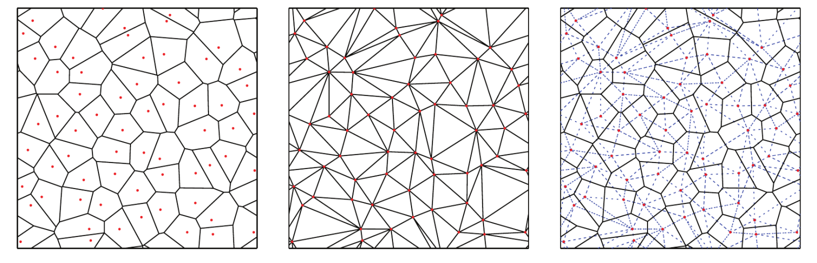
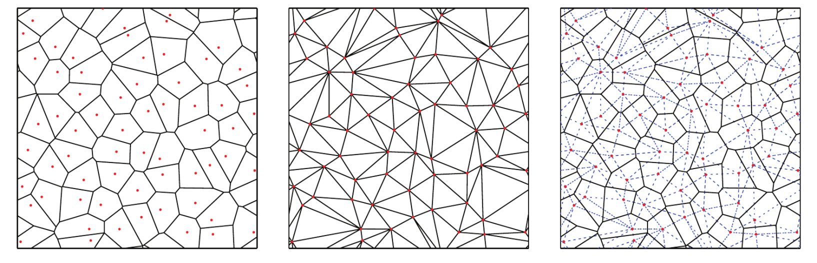


"Modern" approaches: Moving Mesh



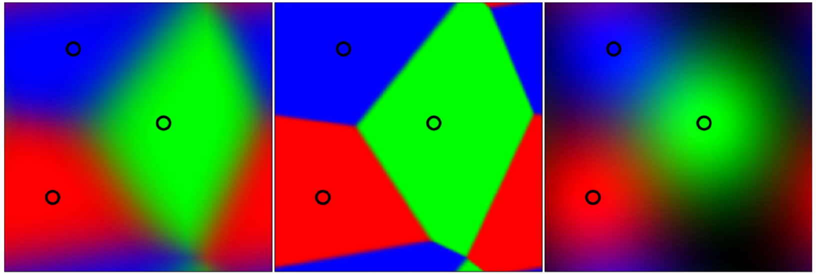
* Arbitrary Lagrangian Eulerian
* Sample fluid with mass points
* Create non regular mesh around particles using Voronoi tessellation / Delauny triangulation
* Solve Riemann problem across cell faces similar to grid code

"Modern" approaches: Moving Mesh



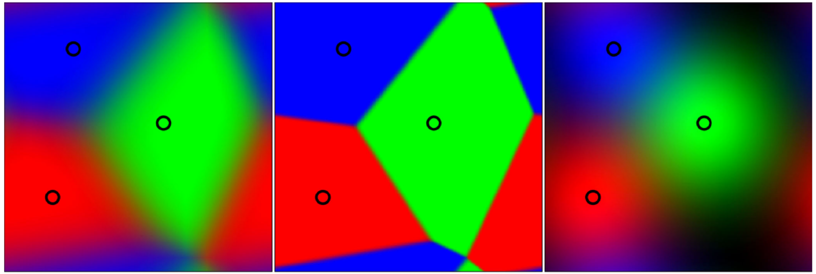
* Let particles move and thereby mesh deform
* Repair / Recreate mesh
* See e.g. Springel 2010

"Modern" approaches: Meshless Finite Mass/Volume



* Sample fluid with mass points
* Partition volume around them using an SPH-like weighting for a smooth transition

"Modern" approaches: Meshless Finite Mass/Volume



- ∗ Solve the Riemann problem with fixed "cells": MFV method
- ∗ Distort Lagrangian volume to keep mass constant: MFM method
- ∗ See e.g. Hopkins 2015

Code Comparisons

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Received 1998 April 9; accepted 1999 June 25

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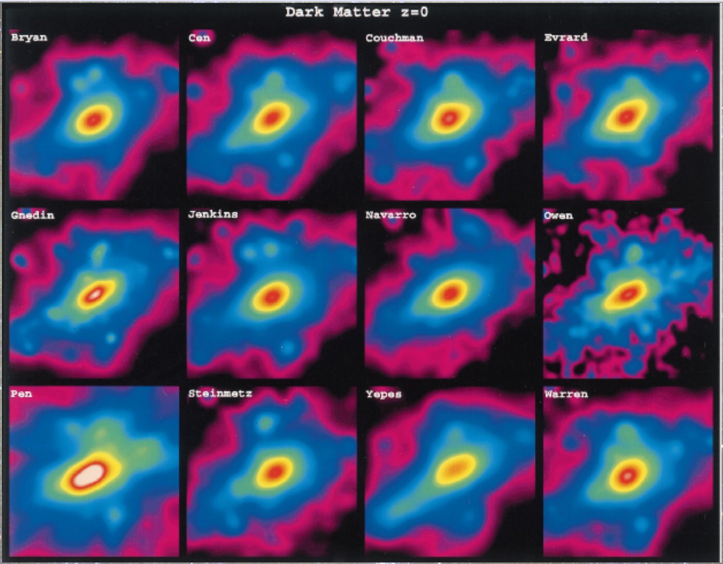
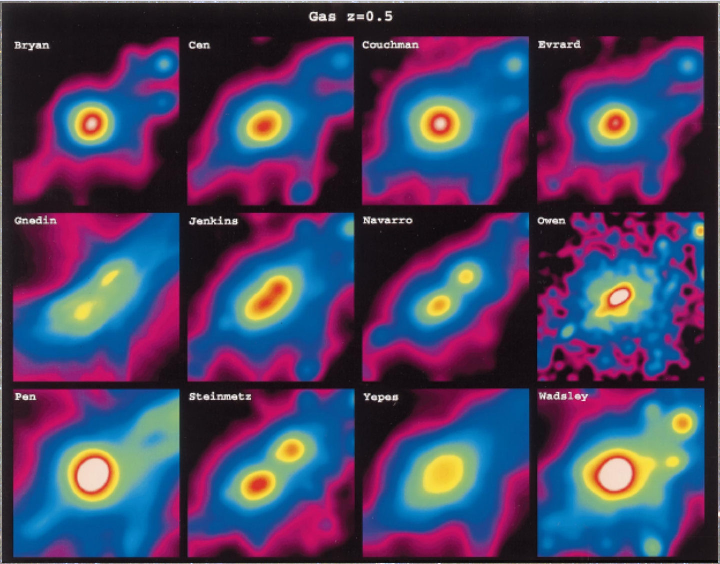


Code Comparisons



Code Comparisons

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nIFTy galaxy cluster simulations – I. Dark matter and non-radiative models

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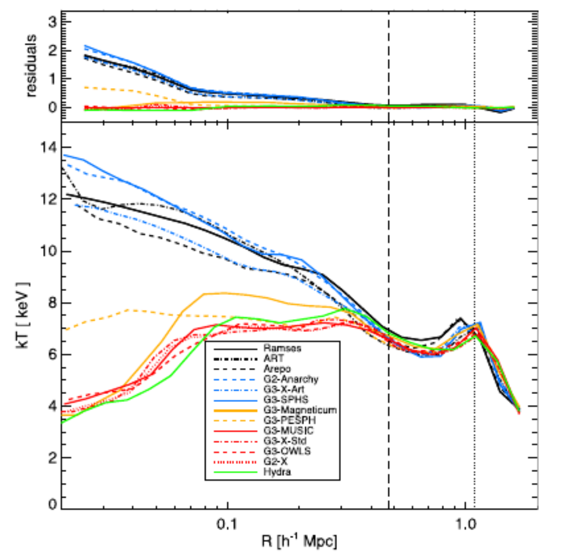
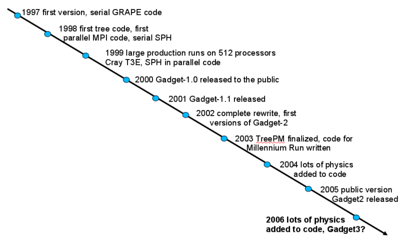
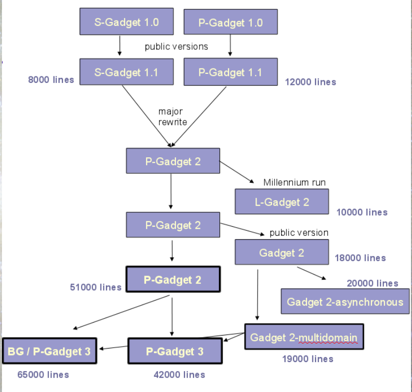
Code Comparisons



Gadget timeline



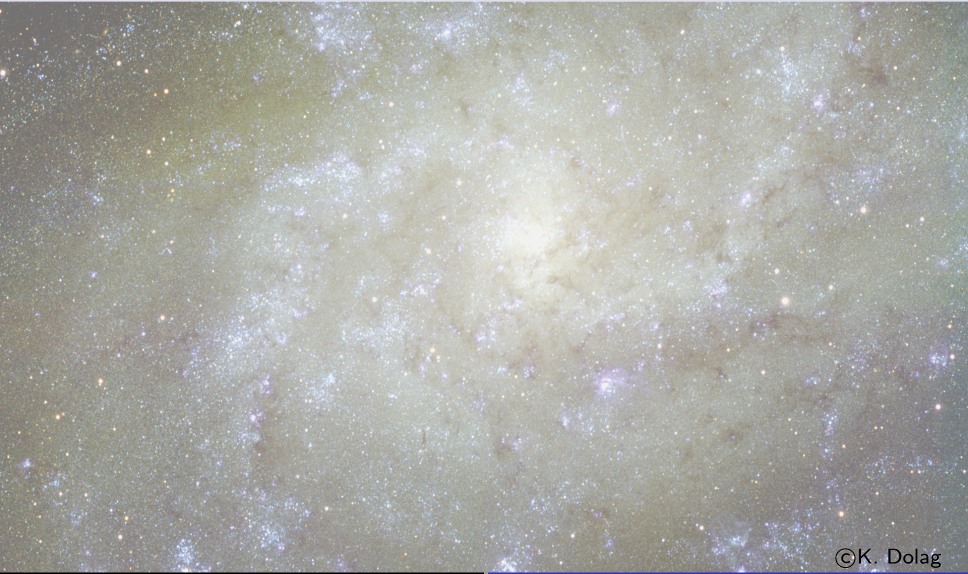


> 200k lines now (©V. Springel)

Gadget features

* Symplectic integration
* Hybrid gravity solver
* Conservative SPH
* Modular
* A lot of subgrid physics
* Different output styles including HDF5
* Hybrid parallelization OpenMP / MPI
* Only fftw and gsl required
* Built in group and halo finder (FoF and Subfind)

Structure formation simulation with gas



Outline

1. What we do we want to solve?
2. Gravity: Solvers & Co.
3. A quick detour: Integrators
4. Is gravity enough?
5. Simulation types $<->$ Initial conditions.

Technical aspects

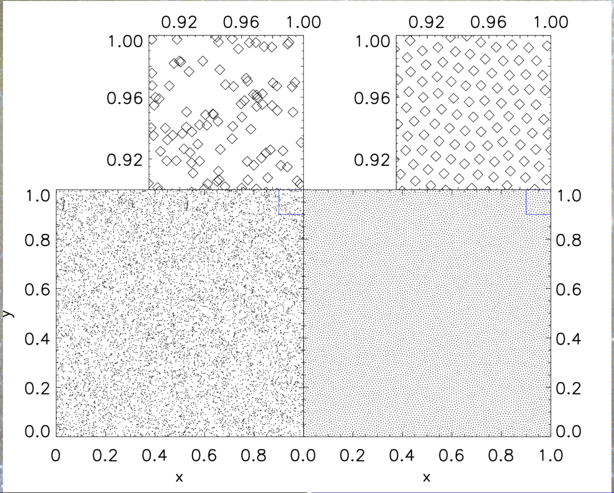
* No simulation without proper initial conditions!
* Need $\rho\,(\vec{x})$, $u\,(\vec{x})$, $\vec{v}\,(\vec{x})$, ...

Technical aspects

* No simulation without proper initial conditions!
* Need $\rho(\vec{x})$, $u(\vec{x})$, $\vec{v}(\vec{x})$, ...
* Easy to translate into a volume discretization, ...
* Mass discretization not so much: Particle configuration needs to resemble $\rho(\vec{x})$
* Adjust particle mass or distribution (or both)

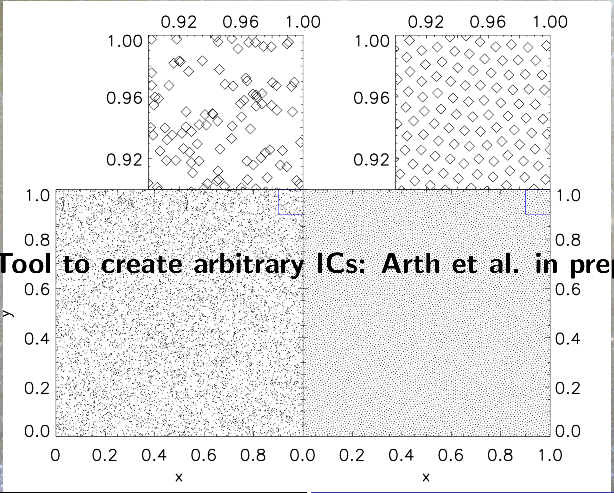
Typical particle configurations

Grid VS Random VS Glass



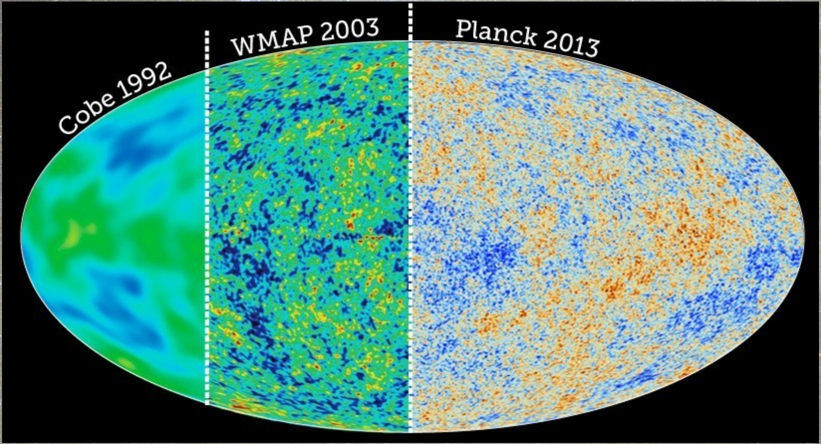
Typical particle configurations

Grid VS Random VS Glass



Tool to create arbitrary ICs: Arth et al. in prep.

Density fluctuations

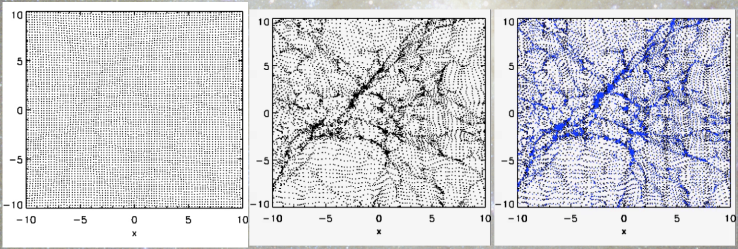


https://briankoberlein.com

Cosmic initial conditions

Gaussian density perturbation

Formation of cosmic structures like voids, filaments and collapsed objects



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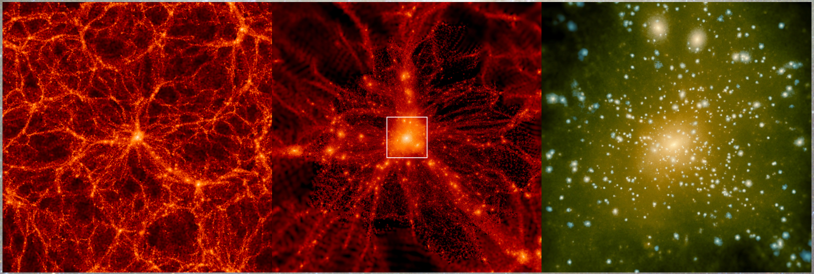
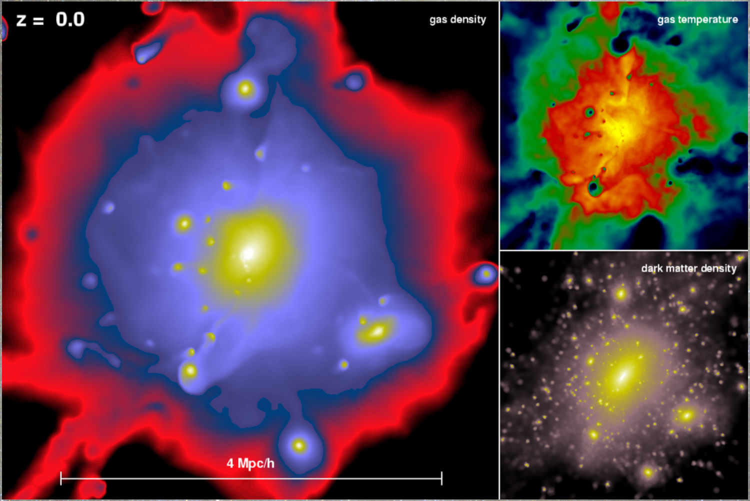
Zoom simulations

- Parent large scale box
- Re-simulation with higher resolution (factor 100-1000 in mass resolution)
- Study internal structures in zoomed region

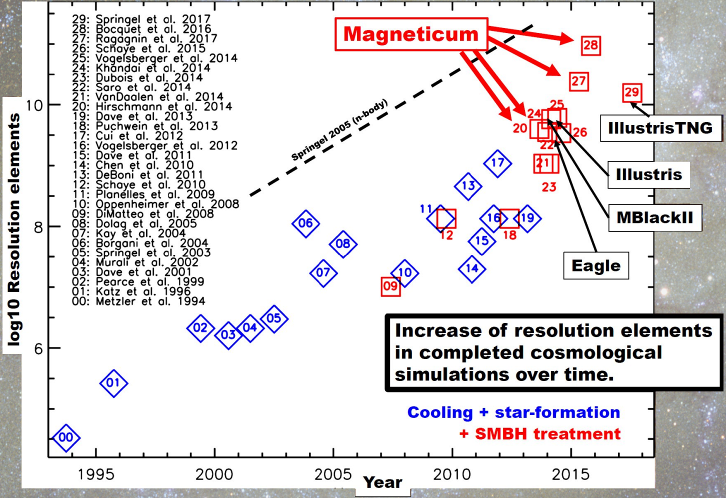


©Springel et al. 2001

Adiabatic gas dynamics



Moore's law: double every 18 months



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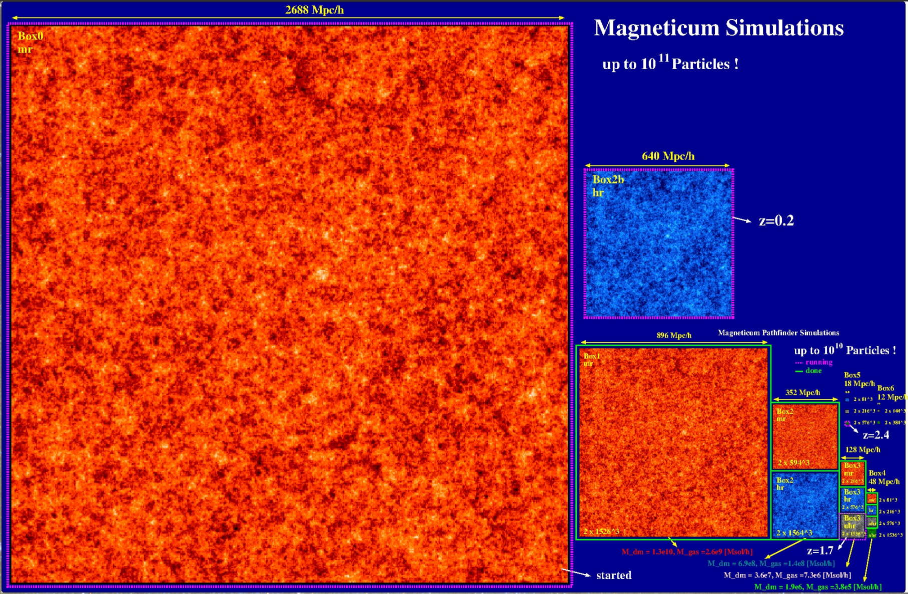
2688 Mpc/h
Box0
mr
Magneticum Simulations
up to 10^{11} Particles !
640 Mpc/h
Box2b
hr
z=0.2
896 Mpc/h
Magneticum Pathfinder Simulations
up to 10^{10} Particles !
running
done
Box1
mr
2 x 1526^3
352 Mpc/h
Box2
mr
2 x 594^3
Box2
hr
2 x 1564^3
Box5
18 Mpc/h
Box6
12 Mpc/h
z=2.4
128 Mpc/h
Box3
mr
Box3
hr
Box3
uhr
Box4
48 Mpc/h
M_dm = 1.3e10, M_gas =2.6e9 [Msol/h]
M_dm = 6.9e8, M_gas =1.4e8 [Msol/h]
z=1.7
M_dm = 3.6e7, M_gas =7.3e6 [Msol/h]
M_dm = 1.9e6, M_gas =3.8e5 [Msol/h]
started

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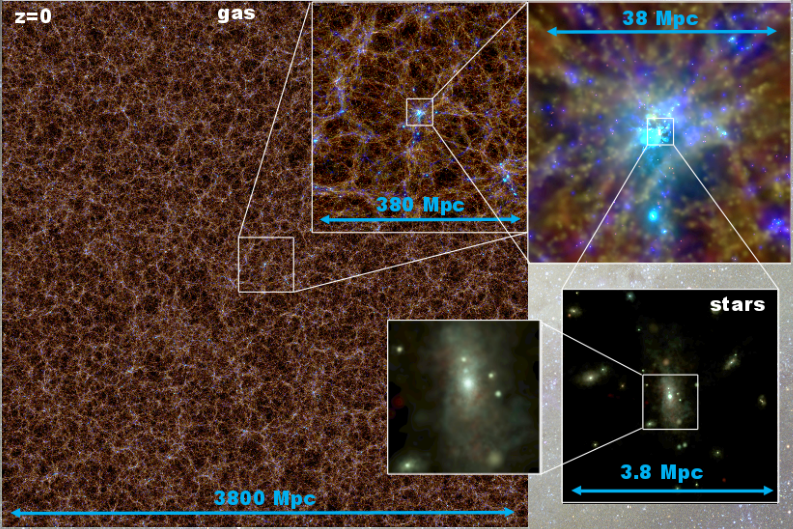
Largest Simulation (Box0/mr)



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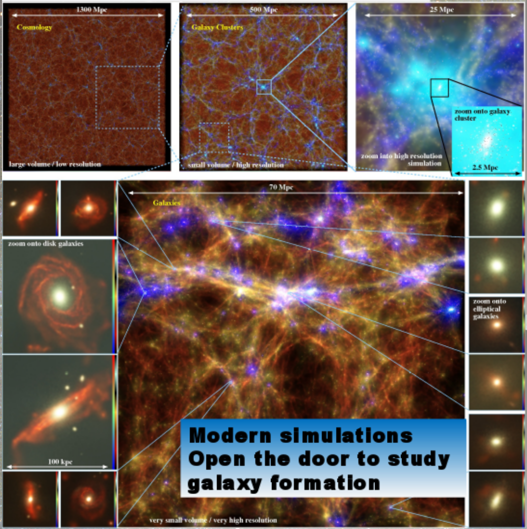
And the rest ...



Physics:

cooling+sfr+winds
Springel & Hernquist 2002/2003

Metals cooling
Wiersma et al. 2009

SNIa,SNII,AGB
Tornatore et al. 2003/2006

BH+AGN feedback
Springel & Di Matteo 2006
Fabjan et al. 2010
Hirschmann et al. 2014 (std)
Steinborn et al. 2015 (new)

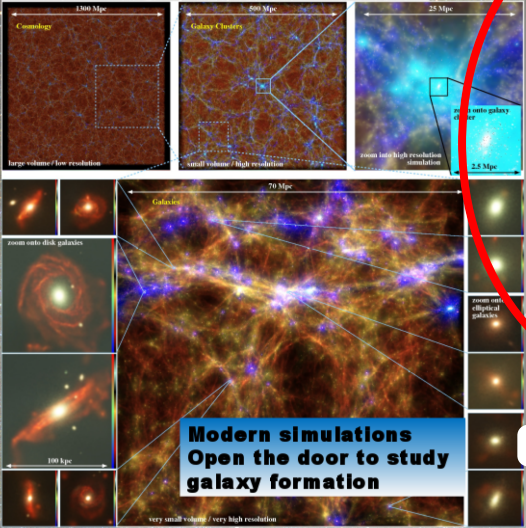
Thermal conduction
1/20th Spitzer
Dolag et al. 2004

Numerics:

New Kernels: WC6
Dehnen et al. 2012

Low visc. scheme
mr/hr (time dep. alpha)
Dolag et al. 2005
uhr (high order grad.)
Beck et al. 2015

And the rest ...



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Next lecture...

Dolag et al. 2005
uhr (high order grad.)
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Sources

* Lecture of Volker Springel
* Lectures of Klaus Dolag
* The Encyclopedia of Cosmology
* My PhD thesis ☺
* Several papers as mentioned ...

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Now, break and tutorials!