Random Matrix Theory and applications in cosmology

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School Analytics, Inference, and Computation in Cosmology: Advanced methods

Outline

1. *Introduction*
2. *The Gaussian Ensemble(s)*
3. *The Coulomb gas approach*
4. *The resolvent approach*
5. *High-energy landscape*

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1. ***Introduction***

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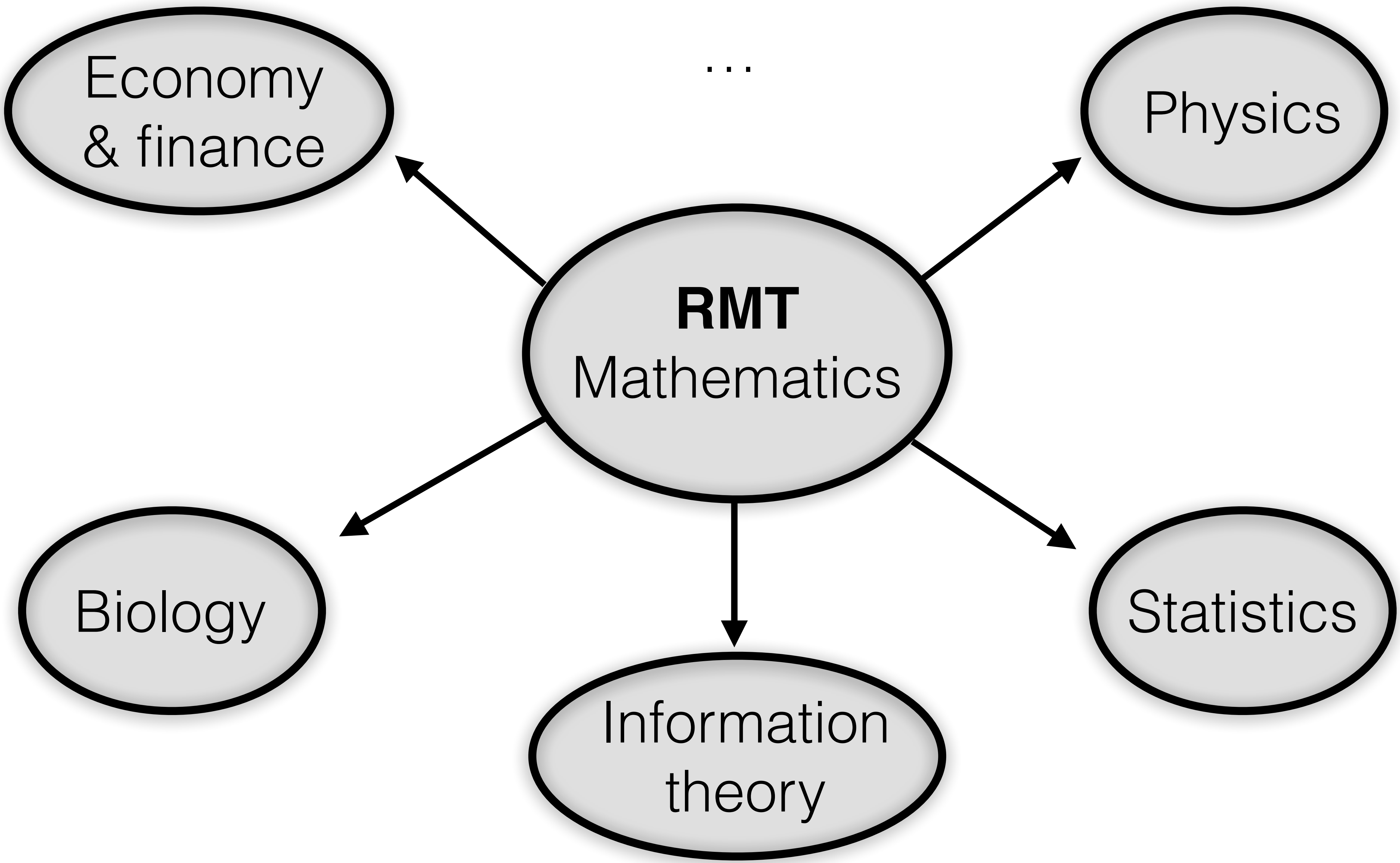
Random Matrix Theory and applications



« The Oxford handbook of random matrix theory »

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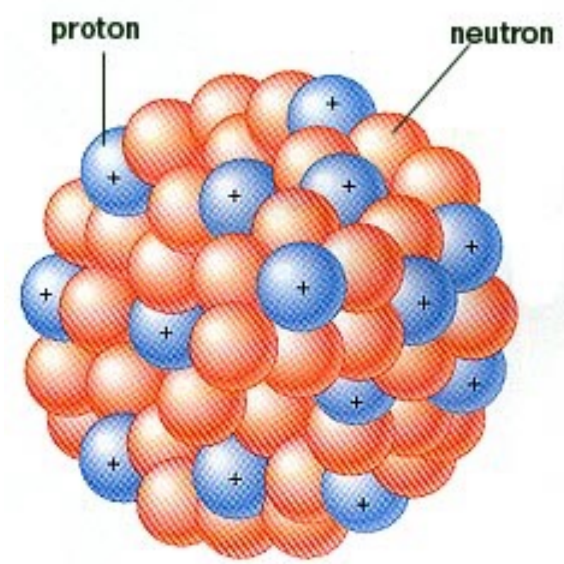
Wigner's surmise

Heavy nucleus



Hard to compute from first principles!

Q: 1956: possible shape of distribution of the spacing of energy levels?

A: Wigner: $$p(s) = \frac{s}{2} e^{-\frac{1}{4}s^2}$$

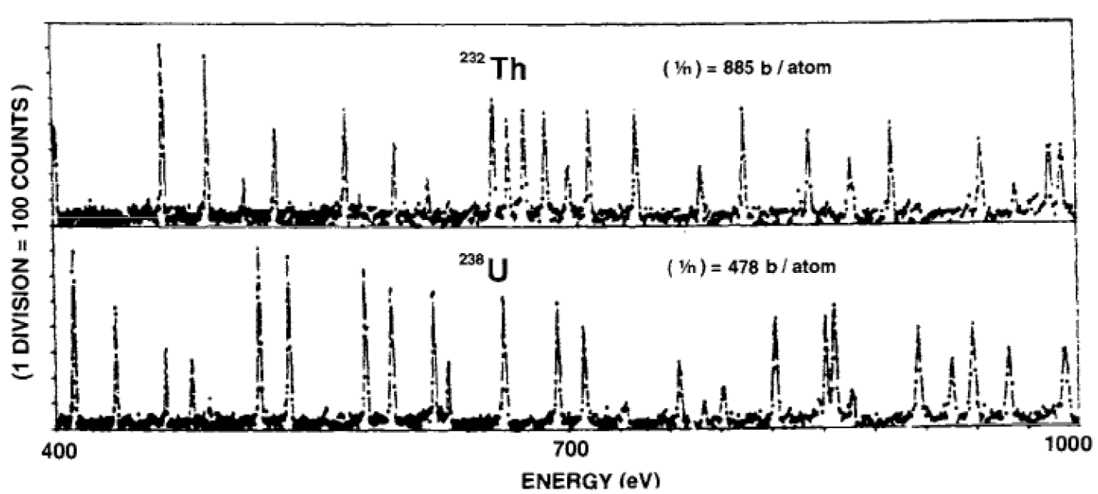
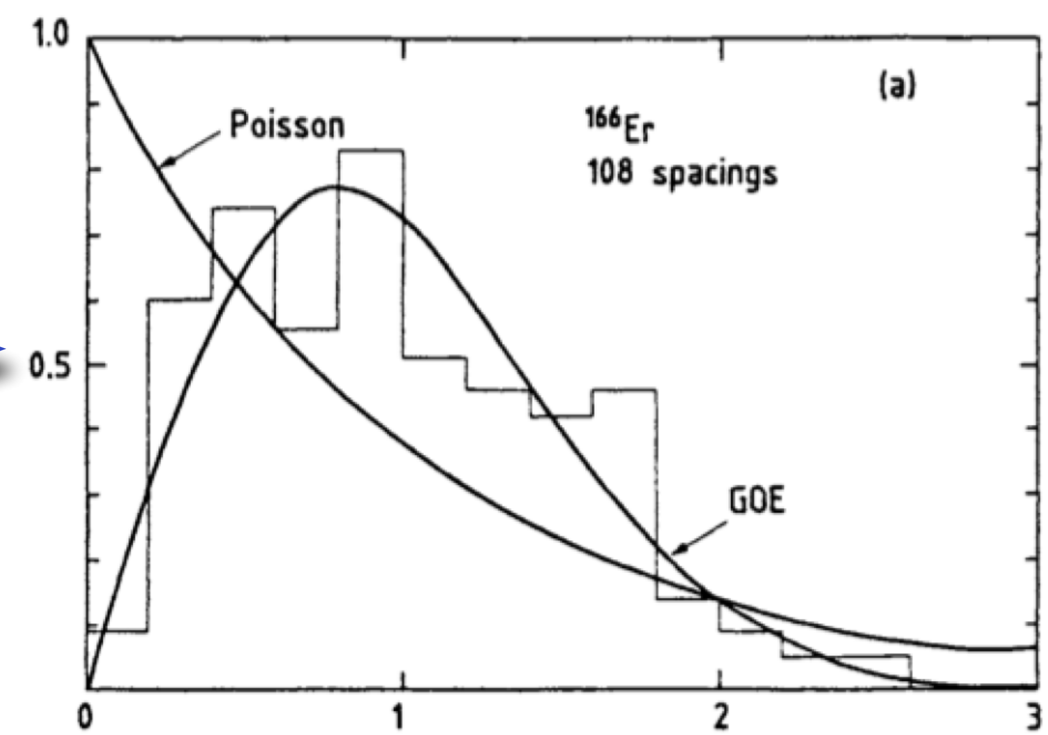


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).



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Wigner and Dyson's idea

Energy levels = eigenvalues of Hamiltonian

Hamiltonian: large complicated Hermitian matrix

Let us model it as a random matrix!

> to develop a "new kind of statistical mechanics in which we renounce exact knowledge not of the state of a system but of the nature of the system itself. We picture a complex nucleus as a "black box" in which a large number of particles are interacting according to unknown laws. The problem is then to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally probable".

Wigner's semicircle law

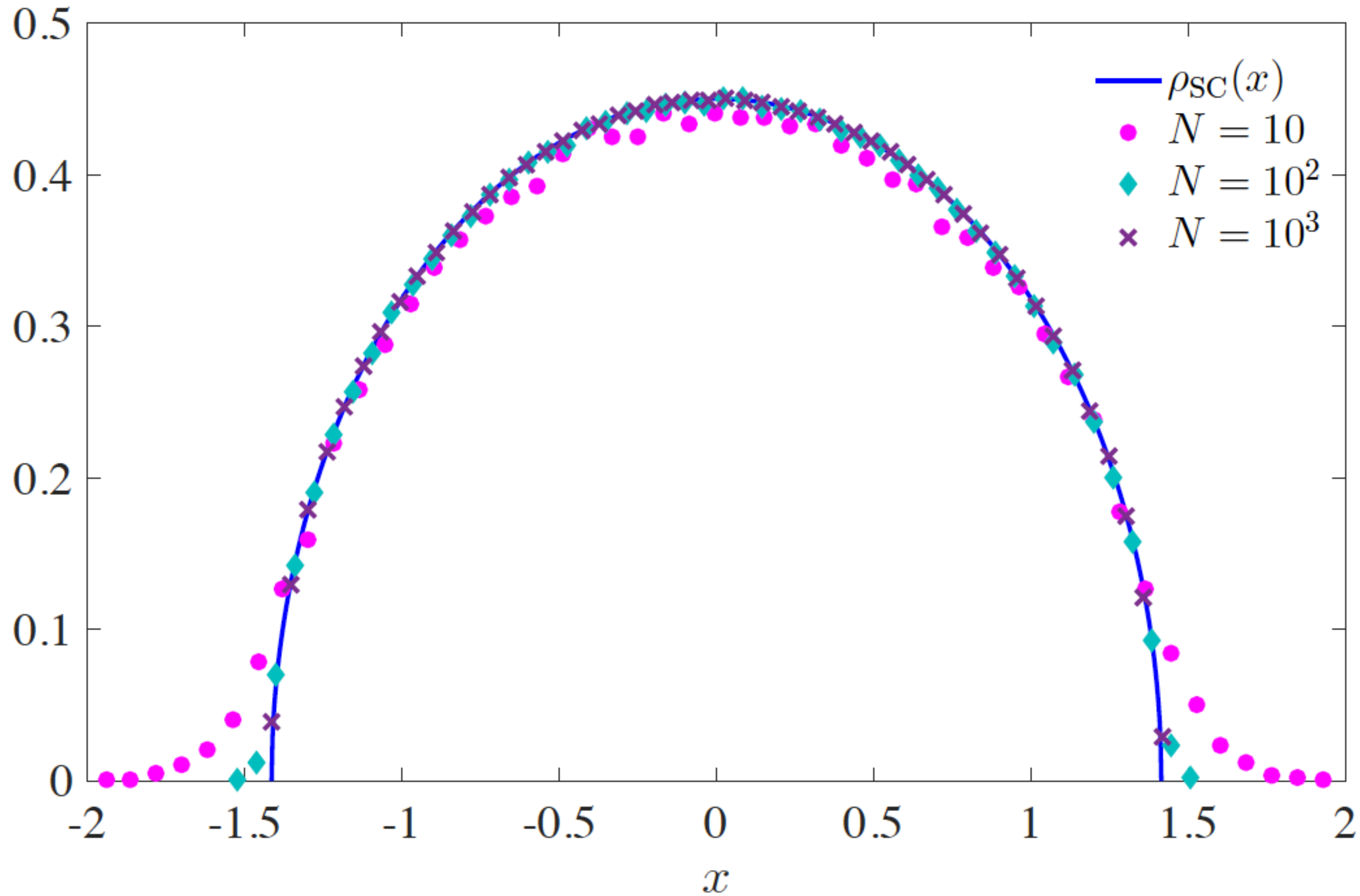

Figure 5.1: Numerical check of the semicircle law for GOE. Increasing the value of N, after a suitable rescaling, the eigenvalue histograms collapse on top of the semicircle curve.

From math-ph:1712.07903

« Introduction to Random Matrices - Theory and Practice »

Universality

Universal behaviors emerge independently of details of distributions of entries

Take real symmetric matrix $H_{ij} : N \times N$

- independent entries $H_{ij}, i \leq j$
- $\langle H_{ij} \rangle = 0$
- entries decay sufficiently fast at infinity
- $\forall i, \sum_{j=1}^{N} \langle H_{ij}^2 \rangle = \frac{1}{2}$ (normalization)

→ $$\rho_N(x) \xrightarrow[N \to \infty]{} \rho_{\mathrm{SC}}(x) = \frac{1}{\pi} \sqrt{2 - x^2}$$

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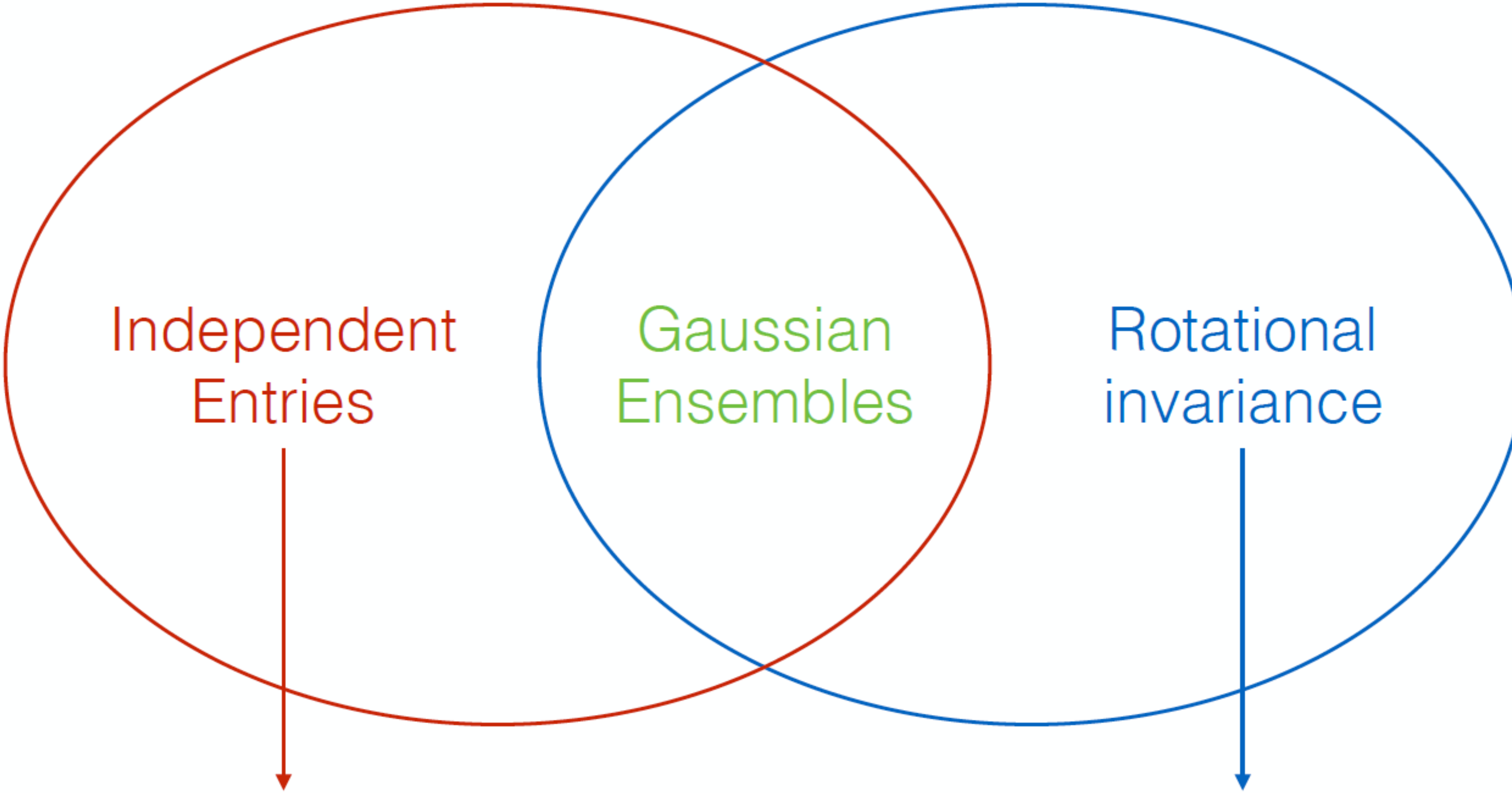
Classification



$$\rho[H] \propto \prod_{i=1}^{N} f_i(H_{ii}) \prod_{i<j} f_{ij}(H_{ij})$$

$$\rho[H] = \rho[UHU^{-1}]$$

Outline

Gaussian ensemble: construction

Simplest starting point:

$$\rho[M] \equiv \rho(M_{11}, \ldots, M_{NN}) = \prod_{i,j=1}^{N} \left[\frac{1}{\sqrt{2\pi}} \exp\left(-M_{ij}^2/2\right) \right]$$

Symmetrization: $H = \frac{1}{2}(M + M^T)$

$$\rho(H) = \prod_{i=1}^{N} \left[\frac{\exp\left(-(H)_{ii}^2/2\right)}{\sqrt{2\pi}} \right] \prod_{i<j} \left[\frac{\exp\left(-(H)_{ij}^2\right)}{\sqrt{\pi}} \right]$$

Independent entries

$$\rho(H) \propto \exp^{-\frac{1}{2}\mathrm{Tr}(H^2)}$$

Rotational invariance

Gaussian ensemble: pdf of eigenvalues

Result:

$$\rho(x_1, \ldots, x_N) = \frac{1}{Z} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta$$

Dyson index $\beta = (1, 2, 4)$ (GOE, GUE, GSE)

Sketch of proof:

$$H = OXO^T \quad \text{with} \quad \begin{cases} X = \text{diag}(x_1, \ldots, x_N) \\ OO^T = 1 \end{cases}$$

Change of variables $H \to \{\boldsymbol{x}, O\}$

$$\hat{\rho}(\boldsymbol{x}, O) = \rho(H(\boldsymbol{x}, O))|J(H \to \{\boldsymbol{x}, O\})|$$

Gaussian ensemble: pdf of eigenvalues

Sketch of proof (ctd):

$$J(H \to \{\boldsymbol{x}, O\}) = \prod_{i<j}(x_j - x_i)$$

Vandermonde determinant

For rotationally invariant ensembles

$$\rho(H) = \varphi(\mathrm{Tr}H, \ldots \mathrm{Tr}(H^N))$$

Integrating over O is trivial

→

$$\rho(x_1, \ldots, x_N) \propto \varphi\left(\sum x_i, \ldots, \sum x_i^N\right) \prod_{j<k} |x_j - x_k|$$

Gaussian ensemble: pdf of eigenvalues

$$\rho(x_1, \ldots, x_N) = \frac{1}{Z} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta$$

Kills configurations with ‘large’ xi

Kills configurations where any two eigenvalues are ‘close’

Interplay between confinement and eigenvalue repulsion

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Spectral density

Counting function
of eigenvalues

$$n(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$$

For random matrix, n becomes a random measure

$$\langle n(x) \rangle = \int \mathrm{d}x_2 \ldots \mathrm{d}x_N \rho(x, x_2, \ldots, x_N) \equiv \rho(x)$$

Spectral density = marginal density of jpdf of eigenvalues

Result:

$$\boxed{\sqrt{\beta N} \rho_N(\sqrt{\beta N} x) \xrightarrow[N \to \infty]{} \rho_{\mathrm{SC}}(x) = \frac{1}{\pi} \sqrt{2 - x^2}}$$

How to prove this? Coulomb gas and resolvent

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The Coulomb gas

Rescaling $x_i \to x_i \sqrt{\beta N}$

$$Z_{N,\beta} \propto \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j \; e^{-\beta N^2 \mathcal{V}[\boldsymbol{x}]}$$

with

$$\mathcal{V}[\boldsymbol{x}] = \frac{1}{2N} \sum_i x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \ln |x_i - x_j|$$

Z = Canonical partition function of

- fluid of particles of positions xi on a line
- in equilibrium at inverse temperature βN^2
- with quadratic potential
- and repulsive logarithmic potential

The Coulomb gas

2D static fluid of charged particles confined on a line, with quadratic potential

Large N ⟷ 0 temperature limit

To find equilibrium position at 0 temperature

Minimize intensive free energy $F = -\frac{1}{\beta N^2} \ln\, Z_{N,\beta}$

Aim:

1) continuum description $Z_{N,\beta} = \int \mathcal{D}n(x)\, e^{-\beta N^2 F[n(x)]}$

Functional integral over counting functions

2) saddle-point → $n^\star(x) = \rho(x)$

Continuum description

1). Coarse-graining

Idea:
- sum over micro states compatible with given macrostate (counting function)
- then sum over all possible counting functions

$$1=\int \mathcal{D}[n(x)]\delta\left[n(x)-\frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\right]$$

→

$$Z\sim\int \mathcal{D}[n(x)]\int_{\mathbb{R}^N}\prod_{j=1}^{N}dx_j\; e^{-\beta N^2\mathcal{V}[\boldsymbol{x}]}\delta\left[n(x)-\frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\right]$$

Continuum description

2). From sums to integrals

$$\int_{\mathbb{R}} n(x) f(x) dx = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

$$\iint_{\mathbb{R}^2} dx dx' n(x) n(x') g(x, x') = \frac{1}{N^2} \sum_{i,j=1}^{N} g(x_i, x_j)$$

➡

$$Z \sim \int \mathcal{D}[n(x)] e^{-\beta N^2 \mathcal{V}[n(x)]} \underbrace{\int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j \; \delta \left[n(x) - \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \right]}_{I_N[n(x)]}$$

with

$$\mathcal{V}[n(x)] = \frac{1}{2} \int_{\mathbb{R}} dx \; x^2 n(x) - \frac{1}{2} \iint_{\mathbb{R}^2} dx dx' n(x) n(x') \ln |x - x'|$$

$$+ \, \text{regularization}$$

Continuum description

3). Compute $I_N[n(x)]$

$$I_N[n(x)] \sim \exp\left[\text{entropy}\right] \;\sim \exp\left[-N\int dx\; n(x)\ln n(x)\right]$$

Summary:

$$Z \sim \int \mathcal{D}[n(x)] \int_{\mathbb{R}} dk\; e^{-\beta N^2 \mathcal{S}[n(x),k] + o(N^2)}$$

$$\mathcal{S}[n(x),\kappa] = \mathcal{V}[n(x)] - k\left(\int dx\; n(x) - 1\right)$$

Saddle-point

$$Z \sim \exp(-\beta N^2 \mathcal{S}[n^\star(x), k^\star])$$

$$\begin{cases} 0 & = \frac{\delta}{\delta n(x)} \mathcal{S}[n(x), k]\Big|_{\substack{n=n^\star \\ k=k^\star}} = \frac{x^2}{2} - \int_{\mathbb{R}} dx' n^\star(x') \ln|x - x'| - k^\star \,, \\ 0 & = \frac{\partial}{\partial k} \mathcal{S}[n(x), k]\Big|_{\substack{n=n^\star \\ k=k^\star}} \Rightarrow \int_{\mathbb{R}} dx \; n^\star(x) = 1 \,, \end{cases}$$

Look for $n^\star(x)$ and support (a, b) !

1) find $n^\star(x; a, b)$

2) find optimal (a,b) by minimizing free energy $F = \mathcal{S}[n^\star(x; a, b)]$

Saddle-point

$$\mathrm{PV}\int dx' \frac{n^\star(x')}{x-x'} = x$$

Tricomi's theorem:

$$\mathrm{PV}\int_a^b dx' \frac{f(x')}{x-x'} = g(x) \Rightarrow f(x) = \frac{C - \mathrm{PV}\int_a^b \frac{dt}{\pi}\frac{\sqrt{(t-a)(b-t)}}{x-t} g(t)}{\pi\sqrt{(x-a)(b-x)}}$$

1) $$n^\star(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}\left[1 - x^2 + \frac{1}{2}(a+b)x + \frac{1}{8}(b-a)^2\right]$$

2) $$-a = b = \sqrt{2}$$

$$n^\star(x) \equiv \rho_{\mathrm{SC}}(x) = \frac{1}{\pi}\sqrt{2-x^2}$$

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Resolvent: generalities

$$G_N(z) \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - x_i}$$

random complex function
with poles at eigenvalues's location

$$\langle G_N(z) \rangle \to \int dx' \, \frac{\rho(x')}{z - x'} \equiv G_\infty^{(av)}(z)$$

Resolvent

Stieltjes transform

Green's function

$$\rho(x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im} \; G_\infty^{(av)}(x - \mathrm{i}\epsilon)$$

Easy to deduce the spectral density from the resolvent

Resolvent for Gaussian Ensemble

$$\frac{\partial \mathcal{V}[\boldsymbol{x}]}{\partial x_i} = 0 \Rightarrow x_i = \frac{1}{N}\sum_{j\neq i}\frac{1}{x_i - x_j} \;\rightarrow\; \frac{1}{N}\sum_i \frac{x_i}{z - x_i} = \frac{1}{N}\sum_i\sum_{j\neq i}\frac{1}{x_i - x_j}\frac{1}{N(z - x_i)}$$

$$\rightarrow \quad -1 + zG_N(z) = \frac{1}{2}G_N^2(z) + \frac{1}{2N}G'_N(z)$$

negligible in large N limit

$$G_\infty^{(av)2}(z) - 2zG_\infty^{(av)}(z) + 2 = 0$$

$$G_\infty^{(av)}(z) = z \pm \sqrt{z^2 - 2}$$

$$\frac{1}{\pi}\mathrm{Im}G_\infty^{(av)}(x - \mathrm{i}\epsilon) \xrightarrow{\epsilon \to 0^+} \begin{cases} \frac{1}{\pi}\sqrt{2 - x^2} & \text{for } x^2 < 2 \\ 0 & \text{otherwise} \end{cases}$$

Another classical ensemble: Wishart-Laguerre

$$W = HH^\dagger \quad \text{with} \quad H: \ N \times M \quad \text{matrix} \quad (M \geq N)$$

positive eigenvalues

Entries:

$$\rho(W) \propto e^{-\frac{1}{2}\mathrm{Tr}W} (\det W)^{\frac{\beta}{2}(M-N+1)-1}$$

Eigenvalues:

$$\rho(x_1, \ldots, x_N) = \frac{1}{Z} e^{-\frac{1}{2}\sum x_i} \prod_{i=1}^{N} x_i^{\beta/2(M-N+1)} \prod_{j<k} |x_j - x_k|^\beta$$

Wishart-Laguerre through the resolvent

Large N,M with $c = N/M \leq 1$ kept fixed

With rescaling $x_i \to \beta N x_i$ → $Z \propto \int_0^\infty \prod_{i=1}^N dx_i \; e^{-\beta N \mathcal{V}[\boldsymbol{x}]}$

$$\mathcal{V}[\boldsymbol{x}] = \frac{1}{2}\sum_i x_i + \left[\frac{2/\beta - 1}{2N} + \frac{1}{2} - \frac{1}{2c}\right]\sum_i \ln x_i - \frac{1}{2N}\sum_{i \neq j} \ln|x_i - x_j|$$

→

$$\frac{1}{2}G_\infty^{(av)}(z) + \frac{1}{2}\left(1 - \frac{1}{c}\right)\frac{K + G_\infty^{(av)}(z)}{z} = \frac{1}{2}G_\infty^{(av)2}(z)$$

where

$$\frac{1}{N}\sum \frac{1}{x_i} \to K = \int dx \frac{\rho(x)}{x}$$

Wishart-Laguerre through the resolvent

$$G_{\infty}^{(av)}(z) = \frac{1}{2}\left(1 - \frac{\gamma}{z} \pm \frac{\sqrt{z^2 - 2\gamma z + \gamma^2 - 4\gamma K z}}{z}\right)$$

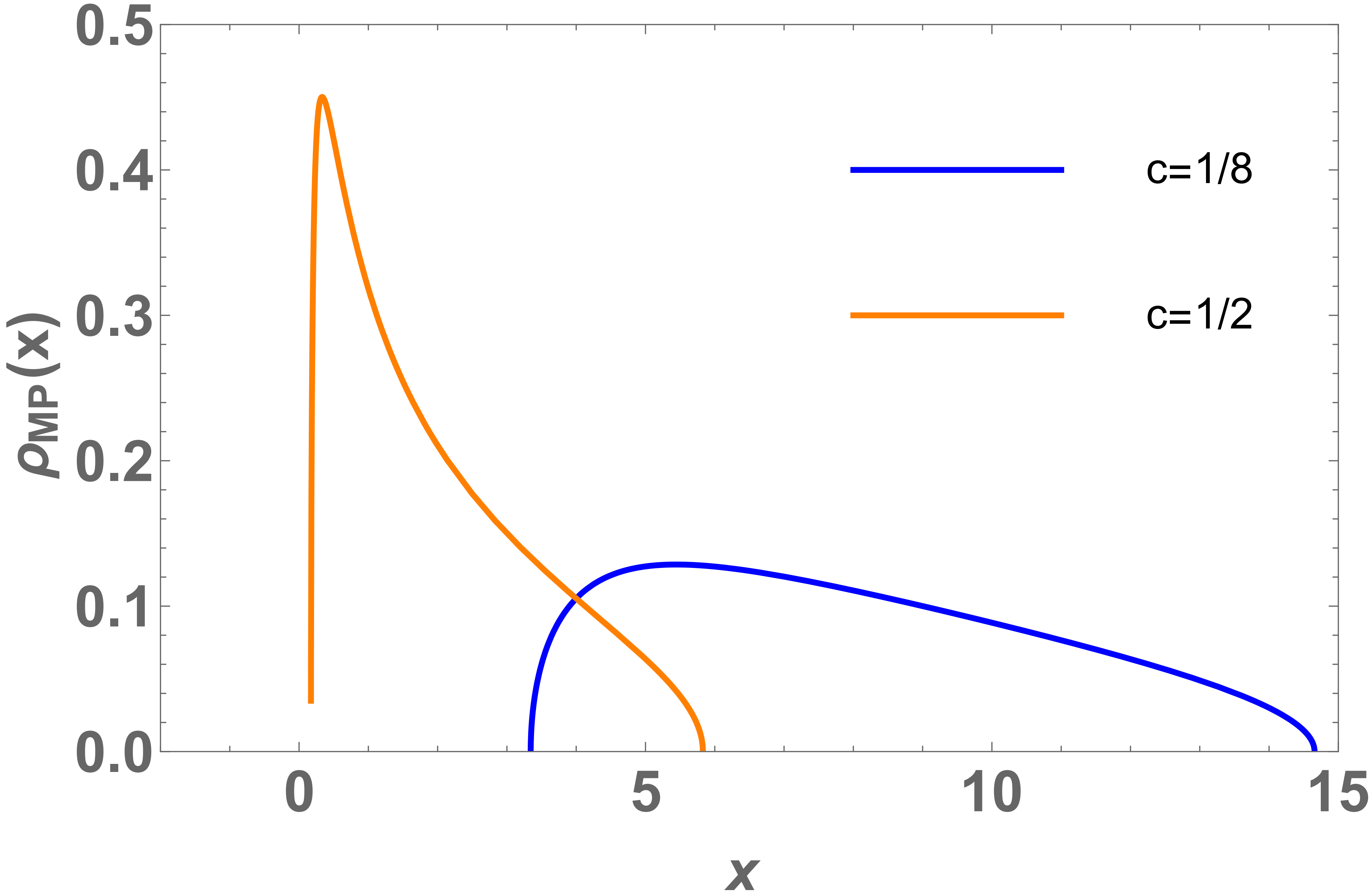
where $\gamma = (1-c)/c > 0$

$$\rho_{\mathrm{MP}}(x) = \frac{1}{2\pi x}\sqrt{(x - x_-)(x_+ - x)}$$

$$x \in (x_-, x_+) \qquad x_{\pm} = \left(\frac{1}{\sqrt{c}} \pm 1\right)^2$$

$$\lim_{N\to\infty} \beta N \rho(\beta N x) = \rho_{\mathrm{MP}}(x)$$

Marcenko-Pastur density



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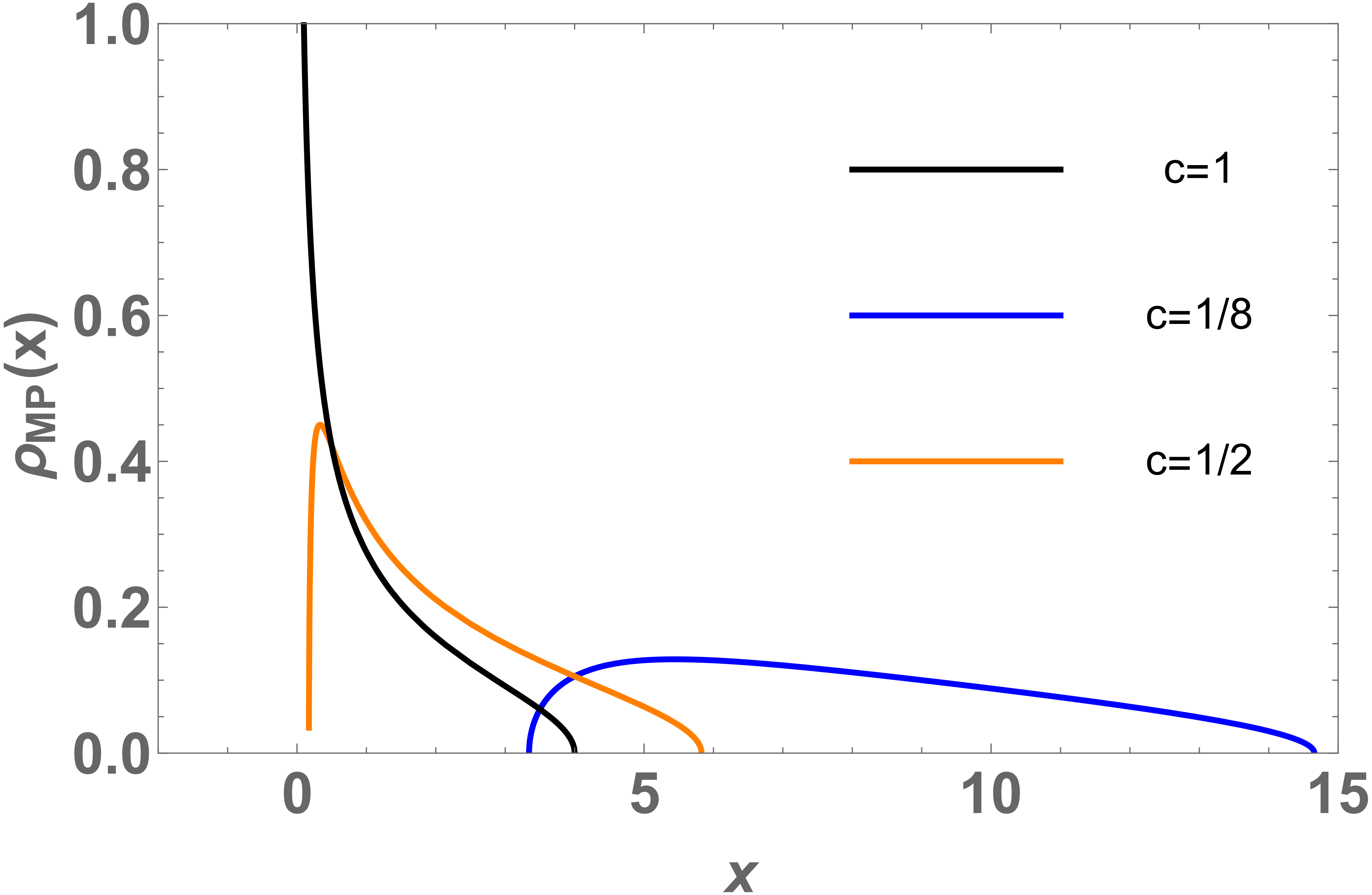
Marcenko-Pastur density



Particular case c=1. Accumulation of eigenvalues near 0

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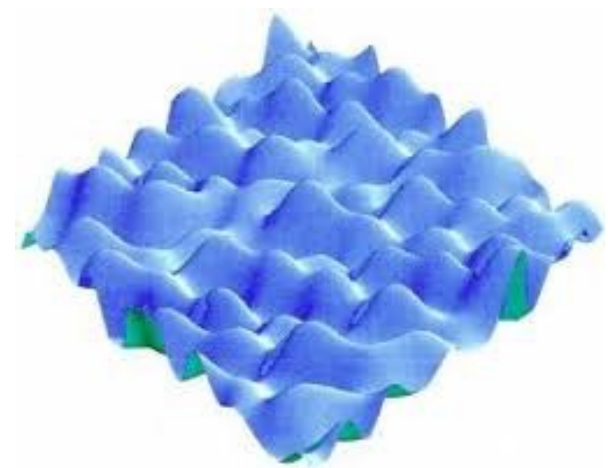
High-energy landscape

High-energy physics: large number of fields, complex interactions

Statistical predictions? Universal behavior in large N limit?

With landscape modeled as
multi-dimensional scalar potential
(say Gaussian random field)



Vacua = local minima of potential =
critical points+ all Hessian eigenvalues positive

Dependence of dynamics of inflation on
smallest eigenvalues of the Hessian

Key Q: **Hessian eigenvalue distribution of Gaussian random fields**

Gaussian Random Fields (GRF)

Gaussian random field, static, statistically homogeneous and isotropic $V(\boldsymbol{\phi})$ with $\langle V(\varphi)\rangle = \bar{V}$

All information in 2 pt correlation function:

$$\langle V(\boldsymbol{\varphi}_1)V(\boldsymbol{\varphi}_2)\rangle - \bar{V}^2 = \frac{1}{(2\pi)^N}\int d^N\boldsymbol{k}\, P(k)e^{i\mathbf{k}\cdot(\boldsymbol{\varphi}_1-\boldsymbol{\varphi}_2)}$$

Define $\sigma_n^2 \equiv \frac{1}{(2\pi)^N}\int d^N\boldsymbol{k}\, k^{2n}P(k)$

For amplitude V_0 and correlation length Λ

$$\sigma_n^2 \sim V_0^2\left(\frac{N}{\Lambda^2}\right)^n$$

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Correlators of Taylor coefficients

Straightforward computations:

$$\langle (V-\bar{V})^2 \rangle = \sigma_0^2$$

$$\langle V_i \rangle = \langle V_{ij} \rangle = \langle V V_i \rangle = \langle V_i V_{jk} \rangle = 0$$

$$\langle V V_{ij} \rangle = -\langle V_i V_j \rangle = -\frac{\sigma_1^2}{N}\delta_{ij}$$

$$\langle V_{ij} V_{kl} \rangle = \frac{\sigma_2^2}{N(N+2)}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

With the large N limit in mind, we take

$$E \equiv \sigma_0^2 \qquad A \equiv \frac{\sigma_2^2}{N(N+2)} \qquad B \equiv \frac{\sigma_1^2}{N}$$

$$(A, B, E) = \mathcal{O}(1)$$

Pdf of Taylor coefficients

Multivariate Gaussian distribution

$$P(V, V_{ij}) \propto \exp\left[-\frac{(V-\bar{V})^2}{2E} - \frac{1}{4A}\left[\mathrm{Tr}(V_s^2) - \frac{AE-B^2}{(N+2)AE-NB^2}(\mathrm{Tr}V_s)^2\right]\right]$$

where $$(V_s)_{ij} = V_{ij} + \frac{B}{E}(V-\bar{V})\delta_{ij}$$

$$\langle V_{ij}\rangle_{|V} = -\frac{B}{E}(V-\bar{V})\delta_{ij}$$

Global shift of the spectrum depending on the value of the potential (expected!)

Discussion about potential minima depend on assumptions about

$$\bar{V} = \begin{cases} 0 \\ \text{fixed} \\ \text{grows with } N \end{cases}$$

Pdf of Taylor coefficients

Irrespective of the value of the potential

$$P(V_{ij}) \propto \exp\left[-\frac{1}{4A}\left[\mathrm{Tr}(V^2) - \frac{1}{N+2}(\mathrm{Tr}V)^2\right]\right]$$

Up to rescaling and possibly shift, we can consider the general model

$$P(H) \propto \exp\left[-\frac{1}{2}\left[\mathrm{Tr}(H^2) - \frac{a}{N}(\mathrm{Tr}H)^2\right]\right]$$

with $a = 1 + \mathcal{O}\left(\frac{1}{N}\right)$

Constrained Coulomb gas

$$P(x_i \geq \lambda_{cr}) = \frac{Z(\lambda_{cr})}{Z_{-\infty}}$$

$$Z(\lambda_{cr}) = \int_{\lambda_{cr}}^{\infty} \mathrm{d}\boldsymbol{x} \exp\left\{ -\frac{1}{2} \left(\sum_i x_i^2 - \frac{a}{N} \left[\sum_i x_i \right]^2 - \sum_{i \neq j} \ln\left(|x_i - x_j|\right) \right) \right\}$$

With rescaled variables $x_i = \mu_i \sqrt{N}$

Coulomb gas machinery:

$$Z(\lambda_{cr}) \sim \int \mathcal{D}[n(\mu)] \int_{\mathbb{R}} dk \; e^{-N^2 \mathcal{S}[n(\mu),k]}$$

counting function with $n(\mu) = 0$ for $\mu \leq \mu_{cr} = \lambda_{cr}/\sqrt{N}$

Constrained Coulomb gas

$$\mathcal{S} = \frac{1}{2}\int_{\mathbb{R}} d\mu\ \mu^2 n(\mu) - \frac{1}{2}\iint_{\mathbb{R}^2} d\mu d\mu' n(\mu) n(\mu') \ln|\mu - \mu'|$$
$$-k\left(\int_{\mathbb{R}} d\mu\ n(\mu) - 1\right) - \frac{a}{2}\left(\int_{\mathbb{R}} d\mu\ \mu n(\mu)\right)^2$$

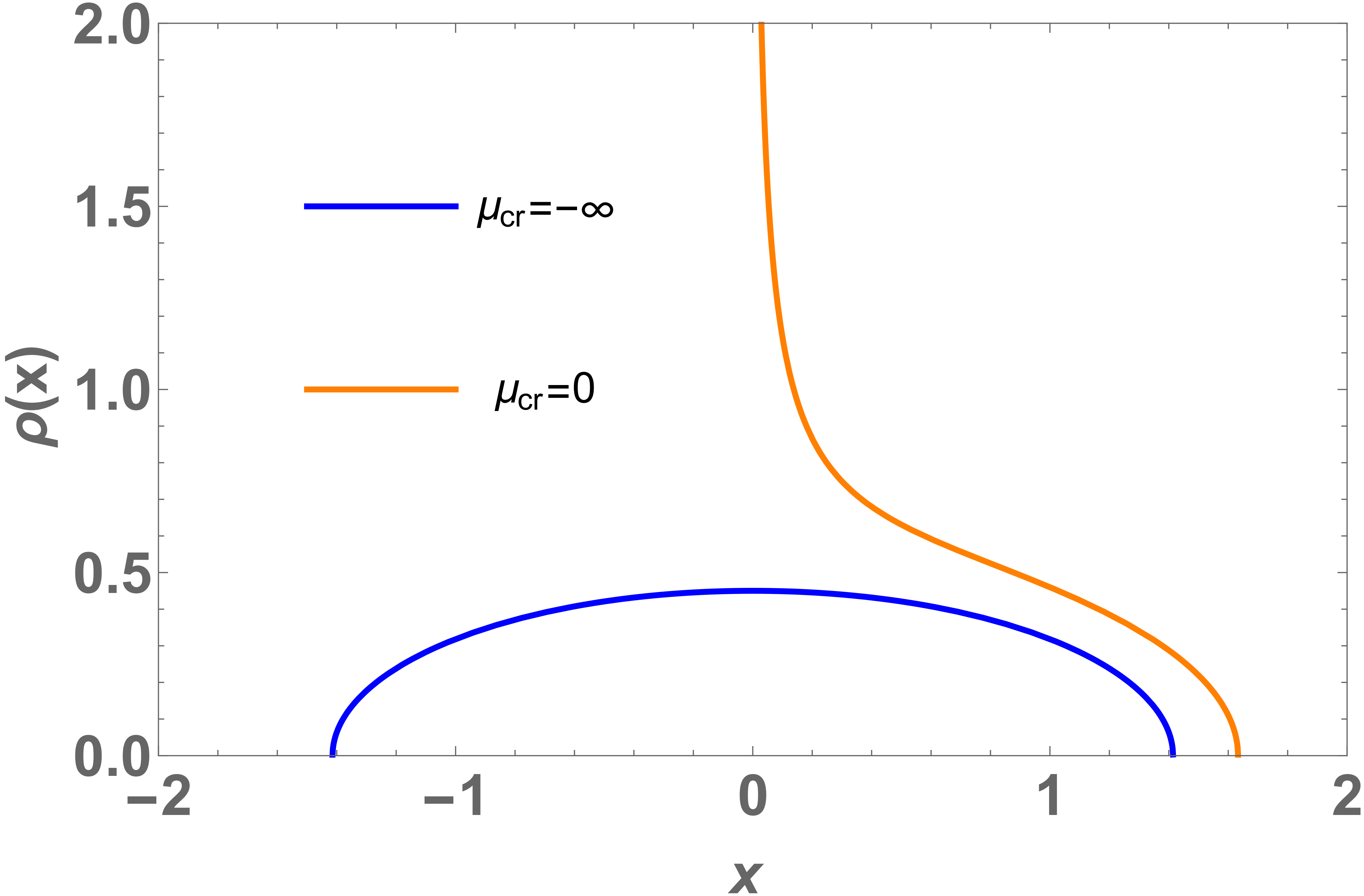
$$a = 0 \quad \leftrightarrow \quad GOE$$

$$a = 1 \quad \leftrightarrow \quad \text{degeneracy} \quad \mu \to \mu + \text{cst}$$

degeneracy broken by inclusion of 1/N effects

Spectral density, GOE

Saddle-point + Tricomi's theorem + Minimization of free energy



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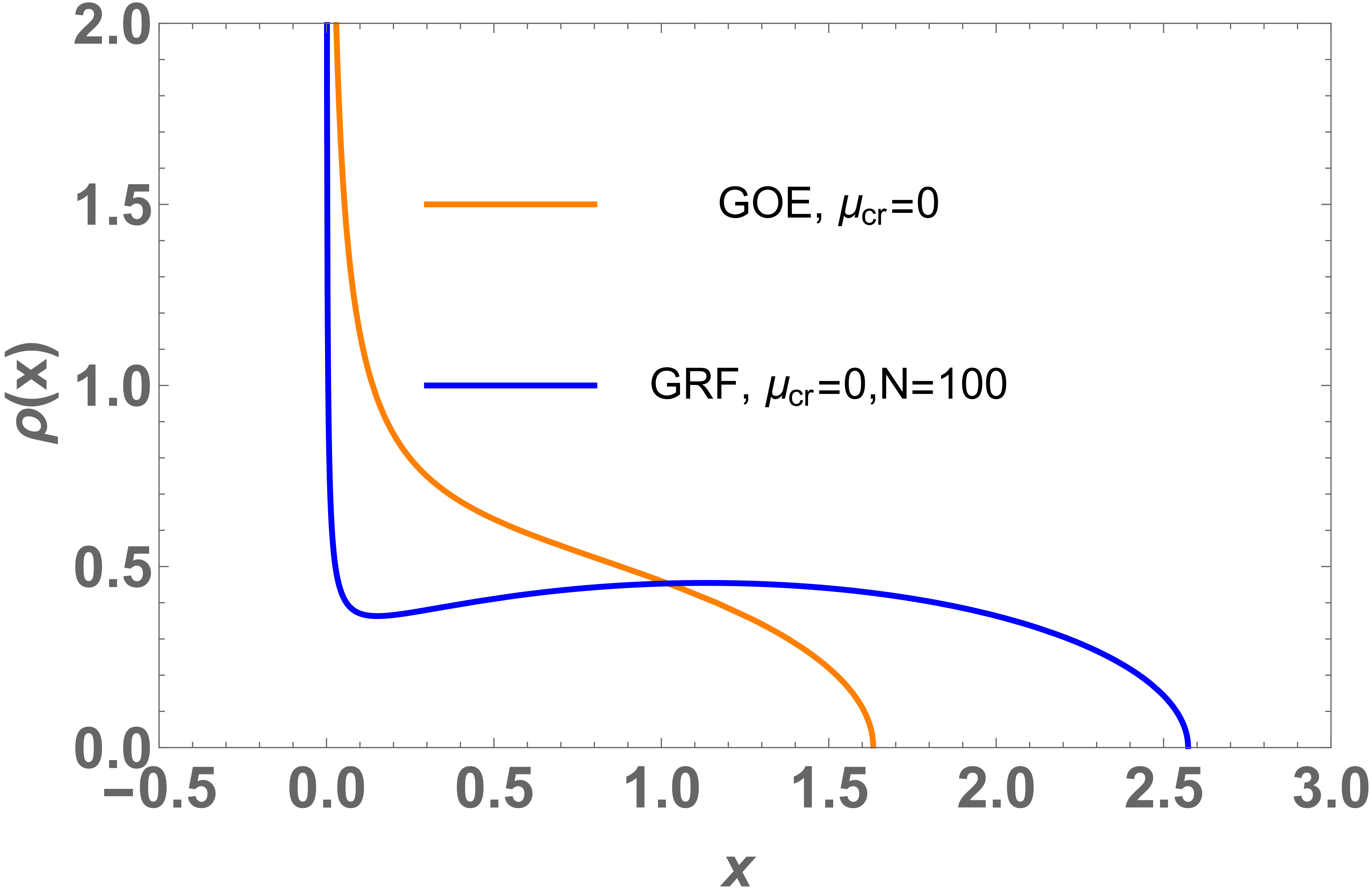
Spectral density, GRF and GOE

Saddle-point + Tricomi's theorem + Minimization of free energy



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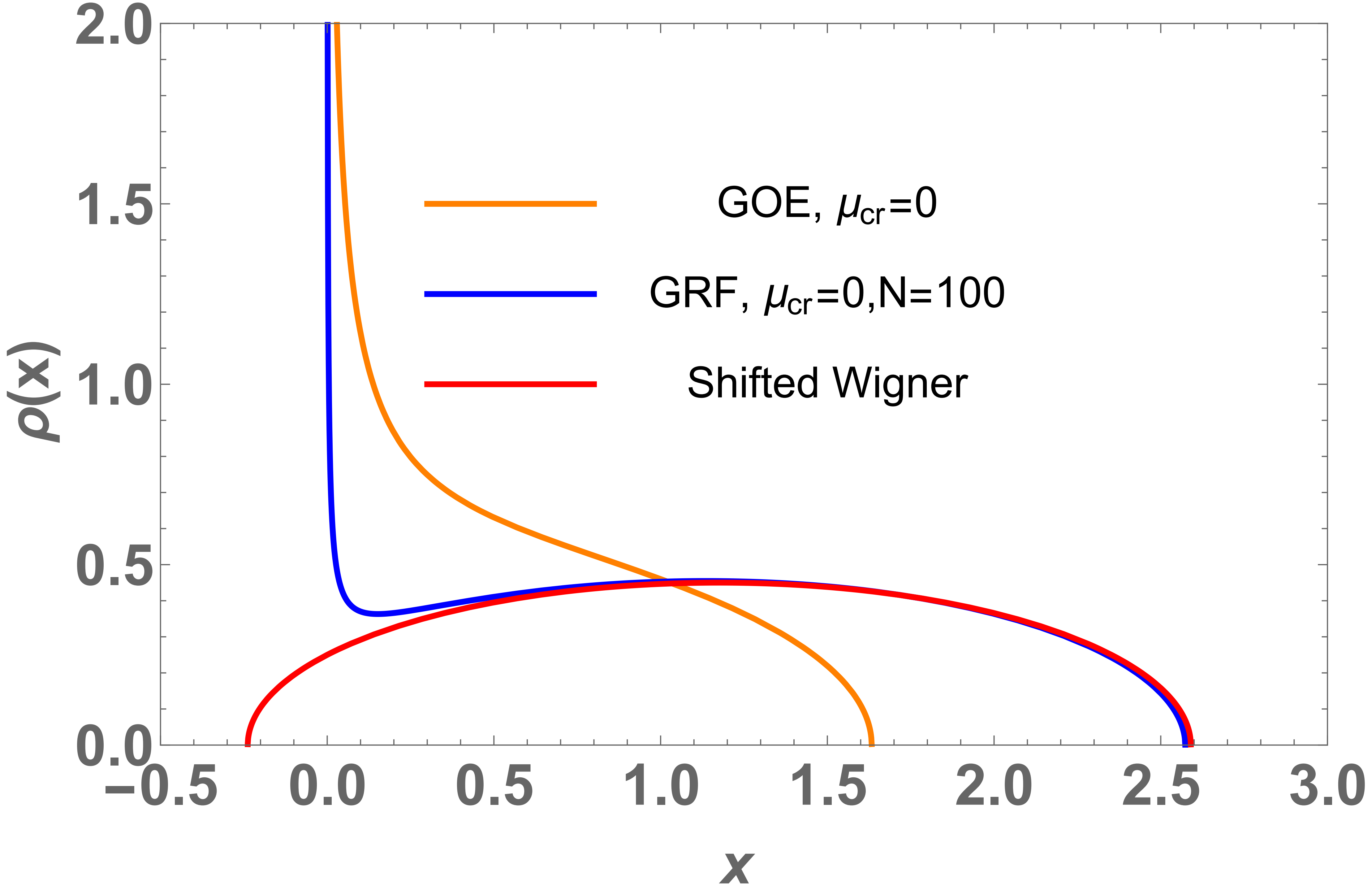
Spectral density, GRF and GOE



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Probability of rare fluctuations to positivity

$$P(\mu_{min} \geq \mu_{cr}) = e^{-N^2\left(S(\mu_{cr})-S(-\sqrt{2})\right)}$$

For $1 - a \ll 1$ $\quad P(\mu_{min} \geq \mu_{cr}) \simeq e^{-\frac{N^2}{2}\left(\mu_{cr}+\sqrt{2}\right)^2(1-a)}$

Contrast

$$P_{\mathrm{GRF}}(\mu_{\mathrm{min}} \geq 0) \simeq e^{-2N}$$

$$P_{\mathrm{GOE}}(\mu_{\mathrm{min}} \geq 0) \simeq e^{-\frac{\ln 3}{4}N^2}$$

Intuitive reason

$$P_{\mathrm{GRF}}(H) \propto \exp\left\{-\frac{1}{2}\left(\sum_i \delta\lambda_i^2 + \frac{2N}{N+2}\lambda_{\mathrm{av}}^2\right)\right\}$$

$$P_{\mathrm{GOE}}(H) \propto \exp\left\{-\frac{1}{2}\left(\sum_i \delta\lambda_i^2 + N\lambda_{\mathrm{av}}^2\right)\right\}$$

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From generic to stationary points

Conditioning on stationary points, and up to rescaling and possibly shift

$$P(H) \propto \exp\left[-\frac{1}{2}\left[\mathrm{Tr}(H^2) - \frac{a}{N}(\mathrm{Tr}H)^2\right]\right] |\mathrm{det}H|$$

→ $$\mathcal{S}[n(\mu)] \to \mathcal{S}[n(\mu)] - \frac{1}{N}\int d\mu\, n(\mu) \ln|\mu|$$

Evaluating this NLO effect on the LO solution (shifted Wigner)

→

Simply $a \to a + 1/N$

$$P_{\mathrm{GRF}}(\text{stationary points are minima}) \simeq e^{-N}$$

Typical smallest eigenvalue at potential minimum

$$P_{\geq}(\mu_{\rm min}) \equiv P(\text{smallest eigenvalue at local minimum} \geq \mu_{\rm min})$$

$$P_{\geq}(\mu_{\rm min}) = e^{-N^2(S(\mu_{\rm min})-S(0))}$$

→

and $$\rho(\mu_{\rm min}) = -\frac{{\rm d}P_{\geq}(\mu_{\rm min})}{{\rm d}\mu_{\rm min}}$$

→

$$\boxed{\mu_{\rm min} \sim \frac{1}{N}}$$

Single or multi field inflation?

Small-field landscape + Inflection-point inflation

Mass spectrum in directions orthogonal to inflationary one?

$$P(x_2,\ldots,x_N) \propto \exp\left\{-\frac{1}{2}\left(\sum_{i\geq 2} x_i^2 - \frac{a}{N}\left[\sum_{i\geq 2} x_i\right]^2 - \sum_{i\neq j\geq 2} \ln\left(|x_i - x_j|\right)\right) - 2\sum_{i\geq 2} \ln|x_i|\right\}$$

conditioning on first derivatives vanishing for orthogonal directions

+

eigenvalue repulsion with $x_1 = 0$

→ $$\mu_2 \sim \frac{1}{N}$$

Renaux-Petel, IAP

Single or multi field inflation?

Rescaling $m_2^2=\sqrt{2AN}\mu_2$ + amplitude $A\sim V_0^2/\Lambda^4$

→

$$m_2^2\sim\frac{V_0}{\Lambda^2\sqrt{N}} \quad \text{versus} \quad H^2\sim\frac{V_0}{M_{\rm Pl}^2}$$

Single-field inflation $\leftrightarrow$ $m_2^2\gtrsim H^2$ $\leftrightarrow$ $\dfrac{\Lambda}{M_{\rm Pl}}\lesssim\dfrac{1}{N^{1/4}}$

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