Doubly curl-free flows and MAK reconstruction

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Monge's mass transportation problem

. . .It is not indifferent that any given molecule of the cuts be transported to this or that place in the fills, but there ought to be a certain distribution of molecules of the former into the latter, according to which the sum of these products will be the least possible, and the cost of transportation will be a ***minimum****.*

Monge transportation with quadratic cost

For given $\rho_{\rm in}(\mathbf{q})$, $\rho_0(\mathbf{x})$ minimize

$$\int |\mathbf{x}(\mathbf{q}) - \mathbf{q}|^2 \rho_{\rm in}(\mathbf{q})\, d\mathbf{q} = \int |\mathbf{x} - \mathbf{q}(\mathbf{x})|^2 \rho_0(\mathbf{x})\, d\mathbf{x}$$

over all $(\mathbf{x}(\mathbf{q}), \mathbf{q}(\mathbf{x}))$ such that $\rho_{\rm in}(\mathbf{q})\, d\mathbf{q} = \rho_0(\mathbf{x})\, d\mathbf{x}$

Theorem (Brenier 1987, 1991) *The minimizing maps are gradients of convex functions:*

$$\mathbf{x}(\mathbf{q}) = \nabla_{\mathbf{q}} \Phi(\mathbf{q}), \qquad \mathbf{q}(\mathbf{x}) = \nabla_{\mathbf{x}} \Theta(\mathbf{x})$$

Φ *and* Θ *solve suitable Monge–Ampère equations*

$$\det\ \left(\nabla_{q_i} \nabla_{q_j} \Phi(q)\right) = \frac{\rho_{\rm in}(q)}{\rho_0(x)}\ ;\ \det\ \left(\nabla_{x_i} \nabla_{x_j} \Theta(x)\right) = \frac{\rho_0(x)}{\rho_{\rm in}(q)}$$

Zeldovich approximation

Euler:

$$\partial_\tau \mathbf{v} + (\mathbf{v}\cdot\nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}(\mathbf{v} + \nabla_{\mathbf{x}}\varphi_{\mathrm{g}})$$

late 1960s

Mass conservation:

$$\partial_\tau \rho + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{v}) = 0$$

Poisson:

$$\nabla_{\mathbf{x}}{}^2\varphi_{\mathrm{g}} = \frac{\rho-1}{\tau}$$

Zeldovich approximation:

$$\mathbf{v} \equiv -\nabla_{\mathbf{x}}\varphi_{\mathrm{g}}$$

$$\mathbf{x} = \mathbf{q} + \tau\mathbf{v}_{\mathrm{in}}(\mathbf{q}) = \mathbf{q} - \tau\nabla\varphi_{\mathrm{g}}^{(\mathrm{in})}(\mathbf{q})$$

$$= \nabla\left[\frac{|\mathbf{q}|^2}{2} - \tau\varphi_{\mathrm{g}}^{(\mathrm{in})}(\mathbf{q})\right]$$

$$= \nabla\Phi(\mathbf{q})$$ ... (graph) invertible if Φ is convex

Bertschinger–Dekel (1989)

Is MAK reconstruction valid beyond Zeldovich?

Reconstruction (in the sense of Peebles 1989): given the present distribution of masses and no information on the peculiar velocities, find the past dynamical history of the universe.

Brenier-Frisch-Hénon-Loeper-Matarese-Mohayaee-Sobolevsky (2003 MNRAS 346, 501) showed that this problem has a unique solution when multi-streaming is negligible and the Euler-Poisson equations are used.

However no practical algorithm has yet been found to calculate this unique solution, unless it is assumed that the Lagrangian map has a convex potential, in which case one can find the initial locations of mass elements very efficiently using Monge-Ampère-Kantorovich (MAK) reconstruction (Frisch-Matarese-Sobolevsky-Mohayaee (2002 Nature 417, 260). This works better than expected (S. Colombi kept telling us).

Non-Zeldovich convex potential flows?

Do there exist flows $\mathbf{q} \mapsto \mathbf{x}(\mathbf{q}, t), \quad t > 0$ such that the map from $\mathbf{x}(\mathbf{q}, t_1)$ to $\mathbf{x}(\mathbf{q}, t_2)$ has a convex potential for any $0 < t_1 < t_2$? (Brenier, Frisch around 2002.)

Defining the velocity $\mathbf{v} \equiv \partial_t \mathbf{x}(\mathbf{q}, t)$ this implies that at any time the velocity is doubly curl-free: $\nabla_{\mathbf{x}} \wedge \mathbf{v} = \nabla_{\mathbf{q}} \wedge \mathbf{v} = 0.$

Can we find flows other than Zeldovich, i.e. with non-straight particle orbits, which satisfy such constraints?

This is here posed as a *kinematical* problem, without reference to any particular dynamical equations of motion.

Frisch-Sobolevsky-Bec (2009): Yes, we can!

$$\phi(q_1, q_2, t) \equiv (1+t)\left(q_1^2 + q_2^2\right)$$

$$+t^2\left(q_1^4 + 2q_1^3 q_2 + 3q_1^2 q_2^2 + 2q_1 q_2^3 + q_2^4\right) \quad x_1 \equiv \frac{\partial \phi}{\partial q_1}, \quad x_2 \equiv \frac{\partial \phi}{\partial q_2}, \quad t > 0$$

is convex (in (q_1, q_2)), has parabolic orbits and all the Hessian matrices $H_{ij} \equiv \frac{\partial^2 \phi}{\partial q_i \partial q_j}$ commute along any orbit.

Commuting Hessian matrices and gradient flows

The gradient of a potential map is the Hessian of the potential, thus a *symmetrical* matrix.

The product of two symmetrical matrices is in general not symmetrical unless they *commute*.

Thus the potentiality of the composition of two potential maps requires the commutation of their Hessian matrices or, equivalently, that they be co-diagonalisable.

For a time-dependent flow the potentiality of the map from any time to any other time, i.e. the double curl-free condition, requires that the Hessian matrices be co-diagonalisable along any particle orbit.

The 2D case

Consider the flow $(q_1,\, q_2) \mapsto (\partial_{q_1}\phi(q_1, q_2, t),\, \partial_{q_1}\phi(q_1, q_2, t))$,
the double curl-free condition requires that, for any given starting point $(q_1,\, q_2)$ all the Hessian matrices

$$H(q_1, q_2, t) \equiv \left[\begin{array}{cc} \phi_{11}(q_1, q_2, t) & \phi_{12}(q_1, q_2, t) \\ \phi_{12}(q_1, q_2, t) & \phi_{22}(q_1, q_2, t) \end{array} \right]$$

commute for different time arguments. Since for a real symmetrical two-by-two matrix ϕ_{ij} with distinct eigenvalues, the eigen-directions depend only on the ratio $(\phi_{11} - \phi_{22})/\phi_{12}$, we obtain the following linear second-order PDE:

$$\phi_{11} - \phi_{22} = \chi(q_1, q_2)\phi_{12}$$

where $\chi(q_1, q_2)$ is a prescribed function of the starting coordinates.
The goal now is to find time-dependent solutions to this equation, reducing initially to the potential of the identity map, which are convex and have non-straight particle orbits.

The method of homogeneous polynomials

When $\chi(q_1, q_2)$ is taken to be the ratio of two homogenous polynomials of degree n the PDE for the 2D double curl-free problem has solutions which are also homogeneous polynomials of various degrees. They can be determined by a purely algebraic method. One of the simplest instances is

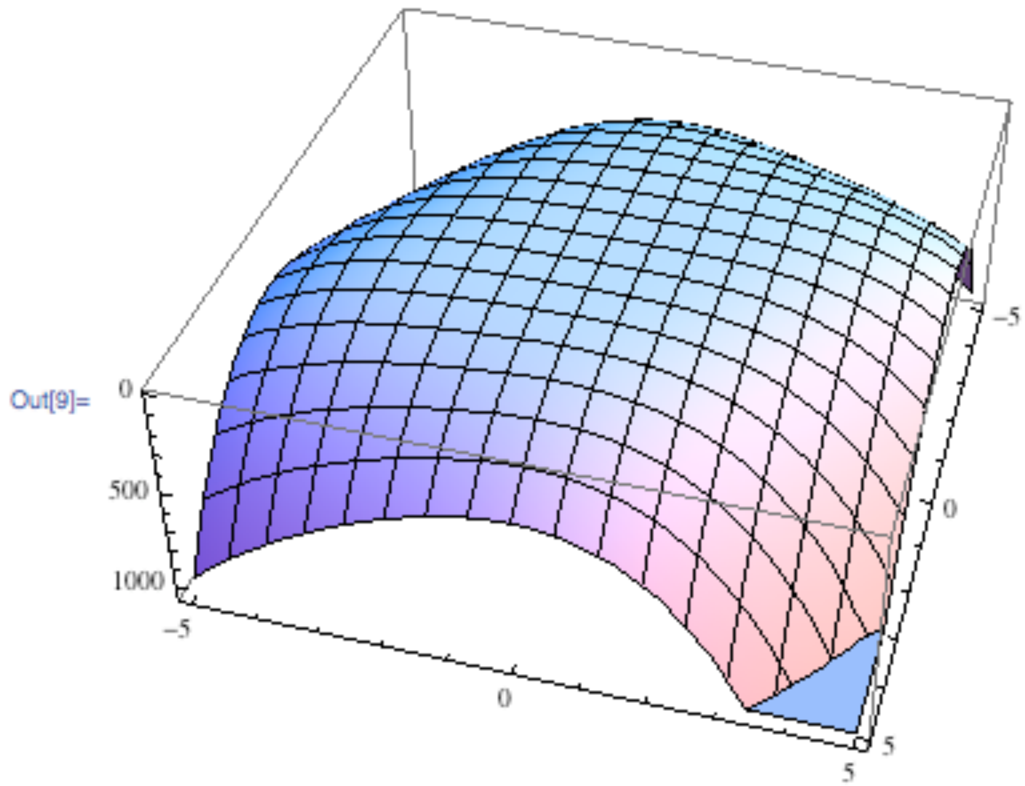
$$\chi(q_1, q_2) \equiv \frac{q_1 - q_2}{q_1 + q_2}$$

There is a quadratic solution $\phi = (1/2)(q_1^2 + q_2^2)$, which just defines the identity map, a cubic solution which is non-convex.

The following combination of quadratic and quartic solutions has all the required properties:

$$\phi(q_1, q_2, t) \equiv (1+t)\left(q_1^2 + q_2^2\right)$$

$$+t^2\left(q_1^4 + 2q_1^3 q_2 + 3q_1^2 q_2^2 + 2q_1 q_2^3 + q_2^4\right) \quad x_1 \equiv \frac{\partial \phi}{\partial q_1}, \quad x_2 \equiv \frac{\partial \phi}{\partial q_2}, \quad t > 0$$



What about 3D? Does the Lagrangian perturbation theory of Moutarde et al. (1991) and Bouchet et al. (1995) provide such doubly curl-free 3D solutions? Is the potential convex?