Aspects of gravity and cosmology in lower dimensions
(Weyl´s integrable space-times)

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1. Gravity in lower dimensions

A) Look for exactly soluble models
B) Simple systems analogous of (3+1)D gravitational systems
C) Look for quantization and interaction with other fields

1. Geometric framework

A) Riemannian manifolds (metric)
B) Non-Riemann manifolds – new degrees of freedom
  i. Riemann-Cartan manifolds (metric and torsion)
  ii. **Weyl´s manifolds** (metric and non-metricity)
  Obs: not Weyl´s space-time (solution of Einstein equations)

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1. **General relativity**

   A) Gravity in **Riemannian space-times**
   B) **Metric** and **Christoffel connection** (determined by metric)
   C) **Space-time properties** determined by **metric**

1. **Gravity in Weyl Integrable space-times (WIST)**

   A) **No second clock effect** (atomic clocks well behaved)

   A) Metric and **Weyl connection** (**geometric** Weyl **scalar field**)

   A) **Space-time properties** determined by 2 geometric objects: **metric** and Weyl **scalar field**

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General relativity in (2+1)D

The gravitational field presents

1. **No curvature outside matter**
2. **No dinamical degrees of freedom**
3. **No Newtonian limit**
4. **No black holes** for vacuum equations (Λ = 0)
5. **No gravitational waves**
6. **Gravity** is determined by **one dynamical geometric object: the metric**

(Giddings et al. GRG 16 (8): 751, 1984)

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Gravity in Weyl integrable (2+1)D space-times

The gravitational field presents

1. **Curvature outside matter**
2. **Dynamical degrees of freedom**
3. **Newtonian limit**
4. **Black holes** for vacuum equations ($\Lambda = 0$)
5. **Gravitational waves**
6. **Gravity** is determined by **two dynamical geometric objects: metric** and **scalar field**

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Gravitational Field Equations

(Melnikov, Salim)

1. Lagrangian (R – curvature scalar of wist)

$$S = \int (R + \xi \omega^{\mu}{}_{;\mu}) \sqrt{-g}\, d^4x \qquad \omega_\mu = \partial_\mu \omega$$

1. Field equations (wist quantities and derivatives)

$$G_{\mu\nu} + \omega_{\mu;\nu} - (2\xi - 1)\, \omega_\mu \omega_\nu + \xi\, g_{\mu\nu}\, \omega_\alpha \omega^\alpha = 0$$

$$\omega^{\alpha}{}_{;\alpha} = -2\omega_\alpha\, \omega^\alpha$$

1. Basic fields (riemannian quantities: denoted by the symbol ^)

$$\lambda = \tfrac{1}{2}(4\xi - 3) \qquad \hat{\Box}\omega = 0$$

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Particles and proper-time: Weyl geodesics

1. Weyl geodesics (auto-parallel) and metric geodesics

$$\delta(\mathrm{ds})=0 \qquad \frac{d^2x^\mu}{ds^2}+\left\{{}^{\mu}_{\alpha\beta}\right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0$$

$$\Gamma^a_{bc}=\left\{{}^{a}_{bc}\right\}-\frac{1}{2}g^{ad}[g_{db}\sigma_c+g_{dc}\sigma_b-g_{bc}\sigma_d]$$

$$\delta(d\tau)=0 \qquad \frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0$$

1. Lenth and proper-time.

$$d\tau^2=\mathrm{e}^{-\varphi}\mathrm{ds}^2$$

1. Weyl geodesic deviation

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Newtonian limit

1. **Non-relativistic weak static fields**

Metric and Weyl´s scalar field

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \qquad \omega = \epsilon\varphi \qquad \varphi_{,0} = 0$$

1. **Weyl´s geodesic equations (auto-parallel)**

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad \Gamma^\alpha_{\mu\nu} = \frac{\epsilon}{2}\eta^{\alpha\lambda}[h_{\lambda\mu,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda} + \eta_{\mu\nu}\varphi_{,\lambda} - \eta_{\lambda\mu}\varphi_{,\nu} - \eta_{\lambda\nu}\varphi_{,\mu}]$$

$$\left(\frac{ds}{dt}\right)^2 \cong c^2(1 + \epsilon h_{00}) \qquad \frac{d^2x^\mu}{dt^2} + c^2\Gamma^\mu_{00} = 0 \qquad \Gamma^i_{00} = -\frac{\epsilon}{2}\eta^{ij}\frac{\partial}{\partial x^j}(h_{00} - \varphi)$$

1. **Newtonian potential U**

$$\frac{d^2\vec{X}}{dt^2} = -\frac{\epsilon}{2}c^2\vec{\nabla}(h_{00} - \varphi) \qquad U = \frac{\epsilon c^2}{2}(h_{00} - \varphi)$$

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Newtonian limit

1. Field equations (n = 3)

$$S = \int d^n x \sqrt{|g|} \left[R + \xi \phi_{,\alpha}\phi^{,\alpha} - \kappa_n e^{-2\phi} L_m\right]$$

$$\tilde{R}_{\mu\nu} = -\kappa_n T_{\mu\nu} e^{-2\phi} + \frac{(n-1)(n-2) - 4\xi}{4}\phi_{,\mu}\phi_{,\nu} + g_{\mu\nu}\frac{\kappa_n}{n-2} T e^{-2\phi}$$

$$\Box\phi = \frac{4T\kappa_n}{(n-1)(n-2) - 4\xi} e^{-2\phi}$$

$$T_{\mu\nu} = (\rho + p)V_\mu V_\nu - p g_{\mu\nu}$$

$$T \simeq \rho$$

$$T e^{-2\phi} \simeq \rho(1 - 2\epsilon\varphi) \simeq \rho$$

$$\nabla^2 \frac{\epsilon}{2} h_{00} = \left(\frac{3-n}{n-2}\right) k_n \rho$$

$$\epsilon\nabla^2\varphi = \frac{-4\kappa_n\rho}{(n-1)(n-2) - 4\xi}$$

1. Poisson´s equation

$$U = \frac{\epsilon c^2}{2}(h_{00} - \varphi)$$

$$\nabla^2 U. = K_n \rho c^2$$

$$K_n = \kappa_n \left(\frac{n-3}{2-n} + \frac{2}{4\xi - (n-1)(n-2)}\right)$$

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Black hole

1. Static circularly symmetric

$$ds^2 = f(r)\,dt^2 - \frac{dr^2}{f(r)} - r^2 d\vartheta^2 \qquad \varphi = \varphi(r,\vartheta)$$

1. Field equations

$$\frac{f(r)\left(\frac{d^2}{dr^2} f(r)\right)}{2} + \frac{f(r)\left(\frac{d}{dr} f(r)\right)}{2\,r} = \lambda\left(\frac{d\varphi}{dt}\right)^2 = 0$$

$$-\frac{\frac{d^2}{dr^2} f(r)}{2\,f(r)} - \frac{\frac{d}{dr} f(r)}{2\,r\,f(r)} = \lambda\left(\frac{d}{dr}\varphi(r,\vartheta)\right)^2$$

$$\varphi = \varphi(\vartheta) \qquad f(r) = a^2 \log\left(\frac{r}{r_h}\right) = a^2 \log(r) - a^2 \log(r_h) \qquad a = \text{const} \quad r_h = \text{const}$$

$$-r\left(\frac{d}{dr} f(r)\right) = \lambda\left(\frac{d}{d\vartheta}\varphi(r,\vartheta)\right)^2 \qquad -a^2 = \lambda\left(\frac{d}{d\vartheta}\varphi(r,\vartheta)\right)^2 \qquad \varphi = \frac{a\,\vartheta}{\ }$$

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Black hole

1. Solution and riemannian curvature scalar

$$ds^2 = f(r)\, dt^2 - \frac{dr^2}{f(r)} - r^2 d\vartheta^2$$

$$\hat{R} = \frac{a^2}{r^2}$$

$$f(r) = a^2 \log\left(\frac{r}{r_h}\right) = a^2 \log(r) - a^2 \log(r_h)$$

$$\varphi = \frac{a\,\vartheta}{\sqrt{-\lambda}}$$

$$a = \text{const}$$

$$r_h = \text{const}$$

1. Weyl's geodesics (particle trajetories)

$$V + e^{\varphi}\left(\frac{d}{d\tau} r(\tau)\right)^2 = E^2$$

$$V = f(r)\left(\frac{L^2}{r^2} + 1\right)$$

$$E = \text{const}$$

$$L = \text{const}$$

1. Radial geodesic

$$e^{\frac{\varphi_0}{2}} = 1$$

$$\frac{d}{d\tau} r(\tau) = \sqrt{E^2 - a^2 \log\left(\frac{r}{r_h}\right)}$$

$$d\tau = |a|\, dt \sqrt{\log\left(\frac{r}{r_h}\right)}$$

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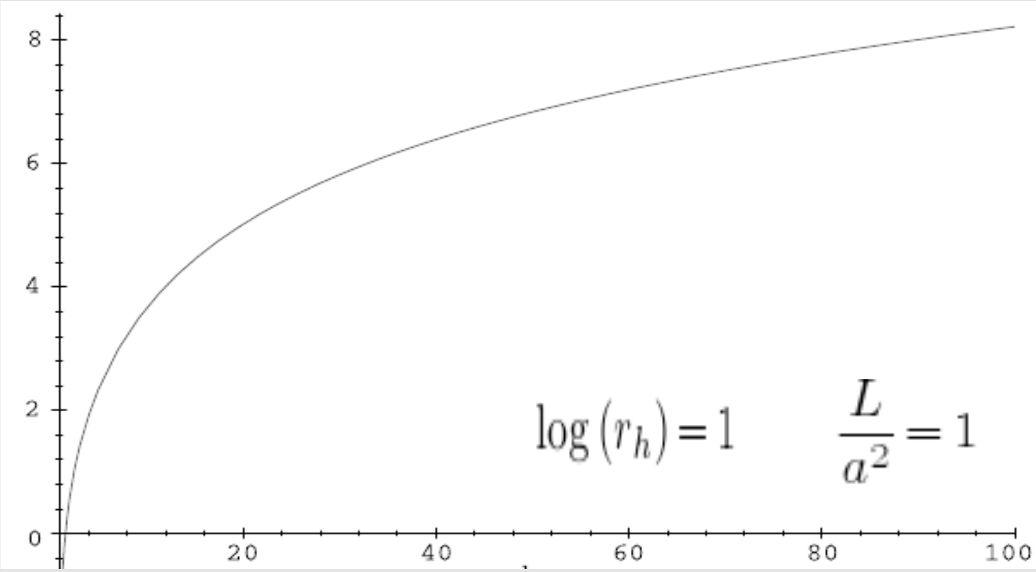
Black hole

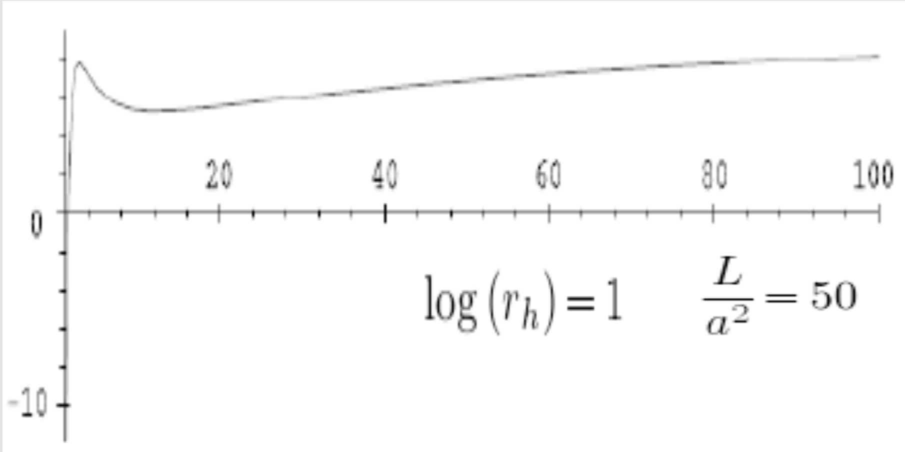
1. Weyl's geodesics – Graph of Potential V(r) depends on $\frac{L}{a^2}$

$$V = f(r)\left(\frac{L^2}{r^2}+1\right) \qquad f(r) = a^2 \log\left(\frac{r}{r_h}\right) = a^2 \log(r) - a^2 \log(r_h)$$

(Clément & Fabbri)



all timelike geodesics are captured



due to the ‘high’ angular momentum, bounded motion (also circular orbits) is possible.

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Black hole

Radial Weyl's geodesics ($L=0$ $e^{\frac{\varphi_0}{2}}=1$

1. Proper-time

$a^2=1$ $V(r_E)=E^2$ $V(r_h)=0$ $$\frac{dr}{d\tau}=\sqrt{E^2-V(r)}$$

$$\tau=-\sqrt{\pi}\, r_h\, e^E \operatorname{erf}(E^2-V(r))$$ $0\leq r\leq r_E$ $r_h\leq r_E$

$\tau(r_E)=0$ $\tau(r=r_h)=\sqrt{\pi}\, r_h\, e^E \operatorname{erf}(E^2)$ $\tau(r=0)=\sqrt{\pi}\, r_h\, e^E$

1. Coordinate time

$$d\tau=e^{\frac{\varphi_0}{2}}|a|\,dt\sqrt{\log\left(\frac{r}{r_h}\right)}$$

$$\frac{dr}{d\tau}=\sqrt{E^2-V(r)}$$

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Gravitational waves (pp waves, λ ω

1. Metric and Weyl´s scalar field

$$ds^2 = H(u,x)\,du^2 + 2\,du\,dv + dx^2 \qquad u = (t+x)/\sqrt{2},\ v = (t-x)/\sqrt{2}$$

$$\omega = \phi(u)$$

1. Field equations

$$\hat{R}_{uu} = -1/2 H_{,xx} = \lambda(\phi_{,u})^2 \qquad \hat{\Box}\omega = 0$$

1. Solution

$$H = A(u) + B(u)x - \lambda(\phi'(u))^2 x^2$$

$A(u)$ and $B(u)$ can both be freely specified

$\phi(u)$ is arbitrary

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Cosmological model

1. Dust (pressureless flluid)

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Conclusion

Questions

1. Causal structure of space-time.
   Singularities.
2. Matter coupling with gravitation
3. Conservation laws
4. Cosmology
5. Weyl´s transformations and symmetries

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Merci Beaucoup
(obrigado)