ABSTRACT

We present a canonical formulation of gravity theories whose Lagrangian is an arbitrary function of the Riemann tensor, which, for example, arises in the low-energy limit of superstring theories. Our approach allows a unified treatment of various subcases and an easy identification of the degrees of freedom of the theory.

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HAMILTONIAN FORMULATION OF f(Riemann) THEORIES OF GRAVITY

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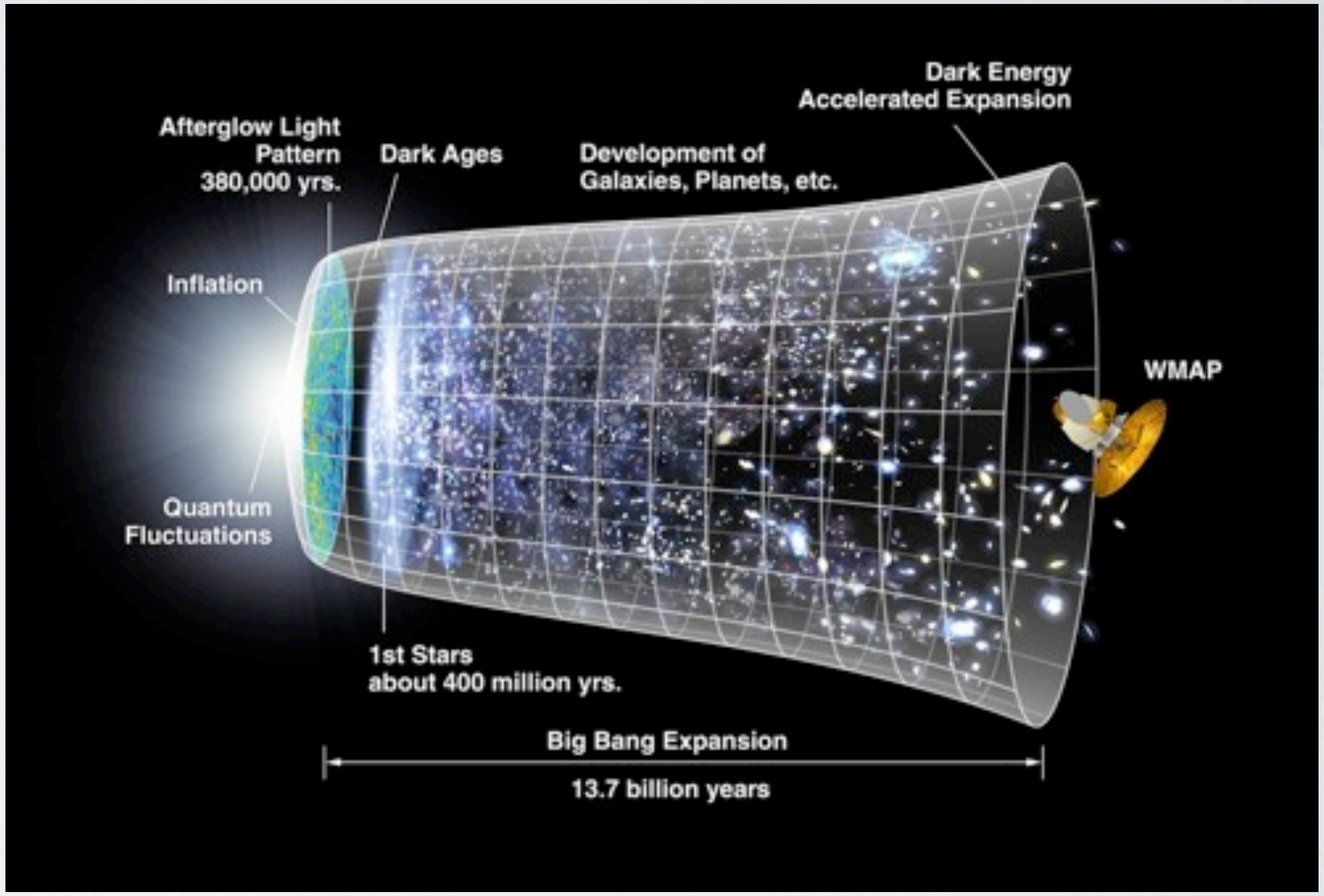
PLAN

- Motivations for "f (Riemann)" gravity
- 1st-order action
- Hamiltonian formulation
- Phase space reduction
- Conclusion

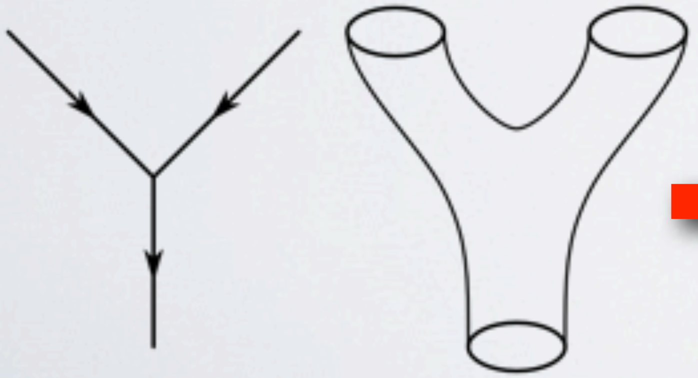
MOTIVATIONS

- Observational evidence
  - The Universe is undergoing accelerated expansion :
  - Not explained by known mechanisms.

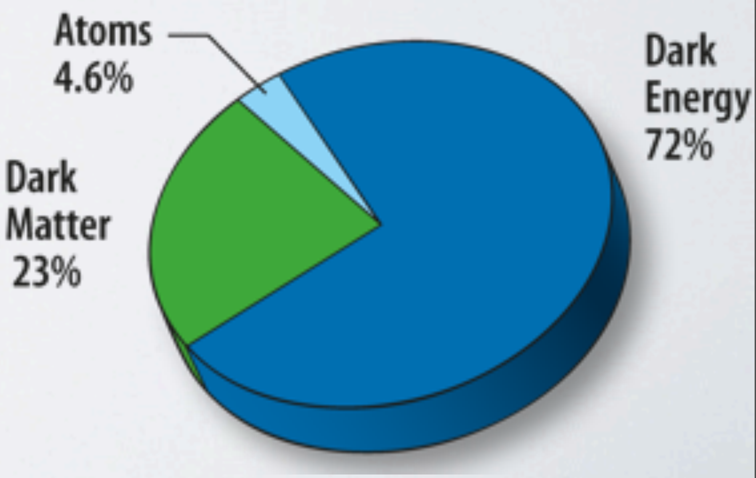


from NASA

- What is responsible for the acceleration? Dark energy? Modified gravity?



$$G_{\mu\nu} = \kappa T_{\mu\nu}$$



$f\,$(Riemann) GRAVITY

- Gravity theories with higher-curvature corrections

$$S = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x\,\sqrt{-g}\,f(\mathcal{R}_{abcd})$$

  - Much more than currently fashionable $f(\mathcal{R})$.

- Various motivations from high-energy (quantum) physics :
  - As counterterms to regularise $\langle T_{ab}\rangle$ on curved spaces [Utiyama & DeWitt (1962)]
  - Einstein-Hilbert action itself is not renormalisable ['t Hooft & Veltman (1974)]
  - String theories predict this kind of modifications ...

- String theories are still under development and their predictions are not secure. Cosmologists may have a chance to determine the true form of gravity from observations prior to particle physicists.

BASIC KNOWLEDGES

- Generally, the eom for the metric is 4th-order :

$$\mathcal{R}^{(a}{}_{cde}\,\frac{\partial f}{\partial \mathcal{R}_{b)cde}} + 2\boxed{\nabla_c \nabla_d \frac{\partial f}{\partial \mathcal{R}_{(a|c|b)d}}} - \frac{1}{2} f\, g^{ab} = \kappa\, T^{ab}$$

  - Because Riemann curvature tensor contains 2nd derivatives of metric.
  - New dynamical dofs other than the metric will appear.
- Some "landscape" of f (Riemann) :
  - $f = f(\mathcal{R})$: Metric+scalar on arbitrary background, any D — Classical argument
    - "Equivalent" to scalar-tensor gravity [Teyssandier & Tourrenc (1983), Maeda (1989)]
    - Used to explain accelerating expansion [$R+R^2$ Starobinsky (1980); $R+R^{-n}$ Capozziello et al. (2003), Carroll et al. (2004); etc.]
  - $f = \mathcal{R}^2 + \mathrm{Weyl}^2$: Metric+scalar+traceless tensor on $D = 4$ Minkowski [Stelle (1978)]
    - Tensor has negative kinetic term : "ghost"
  - Lovelock : 2nd-order eom, no extra dof [Lovelock (198?)] ...

WHAT TO DO

- Questions to answer :
  - How do we treat various sub-classes of f (Riemann) in a unified manner?
  - How is the form of f reflected by the gravitational dynamics?
  - How do we control the ghost? ...

- As a first step, we construct

  Hamiltonian (1st-order) formulation of f (Riemann) gravity

  - Dynamical (time-evolutional) properties become more transparent.
  - Useful for stability analysis ...

- We will keep f to be arbitrary as possible, but let me exclude **Lovelock** terms for a while ...

1ST-ORDER $f\,(\mathrm{Riemann})$ ACTION

BASIC IDEA

- We need an action consists of (at most) 1st derivatives.
- Remove 2nd derivatives

$$L = f(\phi, \dot{\phi}, \ddot{\phi})$$

$$\Rightarrow L = f(\phi, \dot{\phi}, \Omega) + \psi(\Omega - \ddot{\phi})$$ Auxiliary field

$$\Rightarrow L = f(\phi, \dot{\phi}, \Omega) + \psi\Omega + \dot{\psi}\dot{\phi}$$ Integration by parts

- Ω is determined (implicitly) in terms of ψ via

$$\frac{\partial f}{\partial \Omega}[\phi, \dot{\phi}, \Omega(\phi, \dot{\phi}, \psi)] + \psi = 0$$

  - Possible only when f is non-linear (non-degenerate) in the 2nd derivative.
- 2 dynamical dofs φ, ψ

$$L = f[\phi, \dot{\phi}, \Omega(\phi, \dot{\phi}, \psi)] + \psi\Omega(\phi, \dot{\phi}, \psi) + \dot{\psi}\dot{\phi}$$

- Although the real story is a bit more complicated...

Auxiliary fields

- We can first lower the order of derivative from 4 to 2.
- Eom contains higher-derivative (4th-order) due to nonlinearity of curvature (2nd derivative of metric) :

$$S_{\mathrm{g}}[g_{ab}] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x \,\sqrt{-g}\, f(\mathcal{R}_{abcd})$$

- An equivalent action being linear in curvature

$$S\,[g_{ab}, \varrho_{abcd}, \varphi^{abcd}] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x \,\sqrt{-g}\,[f(\varrho_{abcd}) + \varphi^{abcd}\,(\mathcal{R}_{abcd} - \varrho_{abcd})]$$

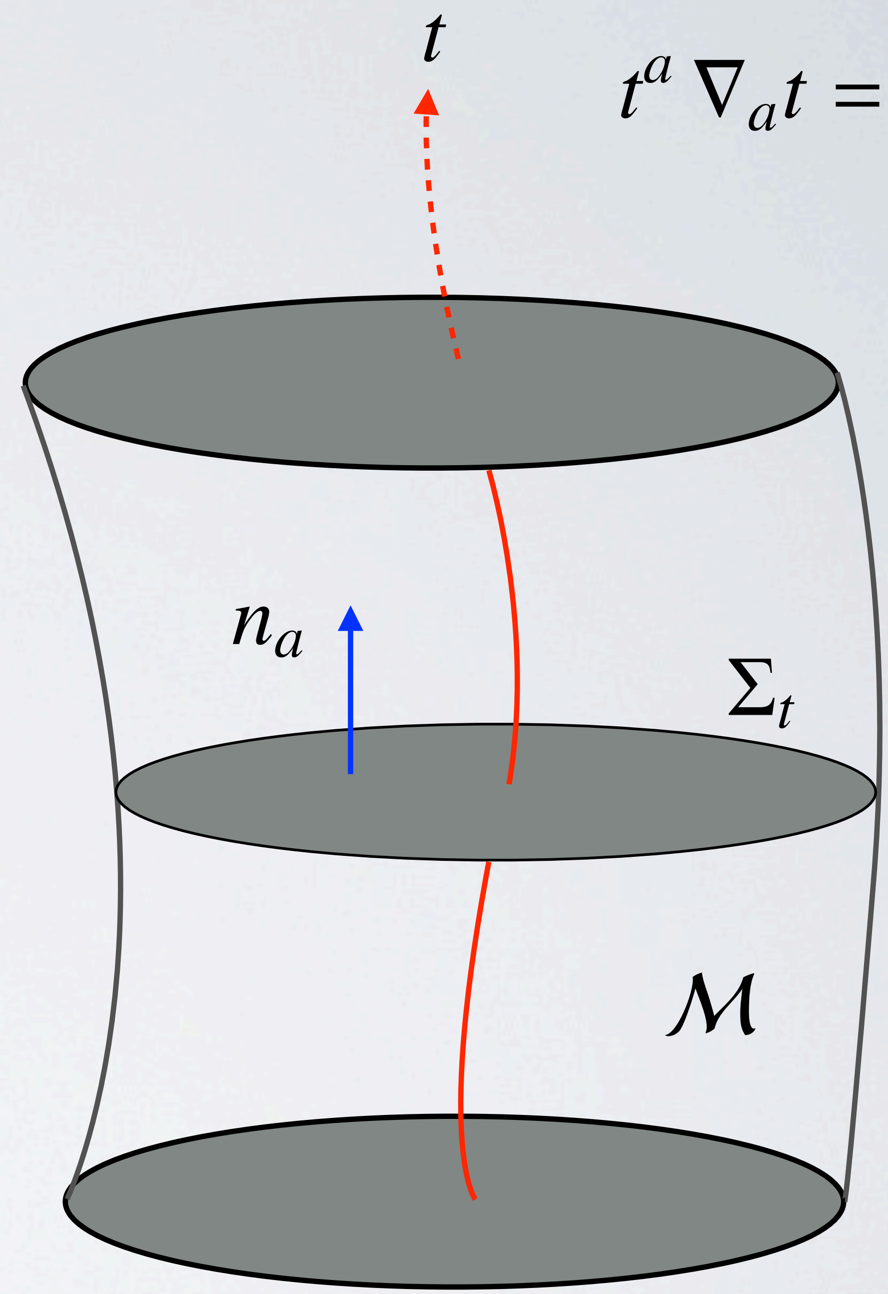
gives two 2nd-order eoms & one constraint equivalent to the 4th-order one :

$$\frac{\delta S}{\delta g_{ab}} = \frac{\delta S}{\delta \varrho_{abcd}} = \frac{\delta S}{\delta \varphi^{abcd}} = 0 \quad \Longleftrightarrow \quad \frac{\delta S_{\mathrm{g}}}{\delta g_{ab}} = 0$$

ADM DECOMPOSITION

- A geometrical way to define time
  [Arnowitt, Deser & Misner (1962)]

$$t^a \nabla_a t = 1$$



- Metric is decomposed into dynamical/non-dynamical parts:

$$g_{ab} \rightarrow \begin{cases} \gamma_{ab} = g_{ab} - \epsilon\, n_a n_b \\ N = \epsilon\, n_a t^a \\ \beta^a = \gamma^a{}_b t^b \\ (t^a = N n^a + \beta^a) \end{cases} \qquad \epsilon = n^a n_a$$

- Spacetime tensors will be orthogonally decomposed using induced metric and normal vector.

DECOMPOSITION OF ACTION

$\perp$: Projection by $\gamma_a{}^b$

$\mathbf{n}$: Contraction with n^a

- ADM decomposition of 2nd-order action :

$$S = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^D x \,\sqrt{-g}\,[f(\varrho_{abcd}) + \varphi^{abcd}\,(\mathcal{R}_{abcd} - \varrho_{abcd})]$$

$$_\perp\varphi^{abcd}\,(_\perp\mathcal{R}_{abcd} - {}_\perp\varrho_{abcd}) + 4\,\epsilon\,{}_\perp\varphi^{abc\mathbf{n}}\,(_\perp\mathcal{R}_{abc\mathbf{n}} - {}_\perp\varrho_{abc\mathbf{n}}) + 2\,\epsilon\,\Psi^{ab}\,(_\perp\mathcal{R}_{a\mathbf{n}b\mathbf{n}} - \Omega_{ab})$$

where $\quad \Psi^{ab} \equiv 2\,\epsilon\,{}_\perp\varphi^{a\mathbf{n}b\mathbf{n}}\,, \quad \Omega_{ab} \equiv {}_\perp\varrho_{a\mathbf{n}b\mathbf{n}}$

- Two eoms immediately determine the redundant components of auxiliary field :

$$\delta_\perp\varphi^{abcd} \;:\; {}_\perp\varrho_{abcd} = {}_\perp\mathcal{R}_{abcd}\,, \qquad \delta_\perp\varphi^{abc\mathbf{n}} \;:\; {}_\perp\varrho_{abc\mathbf{n}} = {}_\perp\mathcal{R}_{abc\mathbf{n}}$$

- Then we get

$$S = \frac{1}{2}\int_{\mathcal{M}} \sqrt{-g}\,[f + 2\,\epsilon\,\Psi^{ab}\,(_\perp\mathcal{R}_{a\mathbf{n}b\mathbf{n}} - \Omega_{ab})]$$

GEOMETRICAL RELATIONS

- Extrinsic curvature is the "velocity" of metric :

$$K_{ab} = \gamma_a{}^c \, \nabla_c n_b = \frac{1}{2\,N} \left(\dot{\gamma}_{ab} + 2\, D_{(a}\beta_{b)}\right)$$

- We have
  - Gauss

$$\gamma_a{}^e \, \gamma_b{}^f \, \gamma_c{}^g \, \gamma_d{}^h \, \mathcal{R}_{efgh} = -2\,\epsilon K_{a[c} \, K_{d]b} + R_{abcd}[\gamma]$$

  - Codazzi

$$\gamma_a{}^d \, \gamma_b{}^e \, \gamma_c{}^f \, n^g \, \mathcal{R}_{defg} = 2\, D_{[a} \, K_{b]c}$$

  - and Ricci relations

$$\gamma_a{}^c \, n^d \, \gamma_b{}^e \, n^f \, \mathcal{R}_{cdef} = -\pounds_n K_{ab} + K_{ac} \, K_b{}^c - \epsilon \, D_a D_b N$$

1st (time) derivatives

2nd (time) derivative

1ST-ORDER ACTION

- Integrating by parts, Ψ appears to be dynamical :

$$S = \int_{\mathcal{M}} \sqrt{|\gamma|}\, N \, [\epsilon \, \Psi^{ab} \, (K \, K_{ab} + K_{ac} \, K_b{}^c - \epsilon \, N^{-1} \, D_a D_b N - \Omega_{ab})$$

$$+ \, \epsilon \, N^{-1} \, K_{ab} \, \pounds_{t-\beta} \Psi^{ab} - \epsilon \, \nabla_c (n^c \, \Psi^{ab} \, K_{ab}) + \frac{1}{2} \, f]$$

- Divergence is canceled by the surface term :

$$\bar{S} = \epsilon \oint_{\partial \mathcal{M}} \mathrm{d}\Sigma_a \, n^a \, \Psi^{bc} \, K_{bc}$$

- No 2nd derivatives in the total action $S + \bar{S}$

- To be discussed : Eom for $\boldsymbol{\Omega}$ determines # of DOFs.

$$2 \, \epsilon \, \Psi^{ab} = \frac{\partial f}{\partial \Omega_{ab}}$$

Hamiltonian formulation

HAMILTONIAN

- Canonical momenta defined as

$$p^{ab} \equiv \frac{\delta(S + \bar{S})}{\delta \dot{\gamma}_{ab}}, \quad \Pi_{ab} \equiv \frac{\delta(S + \bar{S})}{\delta \dot{\Psi}^{ab}}$$

- Canonical action found via Legendre transformation

$$S + \bar{S} = \int \mathrm{d}t\, L = \int \mathrm{d}t \left[\int_{\Sigma_t} \mathrm{d}^{D-1}x\, \sqrt{|\gamma|}\, (p \cdot \dot{\gamma} + \Pi \cdot \dot{\Psi}) - H \right]$$

where

$$H = H[\gamma_{ab}, p^{ab}, \Psi^{ab}, \Pi_{ab}, \Omega_{ab}, N, \beta^a] = \int_{\Sigma_t} (N\, C + \beta^a\, C_a)$$

is the Hamiltonian, where Hamiltonian and momentum constraints are

$$C = -\epsilon\, \frac{2}{\sqrt{|\gamma|}}\, (\gamma^{a(b}\, \Psi^{c)d}\, \Pi_{ab}\, \Pi_{cd} - p \cdot \Pi) - \frac{\sqrt{|\gamma|}}{2}\, (-2\, \epsilon\, \Psi \cdot \Omega + f - 2\, D_a D_b \Psi^{ab})$$

$$C_a = -2\, \sqrt{|\gamma|}\, D_c \left(\gamma_{ab}\, \frac{p^{bc}}{\sqrt{|\gamma|}} - \Psi^{bc}\, \frac{\Pi_{ab}}{\sqrt{|\gamma|}} \right) + \Pi_{bc}\, D_a \Psi^{bc}$$

EOMs and constraints

- Constraints from variations wrt multipliers :

$$\delta N \; : \; \mathcal{C} = 0 \, , \quad \delta\beta^a \; : \; \mathcal{C}_a = 0$$

  - will (after second-class constraints are inserted into the action) turn to be first-class.

- Constraint from auxiliary field

$$\delta\Omega_{ab} \; : \; 2\,\epsilon\,\Psi^{ab} = \frac{\partial f}{\partial \Omega_{ab}}[\gamma_{ab}, \Pi_{ab}, \Omega_{ab}]$$

  - will be used to reduce # of non-dynamical variables.

- Canonical eoms from variations wrt dynamical variables:

$$\dot{\gamma}_{ab} = \frac{\delta H}{\delta p^{ab}} \, , \quad \dot{\Psi}^{ab} = \frac{\delta H}{\delta \Pi_{ab}}$$

$$\dot{p}^{ab} = -\frac{\delta H}{\delta \gamma_{ab}} \, , \quad \dot{\Pi}_{ab} = -\frac{\delta H}{\delta \Psi^{ab}}$$

These 7 eqs recover the original 4th-order eom.

TRACE DECOMPOSITION

- Ψ can be decomposed into the trace and traceless parts :

$$\Phi \equiv \frac{\gamma \cdot \Psi}{D-1}\,, \qquad \psi^{ab} \equiv {}_{\mathbb{T}}\Psi^{ab} \equiv \Psi^{ab} - \frac{\gamma \cdot \Psi}{D-1}\,\gamma^{ab}$$

Traceless

- There are (most generically) a scalar and (traceless) tensor degrees of freedom :

$$\begin{aligned} L &= \int_{\Sigma_t} (p \cdot \dot{\gamma} + \Pi \cdot \dot{\Psi}) - H[\gamma_{ab}, p^{ab}, \Psi^{ab}, \Pi_{ab}, \Omega_{ab}, N, \beta^a] \\ &= \int_{\Sigma_t} (\tilde{p} \cdot \dot{\gamma} + \Pi\,\dot{\Phi} + \pi \cdot \dot{\psi}) - H[\gamma_{ab}, \tilde{p}^{ab}, \Phi, \Pi, \psi^{ab}, \pi_{ab}, \Omega_{ab}, N, \beta^a] \end{aligned}$$

where

$$\tilde{p}^{ab} \equiv p^{ab} - \frac{1}{D-1}\left(\gamma \cdot \Pi\, \Psi^{ab} + \gamma \cdot \Psi\, {}_{\mathbb{T}}(\gamma^{ac}\,\gamma^{bd}\,\Pi_{cd})\right),$$

$$\Pi \equiv \gamma \cdot \Pi\,, \qquad \pi_{ab} \equiv {}_{\mathbb{T}}\Pi_{ab}$$

Done

- We've obtained canonical eoms and constraints for $f\,(\mathrm{Riemann})$ in the most generic form.

What to see below

- Any symmetries of f give rise to additional constraints.
- They are usually 2nd-class and to be inserted into the action to eliminate unnecessary variables.
- One exceptional case is of conformal gravity where conformal (gauge) transformation is generated by a constraint.

Phase space reduction

PROCEDURE

- Our generic Hamiltonian is still reducible in presence of 2nd-class constraints.
- The roles of the constraint eq :

$$\delta\Omega_{ab} \;:\; 2\,\epsilon\,\Psi^{ab} = \frac{\partial f}{\partial \Omega_{ab}}[\gamma_{ab}, \Pi_{ab}, \Omega_{ab}]$$

A) $\mathbf{\Omega}$ determined (f is nonlinear in $\mathbf{\Omega}$) : Ψ dynamical ... nothing happens

B) $\mathbf{\Omega}$ undetermined (f is at most linear in $\mathbf{\Omega}$) : Ψ non-dynamical

- Precisely, a scalar may arise from non-linearity of the trace while a tensor may arise from that of the traceless part of the second derivative.

$$\Omega_{ab} = \frac{\Omega}{D-1}\gamma_{ab} + \omega_{ab}$$

1st class, 2nd class

- Way to reduce action/Hamiltonian :
  1. Take time derivative of the above "primary" constraint to find a "secondary" constraint (Dirac)
  2. If they are 2nd-class, insert them into the canonical action to reduce action/Hamiltonian (Faddeev & Jackiw)

- 1st class constraints
  - commute with all the other constraints (modulo constraints),
  - generate gauge transformations ("Dirac conjecture").
  - should be kept to make gauge symmetries of the system explicit.

- 2nd class constraints
  - do not commute with at least one other constraint,
  - are safely inserted into the action to eliminate non-dynamical dofs.

Example 1: Einstein

- Action

$$f = \mathcal{R} \quad \Rightarrow \quad \delta\Omega_{ab} \; : \; \Psi^{ab} = \gamma^{ab}$$

- Constraints
  - Primary : $\Psi^{ab} = \gamma^{ab}$ (2nd-class)
  - Secondary : $\Pi_{ab} = 2\, p_{ab} - \frac{2\,\gamma \cdot p}{D}\, \gamma_{ab}$ (2nd-class)

- Action/Hamiltonian reduce into ADM's

$$L = \int_{\Sigma_t} \tilde{p} \cdot \dot{\gamma} - H[\gamma_{ab}, \tilde{p}^{ab}, N, \beta^a]$$

where

$$\tilde{p}^{ab} = -p^{ab} + \frac{2\,\gamma \cdot p}{D}\, \gamma^{ab}$$

  - No extra dof.

EXAMPLE 2: $f(\mathcal{R})$

- Action

$$f = f(\mathcal{R}) \quad \Rightarrow \quad \delta\Omega_{ab} \;:\; \Psi^{ab} = f'\,\gamma^{ab}$$

- Constraints
  - Primary : $\quad {}_{\mathbb{T}}\Psi^{ab} = 0 \qquad$ (2nd-class)
  - Secondary : ${}_{\mathbb{T}}\Pi_{ab} = \dfrac{2}{\Phi}\,\gamma_{ac}\,\gamma_{bd}\,{}_{\mathbb{T}}p^{cd}$ (2nd-class) $\qquad \Phi \equiv \dfrac{\gamma\cdot\Psi}{D-1} = f'$

- Reduced action/Hamiltonian

$$L = \int_{\Sigma_t} (\tilde{p}\cdot\dot{\gamma} + \Pi\,\dot{\Phi}) - H[\gamma_{ab}, \tilde{p}^{ab}, \Phi, \Pi, N, \beta^a]$$

where $\qquad \tilde{p}^{ab} = p^{ab} - \Phi\,\gamma^{ac}\,\gamma^{bd}\,\Pi_{cd}$

  - Extra scalar dof.

- Agrees with the independent result [Deruelle, YS, Youssef (2009)].

EXAMPLE 3: C^2

- Action

$$f = C_{abcd}\,C^{abcd} = \mathcal{R}_{abcd}\,\mathcal{R}^{abcd} - \frac{4}{D-2}\,\mathcal{R}_{ab}\,\mathcal{R}^{ab} + \frac{2}{(D-1)\,(D-2)}\,\mathcal{R}^2$$

$$\delta\Omega_{ab} \ : \ \Psi^{ab} = -\frac{4}{D-2}\left[(D-3)\,{}_{\mathbb{T}}\Omega^{ab} + {}_{\mathbb{T}}\rho^{ab}\right]$$

$$\rho_{ab} \equiv \frac{\Pi\,\Pi_{ab} - (\Pi\cdot\Pi)_{ab}}{|\gamma|} + R_{ab}[\gamma]$$

- Constraints
  - Primary : $\gamma \cdot \Psi = 0$ (2nd-class)
  - Secondary : $\gamma \cdot p - \frac{D}{2}\,{}_{\mathbb{T}}\Psi \cdot {}_{\mathbb{T}}\Pi = 0$ (depends on whether $D = 4$ or not)
  - The secondary constraint can be 1st-class depending on # of dimensions. This moment we only use the primary constraint.

- Action is reduced to be

$$L = \int_{\Sigma_t} (\tilde{p} \cdot \dot{\gamma} + \pi \cdot \dot{\psi}) - H[\gamma_{ab}, \tilde{p}^{ab}, \psi^{ab}, \pi_{ab}, \Pi, N, \beta^a]$$

where

$$\tilde{p}^{ab} \equiv p^{ab} - \frac{\gamma \cdot \Pi}{D-1} \psi^{ab} \,, \quad \psi^{ab} \equiv {}_{\mathbb{T}}\Psi^{ab} \,, \quad \pi_{ab} \equiv {}_{\mathbb{T}}\Pi_{ab} \,, \quad \Pi \equiv \gamma \cdot \Pi$$

  - Extra traceless tensor dof (but see below).

- Hamiltonian

$$H = \int_{\Sigma_t} (N\, C + \beta^a\, C_a + \Pi\, C_\Pi)$$

where Π works as a Lagrange multiplier and

$$C_\Pi \propto \gamma \cdot p - \frac{D}{2} \psi \cdot \pi$$

  - In $D = 4$, C_Π is the generator of conformal transformation and commute with other constraints. [Boulware (1984)]
  - If $D > 4$, more secondary constraints may arise. They might be used to further eliminate dofs (undone).

SUMMARY OF THIS PART

- Non-linearity of the second derivative determines what types of extra dofs arise :

	Tr part	Tr-less part	Extra dofs	Extra gauge sym.
$\mathcal{R}$	Linear	-	-	-
$f(\mathcal{R})$	NL	-	Scalar	-
C^2	-	NL	Tensor	Conformal $(D=4)$

Conclusion

CONCLUSION+

- Achievements
  - Hamiltonian formulation of $f\,(\text{Riemann})$ gravity has been established.
  - Effective & simple way to reduce generic Hamiltonian to those of typical sub-cases ($\mathcal{R}$, $f\,(\mathcal{R})$, C^2) was shown.

- Plans for the future [all in progress]
  - Properties of Ψ on various non-trivial backgrounds (e.g. FLRW, black holes) : Is there always "ghost"? If yes, what makes it harmless?
  - Lovelock terms
  - Energy in higher-derivative gravity theories
  - Coupling to matter, Surface term: e.g. Junction conditions for braneworld
  - Feedback to fundamental theories from phenomenological view point of gravity

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