Primordial Non-Gaussianity and Non-linear Formalisms for Multifield Inflation

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Outline

- Introduction
- Non-Gaussianity in the CMB
- Non-Gaussianity in single-field inflation
- $\delta N$ formalism
- Covariant formalism
- Non-Gaussianity in two-field inflation
- Non-Gaussianity in the adiabatic limit
- Fully non-linear equivalence of the two formalisms
- Conclusion
How can we explore the very early Universe if particle accelerators on the Earth cannot do?

Fig. from Baumann arXiv:0907.5424
How do we test inflation?

Can we answer a simple question: 
*How were primordial fluctuations generated?*
A very successful explanation is:

- *Primordial fluctuations were generated by quantum fluctuations of the scalar field that drove inflation.*

- The prediction: a nearly scale-invariant power spectrum in the curvature perturbations, $\zeta$:
  - $P_\zeta(k) = A/k^{4-n_s} \sim A/k^3$
  - where $n_s \sim 1$ and $A$ is a normalization.
  - Two-point function $\langle \hat{\zeta}(\tau, \mathbf{k})\hat{\zeta}(\tau, \mathbf{k}') \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}')P_\zeta(k)$
The latest results from CMB, BAO (SDSS DR7 Percival et al 2010), and SNe Ia (SHOES Riess et al 2009):

- \( n_s = 0.968 \pm 0.012 \) (68% CL)
- \( n_s \neq 1 \): another line of evidence for inflation
Beyond Power Spectrum

- All of these are based on fitting the observed power spectrum.
- Is there any information one can obtain, beyond the power spectrum?
Bispectrum

- Three-point function!

- \( B_\zeta(k_1, k_2, k_3) = \langle \hat{\zeta}(k_1) \hat{\zeta}(k_2) \hat{\zeta}(k_3) \rangle = (\text{amplitude}) \times (2\pi)^3 \delta^3(k_1 + k_2 + k_3) b(k_1, k_2, k_3) \)

  \( \text{shape of triangle} \)
(a) squeezed triangle \( k_1 \approx k_2 >> k_3 \)  
(b) elongated triangle \( k_1 = k_2 + k_3 \)  
(c) folded triangle \( k_1 = 2k_2 = 2k_3 \)  

(d) isosceles triangle \( k_1 > k_2 = k_3 \)  
(e) equilateral triangle \( k_1 = k_2 = k_3 \)
Fig. from Jeong & Komatsu arXiv:0904.0497

(a) squeezed triangle

\( k_1 \approx k_2 >> k_3 \)

Focus on this shape for today’s talk.
Why study bispectrum?

- It probes the **interactions of fields** – new piece of information that cannot be probed by the power spectrum.
- But, above all, it provides us with a **critical test** of the simplest models of inflation: “are primordial fluctuations Gaussian, or non-Gaussian?”
- Bispectrum vanishes for Gaussian fluctuations.
- Detection of the bispectrum = detection of non-Gaussian fluctuations.
Blue spots show directions on the sky where the CMB temperature is \( \sim 10^{-5} \) below the mean, \( T_0 = 2.7 \) K.

Yellow and red indicate hot (underdense) regions.
The one-point function of the CMB anisotropy looks pretty Gaussian.
- Left to right: Q (41GHz), V (61GHz), W (94GHz).
- Deviation from Gaussianity is small, if any.
Inflation likes this result


- According to inflation, the CMB anisotropy was created from **quantum fluctuations of a scalar field in Bunch-Davies vacuum** during inflation.

- Successful inflation (with the expansion factor more than \( e^{60} \)) **demands** the scalar field be almost interaction-free.

- Quantum vacuum fluctuations are Gaussian!
But, not exactly Gaussian

- Of course, there are always corrections to the simplest statement like this.
- Inflaton field *does* have interactions. They are simply weak — they are suppressed by the so-called slow-roll parameter, $\epsilon \sim O(0.01)$, relative to the free-field action.
A non-linear correction to temperature anisotropy

The CMB temperature anisotropy, $\Delta T/T$, is given by the curvature perturbation in the matter-dominated era, $\Phi$.

- On large scales (the Sachs-Wolfe limit), $\Delta T/T = -\Phi/3$.

Add a non-linear correction to $\Phi$:

- $\Phi(x) = \Phi_g(x) + f_{NL}[\Phi_g(x)]^2$ (Komatsu & Spergel 2001)
- $f_{NL}$ was predicted to be small ($\sim 0.01$) for slow-roll inflation.
  (Salopek & Bond 1990; Gangui et al. 1994)
The form of $B_\zeta$ is related to the primordial curvature perturbation, $\zeta$, as $\Phi = (3/5)\zeta$.

\[ \zeta(x) = \zeta_g(x) + (3/5)f_{NL}[\zeta_g(x)]^2 \]

\[ B_\zeta(k_1, k_2, k_3) = (6/5)f_{NL} \times (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \times [P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1)] \]
\( f_{NL} \): Shape of Triangle

- For a scale-invariant spectrum, \( P_\zeta(k) = A/k^3 \),
  \( B_\zeta(k_1, k_2, k_3) = (6A^2/5)f_{NL} \times (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \times [1/(k_1 k_2)^3 + 1/(k_2 k_3)^3 + 1/(k_3 k_1)^3] \)

- Let's order \( k_i \) such that \( k_3 \leq k_2 \leq k_1 \). For a given \( k_1 \), one finds the largest bispectrum when the smallest \( k \), i.e., \( k_3 \), is very small.
  
  - \( B_\zeta(k_1, k_2, k_3) \) peaks when \( k_3 \ll k_2 \sim k_1 \).
  
  - Therefore, the shape of \( f_{NL} \) bispectrum is the \textbf{squeezed triangle}!
    (Babich et al. 2004)

\[ (a) \text{ squeezed triangle} \]
\[ (k_1 \sim k_2 \gg k_3) \]
$B_\zeta$ in the Squeezed Limit

- In the squeezed limit, the $f_{NL}$ bispectrum becomes:

$$B_\zeta(k_1, k_2, k_3) \approx \left(\frac{12}{5}\right) f_{NL} \times (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \times P_\zeta(k_1) P_\zeta(k_3)$$

Why is this important?
For ANY single-field models*, the bispectrum in the squeezed limit is given by

\[ B_\zeta(k_1, k_2, k_3) \approx (1 - n_s) \times (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \times P_\zeta(k_1)P_\zeta(k_3) \]

Therefore, all single-field models predict \( f_{NL} \approx (5/12)(1 - n_s) \).

With the current limit \( n_s = 0.968 \), \( f_{NL} \) is predicted to be 0.013.

* for which the single field is solely responsible for driving inflation and generating observed fluctuations.
Therefore...

- A convincing detection of $f_{NL} \gg O(1)$ would rule out **ALL** of the single-field inflation models, **regardless of**:
  - the form of potential (See, however, Chen, Easther & Lim 2007)
  - the form of kinetic term (or sound speed) (See, e.g., Seery & Lidsey 2005)
  - the form of gravitational coupling (See, e.g., Germani & Watanabe 2011)
  - the initial vacuum state (See, however, Agullo & Parker 2011; Ganc 2011)

- A convincing detection of $f_{NL}$ would be a breakthrough.
Measurements

- **CMB** (WMAP 7-year Komatsu et al 2011)
  - $f_{NL} = 32 \pm 42$ (95% CL)
  - Planck’s expected error bar is $\sim 5$ (68% CL)!

- **CMB and LSS** (Slosar et al 2008)
  - $f_{NL} = 27 \pm 32$ (95% CL)
If $f_{NL}$ is detected, in what kind of models?

- Detection of $f_{NL} = \text{multi-field models}$

- In multi-field inflation models, $\zeta(k)$ can evolve outside the horizon.
  - Curvaton mechanism (Linde & Mukhanov 1997)
  - Inhomogeneous reheating (Dvali, Gruzinov & Zaldarriaga 2004)

- This evolution can give rise to non-Gaussianity; however, causality demands that the form of non-Gaussianity must be \textbf{local}!
  - $\zeta(x) = \zeta_g(x) + (3/5)f_{NL}[\zeta_g(x)]^2 + AS_g(x) + B[S_g(x)]^2 + \cdots$
How to compute 2nd order in $\zeta$?

- Cosmological perturbation theory
  - Very hard because 2nd order
  - Straightforward

- The $\delta N$ formalism
  Starobinsky 1985; Salopek & Bond 1990; Sasaki & Stewart 1996
  - $\zeta = \delta N$ on super-horizon scales
  - Very popular in the literature
  - $\delta N$ is popular and powerful: it gives the statistics of perturbations \textit{without} solving equations for perturbations!! “It’s like a magic.”
The $\delta N$ formalism: $\zeta = \delta N$

- $N =$ number of e-folds counted backward in time (from the end of inflation) $\sim \log[\text{expansion}]
  \bullet a(t_{\text{end}})/a(t) = \exp[N] \Rightarrow N(\varphi) = \int_{t(\varphi)}^{t_{\text{end}}} Hdt = \ln [a(\varphi_{\text{end}})/a(\varphi)]$

\[ N = N(\phi) \]

\[ L = H^{-1} \sim t \]

\[ L = H^{-1} \sim \text{const} \]

\[ t = t(\phi) \quad t = t_{\text{end}} \]
The $\delta N$ formalism: intuitive picture

- Difference in $\log[\text{expansion}]$ is $\zeta$. 

![Diagram showing log[a(x1)] and log[a(x2)] with more expanded region on the right.]

The $\delta N$ formalism: more precise definition

- $\zeta = \delta N$ from an initial \textit{flat} time slice to a final \textit{uniform density} time slice on super-horizon scales.

\[
\delta N = N(t_2, t_1; x) - N_0(t_2, t_1), \quad N_0(t_2, t_1) = \ln \left( \frac{a(t_2)}{a(t_1)} \right)
\]

$\rho(t_2) = \text{const.}$

$\rho(t_1) = \text{const.}$

$\psi(t_1) = 0$
\( \delta N \) for slow-roll inflation

Sasaki & Tanaka 1998; Lyth & Rodriguez 2005

- In slow-roll inflation, the evolution, \( N \), is determined only by the field value, \( \varphi \).
- Non-linear \( \delta N \) for multi-field inflation:

\[
\delta N = N(\varphi^I + \delta\varphi^I) - N(\varphi^I) \approx \sum_I N_I \delta\varphi^I + \frac{1}{2} \sum_{I,J} N_{IJ} \delta\varphi^I \delta\varphi^J,
\]

where derivatives are evaluated at the horizon exit: \( N_I \equiv \frac{\partial N}{\partial \varphi^I} \).
- Non-Gaussianity is given by

\[
\frac{3}{5} f_{NL} = \frac{\sum_{I,J} N_I N_J N_{IJ}}{2[\sum_I N_I N_I]^2}
\]
Linear perturbation theory in multi-field inflation

\[ ds^2 = -(1 + 2A)dt^2 + 2aB_i dx^i dt + a^2 [(1 - 2\psi)\delta_{ij} + 2E_{ij}] dx^i dx^j \]

- \( \delta\varphi^I \)'s determine how curvature perturbations, \( \psi \), evolve.

\[ \dddot{\delta\varphi^I} + 3H\dot{\delta\varphi^I} + \frac{k^2}{a^2} \delta\varphi^I + \sum_J V_{IJ} \delta\varphi^J = -2V_i A + \dot{\varphi}^I \left[ A + 3\psi + \frac{k^2}{a^2} (a^2 \ddot{E} - aB) \right] \]

\[ 3H \left( \dot{\psi} + HA \right) + \frac{k^2}{a^2} \left[ \psi + H(a^2 \ddot{E} - aB) \right] = -4\pi G \delta\rho \]
\[ \dot{\psi} + HA = -4\pi G \delta q \]
\[ \delta\rho = \sum_l \left[ \dot{\varphi}_l \left( \delta\varphi_l - \dot{\varphi}_l A \right) + V_{\varphi_l} \delta\varphi_l \right] \]
\[ \delta q, i = -\sum_l \dot{\varphi}_l \delta\varphi_l, i \]
Adiabatic and entropic perturbations
Gordon, Wands, Bassett & Maartens 2000

\[ \delta \sigma^{(1)} = \cos \theta \delta \phi + \sin \theta \delta \chi, \quad \delta s^{(1)} = - \sin \theta \delta \phi + \cos \theta \delta \chi \]

\[ \ddot{\sigma} + 3H \dot{\sigma} + \left( \frac{k^2}{a^2} + V_{\sigma \sigma} - \dot{\theta}^2 \right) \delta \sigma \]
\[ = -2V_{\sigma} A + \dot{\sigma} \left[ \dot{A} + 3\dot{\psi} + \frac{k^2}{a^2} (a^2 \dot{E} - aB) \right] + 2(\dot{\theta} \delta \dot{s}) - 2 \frac{V_{\sigma}}{\dot{\sigma}} \dot{\theta} \delta s, \]
\[ \ddot{s} + 3H \dot{s} + \left( \frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2 \right) \delta s = \frac{\dot{\theta}}{\dot{\sigma}} \frac{k^2}{2\pi G a^2} \Psi \]

\[ \dot{\sigma} = (\cos \theta) \dot{\phi} + (\sin \theta) \dot{\chi} \quad \dot{\theta} = - \frac{V_{\sigma}}{\dot{\sigma}} \]

- Gauge-invariant curvature perturb. is sourced only by entropy perturb. If the trajectory is curved, it can change on large scales.

\[ - \zeta \equiv \psi + H \frac{\delta \rho}{\dot{\rho}} \quad - \dot{\zeta} = \frac{H k^2}{H a^2} \Psi + \frac{2H}{\dot{\sigma}} \dot{\theta} \delta s \]
\[ \delta \sigma^{(1)} = \cos \theta \delta \phi + \sin \theta \delta \chi , \quad \delta s^{(1)} = -\sin \theta \delta \phi + \cos \theta \delta \chi \]

\[ \delta \dot{\sigma} + 3H \delta \sigma + \left( \frac{k^2}{a^2} + V_{\sigma \sigma} - \dot{\theta}^2 \right) \delta \sigma \]

\[ = -2V_{\sigma} A + \dot{\sigma} \left[ \dot{A} + 3\psi + \frac{k^2}{a^2} (a^2 \dot{E} - aB) \right] + 2(\dot{\theta} \delta s) - 2\frac{V_{\sigma}}{\dot{\sigma}} \dot{\theta} \delta s , \]

\[ \ddot{\delta s} + 3H \dot{\delta s} + \left( \frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2 \right) \delta s = \frac{\dot{\theta}}{\dot{\sigma}} \frac{k^2}{2\pi G a^2 \Psi} \]

- Gauge-invariant curvature perturb. is sourced only by entropy perturb. If the trajectory is curved, it can change on large scales.

\[ -\zeta \equiv \psi + H \frac{\delta \rho}{\dot{\rho}} \quad -\dot{\zeta} = \frac{H}{\dot{H}} \frac{k^2}{a^2} \Psi + \frac{2H}{\dot{\sigma}} \dot{\theta} \delta s \]
Two approaches to non-linear $\zeta$

- **Covariant formalism (valid on all scales)** [Hawking 1966; Ellis, Hwang & Bruni 1989; Langlois & Vernizzi 2005]
  \[
  \dot{\zeta}_\mu \equiv \mathcal{L}_u \zeta_\mu = -\frac{\dot{N}}{\rho + p} \left( \partial_\mu p - \frac{\dot{p}}{\dot{\rho}} \partial_\mu \rho \right)
  \]
  \[
  \zeta_\mu \equiv \partial_\mu N - \frac{\dot{N}}{\dot{\rho}} \partial_\mu \rho,
  \quad N \equiv \frac{1}{3} \int d\tau \Theta
  \]

- **$\delta N$ formalism (valid on large scales)** [Starobinsky 1985; Salopek & Bond 1990; Stewart & Sasaki 1996; Lyth, Malik & Sasaki 2005]
  \[
  \zeta = \delta N - \int_\rho^\rho \frac{N'}{\rho'} d\rho
  \]

Are they equivalent on large scales?
If yes, which approach has more advantages?
2nd order $\zeta$ in two-field inflation

- **Covariant formalism** (Rigopoulos et al 2004; Langlois & Vernizzi 2007)

\[
\dot{\zeta} = -\frac{2H}{\dot{\sigma}} \dot{\theta} (\delta s^{(1)} + \delta s^{(2)}) + \frac{H}{\dot{\sigma}^2} (V_{ss} + 4\dot{\theta}^2) \delta s^{(1)}^2 - \frac{H}{\dot{\sigma}^3} V_{\sigma} \delta s^{(1)} \dot{\delta s}^{(1)}
\]

\[
\ddot{\delta s} + 3H \dot{\delta s} + (V_{ss} + 3\dot{\theta}^2) \delta s = -\frac{\dot{\theta}}{\dot{\sigma}} (\delta s^{(1)})^2 - \frac{2}{\dot{\sigma}} \left( \ddot{\theta} + \frac{\dot{\theta}}{\dot{\sigma}} V_{\sigma} - \frac{3}{2} H \dot{\theta} \right) \delta s^{(1)} \dot{\delta s}^{(1)} - \left( \frac{1}{2} V_{sss} - \frac{5}{\dot{\sigma}} V_{ss} - \frac{9}{\dot{\sigma}^3} \right) (\delta s^{(1)})^2
\]

\[
\zeta(t, x) = \zeta^*_*(x) + \delta s^*_*(x) T^{(1)}_\zeta(t, x) + \delta s^2_*(x) T^{(2)}_\zeta(t, x)
\]

\[
\delta s(t, x) = \delta s^*_*(x) T^{(1)}_\delta s(t, x) + \delta s^2_*(x) T^{(2)}_\delta s(t, x)
\]

- **$\delta N$ formalism** (Sasaki & Tanaka 1998; Lyth & Rodriguez 2005)

\[
\zeta = \delta N = \sum_I N_I \delta \varphi_{I*} + \frac{1}{2} \sum_{I,J} N_{IJ} \delta \varphi_{I*} \delta \varphi_{J*}
\]
\( f_{NL} \) in two-field inflation

- Covariant formalism

\[
f_{NL} = f_{NL}^{\text{transfer}} + f_{NL}^{\text{horizon}} \sim \frac{5}{3} \frac{T_\zeta^{(2)}}{T_\zeta^{(1)}^2}
\]

\[
\frac{3}{5} f_{NL}^{\text{transfer}} = \frac{4\epsilon_*^2 \left[ T_\zeta^{(1)}(t) \right]^2 T_\zeta^{(2)}(t)}{\left[ 1 + 2\epsilon_* \left( T_\zeta^{(1)}(t) \right)^2 \right]^2}
\]

\[
\frac{3}{5} f_{NL}^{\text{horizon}} \approx - (\epsilon \eta_{ss})_* \left[ T_\zeta^{(1)}(t) \right]^2 + \frac{3}{2} \eta_{s} \sigma_s \star T_\zeta^{(1)}(t) + \left( \epsilon - \frac{\eta_{s} \sigma_s}{2} \right)_* \left[ 1 + 2\epsilon_* \left( T_\zeta^{(1)}(t) \right)^2 \right]^2
\]

- \( \delta N \) formalism (Lyth & Rodriguez 2005)

\[
\frac{3}{5} f_{NL} = \frac{\sum_{I,J} N_I N_J N_{IJ}}{2 \left[ \sum_I N_I N_I \right]^2}
\]

\[
Yuki Watanabe, COSMO, Tokyo, 27 Sep. 2010
\]

- Covariant formalism

\[
f_{NL} = f_{NL}^{\text{transfer}} + f_{NL}^{\text{horizon}} \sim \frac{5}{3} \frac{T_\zeta^{(2)}}{T_\zeta^{(1)}^2}
\]

\[
\frac{3}{5} f_{NL}^{\text{transfer}} = \frac{4\epsilon_*^2 \left[ T_\zeta^{(1)}(t) \right]^2 T_\zeta^{(2)}(t)}{\left[ 1 + 2\epsilon_* \left( T_\zeta^{(1)}(t) \right)^2 \right]^2}
\]

\[
\frac{3}{5} f_{NL}^{\text{horizon}} \approx - (\epsilon \eta_{ss})_* \left[ T_\zeta^{(1)}(t) \right]^2 + \frac{3}{2} \eta_{s} \sigma_s \star T_\zeta^{(1)}(t) + \left( \epsilon - \frac{\eta_{s} \sigma_s}{2} \right)_* \left[ 1 + 2\epsilon_* \left( T_\zeta^{(1)}(t) \right)^2 \right]^2
\]

- \( \delta N \) formalism (Lyth & Rodriguez 2005)

\[
\frac{3}{5} f_{NL} = \frac{\sum_{I,J} N_I N_J N_{IJ}}{2 \left[ \sum_I N_I N_I \right]^2}
\]
“Large” NG in two-field inflation

\[ V(\phi, \chi) = \frac{m_1^2}{2} \phi^2 + \frac{m_2^2}{2} \chi^2, \quad m_1 < m_2 \ll H_* \] (1)

- \( f_{NL} \sim \mathcal{O}(1 - 10) \) for \( m_1/m_2 = 1/9 \) [Rigopoulos et al 2005]
- \( f_{NL} \sim \mathcal{O}(0.01) \) [Vernizzi & Wands 2006; Rigopoulos et al 2006; S. Yokoyama et al 2007]

\[ V(\phi, \chi) = \frac{m_2^2}{2} \chi^2 e^{-\lambda \phi^2} \] (2)

- Large negative \( f_{NL} \) [Byrnes et al 2008; Mulryne et al 2009]

\[ V(\phi, \chi) = a_2 \chi^2 + b_0 - b_2 \phi^2 + b_4 \phi^4 \] (3)

- Large \( f_{NL} \) [Tzavara & van Tent 2011]
Numerical estimate: $f_{NL}$ in two-field inflation

$$V(\phi, \chi) = \frac{m_1^2}{2} \phi^2 + \frac{m_2^2}{2} \chi^2$$

$\chi$ is the 1st inflaton.

$\phi$ is the 2nd inflaton.

- A peak in NG shows up at the turn. It is sourced by entropy modes.
- The plateau contribution of NG is from the horizon exit $\sim O(\epsilon) \sim 0.01$.
- $\delta N$ and covariant formalisms match within $\sim 1\%$.
- Slow-roll approx. has been used only for the initial condition (at horizon exit).
How did a peak in $f_{NL}$ show up at the turn?

- Each term in 2nd order perturbations becomes large but almost cancels out!
- The difference in growths of terms makes the peak shape. Only small net effect remains because of symmetry of the potential.
Relation between $\delta N$ and perturbations

$\dot{\zeta} = -\frac{2H}{\dot{\sigma}} \dot{\theta} \delta s$

$\dot{\theta} \delta s < 0$

- A trajectory of $\zeta$ is “kicked” by an entropy mode.
- $\zeta$ is sourced at the turn.
Numerical estimate: $f_{NL}$ in two-field inflation

$$V(\phi, \chi) = \frac{m_1^2}{2} \phi^2 + \frac{m_2^2}{2} \chi^2$$

$\chi$ is the 1st inflaton.

$\phi$ is the 2nd inflaton.

- **A few large peaks** in NG show up at the turn.
- The plateau contribution of NG is from the horizon exit $\sim O(\epsilon) \sim 0.01$.
- $\delta N$ and covariant formalisms match within $\sim 1\%$ except at peaks.
- Discrepancy is from inaccuracies of data-sampling at peaks and the initial condition.
Numerical estimate: $f_{NL}$ in two-field inflation

$$V(\phi, \chi) = \frac{m^2}{2}\chi^2 e^{-\lambda \phi^2}$$

[Byrnes et al 2008; Mulryne et al 2009]

- Large negative NG shows up during the turn. But $f_{NL} \sim -2.1$ after inflation.
Numerical estimate: $f_{\text{NL}}$ in two-field inflation

$$V(\phi, \chi) = a_2 \chi^2 + b_0 - b_2 \phi^2 + b_4 \phi^4$$

[Tzavara & van Tent 2011]

- $f_{\text{NL}} \sim 1.2$
Fate of $f_{NL}$

- It is difficult to have large $f_{NL}$ in two-field inflation. The asymptotic values are model-dependent.
- After all entropy modes decay, the inflationary trajectory approaches to the adiabatic limit in which $\zeta$ is conserved and one can make predictions for observations.
- In this case, there are two regimes:
  - Initially, entropy modes are light and source $\zeta$.
  - Eventually, they get heavy and damps away.

$\Downarrow$

**How fast $f_{NL}$ approaches to its final value?**
Analytic estimate: $f_{NL}$ in the adiabatic limit

$$|\delta s^{(1)}| \approx \frac{2^{\Re(\nu) - 3/2} |\Gamma(\nu)|}{\sqrt{2H}} a^{-3/2} \left( \frac{k}{aH} \right)^{-\Re(\nu)}$$

$$\nu = \sqrt{\frac{9}{4} - \left( 3\eta_{ss} + \frac{3\dot{\theta}^2}{H^2} \right)}$$

In order to answer the fate of $f_{NL}$, we solve the super-Hubble evolution of $\zeta$ in three cases:

- **(A) Overdamped (light)** $\delta s \sim a^{-\eta_{ss}}$: $\eta_{ss} \ll 3/4$ and $(\dot{\theta}/H)^2 \ll 3/4$
- **(B) Underdamped (heavy)** $\delta s \sim a^{-3/2}$: $\eta_{ss} \gg 3/4$ and $(\dot{\theta}/H)^2 \ll 3/4$
- **(C) Underdamped (heavy)** $\delta s \sim a^{-3/2}$: $\eta_{ss} \gg 3/4$ and $(\dot{\theta}/H)^2 \gg 3/4$
Analytic estimate: \( f_{NL} \) in the adiabatic limit

In order to answer the fate of \( f_{NL} \), we solve the super-Hubble evolution of \( \zeta \) in three cases:

- **(A) Overdamped (light):** slow-roll & slow-turn \( \sim \) constant
  \[
  \zeta \simeq \zeta_1 - \frac{\eta \sigma_s}{\eta_{ss}} \sqrt{\frac{2}{\epsilon}} \delta s_1 \left( \frac{a}{a_1} \right)^{-\eta_{ss}} - \frac{\eta^2 \sigma_s}{\epsilon \eta_{ss}} \delta s^2_1 \left( \frac{a}{a_1} \right)^{-2\eta_{ss}}
  \]

- **(B) Underdamped (heavy):** slow-roll & slow-turn

- **(C) Underdamped (heavy):** fast-turn
Analytic estimate: $f_{NL}$ in the adiabatic limit

In order to answer the fate of $f_{NL}$, we solve the super-Hubble evolution of $\zeta$ in three cases:

- **(A) Overdamped (light):** slow-roll & slow-turn

- **(B) Underdamped (heavy):** slow-roll & slow-turn $\dot{\phi}/H \sim \eta_{\sigma s} \sim a^{-\eta_{ss}}$

\[
\zeta \simeq \zeta_1 - \frac{\eta_{\sigma s1}(\eta_{ss}/3 + 1)}{(\eta_{ss} + 3/2)} \sqrt{\frac{2}{\epsilon}} \delta s_1 \left( \frac{a}{a_1} \right)^{-\eta_{ss} - 3/2} - \frac{\eta_{ss} - 3/2}{2\epsilon} \delta s_2 \left( \frac{a}{a_1} \right)^{-3} - \frac{\eta_{\sigma s1}(\eta_{ss}/3 + 1)^2}{(\eta_{ss} + 3/2)\epsilon} \delta s_2 \left( \frac{a}{a_1} \right)^{-2\eta_{ss} - 3}
\]

- **(C) Underdamped (heavy):** fast-turn
Analytic estimate: $f_{NL}$ in the adiabatic limit

In order to answer the fate of $f_{NL}$, we solve the super-Hubble evolution of $\zeta$ in three cases:

- (A) Overdamped (light): slow-roll & slow-turn
- (B) Underdamped (heavy): slow-roll & slow-turn
- (C) Underdamped (heavy): fast-turn $\dot{\theta}/H \sim a^{-3/2}$

\[
\zeta \simeq \zeta_1 - \frac{C_\theta}{3} \sqrt{\frac{2}{\epsilon}} \delta s_1 \left( \frac{a}{a_1} \right)^{-3} - \frac{\eta_{ss} - 3/2}{2\epsilon} \delta s_1^2 \left( \frac{a}{a_1} \right)^{-3} - \frac{C^2_\theta}{3\epsilon} \delta s_1^2 \left( \frac{a}{a_1} \right)^{-6}
\]
Analytic estimate: $f_{NL}$ in the adiabatic limit

$N/m_1/m_2 = 1/4$

$|f_{NL} - f_{NL}(\text{asymp})|

\eta_{ss}, \eta_{os}$
Analytic estimate: $f_{NL}$ in the adiabatic limit

$m_1/m_2=1/20$

Log$_{10}|f_{NL}-f_{NL}(\text{asymp})|$
Fully non-linear equivalence between $\delta N$ and covariant formalisms

Suyama, YW and Yamaguchi 2012

$$\zeta_\mu = \partial_\mu N - \frac{\dot{N}}{\dot{\rho}} \partial_\mu \rho, \quad \dot{\zeta}_\mu = -\frac{\Theta}{3(\rho + P)} \left( \partial_\mu P - \frac{\dot{P}}{\dot{\rho}} \partial_\mu \rho \right)$$

\[\downarrow\] setting the ADM metric with $\beta_i = O(\epsilon = k/aH)$ and on $\Sigma_\rho$

$$ds^2 = -N^2 dt^2 + a^2 e^{2\psi} (e^h)_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\zeta_i = \frac{1}{N} \partial_i \psi' + O(\epsilon^3), \quad -\frac{\Theta}{3(\rho + P)} \left( \partial_i P - \frac{\dot{P}}{\dot{\rho}} \partial_i \rho \right) = -\frac{\tilde{H}}{\dot{\rho} + P} \partial_i P + O(\epsilon^3)$$

\[\downarrow\] integrating over $x^i$ and choosing an integration constant

$$\psi' = -\frac{\rho'}{3(\rho + P)} - \frac{a'}{a} + O(\epsilon^2)$$
Conclusions

- We have re-examined the super-Hubble evolution of the primordial NG in two-field inflation by taking two approaches: the $\delta N$ and the covariant perturbative formalisms.
- The results agree within 1% accuracy in two-field inflation models.
- The peak feature appears on $f_{NL}$ at the turn in the field space, which can be understood as the precise cancellation between terms in the perturbed equation.
- It is difficult to have persistently large NG in two-field inflation.
- NG decays no faster than $a^{-3}$ in the adiabatic limit.