5 DoF Gravity Potentials

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- Building 5 DoF Potentials $V(N,\ N^i,\ \gamma^{ij})$

$$\int d^4x\,\sqrt{\mathrm{g}}\,\Big[\,M_{\mathrm{PL}}^2\,\,\big(R - m^2\,V\big) + \mathcal{L}_{\mathrm{matter}}\Big]$$

  - Dirac's Constrained Hamiltonian Dynamics
  - Constraints $\rightarrow$ Partial Diff Eqs in field space

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  - Dirac's Constrained Hamiltonian Dynamics
  - Constraints $\to$ Partial Diff Eqs in field space
- Phenomenology around Minkowski
- FRW Cosmology of 5 DoF Potentials

Canonical Analysis Review

$$\text{Configuration space} : \phi_i, \qquad i = 1, ..., N, \qquad \textit{Lagrangian} \;\; \mathcal{L}(\phi_i, \; \dot{\phi}_i)$$

$$\text{Phase space} : (\phi_i, \; \Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}) \qquad r = Rank|\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \, \partial \dot{q}_j}| \leq N$$

$$\begin{aligned}
\textit{Primary} : & \qquad \mathcal{P}_a(q, \; \dot{q}, \Pi) = 0, \quad a = N - r \\
\textit{Total Hamiltonian} : & \qquad H_T = \Pi \, \dot{q} - \mathcal{L} + \lambda \, \mathcal{P} \\
\text{conservation of the constraints} : & \qquad \dot{\mathcal{P}} = \{\mathcal{P}, \; H_T\} = 0 \\
\{\textit{Primary}, \; \textit{Secondary}, \; \textit{Tertiary}, \; ...\} & \;\; \rightarrow \;\; \{\{\mathcal{F}.\mathcal{C}.\}, \; \{\mathcal{S}.\mathcal{C}.\}\} \\
\{\mathcal{F}.\mathcal{C}., \; \mathcal{F}.\mathcal{C}.\} = 0, & \quad \{\mathcal{S}.\mathcal{C}., \; \mathcal{S}.\mathcal{C}.\} \neq 0
\end{aligned}$$

$$\boxed{DoF = \frac{2\,N - \#(\mathcal{S}.\mathcal{C}.) - 2\,\#(\mathcal{F}.\mathcal{C}.)}{2}}$$

DoF

- 10 DoF in $g_{\mu\nu}$:

$$\boxed{+4 \text{ scalars}} \quad \boxed{+4 \text{ vectors}} \quad \boxed{+2 \text{ tensors}}$$

DoF

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$+4$ scalars $+4$ vectors $+2$ tensors

- 4 DoF in $g_{0\mu}$ are non dynamical (for non derivative modifications of GR):

-2 scalars -2 vectors 0 tensors

DoF

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$+4$ scalars	$+4$ vectors	$+2$ tensors

- 4 DoF in $g_{0\mu}$ are non dynamical (for non derivative modifications of GR):

-2 scalars	-2 vectors	0 tensors

- In GR, general coordinate invariance $x^\mu \to x^\mu + \xi^\mu$:

-2 scalars	-2 vectors	0 tensors

- In GR DoF=2 (tensors)
- Without Diff. DoF$\leq$ 6 (2 scalars + 2 vectors + 2 tensors)
- A Massive spin 2 in d=4 has 5 DoF (1 scalar + 2 vectors + 2 tensor)

Canonical Analysis for GR

Phase Space 2×10 dim: $(\mathrm{N}, \mathrm{N}^{\mathrm{i}}, \gamma_{\mathrm{ij}}) + (\Pi_0, \Pi_i, \pi^{ij})$, $\quad \mathrm{N}^{\mathrm{A}} \equiv (\mathrm{N}, \mathrm{N}^{\mathrm{i}})$

ADM decomposition

$$\mathrm{g}^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2}\, N^i \\ N^{-2}\, N^i & \gamma^{ij} - N^{-2} N^i N^j \end{pmatrix}, \qquad H = \int d^3x \;\; \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}}(\gamma, \pi)\, \mathrm{N}^{\mathrm{A}}$$

Canonical Analysis for GR

Phase Space 2×10 dim: $(\mathrm{N},\, \mathrm{N^i},\, \gamma_{\mathrm{ij}}) \,+\, (\Pi_0,\, \Pi_i,\, \pi^{ij}),\quad \mathrm{N^A} \equiv (\mathrm{N},\ \mathrm{N^i})$

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- Primary $\quad \Pi_{\mathrm{A}} \approx 0$ and then $H \to H_{\mathrm{T}} = H + \int d^3x\, \lambda^A\, \Pi_A$

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- Secondary $\ \mathcal{S}_{\mathrm{A}} = \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}} \approx 0$

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- Secondary $\ \mathcal{S}_{\mathrm{A}} = \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}} \approx 0$
- Tertiary $\qquad \mathcal{T}_{\mathrm{A}} = \{\mathcal{S}_{\mathrm{A}},\ H\} \propto \underbrace{\{\mathcal{H}^{\mathrm{GR}}_{\mathrm{A}},\ \mathcal{H}^{\mathrm{GR}}_{\mathrm{B}}\} \propto \mathcal{H}^{\mathrm{GR}}_{\mathrm{C}}}_{GR\ Algebra} \approx 0$

$\mathcal{F}.\mathcal{C}. = \{\Pi_A,\ \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}}\}$

$$DoF = \frac{20 - 2\times 4\ (\Pi_{\mathrm{A}}) - 2\times 4\ (\mathcal{S}_{\mathrm{A}})}{2} = 2$$

Canonical Analysis for modified gravity

$$H = \int d^3x \ \left[\mathcal{H}_{\rm A}^{\rm GR}(\gamma,\ \pi)\ {\rm N^A} + \mathcal{V}({\rm N},\ {\rm N^i},\ \gamma_{\rm ij})\right]$$

${\rm N^A} = ({\rm N},\ {\rm N^i})$, $\quad \mathcal{V}_{\rm A} \equiv \frac{\partial \mathcal{V}}{\partial {\rm N^A}}$, $\quad \mathcal{V}_{\rm AB} \equiv \frac{\partial^2 \mathcal{V}}{\partial {\rm N^A} \partial {\rm N^B}}$, $\quad \mathcal{V} = {\rm N}\ \sqrt{\gamma}\ {\rm V}$

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Canonical Analysis for modified gravity

$$H = \int d^3x \left[\mathcal{H}^{\rm GR}_{\rm A}(\gamma,\ \pi)\ {\rm N}^{\rm A} + \mathcal{V}({\rm N},\ {\rm N}^{\rm i},\ \gamma_{\rm ij}) \right]$$

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two cases
(a) $Det|\mathcal{V}_{\rm AB}| \neq 0$
(b) $Det|\mathcal{V}_{\rm AB}| = 0$

Dirac Constraints case (a)

$$\mathcal{T}_{\rm A} = \{\mathcal{S}_{\rm A},\ H\} + \lambda^{\rm B}\,\mathcal{V}_{\rm AB} \approx 0$$

$$\text{If} \quad Det|\mathcal{V}_{\rm AB}| \neq 0 \rightarrow \text{all } \lambda^{\rm A} \text{ are determined}$$

end of the counting of the constraints:

$$DoF = \frac{20 - 4\,(\Pi_{\rm A}) - 4\,(\mathcal{S}_{\rm A})}{2} = 6$$

Around Minkowski: 5 (massive spin 2) +1 (Scalar$\rightarrow$ B.D. Ghost)

Dirac Constraints case (b)

$$\mathcal{T}_{\mathrm{A}} = \{\mathcal{S}_{\mathrm{A}},\ H\} + \lambda^{\mathrm{B}}\ \mathcal{V}_{\mathrm{AB}} \approx 0$$

$$\text{If} \quad Det|\mathcal{V}_{\mathrm{AB}}| = 0 \to \mathrm{DoF} < \ 6$$

$$r \equiv Rank|\mathcal{V}_{\mathrm{AB}}|$$

For $r = 3$ the potential is satisfying a Monge-Ampere equation in the N^A variables (non linear Partial Diff Eq of the second order)

$$\boxed{\partial^2_{\mathrm{N}^0}\mathcal{V} - \left(\partial^2_{\mathrm{N}^0\,\mathrm{N}^i}\mathcal{V}\right)\ \left(\partial^2_{\mathrm{N}^i\,\mathrm{N}^j}\mathcal{V}\right)^{-1}\ \left(\partial^2_{\mathrm{N}^j\,\mathrm{N}^0}\mathcal{V}\right) = 0}$$

NB: No γ^{ij} involved

Dirac Constraints case (b)

$$\mathcal{V}_{\rm AB} \text{ is a } 4\times 4 \text{ matrix with } \; {\rm Det}|\mathcal{V}_{\rm AB}| = 0 \text{ and } {\rm r} = 3$$

one zero eigenstate χ^A and three non zero eigenstate $E^A_{n=1,2,3}$ i.e.

$$\chi^{\rm B}\, \mathcal{V}_{\rm AB} = 0, \quad {\rm E}^{\rm B}_{\rm n}\, \mathcal{V}_{\rm AB} \equiv {\rm k_n}\, {\rm E}^{\rm A}_{\rm n}, \qquad {\rm n} = 1,2,3$$

$$\lambda^{\rm A} = z\, \chi^{\rm A} + \sum_{\rm n} z_{\rm n}\, {\rm E}^{\rm A}_{\rm n} = z\, \chi^{\rm A} + \bar{\lambda}^{\rm A} \;\; \text{from} \;\; \lambda^{A=0,\dots,3} \to (z,\; z_{n=1,2,3})$$

Dirac Constraints case (b)

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Only one $\mathcal{T}_A$ is a true Constraint:

$$\begin{aligned}
\mathcal{T}_{\rm A} &= \{\mathcal{S}_{\rm A},\ H\} + \lambda^{\rm B}\,\mathcal{V}_{\rm AB} \approx 0 \\
& \quad \{\mathcal{S}_{\rm A},\ H\} + \bar{\lambda}^{\rm B}\,\mathcal{V}_{\rm AB} \approx 0 \to \bar{\lambda}^{\rm B} \qquad \mathrm{z}_{\rm n=1,2,3} \ \text{ fixed} \\
\mathcal{T} &\equiv \chi^{\rm A}\,\mathcal{T}_{\rm A} = \chi^{\rm A}\,\{\mathcal{S}_{\rm A},\ H\} \approx 0 \qquad \text{true constraint}
\end{aligned}$$

Dirac Constraints case (b)

$$\mathcal{V}_{\rm AB} \text{ is a } 4\times 4 \text{ matrix with } \ {\rm Det}|\mathcal{V}_{\rm AB}| = 0 \text{ and } {\rm r}=3$$

one zero eigenstate χ^A and three non zero eigenstate $E^A_{n=1,2,3}$ i.e.

$$\chi^{\rm B}\,\mathcal{V}_{\rm AB} = 0, \quad {\rm E}^{\rm B}_{\rm n}\,\mathcal{V}_{\rm AB} \equiv {\rm k_n}\,{\rm E}^{\rm A}_{\rm n}, \qquad {\rm n}=1,2,3$$

$$\lambda^{\rm A} = {\color{red}z}\,\chi^{\rm A} + \sum_{\rm n} z_{\rm n}\,{\rm E}^{\rm A}_{\rm n} = {\color{red}z}\,\chi^{\rm A} + \bar\lambda^{\rm A} \ \text{ from } \ \lambda^{A=0,\dots,3} \to (z,\ z_{n=1,2,3})$$

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$$\begin{aligned}
\mathcal{T}_{\rm A} &= \{\mathcal{S}_{\rm A},\ H\} + \lambda^{\rm B}\,\mathcal{V}_{\rm AB} \approx 0 \\
&\quad\ \{\mathcal{S}_{\rm A},\ H\} + \bar\lambda^{\rm B}\,\mathcal{V}_{\rm AB} \approx 0 \to \bar\lambda^{\rm B} \qquad {\rm z_{n=1,2,3}} \ \text{ fixed} \\
\mathcal{T} &\equiv \chi^{\rm A}\,\mathcal{T}_{\rm A} = \chi^{\rm A}\,\{\mathcal{S}_{\rm A},\ H\} \approx 0 \qquad \text{true constraint} \\
\mathcal{Q} &\propto \{\mathcal{T},\ H\} + \lambda^{\rm A}\,\left(\{\mathcal{V}_{\rm AB},\ \mathcal{H}\} + \{\mathcal{S}_{\rm A},\ \mathcal{S}_{\rm B}\}\right)\,\chi^{\rm B} + \dots \approx 0
\end{aligned}$$

only the $\color{red}z$ component is free

$$\mathcal{Q} \propto \ {\color{red}z}\,\chi^{\rm A}\,\{\mathcal{S}_{\rm A},\ \mathcal{S}_{\rm B}\}\,\chi^{\rm B} + \bar\lambda^{\rm A}\dots$$

Dirac Constraints case (b)

Case (α) $\qquad \chi^{\mathrm{A}} \{\mathcal{S}_{\mathrm{A}},\, \mathcal{S}_{\mathrm{B}}\}\, \chi^{\mathrm{B}} \neq 0$

where $\mathcal{S}_{\mathrm{A}} = \mathcal{H}_{\mathrm{A}} + \mathcal{V}_{\mathrm{A}}$
z is determined and $\mathcal{Q} \approx 0$ is not a constraint on the DoF

$$DoF = \frac{20 - 4\,(\Pi_{\mathrm{A}}) - 4\,(\mathcal{S}_{\mathrm{A}}) - 1\,(\mathcal{T})}{2} = 5.5$$

Dirac Constraints case (b)

Case (α) $\qquad \chi^{\mathrm{A}}\,\{\mathcal{S}_{\mathrm{A}},\,\mathcal{S}_{\mathrm{B}}\}\,\chi^{\mathrm{B}} \neq 0$

where $\mathcal{S}_{\mathrm{A}} = \mathcal{H}_{\mathrm{A}} + \mathcal{V}_{\mathrm{A}}$
z is determined and $\mathcal{Q} \approx 0$ is not a constraint on the DoF

$$DoF = \frac{20 - 4\,(\Pi_{\mathrm{A}}) - 4\,(\mathcal{S}_{\mathrm{A}}) - 1\,(\mathcal{T})}{2} = 5.5$$

Case (β) $\qquad \chi^{\mathrm{A}}\,\{\mathcal{S}_{\mathrm{A}},\,\mathcal{S}_{\mathrm{B}}\}\,\chi^{\mathrm{B}} = 0$

z is not determined and $\mathcal{Q} \approx 0$ is a new constraint

$$DoF = \frac{20 - 4\,(\Pi_{\mathrm{A}}) - 4\,(\mathcal{S}_{\mathrm{A}}) - 1\,(\mathcal{T}) - 1\,(\mathcal{Q})}{2} \leq 5$$

NB: z is determined if $\mathcal{Q}_{,A} \neq 0 \rightarrow$ DoF=5

Note about the last Constraints

Structure of the last constraint: $z\, \chi^{\mathrm{A}}\, \{\mathcal{S}_{\mathrm{A}},\ \mathcal{S}_{\mathrm{B}}\}\, \chi^{\mathrm{B}}$

$$\int dy\, {\color{red}z}(y)\ \underbrace{\chi^{\mathrm{A}}(x)\,\chi^{\mathrm{B}}(y)}_{\text{even } (A,\,x)\leftrightarrow(B,\,y)}\ \underbrace{\{\mathcal{S}_{\mathrm{A}}(x),\ \mathcal{S}_{\mathrm{B}}(y)\}}_{\text{odd } (A,\,x)\leftrightarrow(B,\,y)} \rightarrow \text{always zero?}$$

$$\{\mathcal{S}_{\mathrm{A}}(x),\ \mathcal{S}_{\mathrm{B}}(y)\} = \underbrace{\underbrace{\mathrm{P}_{\mathrm{A\ B}}(x)}_{\text{odd } A\leftrightarrow B}\ \underbrace{\delta(x-y)}_{\text{even } x\leftrightarrow y}}_{\text{odd } (A,\,x)\leftrightarrow(B,\,y)} + \underbrace{\underbrace{\mathrm{R}_{\mathrm{A\ B}}(x)}_{\text{even } A\leftrightarrow B}\ \underbrace{{\color{red}\partial_x\, \delta(x-y)}}_{\text{odd } x\leftrightarrow y}}_{\text{odd } (A,\,x)\leftrightarrow(B,\,y)}$$

$$\int dy\, {\color{red}z}(y)\ \underbrace{\chi^{\mathrm{A}}(x)\,\chi^{\mathrm{B}}(y)\mathrm{R}_{\mathrm{A\ B}}(x,\ y)}_{F(x,y)\ \text{ even } x\leftarrow y}\ \partial_x\, \delta(x-y) = -\frac{1}{2\, z(x)}\partial_x(z(x)^2\ F(x,\ x))$$

5 DoF Conditions

Conditions for the potential $\mathcal{V}$ to have 5 DoF

- Monge-Ampere equation

$$Det|\mathcal{V}_{AB}| = 0 \rightarrow \boxed{\partial^2_{N^0}\tilde{\mathcal{V}} - \partial^2_{N^0\,N^i}\tilde{\mathcal{V}}\ \left(\partial^2_{N^i\,N^j}\tilde{\mathcal{V}}\right)^{-1}\ \partial^2_{N^j\,N^0}\tilde{\mathcal{V}} = 0} \Leftrightarrow \mathrm{DoF} = 5.5$$

- Extra Differential equation

$$\chi^A\,\{\mathcal{S}_A,\ \mathcal{S}_B\}\,\chi^B = 0 \rightarrow \boxed{\frac{\partial\tilde{\mathcal{V}}}{\partial N^i} + 2\,\xi^A\xi^j\frac{\partial^2\tilde{\mathcal{V}}}{\partial N^A\,\partial\gamma^{ij}} = 0} \Leftrightarrow \mathrm{DoF} = 5$$

where $\tilde{\mathcal{V}} = \gamma^{1/2}\mathcal{V}, \quad \chi^A\,\mathcal{V}_{AB} = 0, \quad \xi^A = \chi^A/\chi^0$

Solution of the Monge-Ampere eq

D.B. Fairlie and A.N. Leznov, (1995)

Implicit change of variables $N^i \to \xi^i$ such that

$$\boxed{N^{\rm i} = N\,\xi^{\rm i} + \mathcal{Q}^{\rm i}(\xi,\gamma)} \to \xi^{\rm i} = \xi^{\rm i}(N,\, N^i,\, \gamma)$$

The solution is given by:

$$\boxed{\mathrm{V}(\mathrm{N},\, \xi^{\rm i},\, \gamma^{\rm ij}) = \,\mathcal{U} + \mathrm{N}^{-1}\,\left(\mathcal{E} + \partial_{\xi^{\rm i}}\mathcal{U}\,\,\mathcal{Q}^{\rm i}\right)}$$

Where the two free functions $\mathcal{U}(\xi^{\rm i},\, \gamma^{\rm ij})$ and $\mathcal{E}(\xi^{\rm i},\, \gamma^{\rm ij})$ fix also

$$\mathcal{Q}^{\rm i}(\xi,\gamma) \equiv -(\,\partial^2_{\xi^{\rm i}\,\xi^{\rm j}}\mathcal{U})^{-1}\,\partial_{\xi^{\rm j}}\mathcal{E}$$

$$\boxed{\boxed{\mathrm{V} = \,\mathcal{U} + \mathrm{N}^{-1}\,\left(\mathcal{E} - \mathcal{U}_\xi \cdot \mathcal{U}^{-1}_{\xi\,\xi} \cdot\,\mathcal{E}_\xi\right)}}$$

Solution Monge Ampere + Extra eq

$$\boxed{\frac{\partial \tilde{\mathcal{V}}}{\partial N^{\rm i}} + 2\,\xi^{\rm A}\xi^{\rm j}\frac{\partial^2 \tilde{\mathcal{V}}}{\partial N^{\rm A}\partial\gamma^{\rm ij}} = 0} \to \frac{\partial \mathcal{U}}{\partial \xi^{\rm i}} + 2\,\xi^{\rm j}\frac{\partial \mathcal{U}}{\partial \gamma^{\rm ij}} = 0$$

$$\mathcal{U}(\xi^{\rm i}, \gamma^{\rm ij}) = \boxed{\mathcal{U}(\gamma^{\rm ij} - \xi^{\rm i}\,\xi^{\rm j})}$$

Solution Monge Ampere + Extra eq

$$\boxed{\frac{\partial \tilde{\mathcal{V}}}{\partial N^{\rm i}} + 2\,\xi^{\rm A}\xi^{\rm j}\frac{\partial^2 \tilde{\mathcal{V}}}{\partial N^{\rm A}\partial\gamma^{\rm ij}} = 0} \to \frac{\partial \mathcal{U}}{\partial \xi^{\rm i}} + 2\,\xi^{\rm j}\frac{\partial \mathcal{U}}{\partial \gamma^{\rm ij}} = 0$$

$$\mathcal{U}(\xi^{\rm i},\gamma^{\rm ij}) = \boxed{\mathcal{U}(\gamma^{\rm ij} - \xi^{\rm i}\,\xi^{\rm j})} \equiv \mathcal{U}(\mathcal{K}^{\rm ij})$$

$$\mathrm{V}(\mathrm{N},\ \mathrm{N}^{\rm i},\ \gamma^{\rm ij}) = \mathcal{U}(\mathcal{K}^{\rm ij}) + \mathrm{N}^{-1}\left(\mathcal{E}(\xi^{\rm i},\gamma^{\rm ij}) + \partial_{\xi^{\rm a}}\mathcal{U}(\mathcal{K}^{\rm ij})\ \mathcal{Q}^{\rm a}(\xi^{\rm i},\gamma^{\rm ij})\right)$$

$$\boxed{\mathrm{V}(\mathrm{N},\ \mathrm{N}^{\rm i},\ \gamma^{\rm ij}) = \mathcal{U} + \mathrm{N}^{-1}\ \left(\mathcal{E} + \partial_{\xi^{\rm i}}\mathcal{U}\ \mathcal{Q}^{\rm i}\right)}$$

$$\boxed{\mathrm{N}^{\rm i} = \mathrm{N}\,\xi^{\rm i} + \mathcal{Q}^{\rm i},\quad \mathcal{Q}^{\rm i} = -\,\mathcal{U}^{-1}_{\xi^{\rm i}\xi^{\rm j}}\ \mathcal{E}_{\xi^{\rm j}}}$$

with $\mathcal{K}^{\rm ij} = \gamma^{\rm ij} - \xi^{\rm i}\,\xi^{\rm j}$

The Energy for the 5 DoF Potential

$$H = H_{\text{Back.}} + H_{\text{Surf.}} = \int d^3x \, \left(\mathcal{H}\, N + \mathcal{H}_{\rm i}\, N^{\rm i} + \mathcal{V}\right) + H_{ADM}$$

$$H|_{\text{on shell}} = \int d^3x \, \sqrt{\gamma} \, \underbrace{\mathcal{E}(\xi, \gamma)}_{\textit{Bulk Energy Density}} + H_{ADM}$$

Example of 5 DoF Potential

$$\boxed{V = \mathcal{U} + N^{-1}\,\left(\mathcal{E} + \mathcal{U}_{\xi^i}\,\mathcal{Q}^i\right)} \quad \boxed{N^i = N\,\xi^i + \mathcal{Q}^i, \quad \mathcal{Q}^i = -\,\mathcal{U}^{-1}_{\xi^i\xi^j}\,\mathcal{E}_{\xi^j}}$$

$$\text{If}\;\; \mathcal{E}(\xi,\,\gamma) = \mathcal{E}(\gamma) \to \;\; \xi^i = N^i/N \;\to\;\; \mathcal{K}^{ij} = \gamma^{ij} - \frac{N^i\,N^j}{N^2} = g^{ij}$$

$$\boxed{V = \mathcal{U}(g^{ij}) + N^{-1}\;\mathcal{E}(\gamma^{ij})}$$

$$H\big|_{\text{on shell}} = \int d^3x\,\sqrt{\gamma}\,\mathcal{E}(\gamma) + \mathit{Boundaries}$$

Lorentz Invariant ghost free DeRGT potential

$$\boxed{\mathrm{V} = \mathcal{U} + \mathrm{N}^{-1} \left(\mathcal{E} + \mathcal{U}_{\xi^{\mathrm{i}}}\ \mathcal{Q}^{\mathrm{i}}\right)} \quad \boxed{\mathrm{N}^{\mathrm{i}} = \mathrm{N}\ \xi^{\mathrm{i}} + \mathcal{Q}^{\mathrm{i}},\quad \mathcal{Q}^{\mathrm{i}} = -\ \mathcal{U}^{-1}_{\xi^{\mathrm{i}}\xi^{\mathrm{j}}}\ \mathcal{E}_{\xi^{\mathrm{j}}}} \quad \mathcal{K}^{\mathrm{ij}} = \gamma^{\mathrm{ij}} - \xi^{\mathrm{i}}\xi^{\mathrm{j}}$$

$$\mathcal{U} = \mathrm{Tr}(\mathcal{K}^{1/2}) - 3, \qquad \mathcal{E} = \frac{1}{\sqrt{1 - \xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}}}, \qquad \mathcal{U}_{\xi^{\mathrm{i}}}\,\mathcal{Q}^{\mathrm{i}} = -\frac{\xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}}{\sqrt{1 - \xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}}}$$

$$\mathrm{V_{LI}} = (\mathrm{Tr}(\mathcal{K}^{1/2}) - 3) + \mathrm{N}^{-1}\sqrt{1 - \xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}} = \mathrm{Tr}\left[(\mathrm{g}^{\mu\alpha}\ \eta_{\alpha\nu})^{1/2}\right] - 3$$

$$H|_{\text{on shell}} = \int d^3x\ \sqrt{\gamma}\ \mathcal{E}(\xi, \gamma) + \mathit{Boundaries}$$

Structure of the Potential

Any scalar diff invariant function of g is trivial so to build a gravitational potential $V(g_{\mu\nu})$ we need to break diff. invariance.

- *Lorentz invariant Theories*: Add an external tensor $\eta_{\mu\nu}$:

$$V(X^{\mu}_{\nu} = g^{\mu\alpha}\ \eta_{\alpha\nu}) = V(Tr[X^a]), \qquad a = 1, .., 4$$

We can preserve some symmetry in some specific background: Global Lorentz symmetry of the frozen metric $\Lambda^t_\eta \cdot \eta \cdot \Lambda_\eta = \eta$

$$V(Tr[(\eta + h)^{-1} \cdot \eta]) \propto Tr[(h \cdot \eta))], \qquad g = \eta + h$$

symmetry of the perturbations: $h \to \Lambda^t_\eta \cdot h \cdot \Lambda_\eta$. But for $g = \bar{g} + h$ with $\bar{g} \neq \eta$ no Λ_η Symmetry!!
No Local Lorentz symmetry: given g and η in the *Local Lorentz frame*

$$g \to \eta \qquad and \qquad \eta \to Diag||\eta_0,\ \eta_1,\ \eta_2,\ \eta_3||$$

- *Rotational SO(3) Invariance*: Add an external spatial metric δ_{ij}

$$V(N\ scalar,\ N^i\ vector,\ \gamma_{ij}\ tensor) \propto Rotational\ scalars$$

Unbroken rotations only around $\gamma_{ij} \propto \delta_{ij}$ as for homogeneous representation of FRW.

Both model can became diff. invariants:

Add Fields (Stuckelberg trick) $\Leftrightarrow$ More symmetries

Quadratic Perturb. in Minkowsky

Quadratic fluctuation's Lagrangian (the most general SO(3) invariant case) for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\mathcal{L}_{(2)} = \mathcal{L}^{GR}_{Kin} + \frac{M^2_{Pl}}{2} \left(m_0^2\; h_{00}^2 + 2\; m_1^2\; h_{0i}^2 - m_2^2\; h_{ij}^2 + m_3^2\; h_{ii}^2 - 2\; m_4^2\; h_{00}\; h_{ii} \right)$$

- $\boxed{m_0 = 0}$ & $m_1 \neq 0$ $\qquad 2_{Tensor} + 2_{Vector} + 1_{Scalar} = 5$ DoF
- $\boxed{m_1 = 0}$ $\qquad 2_{Tensor} = 2$ DoF

Phenomenology

- No VDVZ discontinuity

$$\Phi \sim 1 - \frac{r_S}{r} + m^2 r^2 (c_1 \frac{r_S}{r} + c_2 \frac{r_S^2}{r^2} \log mr), \qquad mr \ll 1$$

$$\Phi \sim \frac{r_S}{r} (A_1 e^{-M_1 r} + A_2 e^{-M_2 r}), \qquad mr \gg 1$$

- Strong coupling scale $\Lambda_2 \sim \sqrt{m\, M_{Pl}}$

$$\phi \sim M_{Pl}\, \frac{m_4}{m_1} \sqrt{m_4^2 - m_1^2}\, \nabla\, \phi_c \sim M_{Pl}\, m\, \nabla\, \phi_c$$

$$A_i \sim M_{pl}\; m_1\; A_{i\;c}, \quad h \sim M_{Pl}\; h_c$$

Minkowski perturbations

$$\sqrt{g}\left(R - m^2\;\left(\mathcal{U} + \mathrm{N}^{-1}\left(\mathcal{E} + \mathcal{Q}^{\mathrm{i}}\,\mathcal{U}_{\mathrm{i}}\right)\right)\right) \underbrace{\rightarrow}_{\text{Minimum}} \mathcal{U}\Big|_{\eta} = 0, \quad \mathcal{U}' + \mathcal{E}' - \frac{1}{2}\mathcal{E}\Big|_{\eta} = 0$$

$$(\partial_{\gamma^{\mathrm{ij}}}\mathcal{U} \equiv \gamma_{\mathrm{ij}}\;\mathcal{U}',\;\; \partial_{\gamma^{\mathrm{ij}}}\mathcal{E} \equiv \gamma_{\mathrm{ij}}\;\mathcal{E}')$$

Minkowski perturbations

$$\sqrt{g}\left(R - m^2\ \left(\mathcal{U} + N^{-1}\left(\mathcal{E} + \mathcal{Q}^i\,\mathcal{U}_i\right)\right)\right) \underbrace{\rightarrow}_{\text{Minimum}} \mathcal{U}|_\eta = 0, \quad \mathcal{U}' + \mathcal{E}' - \frac{1}{2}\mathcal{E}\Big|_\eta = 0$$

$$(\partial_{\gamma^{ij}}\mathcal{U} \equiv \gamma_{ij}\,\mathcal{U}', \quad \partial_{\gamma^{ij}}\mathcal{E} \equiv \gamma_{ij}\,\mathcal{E}')$$

Masses

$$\boxed{m_0^2 = 0}, \qquad m_1^2 = m^2\,\mathcal{U}'\,\frac{\partial\xi^k}{\partial N^k}\Big|_\eta, \qquad m_4^2 = \frac{3}{2}\,m^2\,\mathcal{U}'\Big|_\eta, \quad m_{2,\,3}^2 \neq 0$$

$N^i = N\,\xi^i + \mathcal{Q}^i(\xi)$

$$\boxed{m_1^2 = \frac{2}{3}\,m_4^2\,\frac{\partial\xi^k}{\partial N^k}\Big|_\eta}$$

$$\boxed{5 \textit{ perturbative DoF } \Leftrightarrow \mathcal{U}'|_\eta \neq 0 \Leftrightarrow \mathcal{E} \neq 0 \Leftrightarrow \text{Energy}_{\text{Bulk}} \neq 0}$$

Ex: (1) $\mathcal{E} = \mathcal{E}(\gamma) \rightarrow \ \xi^i = N^i/N \rightarrow m_1^2 = 2\,m_4^2$

(2) LI implies $m_1^2 = m_4^2 \rightarrow \frac{\partial\xi^k}{\partial N^k}\Big|_\eta = \frac{3}{2}$

Cosmology of Massive Gravity

$$g_{\mu\nu} = \mathrm{Diag}||\mathrm{N}^2(t), -\alpha^2(t), -\alpha^2(t), -\alpha^2(t)||$$

$\mathrm{N}^{\mathrm{i}} = \xi^{\mathrm{i}} = 0, \;\; \partial_{\gamma^{ij}}\mathcal{E} \equiv \gamma_{\mathrm{ij}}\mathcal{E}', \;\; \mathcal{E}(\alpha), \;\; \mathcal{U}(\alpha), \qquad \dot{\mathcal{E}} = -6\,\frac{\dot{\alpha}}{\alpha}\,\mathcal{E}', \;\; \mathcal{E}' = -\frac{\partial\mathcal{E}}{6\,\partial\log\alpha}$

$$g_{\mu\nu} = \mathrm{Diag}||\mathrm{N}^2(t), -\alpha^2(t), -\alpha^2(t), -\alpha^2(t)||$$

$\mathrm{N^i} = \xi^{\mathrm{i}} = 0, \;\; \partial_{\gamma ij}\mathcal{E} \equiv \gamma_{\mathrm{ij}}\mathcal{E}', \;\; \mathcal{E}(\alpha), \;\; \mathcal{U}(\alpha), \qquad \dot{\mathcal{E}} = -6\,\frac{\dot\alpha}{\alpha}\,\mathcal{E}', \;\; \mathcal{E}' = -\frac{\partial\mathcal{E}}{6\;\partial\log\alpha}$

$$\left(\frac{\dot\alpha}{\alpha}\right)^2 \equiv \mathcal{H}^2 = \mathrm{N}^2\left(\frac{\rho_{\mathrm{matter}}}{6\;M_{Pl}^2} + m^2\mathcal{U}\right), \; \boxed{\mathcal{H}\left(\mathcal{E}' - \frac{\mathcal{E}}{2}\right)} = 0 \quad (\textit{Bianchi id.})$$

$$g_{\mu\nu} = \mathrm{Diag}||\mathrm{N}^2(t), -\alpha^2(t), -\alpha^2(t), -\alpha^2(t)||$$

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Algebraic constraint on α with two possibilities

- $\alpha = \alpha_0$ const i.e. no FRW
- $\mathcal{E}' - \frac{\mathcal{E}}{2} \equiv 0$ *Automatically* ex: $\mathcal{E} = 0$ or $\mathcal{E} = \frac{\mathcal{E}_0}{\sqrt{\gamma}}$

$$g_{\mu\nu} = \mathrm{Diag}||\mathrm{N}^2(t), -\alpha^2(t), -\alpha^2(t), -\alpha^2(t)||$$

$$\mathrm{N^i} = \xi^{\mathrm{i}} = 0, \quad \partial_{\gamma^{ij}}\mathcal{E} \equiv \gamma_{\mathrm{ij}}\mathcal{E}', \quad \mathcal{E}(\alpha), \quad \mathcal{U}(\alpha), \qquad \dot{\mathcal{E}} = -6\,\frac{\dot\alpha}{\alpha}\,\mathcal{E}', \quad \mathcal{E}' = -\frac{\partial\mathcal{E}}{6\,\partial\log\alpha}$$

$$\left(\frac{\dot\alpha}{\alpha}\right)^2 \equiv \mathcal{H}^2 = \mathrm{N}^2\left(\frac{\rho_{\mathrm{matter}}}{6\,M_{Pl}^2} + m^2\mathcal{U}\right), \quad \boxed{\mathcal{H}\left(\mathcal{E}' - \frac{\mathcal{E}}{2}\right)} = 0 \quad (\textit{Bianchi id.})$$

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but on Minkowski

$$\mathcal{U}\big|_\eta = 0, \quad \mathcal{U}' + \underbrace{\left(\mathcal{E} - \frac{\mathcal{E}'}{2}\right)}_{=0}\Bigg|_\eta = \mathcal{U}'\big|_\eta = 0$$

$$g_{\mu\nu} = \mathrm{Diag}||\mathrm{N}^2(t), -\alpha^2(t), -\alpha^2(t), -\alpha^2(t)||$$

$\mathrm{N}^{\mathrm{i}} = \xi^{\mathrm{i}} = 0, \;\; \partial_{\gamma^{ij}} \mathcal{E} \equiv \gamma_{\mathrm{ij}} \mathcal{E}', \;\; \mathcal{E}(\alpha), \;\; \mathcal{U}(\alpha), \qquad \dot{\mathcal{E}} = -6\,\frac{\dot\alpha}{\alpha}\,\mathcal{E}', \;\; \mathcal{E}' = -\frac{\partial \mathcal{E}}{6\,\partial \log \alpha}$

$$\left(\frac{\dot\alpha}{\alpha}\right)^2 \equiv \mathcal{H}^2 = \mathrm{N}^2 \left(\frac{\rho_{\mathrm{matter}}}{6\,M_{Pl}^2} + m^2 \mathcal{U}\right), \quad \boxed{\mathcal{H}\left(\mathcal{E}' - \frac{\mathcal{E}}{2}\right)} = 0 \quad (\textit{Bianchi id.})$$

Algebraic constraint on α with two possibilities

- $\alpha = \alpha_0$ const i.e. no FRW
- $\mathcal{E}' - \frac{\mathcal{E}}{2} \equiv 0$ <u>*Automatically*</u> ex: $\mathcal{E} = 0$ or $\mathcal{E} = \frac{\mathcal{E}_0}{\sqrt{\gamma}}$

but on Minkowski

$$\mathcal{U}|_\eta = 0, \quad \mathcal{U}' + \underbrace{\left(\mathcal{E} - \frac{\mathcal{E}'}{2}\right)}_{=0}\Bigg|_\eta = \mathcal{U}'\big|_\eta = 0 \to \mathrm{m}_1^2 = 0 \quad \textit{strong coupling}$$

FRW versus Minkowski

$$\mathcal{U}|_{\text{Mink}} = 0, \qquad \mathcal{U}' + \mathcal{E}' - \frac{1}{2}\mathcal{E}\Big|_{\text{Mink}} = 0 \quad \Leftrightarrow \quad \mathcal{E}' - \frac{1}{2}\mathcal{E}\Big|_{\text{FRW}} = 0$$

If we require that $\rho_{\text{matter}} \to 0 \Leftrightarrow g_{\mu\nu} \to \eta_{\mu\nu}$ then

$$\mathcal{U}'\Big|_{\text{Mink}} = 0 \Rightarrow m_1^2 = 0 \to \text{Strong Coupling}$$

The way out is: No Minkowski space in absence of matter: $\mathcal{U}|_\eta \neq 0$

$$\rho_{\text{eff}} = m^2 \; M_{Pl}^2 \, \mathcal{U}$$

$$p_{\text{eff}} = \; m^2 \; M_{Pl}^2 \; \left(-\mathcal{U} + 2\,\mathcal{U}'\right)$$

$$w_{\text{eff}} = -1 + \frac{2\,\mathcal{U}'}{\mathcal{U}} = -1 - \frac{\alpha}{3}\partial_\alpha \log \mathcal{U}$$

$$\rho_{\rm eff} = m^2\ M_{Pl}^2\, \mathcal{U}$$

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$$p_{\rm eff} = \ m^2\ M_{Pl}^2\ \left(-\mathcal{U} + 2\,\mathcal{U}'\right)$$

$$m_1^2 = \frac{m^2}{N^2}\,\mathcal{U}'\ \xi^{\rm i}_{,{\rm N^i}} \qquad {\rm m}_4^2 = \frac{{\rm m}^2}{2\ {\rm N}^2}\ \left(\mathcal{U}' - \frac{\mathcal{U}}{2}\right)$$

$$\rho_{\text{eff}} = m^2\ M_{Pl}^2\,\mathcal{U}$$

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$$m_1^2 = \frac{m^2}{N^2}\,\mathcal{U}'\ \xi^{\rm i}_{,\rm N^i} \qquad \mathrm{m}_4^2 = \frac{\mathrm{m}^2}{2\ \mathrm{N}^2}\ \left(\mathcal{U}' - \frac{\mathcal{U}}{2}\right)$$

$$\boxed{w_{\text{eff}} = -1 + \frac{m_1^2}{m_1^2 - 2\ m_4^2\ \xi^{\rm i}_{,\rm N^i}}}$$

DeSitter $\Leftrightarrow w_{eff} = -1 \quad \Rightarrow \quad m_1^2 = 0 \rightarrow$ Strong Coupling

Present Cosmology

For $\alpha \leq 1$, $\quad \mathcal{U} = \sum_{\rm n} \bar{\mathcal{U}}_{\rm n}\, (\alpha - 1)^{\rm n}$

$$w_{eff} = -1 - \frac{\bar{\mathcal{U}}_1}{3\,\bar{\mathcal{U}}_0} - (1-\alpha)\frac{\bar{\mathcal{U}}_1^2 - \bar{\mathcal{U}}_0(2\,\bar{\mathcal{U}}_2 + \bar{\mathcal{U}}_1)}{3\,\bar{\mathcal{U}}_0^2} + \ldots$$

From Planck+BAO+WP

$$w = w_0 + w_\alpha(1-\alpha) \to w_0 = -1.04^{+0.7}_{-0.7}, \quad w_\alpha < 1.3$$

$$\left|\frac{\bar{\mathcal{U}}_1}{\bar{\mathcal{U}}_0}\right| = 0.12^{+2.1}_{-2.1} \qquad \left|\frac{\bar{\mathcal{U}}_2}{\bar{\mathcal{U}}_0}\right| \leq 3.3$$

$$\frac{m_1^2}{m_4^2} = \frac{3\,\bar{\mathcal{U}}_1}{\bar{\mathcal{U}}_1 + 3\,\bar{\mathcal{U}}_0} + \ldots \sim 0.1^{+1.2}_{-5.9} \qquad \textit{for}\ \ \xi^i_{N^i} = 3/2\ (\textit{as in L.I.})$$

Conclusion

- Given two arbitrary functions $\mathcal{U}(\mathcal{K}^{ij})$ and $\mathcal{E}(\xi^i, \gamma^{ij})$ (of the specific arguments) the deformation potential V of any massive gravity theory with five propagating DoFs can be written as

$$\boxed{V = \mathcal{U} + \mathcal{N}^{-1}(\mathcal{E} - \mathcal{U}_i\, \mathcal{U}_{ij}^{-1}\, \mathcal{E}_j)}$$

with $N^i = N\, \xi^i - \mathcal{U}_{ij}^{-1}\, \mathcal{E}_j$ and $\mathcal{K}^{ij} = \gamma^{ij} - \xi^i \xi^j$

- We analyzed in a model independent way FRW cosmology of any massive gravity theory with five degrees of freedom
- The existence of a FRW background requires the selection of a particular subclass of massive gravity theories
- Tension between FRW versus Minkowski - DeSitter