Stability of a string in a thermal bath of photons
Spontaneous Workshop VII, Cargèse

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Outline

1. Introduction: Goal and Motivation
   What are cosmic strings?

2. PART 1 Effects of a Thermal Bath of Photons on Stability

3. PART 2 Bounce and instanton computations

REFERENCES


Goal: To stabilize cosmic string by a thermal bath of photons

Why?: Stabilized embedded defects applications in cosmology.

- Explanation for the origin and coherence of cosmological magnetic fields on galaxy scale.
- CMB: temperature fluctuations, non-gaussianity.
- can contribute to structure formation.
- may play a role in baryogenesis
- loops can contribute to ultra-high-energy cosmic rays
Cosmic strings

- Topological defects: commonly formed in laboratories and seen in condensed matter systems during phase transition.
Strings arise from spontaneous symmetry breaking occurring when a scalar field, usually called the Higgs field, takes on its vacuum expectation value.

GOAL Strings can come from fields present in the Standard Model of particle physics: eg: pion string, electroweak string.
Symmetry Breaking

Figure: A simple potential in 3-dimensions, the Mexican-hat potential, can give rise to strings through symmetry breaking.
Topological defects correspond to boundaries between regions with different choices of minima.
In particular, there is a non-trivial winding of the phase around a string.
Thermal field theory

Thermal bath on a system $\xrightarrow{\text{finite-temperature field theory}}$ to compute physical observables. $\xrightarrow{\text{time imaginary and wrapped on itself}}$ with a period $\beta = 1/k_b T$

Figure: Euclideanized spacetime: cylinder of radius $r = \frac{1}{2\pi k_b T}$ and of infinite height.
The new time variable, $\tau = it$, becomes compactified. As a result, spacetime becomes Euclidean the metric goes from Minkowski $(-, +, +, +)$ to Euclidean geometry $(+, +, +, +)$

$t : -\infty \rightarrow +\infty \Rightarrow \tau : 0 \rightarrow \beta$

The Euclidean action, $S_E$,:

$$S_E = \int_0^\beta d\tau \int d^3x L_E.$$  

- thermal bath of photons: temperature $T$.
- scalar fields: out of equilibrium since we are below the critical temperature.
In the imaginary time formalism of thermal field theory, the integration over four-momenta is carried out in Euclidean space \( \int \frac{d^4 k}{(2\pi)^4} \rightarrow i \int \frac{d^4 k_E}{(2\pi)^4} \). Frequencies take discrete values, namely \( \omega_n = 2n\pi T \) with \( n \) an integer.

\[
\int \frac{d^4 k_E}{(2\pi)^4} \rightarrow T \sum_n \int \frac{d^3 k}{(2\pi)^3}
\]

We use this Matsubara mode decomposition for the thermal field

\[
A_{\mu}(\tau, x) = T \sum_{n=-\infty}^{+\infty} \tilde{A}_{\mu}(\omega_n, x) e^{i\omega \tau}
\]

Fourier transforming

\[
A_{\mu}(\tau, x) = T \sum_{\omega_n} \frac{1}{V} \sum_k \tilde{A}_{\mu}(\omega_n, k) e^{i\omega \tau + k \cdot x}
\]
The Pion String

PART 1
The linear sigma model: low energy description of QCD after chiral symmetry breaking $m_u = m_d = 0$

- Symmetry breaking occurs when the sigma field takes on its vacuum expectation value
- Gives rise to a triplet of massless pions $\vec{\pi} = (\pi^0, \pi^+, \pi^-)$.

Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - \frac{\lambda}{4} \left( \sigma^2 + \vec{\pi}^2 - \eta^2 \right)^2,$$

- symmetry of the vacuum manifold = $O(4)$
- vacuum manifold is a 3-sphere: $\mathcal{M} = S^3$
- topologically unstable strings since $\Pi_1(S^3) = 1$.
- Effectively reducing the vacuum manifold to $S^1$ strings.
Effective Lagrangian

Electric charge \(\Rightarrow\) charged pions fields are coupled to electromagnetism \(\Rightarrow\) Lagrangian can be promoted to a Lagrangian with covariant derivatives.

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 + D^+ \pi^+ D^- \pi^- - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + V_0 ,
\]

where \(D^+ = \partial_\mu + ieA_\mu\), \(D^- = \partial_\mu - ieA_\mu\).

• 2 complex scalar fields: \(\pi_c = \pi^1 + i\pi^2\) and \(\phi = \sigma + i\pi_0\).
Effective Potential

\( V_{\text{eff}}(\Phi, \pi_c) \) : defined via the partition function of the system, considering thermal \( A_\mu \)

- String configurations : out-of-equilibrium states below \( T_G \)
- Scalar fields out of thermal equilibrium since \( M \gg T_c \)

**STEPS :**

- Treat the scalar fields as external out-of-equilibrium ones.
- Compute the finite temperature functional integral over \( A_\mu \)
- Partition function of the system, \( Z[T] \)

\[
Z[T] = \int \mathcal{D}\Phi \mathcal{D}\pi_c \mathcal{D}A_\mu e^{-S[A^\mu, \Phi, \pi_c]} = \int \mathcal{D}\Phi \mathcal{D}\pi_c e^{-S[\Phi, \pi_c]} e^{-\frac{V_{\text{eff}}(\Phi, \pi_c) V}{T}}
\]

\( S[\Phi, \pi_c] \) is the gauge field independent part

\( V \) : volume : \( \int d\tau d^3x = \frac{V}{T} \).
Partition function at finite temperature $T$

The partition function of the system can be written in terms of gauge fields, Faddeev-Popov ghosts and scalar fields:

$$Z[T] = \int D\Phi D\pi_c Dc D\bar{c} DA_\mu e^{-S[\Phi,\pi_c]} e^{-\int_0^\beta d\tau \int d^3x \bar{c} \left(-\partial^2 - e^2|\pi_c|^2\right) c}$$

$$\times e^{-\int_0^\beta d\tau \int d^3x \frac{1}{2} A_\mu \left(\partial^2 + e^2|\pi_c|^2\right) A_\mu}$$

Define $\omega = \sqrt{k^2 + m_{\text{eff}}^2}$ and $m_{\text{eff}} = e|\pi_c|$. Here the summation of $A_\mu A_\mu$ is in Euclidean space since $A_0 \rightarrow iA_0$. Gaussian integration over the gauge field and the ghost fields.

$$Z[T] = \int D\Phi D\pi_c e^{-S[\Phi,\pi_c]}$$

$$\times e^{\frac{1}{2} Tr[\ln(\omega_n^2 + k^2 + m_{\text{eff}}^2)]} e^{-4\frac{1}{2} Tr[\ln(\omega_n^2 + k^2 + m_{\text{eff}}^2)]}$$

$$Z[T] = \int D\Phi D\pi_c e^{-S[\Phi,\pi_c]} e^{-Tr[\ln(\omega_n^2 + \omega^2)]}$$
Effective potential : Result

One can deduce the effective potential from the partition function:

\[ V_{\text{eff}}(\Phi, \pi_c, T) = V_0 + \lim_{V \to \infty} \frac{T}{V} \sum_{n \in \mathbb{Z}} \ln(\omega_n^2 + \omega^2) + \text{cst} \]

\[ = \frac{\lambda}{4} (|\Phi|^2 + |\pi_c|^2 - \eta^2)^2 + 2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\omega}{2} + T \ln(1 - e^{-\frac{\omega}{T}}) \right] \]

At high-temperature, we can truncate the series above and get

\[ V_{\text{eff}}(\Phi, \pi_c, T) = \frac{\lambda}{4} (|\Phi|^2 + |\pi_c|^2 - \eta^2)^2 - \frac{\pi^2 T^4}{45} + \frac{e^2 |\pi_c|^2 T^2}{12} - \frac{e^3 |\pi_c|^3 T}{6\pi} - \frac{e^4 |\pi_c|^4}{16\pi^2} \left[ \ln \left( \frac{e |\pi_c| e^{\gamma E}}{4\pi T} \right) - \frac{3}{4} \right] \]

The effective vacuum manifold is now reduced:

\[ M = S^1 \quad \Rightarrow \quad \text{stable string.} \]
Quantum tunneling

PART 2: Computation of the decay of a metastable cosmic string
Vacuum seems metastable but this depends on the value of $\lambda$ and $\eta$.

Figure: Finite temperature effective potential in the core of the string.
Quantum tunneling

- Take a metastable string with 2 complex fields
- Study decay within its core
- Problem with QCD string: first order not reliable from perturbation theory
- Expansion parameter $\sim \frac{\lambda^2}{\epsilon^4}$ which is big for realistic value of $\lambda_{QCD}$ in fact second order phase transition
- Electroweak string: other problem of stability for realistic values of parameters
- However it may still be useful to see how this work, see for example Landau Ginzburg superconducting strings. In the high-temperature expansion, $\frac{|\pi_c|}{T} \ll 1$

$$V_{eff}(\phi, \pi_c, T) \simeq \frac{\lambda}{4} (|\phi|^2 + |\pi_c|^2 - \eta^2)^2 + \frac{e^2|\pi_c|^2}{12} T^2 - \frac{e^3|\pi_c|^3}{6\pi} T$$  \hspace{1cm} (0)
Instanton computations

Quantum tunneling

- Potential $V_{\text{eff}}$:

$$V(\pi_c, T) = D(T^2 - T_o^2)\pi_c^2 - ET\pi_c^3 + \frac{\lambda}{4}\pi_c^4$$

where the coefficients are given by

$$D = \frac{e^2}{12}, \quad E = \frac{e^3}{6\pi}, \quad T_o^2 = \frac{6\lambda\eta^2}{e^2}$$

Figure: Bubble. Action has $O(4)$ symmetry
Figure: string breaking

\begin{equation}
t : \quad \pm \infty \rightarrow 0
\end{equation}

\begin{equation}
(\phi_i, \pi_{ci}) \rightarrow (\phi_b, \pi_{cb})
\end{equation}

\[ \text{bounces at } t = 0 \]

[Coleman:1977] Nielsen and Olesen static string at \( t = \pm \infty \)

\[ = \text{initial configuration} : \]

\[ (\phi_i, \pi_{ci}) = (\eta f(\rho) e^{i \theta}, 0) \]
Configuration at bouncing point:

\[ \phi_b = \begin{cases} 
\eta f(\rho)e^{in\theta} & \text{for } z \neq 0 \text{ and } z = 0, \rho \geq \rho_0 \\
0 & \text{for } z = 0 \text{ and } \rho < \rho_0 
\end{cases} \]

\[ \pi_{cb} = \begin{cases} 
\eta & \text{for } z = 0, \rho < \rho_0 \\
0 & \text{for } z \neq 0 \text{ and } z = 0, \rho > \rho_0 
\end{cases} \]

Full function for neutral string

\[ \phi(\tau, \rho, z, \theta) = \eta f(\rho)e^{in\theta}[g_1(\rho) + \sqrt{1 - g_1(\rho)^2}g_2(s)] \]

\[ \pi_c(\tau, \rho, z, \theta) = \eta \sqrt{1 - g_1(\rho)^2} \sqrt{1 - g_2(s)^2} \]

- \[ s = \sqrt{z^2 + \tau^2} \rightarrow \text{two O(2) symmetries of the bounce solution} \]
• **General boundary conditions** to get a bounce:

\[
\partial_\tau (\phi(\tau), \pi_c(\tau))_{\tau=0} = \partial_\tau (\phi_b, \pi_{cb}) = (0, 0)
\]

\[
\lim_{\tau \to \pm\infty} (\phi, \pi_c) = (\phi_i, \pi_{ci})
\]

• **Boundary conditions for** \(g_1(\rho)\) **and** \(g_2(s)\):

\[
g_1(0) = g_1(\rho \leq \rho_0) = 0, \quad g_1(\rho > \rho_0) = 1, \quad g'_1(0) = 0
\]

\[
g_2(0) = 0, \quad g_2(\pm\infty) = 1, \quad g'_2(0) = 0
\]

Figure: Profile function for \(g_1(\rho)\) and for \(g_2(s)\)
Instanton computations

Quantum tunneling and no Thermal effects

- Potential $V_{\text{eff}}$:

$$V(\pi_c, T) = D(T^2 - T_o^2)\pi_c^2 - ET\pi_c^3 + \frac{\lambda}{4}\pi_c^4$$

where the coefficients are given by

$$D = \frac{e^2}{12} \quad E = \frac{e^3}{6\pi} \quad T_o^2 = \frac{6\lambda\eta^2}{e^2}$$

Figure: Spherical symmetric instanton
In this section, we discuss the stability of a string in a thermal bath. We focus on instantaneous computations and the comparison between quantum tunneling effects and thermal effects.

**Quantum tunneling Vs Thermal effects**

![Wiggly cylinder with O(3) symmetry](image1)

**Figure**: Wiggly cylinder with O(3) symmetry in 3 spatial dimensions.

![Cylindrical instanton](image2)

**Figure**: Cylindrical instanton with O(3) symmetry in 3 spatial dimensions.
Thermal photons + out of eq. scalar fields

- Assume $\phi = 0 \rightarrow$ we study the core of the neutral string
- Thin-wall approximation: almost degenerated $V(\pi_c) \sim V_D(\pi_c)$

$$\frac{d^2 \pi_c}{dx^2} + \frac{1}{x} \frac{d \pi_c}{dx} = V'(\pi_c) \rightarrow \frac{d^2 \pi_c}{dx^2} = V'(\pi_c) \simeq V_D'(\pi_c)$$

where $V_D(\pi_c)$ is the potential in the limit where the potential has an exact degeneracy
- One-dimensional Euclidean action:

$$S_1 = \int dx \left[ \frac{1}{2} \left( \frac{\partial \pi_c}{\partial x} \right)^2 + V(\pi_c) \right] = \int_{\pi_c}^{0} d\pi_c \left[ 2V_D(\pi_c) \right]^{\frac{1}{2}} = \frac{\left( \pi_c^D \right)^3 \sqrt{\lambda}}{6 \sqrt{2}}$$
with the thin-wall parameter $\epsilon$

$$V(\pi_c) = \frac{\lambda}{4} \pi_c^2 (\pi_c - \pi_c^D)^2 - \frac{\lambda}{2} \epsilon \pi_c D \pi_c^3$$  \hspace{1cm} (3)$$

where

$$\epsilon = \frac{ET}{\sqrt{\lambda D}} \frac{1}{\sqrt{T^2 - T_0^2}} - 1 = \sqrt{\frac{T_c^2 - T_0^2}{T^2 - T_0^2}} \frac{T}{T_c} - 1$$  \hspace{1cm} (4)$$

and

$$\pi_c^D(T) = 2 \sqrt{\frac{D}{\lambda} (T^2 - T_0^2)}$$  \hspace{1cm} (5)$$

T-dependant Action:

$$S_1(T) = \frac{4}{3\sqrt{2\lambda}} [D(T^2 - T_0^2)]^{3/2} = \frac{e^3}{18\sqrt{6\lambda}} [(T^2 - T_0^2)]^{3/2}$$

$S_1$ has mass dimension 3 since it is the one-dimensional action.
Potential energy density difference between the two minima

\[ \Delta V = V(\pi_{cmin}) = \frac{\lambda}{2} \epsilon (\pi_c^D)^4 \] (6)

\[ \Delta V = 8(ET - \sqrt{\lambda D} \sqrt{T^2 - T_0^2})\left[\frac{D}{\lambda} (T^2 - T_0^2)\right]^{\frac{3}{2}} \]

Here, contrary to the Mexican hat potential case with a linear term [Coleman:1977], \( \Delta V \neq \epsilon \)

T-dependant Thin-wall parameter

\[ T = \frac{T_0}{\sqrt{1 - \frac{e^4}{3\pi^2 \lambda (\epsilon + 1)^2}}} \] and \( \epsilon(T) = \sqrt{\frac{e^4}{3\pi^2 \lambda} \frac{1}{1 - \frac{T_0^2}{T^2}}} - 1 \)
- Equation of motion for $r \sim R$:

$$
\frac{d^2 \pi_c}{dr^2} + \frac{3}{r} \frac{d\pi_c}{dr} = V'(\pi_c)
$$

- $S_{\text{sphere}} = \pi^2 \int r^3 dr \left[ \frac{1}{2} \left( \frac{\partial \pi_c}{\partial r} \right)^2 + V(\pi_c) \right]
= -\frac{\pi^2}{2} R^4 \Delta V + 2\pi^2 R^3 S_1$

- Extremizing $S_{\text{sphere}}$: \( \frac{\partial S_E}{\partial R} = 0 \)

$$
R(T) = \frac{3S_1}{\Delta V} = \sqrt{\frac{3}{2e}} \frac{1}{\epsilon \sqrt{T^2 - T_0^2}}
$$

Figure: Bubble
• **T** dependent decay rate: 

\[ \frac{\Gamma_{\text{sphere}}}{V} \sim P_4 \exp\left[-\pi^2 \frac{1}{48 \lambda \left( \sqrt{\frac{e^4}{3\pi^2\lambda}} \frac{T^2}{T^2 - T_0^2} - 1 \right)^3} \right] \]

• After tunneling in vacuum, bubble radially expands at 

\[ \nu = \frac{d|\vec{x}|}{dt} = \frac{\sqrt{|\vec{x}|^2 - R^2}}{|\vec{x}|} \]

(\( \nu \sim c \)) but plasma pressure slows down this expansion.

• Energy of the bubble wall: 

\[ E_{\text{wall}} = 4\pi |\vec{x}|^2 (S^{\pi c}_1)(1 - \nu^2)^{-\frac{1}{2}} \]

which finally reduces to 

\[ E_{\text{wall}} = 4\pi |\vec{x}|^3 \frac{S^{\pi c}_1}{R} = \frac{2\pi \epsilon}{27\lambda} |\vec{x}|^3 e^{4(T^2 - T_0^2)^2} \]
String Stability from thermal effects

Thermal bath of photons can make string stable. Instantons productions quantify the stability of strings against breaking.

PART 1

- The plasma effects lift the potential in direction of the charged pion fields.
- This lead to an effective vacuum manifold which admits cosmic string solutions, the pion strings.
- Our arguments are general and apply to many theories beyond the Standard Model.
- Topological defects embedded defects of the full theory with the property that they are stabilized in the early Universe for the right values of the parameters.
- For realistic values of QCD parameters, the string is stabilized and is not metastable.
PART 2

- 1st order phase transition for strings: eg superconducting strings can decay into 2 strings.
- We found the temperature dependent radius and the decay rate of a string into two strings.

Effects of a Thermal Bath of Photons on Embedded String Stability: Topologically unstable defects can become stable in a thermal bath of photons.