Cosmological Scenarios in Horndeski Theory

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Based on

- J. Matsumoto, SVS, JCAP 1511, 047 (2015)
- A. Starobinsky, SVS, M. Volkov, JCAP 1606, 007 (2016)
- J. Matsumoto, SVS, arXiv:1703.04966

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Motivation

Scalar fields in gravitational physics

Scalar fields *minimally coupled* to gravity

Scalar fields *nonminimally coupled* to gravity

Scalar fields with *nonminimal derivative coupling*

Horndeski model

Cosmological models with nonminimal derivative coupling
  - No potential
  - Cosmological constant
  - Power-law potential
  - Higgs-like potential
  - Role of matter

The screening Horndeski cosmologies

Summary
Scalar fields in gravitational physics:

- gravitational potential in Newtonian gravity
- variation of “fundamental” constants
- Brans-Dicke theory initially elaborated to solve the Mach problem
- various compactification schemes
- the low-energy limit of the superstring theory
- scalar field as inflaton
- scalar field as dark energy and/or dark matter
- fundamental Higgs bosons, neutrinos, axions, . . .
- etc...
Scalar fields minimally coupled to gravity

\[ S = \int d^4 x \sqrt{-g} [L_{GR} + L_s] \]

\( L_{GR} \) – gravitational Lagrangian
- general relativity; \( L_{GR} = R \)
- square gravity; \( L_{GR} = R + cR^2 \)
- \( f(R) \)-theories; \( L_{GR} = f(R) \)
- etc...

\( L_s \) – scalar field Lagrangian;
- ordinary STT; \( L_s = -\epsilon (\nabla \phi)^2 - 2V(\phi) \)
  - \( \epsilon = +1 \) – canonical scalar field
  - \( \epsilon = -1 \) – phantom or ghost scalar field with negative kinetic energy
- \( V(\phi) \) – potential of self-action
- \( K \)-essence; \( L_s = K(X) [X = (\nabla \phi)^2] \)
- etc...
Scalar fields nonminimally coupled to gravity

Bergmann-Wagoner-Nordtvedt scalar-tensor theories

\[ S = \int d^4x \sqrt{-g} \left[ f(\phi)R - h(\phi)(\nabla\phi)^2 - 2V(\phi) \right] \]

\[ f(\phi)R \implies \text{nonminimal coupling between } \phi \text{ and } R \]
Scalar fields nonminimally coupled to gravity

Bergmann-Wagoner-Nordtvedt scalar-tensor theories

\[ S = \int d^4x \sqrt{-g} \left[ f(\phi) R - h(\phi) (\nabla \phi)^2 - 2V(\phi) \right] \]

\( f(\phi) R \rightarrow \) nonminimal coupling between \( \phi \) and \( R \)

Conformal transformation to the Einstein frame (Wagoner, 1970):

\[ \tilde{g}_{\mu\nu} = f(\phi) g_{\mu\nu}; \quad \frac{d\phi}{d\tilde{\psi}} = f \left| f h + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2 \right|^{-1/2} \]

\[ S = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \epsilon (\tilde{\nabla} \psi)^2 - 2U(\psi) \right] \]

\( \psi \rightarrow \) new scalar field \[ \epsilon = \text{sign} \left[ f h + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2 \right] \]

\( U(\psi) \rightarrow \) new effective potential
Scalar fields nonminimally coupled to gravity

General scalar-tensor theories

\[ S = \int d^4x \sqrt{-g} \left[ F(\phi, R) - (\nabla \phi)^2 - 2V(\phi) \right] \]

\[ F(\phi, R) \implies \text{generalized nonminimal coupling between } \phi \text{ and } R \]
Scalar fields nonminimally coupled to gravity

General scalar-tensor theories

\[ S = \int d^4 x \sqrt{-g} \left[ F(\phi, R) - (\nabla \phi)^2 - 2V(\phi) \right] \]

\[ F(\phi, R) \implies \text{generalized nonminimal coupling between } \phi \text{ and } R \]

Conformal transformation to the Einstein frame (Maeda, 1989):

\[ \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}; \quad \frac{\Omega^2}{16\pi} \equiv \left| \frac{\partial F(\phi, R)}{\partial R} \right| \]

\[ S = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi} - h(\phi)\psi^{-1}(\tilde{\nabla} \phi)^2 - \frac{3}{32\pi} \psi^{-2}(\tilde{\nabla} \psi)^2 + U(\phi, \psi) \right] \]

\[ \psi \equiv \Omega^2 \implies \text{new (second!) scalar field} \]

\[ U(\phi, \psi) \implies \text{new effective potential} \]
Some remarks:

- A nonminimal scalar field is conformally equivalent to the minimal one possessing some effective potential $V(\phi)$.
- A behavior of the scalar field is “encoded” in the potential $V(\phi)$.
- The potential $V(\phi)$ is a very important ingredient of various cosmological models: a slowly varying potential behaves like an effective cosmological constant providing one or more than one inflationary phases.
  An appropriate choice of $V(\phi)$ is known as a problem of fine tuning of the cosmological constant.
Nonminimal derivative coupling generalization

\[ S = \int d^4 x \sqrt{-g} \left[ R - g^\mu_\nu \phi,\mu \phi,\nu - 2V(\phi) \right] \]

- \( F(\phi, R, R_{\mu\nu}, \ldots) \) nonminimal coupling generalization!
- \( K(\phi,\mu, \phi; \mu\nu, \ldots, R, R_{\mu\nu}, \ldots) \) nonminimal derivative coupling generalization!
Nonminimal derivative coupling generalization

\[ S = \int d^4 x \sqrt{-g} \left[ R - g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} - 2V(\phi) \right] \]

---

\[ F(\phi, R, R_{\mu\nu}, \ldots) \]

**nonminimal coupling generalization!**

\[ K(\phi,_{\mu}, \phi,_{\mu\nu}, \ldots, R, R_{\mu\nu}, \ldots) \]

**nonminimal derivative coupling generalization!**
\[ S = \int d^4 x \sqrt{-g} \left[ R - g^{\mu \nu} \phi,_{\mu} \phi,_{\nu} - 2V(\phi) \right] \]

- Nonminimal coupling generalization:
  \[ F(\phi, R, R_{\mu \nu}, \ldots) \]
  nonminimal coupling generalization!

- Nonminimal derivative coupling generalization:
  \[ K(\phi,_{\mu}, \phi;_{\mu \nu}, \ldots, R, R_{\mu \nu}, \ldots) \]
  nonminimal derivative coupling generalization!
$$S = \int d^4 x \sqrt{-g} \left[ R - g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) \right]$$

Theories with higher derivatives!

\[ F(\phi, R, R_{\mu \nu}, \ldots) \]

nonminimal coupling generalization!

\[ K(\phi, \phi_{,\mu}, \phi_{,\mu \nu}, \ldots, R, R_{\mu \nu}, \ldots) \]

nonminimal derivative coupling generalization!
In 1974, Horndeski derived the action of the most general scalar-tensor theories with second-order equations of motion


**Horndeski Lagrangian:**

\[
L_H = \sqrt{-g} \left( \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right)
\]

\[
\mathcal{L}_2 = G_2(X, \Phi) ,
\]

\[
\mathcal{L}_3 = G_3(X, \Phi) \Box \Phi ,
\]

\[
\mathcal{L}_4 = G_4(X, \Phi) R + \partial_X G_4(X, \Phi) \delta_{\alpha \beta}^{\mu \nu} \nabla_\mu \Phi \nabla_\nu \Phi ,
\]

\[
\mathcal{L}_5 = G_5(X, \Phi) G_{\mu \nu} \nabla^{\mu \nu} \Phi - \frac{1}{6} \partial_X G_5(X, \Phi) \delta_{\alpha \beta \gamma}^{\mu \nu \rho} \nabla_\mu \Phi \nabla_\nu \Phi \nabla_\rho \Phi ,
\]

where \( X = -\frac{1}{2} (\nabla \phi)^2 \), and \( G_k(X, \Phi) \) are arbitrary functions,

and \( \delta^{\lambda \rho}_{\nu \alpha} = 2! \delta_{\nu}^{\lambda} \delta_{\alpha}^{\rho} \), \( \delta^{\lambda \rho \sigma}_{\nu \alpha \beta} = 3! \delta_{\nu}^{\lambda} \delta_{\alpha}^{\rho} \delta_{\beta}^{\sigma} \)
There is a special subclass of the theory, sometimes called Fab Four (F4), for which the coefficients are chosen such that the Lagrangian becomes

\[ L_{F4} = \sqrt{-g} \left( \mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda \right) \]

with

\[
\begin{align*}
\mathcal{L}_J &= V_J(\Phi) \, G_{\mu\nu} \nabla^\mu \Phi \nabla^\nu \Phi, \\
\mathcal{L}_P &= V_P(\Phi) \, P_{\mu\nu\rho\sigma} \nabla^\mu \Phi \nabla^\rho \Phi \nabla^\nu \Phi \nabla^\sigma \Phi, \\
\mathcal{L}_G &= V_G(\Phi) \, R, \\
\mathcal{L}_R &= V_R(\Phi) \left( R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right).
\end{align*}
\]

Here the double dual of the Riemann tensor is

\[ P_{\mu\nu}^{\alpha\beta} = -\frac{1}{4} \, \delta_{\mu\nu\gamma\delta} \, R^{\sigma\lambda} \, \gamma_{\delta} = -R_{\mu\nu}^{\alpha\beta} + 2R_{[\alpha}^{\mu} \delta_{\beta]}^{\nu} - 2R^{\nu}_{[\alpha} \delta_{\beta]}^{\mu} - R_{[\alpha}^{\mu} \delta_{\beta]}^{\nu} , \]

whose contraction is the Einstein tensor, \[ P_{\mu\nu}^{\alpha\nu} = G^{\mu}_{\nu} . \]
Fab Four subclass of the Horndeski theory

**Fab Four Lagrangian:**

\[ L_{F4} = \sqrt{-g} \left( \mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda \right) \]

- The Fab Four model is distinguished by the *screening property* — it is the most general subclass of the Horndeski theory in which flat space is a solution, despite the presence of the cosmological term \( \Lambda \).
- This property suggests that \( \Lambda \) is actually irrelevant and hence there is no need to explain its value.
- Indeed, however large \( \Lambda \) is, Minkowski space is always a solution and so one may hope that a slowly accelerating universe will be a solution as well.
Action:

\[ S = \frac{1}{2} \int \left( M_{\text{Pl}}^2 R - (\eta G_{\mu\nu} + \varepsilon g_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi - 2V(\phi) \right) \sqrt{-g} \, d^4x + S_m \]

Field equations:

\[
M_{\text{Pl}}^2 G_{\mu\nu} = \eta \mathcal{T}_{\mu\nu} + \epsilon T^{(\Phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}
\]

\[
[\epsilon g^{\mu\nu} + \eta G^{\mu\nu}] \nabla_\mu \nabla_\nu \phi = V'_{\phi}
\]

\[
T^{(\Phi)}_{\mu\nu} = \epsilon \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \epsilon g_{\mu\nu} (\nabla \phi)^2 - g_{\mu\nu} V(\phi),
\]

\[
\mathcal{T}_{\mu\nu} = -\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R + 2\nabla_\alpha \phi \nabla_{(\mu} \phi R^{\alpha}_{\nu)} - \frac{1}{2} (\nabla \phi)^2 G_{\mu\nu} + \nabla^\alpha \phi \nabla^\beta \phi R_{\mu\alpha\nu\beta}
\]

\[
+ \nabla_\mu \nabla^\alpha \phi \nabla_\nu \nabla_\alpha \phi - \nabla_\mu \nabla_\nu \phi \Box \phi + g_{\mu\nu} \left[ -\frac{1}{2} \nabla^\alpha \nabla^\beta \phi \nabla_\alpha \nabla_\beta \phi + \frac{1}{2} (\Box \phi)^2 - \nabla_\alpha \phi \nabla_\beta \phi R^{\alpha\beta} \right]
\]

\[
T^{(m)}_{\mu\nu} = (\rho + p) U_\mu U_\nu + pg_{\mu\nu},
\]

**Notice:** *The field equations are of second order!*
Cosmological models: General formulas

Ansatz:

\[ ds^2 = -dt^2 + a^2(t)dx^2, \]
\[ \phi = \phi(t) \]

\( a(t) \) cosmological factor, \( H = \dot{a}/a \) Hubble parameter

Field equations:

\[ 3M_{Pl}^2 H^2 = \frac{1}{2} \dot{\phi}^2 \left( \epsilon - 9\eta H^2 \right) + V(\phi), \]
\[ M_{Pl}^2 (2\dot{H} + 3H^2) = -\frac{1}{2} \dot{\phi}^2 \left[ \epsilon + \eta \left( 2\dot{H} + 3H^2 + 4H \dddot{\phi} \phi^{-1} \right) \right] + V(\phi), \]
\[ \frac{d}{dt} \left[ (\epsilon - 3\eta H^2)a^3 \dot{\phi} \right] = -a^3 \frac{dV(\phi)}{d\phi} \]
Cosmological models: General formulas

Ansatz:

\[ ds^2 = -dt^2 + a^2(t)dx^2, \]
\[ \phi = \phi(t) \]

\( a(t) \) cosmological factor, \( H = \dot{a}/a \) Hubble parameter

Field equations:

\[ 3M^2_{Pl} H^2 = \frac{1}{2} \dot{\phi}^2 \left( \epsilon - 9\eta H^2 \right) + V(\phi), \]
\[ M^2_{Pl}(2\dot{H} + 3H^2) = -\frac{1}{2} \dot{\phi}^2 \left[ \epsilon + \eta \left( 2\dot{H} + 3H^2 + 4H \ddot{\phi} \dot{\phi}^{-1} \right) \right] + V(\phi), \]
\[ \frac{d}{dt} [(\epsilon - 3\eta H^2)a^3 \dot{\phi}] = -a^3 \frac{dV(\phi)}{d\phi} \]

\[ V(\phi) \equiv \text{const} \implies \dot{\phi} = \frac{Q}{a^3(\epsilon - 3\eta H^2)} \quad \text{Q is a scalar charge} \]
Trivial model without kinetic coupling, i.e. $\eta = 0$

\[
S = \int d^4x \sqrt{-g} \left[ M_{\text{Pl}}^2 R - (\nabla \phi)^2 \right]
\]
Trivial model without kinetic coupling, i.e. $\eta = 0$

$$S = \int d^4 x \sqrt{-g} \left[ M_{Pl}^2 R - (\nabla \phi)^2 \right]$$

Solution:

$$a_0(t) = t^{1/3}; \quad \phi_0(t) = \frac{1}{2\sqrt{3\pi}} \ln t$$

$$ds_0^2 = -dt^2 + t^{2/3} dx^2$$

$t = 0$ is an initial singularity
Model without free kinetic term, i.e. $\epsilon = 0$

$$S = \int d^4 x \sqrt{-g} \left[ M_{P1}^2 R - \eta G^{\mu\nu} \phi,_{\mu} \phi,_{\nu} \right]$$
Model without free kinetic term, i.e. \( \epsilon = 0 \)

\[
S = \int d^4x \sqrt{-g} \left[ M_{P1}^2 R - \eta G^{\mu\nu} \phi,_{\mu} \phi,_{\nu} \right]
\]

Solution:

\[
a(t) = t^{2/3}; \quad \phi(t) = \frac{t}{2 \sqrt{3\pi |\eta|}}, \quad \eta < 0
\]

\[
ds_0^2 = -dt^2 + t^{4/3} dx^2
\]

\( t = 0 \) is an initial singularity
Model for an ordinary scalar field ($\epsilon = 1$) with nonminimal kinetic coupling $\eta \neq 0$

$$S = \int d^4 x \sqrt{-g} \left\{M_{\text{Pl}}^2 R - (g^{\mu\nu} + \eta G^{\mu\nu}) \phi,_{\mu} \phi,_{\nu}\right\}$$
Model for an ordinary scalar field $\left(\epsilon = 1\right)$ with nonminimal kinetic coupling $\eta \neq 0$

$$S = \int d^4x\sqrt{-g}\left\{M_{\text{Pl}}^2R - (g^{\mu\nu} + \eta G^{\mu\nu})\phi,_{\mu}\phi,_{\nu}\right\}$$

Asymptotic for $t \to \infty$:

$$a(t) \sim a_0(t) = t^{1/3}; \quad \phi(t) \sim \phi_0(t) = \frac{1}{2\sqrt{3\pi}} \ln t$$

Notice: At large times the model with $\eta \neq 0$ has the same behavior like that with $\eta = 0$
Asymptotics for early times

The case $\eta < 0$:

$$a_{t \to 0} \approx t^{2/3}; \quad \phi_{t \to 0} \approx \frac{t}{2 \sqrt{3 \pi |\eta|}}$$

$$ds_{t \to 0}^2 = -dt^2 + t^{4/3} dx^2$$

$t = 0$ is an initial singularity

The case $\eta > 0$:

$$a_{t \to -\infty} \approx e^{H_\eta t}; \quad \phi_{t \to -\infty} \approx Ce^{-t/\sqrt{\eta}}$$

$$ds_{t \to -\infty}^2 = -dt^2 + e^{2H_\eta t} dx^2$$

de Sitter asymptotic with $H_\eta = 1/\sqrt{9\eta}$
Plots of $\alpha = \ln a$ in case $\eta \neq 0$, $\epsilon = 1$, $V = 0$.

(a) $\eta < 0$; $\eta = 0; -1; -10; -100$

(b) $\eta > 0$; $\eta = 0; 1; 10; 100$

De Sitter asymptotics: $\alpha(t) = \frac{t}{\sqrt{9\epsilon}} \quad \Rightarrow \quad H = \frac{1}{\sqrt{9\epsilon}}$

Notice: In the model with nonminimal kinetic coupling one get de Sitter phase without any potential!
Models with the constant potential $V(\phi) = M_{Pl}^2 \Lambda = \text{const}$

$$S = \int d^4x \sqrt{-g} \left[ M_{Pl}^2 (R - 2\Lambda) - \epsilon g^{\mu\nu} + \eta G^{\mu\nu} \right] \phi,_{\mu} \phi,_{\nu}$$
Models with the constant potential $V(\phi) = M_{Pl}^2 \Lambda = \text{const}$

$$S = \int d^4 x \sqrt{-g} \left[ M_{Pl}^2 (R - 2\Lambda) - \epsilon g^{\mu\nu} + \eta G^{\mu\nu} \right] \phi_{,\mu} \phi_{,\nu}$$

There are two exact de Sitter solutions:

I. $\alpha(t) = H_\Lambda t$, $\phi(t) = \phi_0 = \text{const}$,

II. $\alpha(t) = \frac{t}{\sqrt{3|\eta|}}$, $\phi(t) = M_{Pl} \left| \frac{3\eta H_\Lambda^2 - 1}{\eta} \right|^{1/2} t$,

$$H_\Lambda = \sqrt{\Lambda/3}$$
Plots of $\alpha(t)$ in case $\eta > 0$, $\epsilon = 1$, $V = M_{Pl}^2 \Lambda$

(a) $H^2_{\Lambda} < \dot{\alpha}^2 < 1/9\eta$

(b) $1/9\eta < \dot{\alpha}^2 < 1/3\eta < H^2_{\Lambda}$

De Sitter asymptotics:

$\alpha_1(t) = H_{\Lambda} t$ (dashed),

$\alpha_2(t) = t/\sqrt{9\eta}$ (dash-dotted),

$\alpha_3(t) = t/\sqrt{3\eta}$ (dotted).
Plots of $\alpha(t)$ in cases $\eta > 0$, $\epsilon = -1$ and $\eta < 0$, $\epsilon = 1$

(a) $\eta < 0$, $\epsilon = 1$

(b) $\eta > 0$, $\epsilon = -1$

De Sitter asymptotic:

$\alpha_1(t) = H \Lambda t$ (dashed).
Role of potential

\[ S = \int d^4 x \sqrt{-g} \left\{ M_{Pl}^2 R - [g^{\mu\nu} + \eta G^{\mu\nu}] \phi_{,\mu} \phi_{,\nu} - 2V(\phi) \right\} \]

What a role does a potential play in cosmological models with the nonminimal kinetic coupling?
Models with the quadratic potential $V(\phi) = \frac{1}{2} m^2 \phi^2$

Primary (early-time) “kinetic” inflation:

$$H_{t \to -\infty} \approx \frac{1}{\sqrt{9\eta}} (1 + \frac{1}{2} \eta m^2)$$

Late-time cosmological scenarios:

Oscillatory asymptotic or “graceful” exit from inflation

$$H_{t \to \infty} \approx \frac{2}{3t} \left[ 1 - \frac{\sin 2mt}{2mt} \right]$$

quasi-de Sitter asymptotic or secondary inflation

$$H_{t \to \infty} \approx \frac{1}{\sqrt{3\eta}} \left( 1 \pm \sqrt{\frac{1}{6} \eta m^2} \right)$$
Cosmological models: Power-law potential

Initial conditions
\[ \phi_0 = \dot{\phi}_0 \]

De Sitter asymptotics: \[ H_{t \to -\infty} \approx 1/\sqrt{9\eta}(1 + \frac{1}{2}\eta m^2), \]
\[ H_{t \to \infty} \approx 1/\sqrt{3\eta} \left( 1 \pm \sqrt{\frac{1}{6}\eta m^2} \right). \]
Higgs potential \( V(\phi) = \frac{\lambda}{4}(\phi^2 - \phi_0^2)^2 \)

Matsumoto, Sushkov, JCAP2015

Higgs field: \( \lambda \approx 0.14, \quad \phi_0 \approx 246 \text{ GeV} \)

Field equations

\[
H^2 = \frac{\frac{1}{2} \dot{\phi}^2 + V(\phi)}{3(M_{Pl}^2 + \frac{3}{2} \eta \dot{\phi}^2)}
\]

\[
\ddot{\phi} = [1 + 12\pi \eta \dot{\phi}^2 + 96\pi^2 \eta^2 \dot{\phi}^4 + 8\pi \eta V(\phi)(12\pi \eta \dot{\phi}^2 - 1)]^{-1} \\
\times \left\{ -2\sqrt{3}\pi \phi [1 + 8\pi \eta \dot{\phi}^2 - 8\pi \eta V(\phi)] \\
\sqrt{[\dot{\phi}^2 + 2V(\phi)](12\pi \eta \dot{\phi}^2 + 1) - (12\pi \eta \dot{\phi}^2 + 1)(4\pi \eta \dot{\phi}^2 + 1)V_\phi} \right\}
\]
Dimensionless variables and parameters:

\[ x = \frac{\phi}{\phi_0}, \quad y = \sqrt{8\pi G \eta \dot{\phi}}, \quad \tau = \phi_0 t, \quad V_0 = 2\pi G \eta \lambda \phi_0^4, \quad \gamma \equiv G \phi_0^2 \]

Autonomous dynamical system:

\[
\frac{dx}{d\tau} = \sqrt{\frac{\lambda}{4V_0}} y,
\]
\[
\frac{dy}{d\tau} = \frac{1}{\Delta} \left\{ -\sqrt{3\pi \gamma \lambda V_0^{-1}} y [1 + y^2 - V_0(x^2 - 1)^2] \right. \\
\times \sqrt{[y^2 + 2V_0(x^2 - 1)^2]} \left( \frac{3}{2} y^2 + 1 \right) \\
- 2\sqrt{\lambda V_0} \left( \frac{3}{2} y^2 + 1 \right) \left( \frac{1}{2} y^2 + 1 \right) x(x^2 - 1) \right\},
\]

where \(\Delta = 1 + \frac{3}{2} y^2 + \frac{3}{2} y^4 + V_0(x^2 - 1)^2 \left( \frac{3}{2} y^2 - 1 \right)\).
Stationary points:

\((\pm 1, 0)\) global minima of \(V(\phi)\); \(\phi = \pm \phi_0, V(\pm \phi_0) = 0\)

\((0, 0)\) local maximum of \(V(\phi)\); \(\phi = 0, V(0) = V_0 = \frac{\lambda}{4} \phi_0^4\)

\((\pm \infty, 0)\) “wings” of \(V(\phi)\); \(\phi \to \pm \infty\)

Phase diagrams:

\[
\frac{1}{4} \eta \lambda \phi_0^4 \leq M_{Pl}^2
\]

\((0, 0)\) – saddle point

\[
\frac{1}{4} \eta \lambda \phi_0^4 > M_{Pl}^2
\]

\((0, 0)\) – stable node!
Accelerated cosmological scenarios: Quasi-de Sitter

Quasi-de Sitter scenario

$t \to \infty$ distant future asymptotic

$V(\phi) \to V_0 = \lambda \phi_0^4/4$ (maximum)

$H(t) \to H_\infty = \sqrt{\frac{2}{3} \pi G \lambda \phi_0^4} = \text{const}$
Big Rip scenario

\[ t \to t_* \quad \text{finite time asymptotic} \]

\[
\phi(t) \simeq \sqrt{\frac{392\eta}{\lambda}} \frac{1}{(t_* - t)^2} \\
\phi \to \infty, \quad \dot{\phi} \to \infty \\
V(\phi) \simeq \frac{\lambda\phi^4}{4} \to \infty \\
\]

\[
H^2(t) \simeq \frac{49}{9} \frac{\tau^2}{(t_* - t)^2} \to \infty 
\]
Accelerated cosmological scenarios: Little Rip

**Little Rip scenario**

\[ t \to \infty \quad \text{distant future asymptotic} \]

\[ \phi(t) \simeq \left( \frac{2}{3\pi^3} G^3 \lambda \eta^2 \right)^{1/8} t^{1/4} \]

\[ \phi \to \infty, \quad \dot{\phi} \to 0 \]

\[ V(\phi) \simeq \lambda \phi^4 / 4 \to \infty \]

\[ H(t) \simeq \left( \frac{8\lambda}{27\pi G \eta^2} \right)^{1/4} t^{1/2} \to \infty \]
Role of matter?

\[ S = \int d^4 x \sqrt{-g} \left\{ M_{Pl}^2 (R - 2\Lambda) - [g^{\mu\nu} + \eta G^{\mu\nu}] \phi,_{\mu} \phi,_{\nu} \right\} + S_{\text{matter}} \]
Role of matter?

\[ S = \int d^4x \sqrt{-g} \left\{ M_{Pl}^2 (R - 2\Lambda) - [g^{\mu\nu} + \eta G^{\mu\nu}] \phi,_{\mu} \phi,_{\nu} \right\} + S_{\text{matter}} \]

**Stress-energy tensor:** \( T^{(m)}_{\mu\nu} = \text{diag}(\rho, p, p, p) \)

**Field equations:**

\[
3 M_{Pl}^2 H^2 = \frac{1}{2} \dot{\phi}^2 \left( 1 - 9 \eta H^2 \right) + M_{Pl}^2 \Lambda + \rho, \\
M_{Pl}^2 (2 \dot{H} + 3H^2) = -\frac{1}{2} \dot{\phi}^2 \left[ 1 + \eta \left( 2 \dot{H} + 3H^2 + 4H \ddot{\phi} \phi^{-1} \right) \right] + M_{Pl}^2 \Lambda - p \\
\frac{dt}{dt} \left[ (1 - 3\eta H^2) a^3 \dot{\phi} \right] = 0
\]
Role of matter?

\[ S = \int d^4x \sqrt{-g} \left\{ M_{Pl}^2 (R - 2\Lambda) - [g^{\mu\nu} + \eta G^{\mu\nu}] \phi,_{\mu} \phi,_{\nu} \right\} + S_{\text{matter}} \]

**Stress-energy tensor:** \( T_{\mu\nu}^{(m)} = \text{diag} (\rho, p, p, p) \)

**Field equations:**

\[ 3M_{Pl}^2 H^2 = \frac{1}{2} \dot{\phi}^2 \left( 1 - 9\eta H^2 \right) + M_{Pl}^2 \Lambda + \rho, \]

\[ M_{Pl}^2 (2\dot{H} + 3H^2) = -\frac{1}{2} \dot{\phi}^2 \left[ 1 + \eta \left( 2\dot{H} + 3H^2 + 4H \ddot{\phi}\phi^{-1} \right) \right] + M_{Pl}^2 \Lambda - p \]

\[ \frac{d}{dt} \left[ (1 - 3\eta H^2) a^3 \dot{\phi} \right] = 0 \quad \Rightarrow \quad \dot{\phi} = \frac{Q}{a^3 (1 - 3\eta H^2)} \]
Cosmological scenarios with nonminimal kinetic coupling and matter

Modified Friedmann equation:

\[ H^2 = H_0^2 \left[ \Omega_{\Lambda 0} + \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{\phi 0}(1 - 9\eta H^2)}{a^6(1 - 3\eta H^2)^2} \right] \]

Constraint for parameters:

\[ \Omega_{\Lambda 0} + \Omega_{m0} + \frac{\Omega_{\phi 0}(1 - 9\eta H_0^2)}{(1 - 3\eta H_0^2)^2} = 1 \]

Universal asymptotic:

\[ H \rightarrow H_\eta = \frac{1}{\sqrt{9\eta}} \quad \text{at} \quad a \rightarrow 0 \]

Notice: The asymptotic \( H \approx H_\eta \) at early cosmological times is only determined by the coupling parameter \( \eta \) and does not depend on other parameters!
Cosmological scenarios: Numerical solutions

Scale factor $a(t)$

Hubble parameter $H(a)$  Acceleration parameter $q$
Cosmological scenarios: Estimations

\[ H_\eta t_f \sim 60 \text{ e-folds} \]
\[ t_f \simeq 10^{-35} \text{ sec} \quad \text{the end of initial inflationary stage} \]

\[ \Rightarrow H_\eta = \frac{1}{\sqrt{9\eta}} \simeq 6 \times 10^{36} \text{ sec}^{-1} \]

\[ \eta \simeq 10^{-74} \text{ sec}^2 \quad \text{or} \quad l_\eta = \eta^{1/2} \simeq 10^{-27} \text{ cm} \]
Cosmological scenarios: Estimations

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\[ \eta \simeq 10^{-74} \text{ sec}^2 \quad \text{or} \quad l_\eta = \eta^{1/2} \simeq 10^{-27} \text{ cm} \]

\[ H_0 \sim 70 \text{ (km/sec)/Mpc} \simeq 10^{-18} \text{ sec}^{-1} \quad \text{Present Hubble parameter} \]
\[ \gamma = 3\eta H_0^2 \simeq 10^{-109} \quad \text{Extremely small!} \]

\[ H^2 = H_0^2 \left[ \Omega_{\Lambda 0} + \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{\phi 0}(1 - 9\eta H^2)}{a^6(1 - 3\eta H^2)^2} \right] \Rightarrow \Omega_{\Lambda 0} + \Omega_{m0} + \Omega_{\phi 0} \approx 1 \]

\[ \Omega_{\Lambda 0} = 0.73, \Omega_{\phi 0} = 0.23, \Omega_{m0} = 0.04 \quad \Rightarrow \quad q_0 = 0.25 \]
The FLRW ansatz for the metric:

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right], \]

\( a(t) \) cosmological factor, \( H = \dot{a}/a \) Hubble parameter

Gravitational equations:

\[-3M_{\text{Pl}}^2 \left( H^2 + \frac{K}{a^2} \right) + \frac{1}{2} \varepsilon \psi^2 - \frac{3}{2} \eta \psi^2 \left( 3H^2 + \frac{K}{a^2} \right) + \Lambda + \rho = 0,\]
\[-M_{\text{Pl}}^2 \left( 2\dot{H} + 3H^2 + \frac{K}{a^2} \right) - \frac{1}{2} \varepsilon \psi^2 - \eta \psi^2 \left( \dot{H} + \frac{3}{2}H^2 - \frac{K}{a^2} + 2H \frac{\dot{\psi}}{\psi} \right) + \Lambda - p = 0.\]

The scalar field equation:

\[ \frac{1}{a^3} \frac{d}{dt} \left( a^3 \left( 3\eta \left( H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi \right) = 0, \]

where \( \psi = \dot{\phi} \), and \( \phi = \phi(t) \) is a homogeneous scalar field
The first integral of the scalar field equation:

\[ a^3 \left( 3\eta \left( H^2 + \frac{K}{a^2} \right) - \varepsilon \right) \psi = Q, \]

where \( Q \) is the Noether charge associated with the shift symmetry \( \phi \to \phi + \phi_0 \).

Let \( Q = 0 \). One finds in this case two different solutions:

**GR branch:** \( \psi = 0 \quad \implies \quad H^2 + \frac{K}{a^2} = \frac{\Lambda + \rho}{3M_{\text{Pl}}^2} \)

**Screening branch:** \( H^2 + \frac{K}{a^2} = \frac{\varepsilon}{3\eta} \quad \implies \quad \psi^2 = \frac{\eta(\Lambda + \rho) - \varepsilon M_{\text{Pl}}^2}{\eta(\varepsilon - 3\eta K/a^2)} \)

**NOTICE:** The role of the cosmological constant in the screening solution is played by \( \varepsilon/3\eta \) while the \( \Lambda \)-term is screened and makes no contribution to the universe acceleration.

Note also that the matter density \( \rho \) is screened in the same sense.
Let $Q \neq 0$, then

$$\psi = \frac{Q}{a^3 \left[ 3\eta (H^2 + \frac{K}{a^2}) - \varepsilon \right]},$$

and the modified Friedmann equation reads

$$3M_{\text{Pl}}^2 \left( H^2 + \frac{K}{a^2} \right) = \frac{Q^2 \left[ \varepsilon - 3\eta (3H^2 + \frac{K}{a^2}) \right]}{2a^6 \left[ \varepsilon - 3\eta (H^2 + \frac{K}{a^2}) \right]^2} + \Lambda + \rho.$$

Introducing dimensionless values and density parameters

$$H^2 = H_0^2 y, \quad a = a_0 a, \quad \rho_{\text{cr}} = 3M_{\text{Pl}}^2 H_0^2, \quad \zeta = \frac{\varepsilon}{3\eta H_0^2},$$

$$\Omega_0 = \frac{\Lambda}{\rho_{\text{cr}}}, \quad \Omega_2 = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_6 = \frac{Q^2}{6\eta a_0^6 H_0^2 \rho_{\text{cr}}}, \quad \rho = \rho_{\text{cr}} \left( \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} \right)$$

gives

the master equation:

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6 \left[ \zeta - 3y + \frac{\Omega_2}{a^2} \right]}{a^6 \left[ \zeta - y + \frac{\Omega_2}{a^2} \right]^2}$$
Asymptotical behavior: Late time limit $a \to \infty$

**GR branch:**

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{(\zeta - 3\Omega_0)}{(\Omega_0 - \zeta)^2} \frac{\Omega_6}{a^6} + \mathcal{O}\left(\frac{1}{a^7}\right)$$

$$\implies H^2 \to \Lambda/3$$

**Notice:** The GR solution is stable (no ghost) if and only if $\zeta > \Omega_0$.

**Screening branches:**

$$y_{\pm} = \zeta + \frac{\Omega_2}{a^2} \pm \frac{\chi}{(\Omega_0 - \zeta)} \frac{\Omega_6}{a^6} - \frac{\Omega_3 \chi}{2(\Omega_0 - \zeta)^2} + \mathcal{O}\left(\frac{1}{a^7}\right)$$

$$\implies H^2 \to \varepsilon/3\alpha$$

**Notice:** The screening solutions are stable (no ghost) if and only if $0 < \zeta < \Omega_0$. 
Asymptotical behavior: The limit $a \to 0$

**GR branch:**

$$y = \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} + \frac{\Omega_2 \Omega_4 - 3\Omega_6}{\Omega_4 a^2} + \frac{3\Omega_3 \Omega_6}{\Omega_4 a} + \mathcal{O}(1)$$

**Notice:** The GR solution is unstable

**Screening branch:**

$$
\begin{align*}
y_+ &= \frac{3\Omega_6}{\Omega_4 a^2} - \frac{3\Omega_3 \Omega_6}{\Omega_4 a} + \frac{5}{3} \zeta + \frac{3\Omega_6 \Omega_3^2 + 9\Omega_6^2}{\Omega_4^3} + \mathcal{O}(a), \\
y_- &= \frac{\zeta}{3} + \frac{4 \zeta^2}{27 \Omega_6} \left( \Omega_4 a^2 + \Omega_3 a^3 \right) + \mathcal{O}(a^4)
\end{align*}
$$

**Notice:** Both screening solutions are stable
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left( \zeta - 3y + \frac{\Omega_2}{a^2} \right) \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \Omega_2 = 0, \Omega_3 = \Omega_4 = 0 \) and for \( \zeta = 6 \)
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left( \zeta - 3y + \frac{\Omega_2}{a^2} \right) \left( \zeta - y + \frac{\Omega_2}{a^2} \right)^2 \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \Omega_2 = 0, \Omega_3 = \Omega_4 = 0, \zeta = 0.2 \)
Global behavior

\[ y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6}{a^6} \left[ \zeta - 3y + \frac{\Omega_2}{a^2} \right] \]

Solutions \( y(a) \) for \( \Omega_0 = \Omega_6 = 1, \Omega_3 = 5, \Omega_4 = 0, \zeta = 0.2 \). One has \( \Omega_2 = 0 \).
The nonminimal kinetic coupling provides an *essentially new* inflationary mechanism which does not need any fine-tuned potential.

At early cosmological times the coupling $\kappa$-terms in the field equations are dominating and provide the quasi-De Sitter behavior of the scale factor: $a(t) \propto e^{H_\kappa t}$ with $H_\kappa = 1/\sqrt{9\kappa}$ and $\kappa \approx 10^{-74}$ sec$^2$ (or $l_\kappa \equiv \kappa^{1/2} \approx 10^{-27}$ cm)

The model provides a natural mechanism of epoch change without any fine-tuned potential.

The nonminimal kinetic coupling crucially changes a role of the scalar potential. Power-law and Higgs-like potentials with kinetic coupling provide accelerated regimes of the Universe evolution.
The theory with nonminimal kinetic coupling admits various cosmological solutions.

Ghost-free solutions exist if $\eta \geq 0$ and $\varepsilon \geq 0$.

The no-ghost conditions eliminate many solutions, as for example the bounces or the “emerging time” solutions.

For $\zeta > \Omega_0$ there exists a ghost-free solution. It describes a universe with the standard late time dynamic dominated by the $\Lambda$-term, radiation and dust. At early times the matter effects are totally screened and the universe expands with a constant Hubble rate determined by $\varepsilon/\eta$. Since it contains two independent parameters $\zeta$ and $\Omega_0 \sim \Lambda$ in the asymptotics, this solution can have an hierarchy between the Hubble scales at the early and late times. However, at late times it is not screening and dominated by $\Lambda$, thus invoking again the cosmological constant problem.
For $0 < \zeta < \Omega_0$ there exist two ghost-free solutions, A and B. The solution A is sourced by the scalar field, with or without the matter, while the solution B exists only when the matter is present. They both show the screening because their late time behaviour is controlled by $\zeta \sim \varepsilon/\eta$ and not by $\Lambda$. Therefore, they could in principle describe the late time acceleration while circumventing the cosmological constant problem, and one might probably find arguments justifying that $\varepsilon/\eta$ should be small. At the same time, these solutions cannot describe the early inflationary phase. Indeed, the near singularity behaviour of the solution B does not correspond to inflation, while the solution A does show an inflationary phase, but with essentially the same Hubble rate as at late times, hence there is no hierarchy between the two Hubble scales.
THANKS FOR YOUR ATTENTION!