Correction DS du 10/12/11

Exercice 1 – Questions de cours

1) Loi de transformation de la vitesse

$$\vec{r}^* = A\vec{r} + \vec{b}$$

$$\vec{v}^* = \frac{d\vec{r}^*}{dt} = \dot{A}\vec{r} + A\vec{v} + \dot{\vec{b}} = \dot{A}A^{-1}A\vec{r} + A\vec{v} + \dot{\vec{b}} \quad \text{car } A^{-1}A = \mathbb{1}.$$

Or, nous avons $j(\vec{w}) = \dot{A}A^{-1}$ et $j(\vec{w})\vec{u} = \vec{w}\times\vec{u}$

Conclusion : $\boxed{\vec{v}^* = \vec{w}\times(A\vec{r}) + A\vec{v} + \dot{\vec{b}}}$

2) Loi de transformation de l'accélération

$$\vec{a}^* = \frac{d\vec{v}^*}{dt} = \dot{\vec{w}}\times(A\vec{r}) + \vec{w}\times(\dot{A}\vec{r}) + \vec{w}\times(A\vec{v}) + \dot{A}\vec{v} + A\dot{\vec{v}} + \ddot{\vec{b}}$$

$$= \dot{\vec{w}}\times(A\vec{r}) + \vec{w}\times(\dot{A}A^{-1}A\vec{r}) + \vec{w}\times(A\vec{v}) + \dot{A}A^{-1}A\vec{v} + A\vec{a} + \ddot{\vec{b}}$$

$$= \dot{\vec{w}}\times(A\vec{r}) + \vec{w}\times(\vec{w}\times(A\vec{r})) + \vec{w}\times(A\vec{v}) + \vec{w}\times(A\vec{v}) + A\vec{a} + \ddot{\vec{b}}$$

Conclusion : $\boxed{\vec{a}^* = \dot{\vec{w}}\times(A\vec{r}) + \vec{w}\times(\vec{w}\times(A\vec{r})) + 2\vec{w}\times(A\vec{v}) + A\vec{a} + \ddot{\vec{b}}}$

3) Première application : $A = \begin{pmatrix} \cos w_0 t & -\sin w_0 t & 0 \\ \sin w_0 t & \cos w_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{b} = b_0 t\,\vec{e}_z$ et $\vec{r} = r_0\vec{e}_x$ objet immobile dans R.

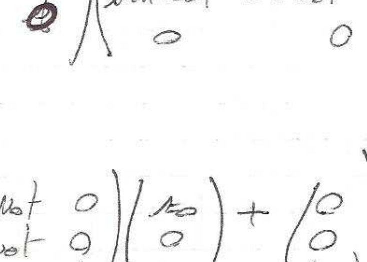
$$j(w) = \dot{A}A^{-1} = w_0\begin{pmatrix} -\sin w_0 t & -\cos w_0 t & 0 \\ \cos w_0 t & -\sin w_0 t & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos w_0 t & \sin w_0 t & 0 \\ -\sin w_0 t & \cos w_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix} = w_0\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Donc $\boxed{\vec{w} = w_0\vec{e}_z.}$

Calcul de $\vec{r}^* = A\vec{r} + \vec{b} = \begin{pmatrix} \cos w_0 t & -\sin w_0 t & 0 \\ \sin w_0 t & \cos w_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b_0 t \end{pmatrix} = \begin{pmatrix} r_0\cos w_0 t \\ r_0 \sin w_0 t \\ b_0 t \end{pmatrix} = \vec{r}^*$

C'est l'équation d'une hélice :

$$\|\vec{r}^*\|^2 = r_0^2 + b_0^2 t^2$$



Calcul de $\vec{v}^* = \begin{pmatrix}0\\0\\w_0\end{pmatrix} \times \begin{pmatrix}r_0\cos w_0t\\r_0\sin w_0t\\0\end{pmatrix} + A\vec{0} + \begin{pmatrix}0\\0\\b_0\end{pmatrix}$ car $\vec{v} = \vec{0}$ et $\dot{\vec{b}} = b_0\vec{e}_z$.

$\Rightarrow \vec{v}^* = \begin{pmatrix}-w_0r_0\sin w_0t\\w_0r_0\cos w_0t\\b_0\end{pmatrix}$ elle est bien égale à $\frac{d\vec{r}^*}{dt}$.

$\boxed{\|\vec{v}^*\|^2 = r_0^2w_0^2 + b_0^2}$

Calcul de $\vec{a}^* = \vec{0}\times(A\vec{r}) + \vec{w}\times(\vec{w}\times(A\vec{r})) + 2\vec{w}\times(\vec{0}) + \vec{0}$ car $\vec{v} = \vec{0}$, $\dot{\vec{w}} = \vec{0}$ et $\ddot{\vec{b}} = \vec{0}$

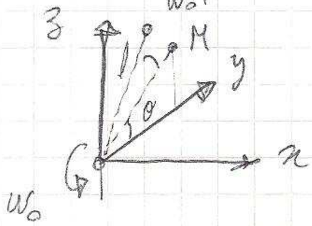
$= \begin{pmatrix}0\\0\\w_0\end{pmatrix} \times \left(\begin{pmatrix}0\\0\\w_0\end{pmatrix} \times \begin{pmatrix}r_0\cos w_0t\\r_0\sin w_0t\\0\end{pmatrix}\right) =$

$= \begin{pmatrix}0\\0\\w_0\end{pmatrix} \times \begin{pmatrix}-w_0r_0\sin w_0t\\w_0r_0\cos w_0t\\0\end{pmatrix} = \begin{pmatrix}-w_0^2r_0\cos w_0t\\-w_0^2r_0\sin w_0t\\0\end{pmatrix}$ C'est bien égal à $\frac{d\vec{v}^*}{dt}$.

$\boxed{\|\vec{a}^*\|^2 = r_0^2w_0^4}$

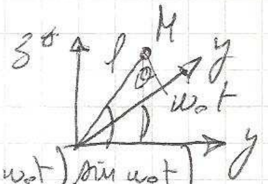
4) Deuxième application : on a $\vec{w} = w_0\vec{e}_x$ et $\vec{b} = \vec{0}$ ainsi que $\vec{r} = v_0t\,\vec{e}_z$.

Calcul de A ?



$j(\vec{w}) = \begin{pmatrix}0&0&0\\0&0&-w_0\\0&w_0&0\end{pmatrix}$ pour $\vec{w} = w_0\vec{e}_x$

$\vec{r}_M^* = \begin{pmatrix}x\\l\cos\theta\\l\sin\theta\end{pmatrix}$ et $\vec{r}_M = \begin{pmatrix}x\\l\cos(\theta-w_0t)\\l\sin(\theta-w_0t)\end{pmatrix}$



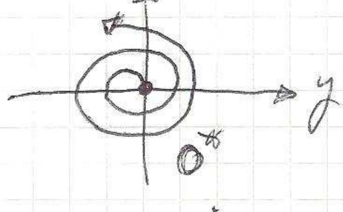
$\Rightarrow \vec{r}_M^* = \begin{pmatrix}x\\l\cos(\theta-w_0t+w_0t)\\l\sin(\theta-w_0t+w_0t)\end{pmatrix} = \begin{pmatrix}x\\l\cos(\theta-w_0t)\cos w_0t - l\sin(\theta-w_0t)\sin w_0t\\l\sin(\theta-w_0t)\cos w_0t + l\cos(\theta-w_0t)\sin w_0t\end{pmatrix}$

$\Rightarrow \boxed{A = \begin{pmatrix}1&0&0\\0&\cos w_0t&-\sin w_0t\\0&\sin w_0t&\cos w_0t\end{pmatrix}}$

Calcul de $\vec{r}^* = A\vec{r} = \begin{pmatrix}1&0&0\\0&\cos w_0t&-\sin w_0t\\0&\sin w_0t&\cos w_0t\end{pmatrix}\begin{pmatrix}0\\0\\v_0t\end{pmatrix} = \begin{pmatrix}0\\-v_0t\sin w_0t\\v_0t\cos w_0t\end{pmatrix} = \vec{r}^*$

$\|\vec{r}^*\|^2 = v_0^2t^2$

C'est l'équation d'une spirale d'Archimède :



Calcul de $\vec{v}^* = \vec{w}\times(A\vec{r}) + A\vec{v}$ car $\dot{\vec{b}} = \vec{0}$ avec $\vec{v} = v_0\vec{e}_z$

$= \begin{pmatrix}w_0\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\-v_0t\sin w_0t\\v_0t\cos w_0t\end{pmatrix} + \begin{pmatrix}1&0&0\\0&\cos w_0t&-\sin w_0t\\0&\sin w_0t&\cos w_0t\end{pmatrix}\begin{pmatrix}0\\0\\v_0\end{pmatrix}$

$$\Longrightarrow \quad \vec{v}^* = \begin{pmatrix} 0 \\ -v_0 t w_0 \cos w_0 t - v_0 \sin w_0 t \\ -v_0 t w_0 \sin w_0 t + v_0 \cos w_0 t \end{pmatrix}$$

C'est bien égal à $\dfrac{d\vec{r}^*}{dt}$.

$$\boxed{\|\vec{v}^*\|^2 = v_0^2 t^2 w_0^2 + v_0^2 = (1 + t^2 w_0^2) v_0^2}$$

Calcul de $\vec{a}^*$ avec $\dot{\vec{w}} = \vec{0}$, $\vec{a} = \vec{0}$ et $\ddot{\vec{b}} = \vec{0}$ et $\vec{v} = v_0 \vec{e}_3$

$$\vec{a}^* = \begin{pmatrix} w_0 \\ 0 \\ 0 \end{pmatrix} \times \left(\begin{pmatrix} w_0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -v_0 t \sin w_0 t \\ v_0 t \cos w_0 t \end{pmatrix} \right) + 2 \begin{pmatrix} w_0 \\ 0 \\ 0 \end{pmatrix} \times \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos w_0 t & -\sin w_0 t \\ 0 & \sin w_0 t & \cos w_0 t \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_0 \end{pmatrix} \right)$$

$$\begin{pmatrix} 0 \\ -v_0 t w_0 \cos w_0 t - v_0 \sin w_0 t \\ -v_0 t w_0 \sin w_0 t + v_0 \cos w_0 t \end{pmatrix}$$

$$\Longrightarrow \vec{a}^* = \begin{pmatrix} w_0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -v_0 w_0 t \cos w_0 t \\ -v_0 w_0 t \sin w_0 t \end{pmatrix} + 2 \begin{pmatrix} w_0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -v_0 \sin w_0 t \\ v_0 \cos w_0 t \end{pmatrix}$$

$$\Longrightarrow \vec{a}^* = \begin{pmatrix} 0 \\ v_0 w_0^2 t \sin w_0 t - 2 w_0 v_0 \cos w_0 t \\ -v_0 w_0^2 t \cos w_0 t - 2 w_0 v_0 \sin w_0 t \end{pmatrix}$$

C'est bien égal à $\dfrac{d\vec{v}^*}{dt}$.

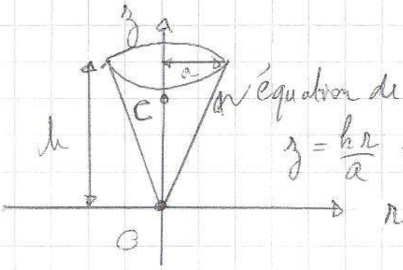
$$\boxed{\|\vec{a}^*\|^2 = v_0^2 w_0^4 t^2 + 4 w_0^2 v_0^2 = (4 + w_0^2 t^2) w_0^2 v_0^2}$$

Angle entre $\vec{v}^*$ et $\vec{a}^*$? $\vec{a}^* . \vec{v}^* = \|\vec{a}^*\| \|\vec{v}^*\| \cos\theta$.

question supplémentaire : calculer les normes $\|\vec{v}^*\|$ et $\|\vec{a}^*\|$, et déterminer l'angle θ entre $\vec{v}^*$ et $\vec{a}^*$ à partir de la relation $\vec{v}^* . \vec{a}^* = \|\vec{v}^*\| \|\vec{a}^*\| \cos\theta$ pour les 2 applications.

Réponse : $\boxed{\theta = \frac{\pi}{2}}$ dans les deux cas car $\vec{v}^* . \vec{a}^* = 0$

P2 1) $V = \iiint r\,dr\,d\theta\,dz$



équation de la droite : $z = \frac{hr}{a} \Rightarrow r = \frac{az}{h}$

Intervalles de variation : $\begin{cases} r \in [0, \frac{az}{h}] \\ \theta \in [0, 2\pi] \\ z \in [0, h] \end{cases}$

La valeur maximale de r dépend de z.

$$\Rightarrow V = \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r\,dr = 2\pi \int_0^h dz\, \frac{1}{2}\left(\frac{az}{h}\right)^2 = \frac{\pi a^2}{3h^2} h^3 = \boxed{\frac{\pi a^2 h}{3} = V}$$

2) $\vec{R} = \frac{\rho}{M} \iiint \vec{r}\, r\,dr\,d\theta\,dz$ avec $\rho = \frac{M}{V}$

$$R_x = \frac{1}{V} \iiint r^2 \cos\theta\,dr\,d\theta\,dz = 0 \text{ car } \int_0^{2\pi} \cos\theta\,d\theta = 0$$

$$R_y = \frac{1}{V} \iiint r^2 \sin\theta\,dr\,d\theta\,dz = 0 \text{ car } \int_0^{2\pi} \sin\theta\,d\theta = 0$$

$$R_z = \frac{1}{V} \int_0^h z\,dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r\,dr = \frac{2\pi}{V} \int_0^h \frac{z\,dz}{2} \left(\frac{az}{h}\right)^2$$

$$= \frac{\pi a^2}{V h^2} \frac{h^4}{4} = \frac{\pi a^2 h^2}{4V} = \frac{3h}{4} \text{ donc } \boxed{\vec{R} = \frac{3h}{4} \vec{e}_z}$$

3) Définition du tenseur d'inertie par rapport à l'origine O :

$$I = \frac{M}{V} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r\,dr \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

Eléments non-diagonaux :

$$\left.\begin{aligned} \cdot\, I_{xy} &= \frac{-M}{V} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r^3 dr \sin\theta\cos\theta = 0 = I_{yx} \\ \cdot\, I_{xz} &= -\frac{M}{V} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r^2 dr\, z\cos\theta = 0 = I_{zx} \\ \cdot\, I_{yz} &= -\frac{M}{V} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r^2 dr\, z \sin\theta = 0 = I_{zy} \end{aligned}\right\} I \text{ est diagonal.}$$

Eléments diagonaux :

$$\cdot\, I_{xx} = \frac{M}{V} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r\,dr\,(r^2 \sin^2\theta + z^2)$$

$$= \frac{M}{V} \int_0^h dz \underbrace{\int_0^{2\pi} \sin^2\theta\,d\theta}_{=\pi} \int_0^{\frac{az}{h}} r^3\,dr + \frac{M}{V} \int_0^h z^2 dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r\,dr$$

$$\Rightarrow I_{xx} = \frac{M\pi}{V}\int_0^h \frac{dz}{4}\left(\frac{az}{h}\right)^4 + \frac{2\pi M}{V}\int_0^h \frac{z^2}{2}\left(\frac{az}{h}\right)^2 dz$$

$$= \frac{\pi M a^4}{4Vh^4}\frac{h^5}{5} + \frac{\pi M a^2}{Vh^2}\frac{h^5}{5} = \frac{3Ma^2}{20} + \frac{3Mh^2}{5}$$

$$\Rightarrow \boxed{I_{xx} = \frac{3M}{5}\left(\frac{a^2}{4} + h^2\right)}$$

• $I_{yy} = \frac{M}{V}\int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r\,dr\,(r^2\cos^2\theta + z^2)$

Comme $\int_0^{2\pi}\cos^2\theta\, d\theta = \int_0^{2\pi}\sin^2\theta\, d\theta$, le calcul est en tout point similaire à celui de I_{xx} $\Rightarrow \boxed{I_{yy} = \frac{3M}{5}\left(\frac{a^2}{4} + h^2\right) = I_{xx}}$

• $I_{zz} = \frac{M}{V}\int_0^h dz \int_0^{2\pi} d\theta \int_0^{\frac{az}{h}} r^3 dr = \frac{2\pi M}{V}\int_0^h \frac{dz}{4}\left(\frac{az}{h}\right)^4 = \frac{\pi M a^4}{2Vh^4}\frac{h^5}{5}$

$$\Rightarrow \boxed{I_{zz} = \frac{3Ma^2}{10}}$$