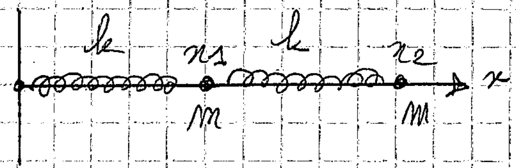
Correction partiel PHY – 2011/2012

Problème 1 : Masses fixées à des ressorts



1) **Energie cinétique** :

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

2) **Energie potentielle** :

$$V = \frac{1}{2} k (x_1 - x_0)^2 + \frac{1}{2} k (x_2 - x_1 - x_0)^2$$ où x_0 = longueur du ressort au repos

3) **Lagrangien** $\boxed{\mathcal{L} = T - V = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k (x_1 - x_0)^2 - \frac{1}{2} k (x_2 - x_1 - x_0)^2}$

$$-\frac{1}{2} k \left(x_1^2 + x_0^2 - 2x_0 x_1 + x_2^2 + x_1^2 + x_0^2 - 2x_1 x_2 - x_0 x_2 \right)$$

4) **Equations de Lagrange** :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow m \ddot{x}_1 + k(x_1 - \cancel{x_0}) - k(x_2 - x_1 - \cancel{x_0}) = 0$$

$$\Rightarrow \boxed{m \ddot{x}_1 + k(2x_1 - x_2) = 0}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow \boxed{m \ddot{x}_2 + k(x_2 - x_1 - x_0) = 0}$$

5) On pose $\begin{cases} X_1 = x_1 - x_0 \\ X_2 = x_2 - 2x_0 \end{cases} \Rightarrow \begin{cases} x_1 = X_1 + x_0 \\ x_2 = X_2 + 2x_0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 = \ddot{X}_1 \\ \ddot{x}_2 = \ddot{X}_2 \end{cases}$

Les équations du mouvement deviennent :

$$m \ddot{X}_1 + k(2X_1 + \cancel{2x_0} - X_2 - \cancel{2x_0}) = 0 \Rightarrow \ddot{X}_1 + \omega_0^2 (2X_1 - X_2) = 0$$

$$m \ddot{X}_2 + k(X_2 + \cancel{2x_0} - X_1 - \cancel{x_0} - \cancel{x_0}) = 0 \Rightarrow \ddot{X}_2 + \omega_0^2 (X_2 - X_1) = 0$$

6) $X_1(t) = A_1 \cos(\omega t)$ et $X_2(t) = A_2 \cos(\omega t)$ avec $\omega_0 = \sqrt{\frac{k}{m}}$

On a $\begin{cases} -\omega^2 A_1 + \omega_0^2 (2A_1 - A_2) = 0 \\ -\omega^2 A_2 + \omega_0^2 (A_2 - A_1) = 0 \end{cases} \Rightarrow \begin{cases} (2\omega_0^2 - \omega^2) A_1 - \omega_0^2 A_2 = 0 \\ -\omega_0^2 A_1 + (\omega_0^2 - \omega^2) A_2 = 0 \end{cases}$

Déterminant $\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0 \Rightarrow (2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = 0$

$$\Rightarrow 2\omega_0^2 - 3\omega_0^2 \omega^2 + \omega^4 - \omega_0^4 = 0$$

$$\Rightarrow \omega^4 - 3\omega_0^2 \omega^2 + \omega_0^2 = 0$$

<u>Solutions :</u> $\omega^2 = \dfrac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^2}}{2} = \dfrac{3\pm\sqrt{5}}{2}\omega_0^2$

$\Rightarrow \boxed{\omega^2 = \dfrac{3\pm\sqrt{5}}{2}\omega_0^2}$ correspond aux fréquences propres

- Solution $\dfrac{3+\sqrt{5}}{2}\omega_0^2$ donne $\begin{cases} \left(2\omega_0^2 - \dfrac{3+\sqrt{5}}{2}\omega_0^2\right)A_1 - \omega_0^2 A_2 = 0 \\ -\omega_0^2 A_1 + \left(\omega_0^2 - \dfrac{3+\sqrt{5}}{2}\omega_0^2\right)A_2 = 0 \end{cases}$

$\Rightarrow \begin{cases} \dfrac{1-\sqrt{5}}{2}A_1 - A_2 = 0 \\ -A_1 - \dfrac{1+\sqrt{5}}{2}A_2 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = \dfrac{1-\sqrt{5}}{2}A_1 \\ A_1 = -\dfrac{1+\sqrt{5}}{2}A_2 \end{cases}$ A_1 et A_2 sont de signes différents

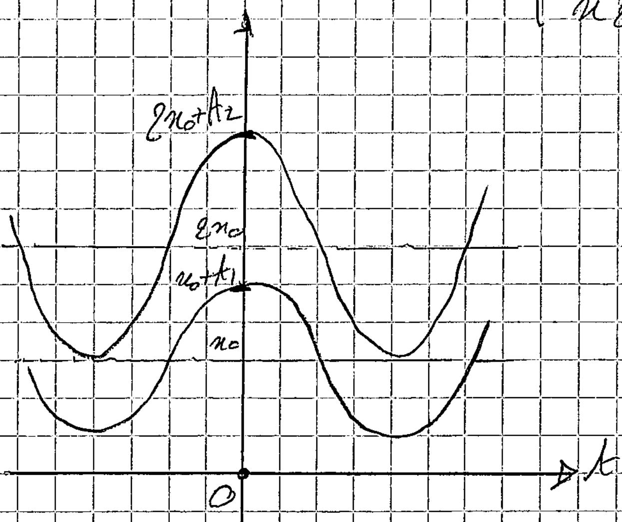
$\boxed{A_2 = \dfrac{1-\sqrt{5}}{2}A_1}$ <u>Les ressorts oscillent en opposition de phase.</u>

- Solution $\dfrac{3-\sqrt{5}}{2}\omega_0^2$ donne $\begin{cases} \left(2\omega_0^2 - \dfrac{3-\sqrt{5}}{2}\omega_0^2\right)A_1 - \omega_0^2 A_2 = 0. \\ -\omega_0^2 A_1 + \left(\omega_0^2 - \dfrac{3-\sqrt{5}}{2}\omega_0^2\right)A_2 = 0 \end{cases}$

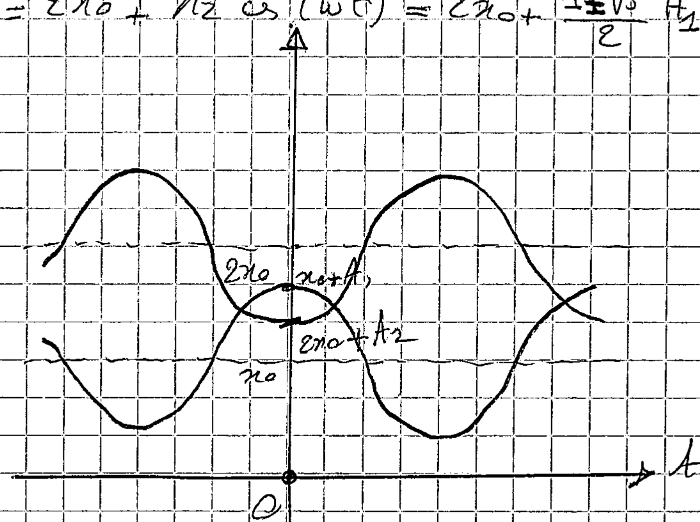
$\Rightarrow \begin{cases} \dfrac{1+\sqrt{5}}{2}A_1 - A_2 = 0 \\ -A_1 + \dfrac{-1+\sqrt{5}}{2}A_2 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = \dfrac{1+\sqrt{5}}{2}A_1 \\ A_1 = -\dfrac{1-\sqrt{5}}{2}A_2 \end{cases}$ A_1 et A_2 sont de même signe.

$\boxed{A_2 = \dfrac{1+\sqrt{5}}{2}A_1}$ <u>Les ressorts oscillent en phase</u>

<u>Représentation graphique :</u> $\begin{cases} x_1(t) = x_0 + A_1\cos(\omega t) \\ x_2(t) = 2x_0 + A_2\cos(\omega t) = 2x_0 + \dfrac{1\pm\sqrt{5}}{2}A_1\cos(\omega t) \end{cases}$



En phase : $A_1 > 0$ et $A_2 > 0$



En opposition de phase : $A_1 > 0$ et $A_2 < 0$

Problème 2 : Particule relativiste

$$\mathcal{L} = -mc\sqrt{c^2 - v^2} - V = -mc\sqrt{c^2 - (v_x^2 + v_y^2 + v_z^2)} - V$$

1) Impulsion $\vec{p}$

$$p_x = \frac{\partial \mathcal{L}}{\partial v_x} = \frac{mc\, v_x}{\sqrt{c^2 - v^2}}$$

$$p_y = \frac{\partial \mathcal{L}}{\partial v_y} = \frac{mc\, v_y}{\sqrt{c^2 - v^2}} \Longrightarrow \boxed{\vec{p} = \frac{mc\,\vec{v}}{\sqrt{c^2 - v^2}}}$$

$$p_z = \frac{\partial \mathcal{L}}{\partial v_z} = \frac{mc\, v_z}{\sqrt{c^2 - v^2}}$$

2) Expression de v

Nous avons $p^2 = \dfrac{m^2c^2v^2}{c^2 - v^2}$ $\Longrightarrow p^2c^2 - p^2v^2 = m^2c^2v^2$

$$\Longrightarrow p^2c^2 = m^2c^2v^2 + p^2v^2$$

$$\Longrightarrow p^2c^2 = v^2(p^2 + m^2c^2)$$

$$\Longrightarrow \boxed{v = \frac{pc}{\sqrt{p^2 + m^2c^2}}}$$

3) Produit scalaire $\vec{p}\cdot\vec{v}$

$$\vec{p}\cdot\vec{v} = \sqrt{c^2 - v^2}\,(mc)^{-1} p^2$$

or $\sqrt{c^2 - v^2} = \sqrt{c^2 - \dfrac{p^2c^2}{p^2 + m^2c^2}} = \sqrt{\dfrac{m^2c^4}{p^2 + m^2c^2}} = \dfrac{mc^2}{\sqrt{p^2 + m^2c^2}}$

Donc $\boxed{\vec{p}\cdot\vec{v} = \dfrac{p^2c}{\sqrt{p^2 + m^2c^2}}}$

4) Hamiltonien $\mathcal{H}$

$$\mathcal{H} = \vec{p}\cdot\vec{v} - \mathcal{L} = \vec{p}\cdot\vec{v} + mc\sqrt{c^2 - v^2} + V$$

$$= \frac{p^2c}{\sqrt{p^2 + m^2c^2}} + \frac{m^2c^3}{\sqrt{p^2 + m^2c^2}} + V = \frac{c(p^2 + m^2c^2)}{\sqrt{p^2 + m^2c^2}} + V = c\sqrt{p^2 + m^2c^2} + V$$

$$\Longrightarrow \boxed{\mathcal{H} = \sqrt{p^2c^2 + m^2c^4} + V}$$

5) Energie totale E

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \cdot \frac{d\vec{p}}{dt} + \frac{\partial \mathcal{H}}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} + \frac{\partial \mathcal{H}}{\partial t}$$

or, $\frac{\partial \mathcal{H}}{\partial \vec{p}} = +\vec{v}$ et $\frac{\partial \mathcal{H}}{\partial \vec{r}} = -\dot{\vec{p}}$ et $\frac{\partial \mathcal{H}}{\partial t} = 0$ car V ne dépend pas explicitement du temps.

$$\Longrightarrow \frac{d\mathcal{H}}{dt} = \vec{v} \cdot \dot{\vec{p}} - \dot{\vec{p}} \cdot \vec{v} = 0 \Longrightarrow \boxed{\mathcal{H} = E = C^{te}}$$

6) Energies cinétique et potentielle

Nous avons $\begin{cases} \mathcal{L} = T - E_p \\ E = T + E_p \end{cases} \Longrightarrow \begin{cases} 2T = \mathcal{L} + E \\ 2E_p = E - \mathcal{L} \end{cases}$

$$T = \frac{1}{2}\left(-mc\sqrt{c^2 - v^2} - V + \sqrt{\frac{m^2c^2v^2}{c^2 - v^2} + m^2c^4} + V\right)$$

$$= \frac{1}{2}\left(-\sqrt{m^2c^4 - m^2c^2v^2} + \sqrt{\frac{m^2c^6}{c^2 - v^2}}\right)$$

$$= \frac{1}{2}\left(-mc\sqrt{c^2 - v^2} + \frac{mc^3}{\sqrt{c^2 - v^2}}\right) = \frac{mc}{2\sqrt{c^2 - v^2}}\left(-\cancel{c^2} + v^2 + \cancel{c^2}\right)$$

$$\Longrightarrow \boxed{T = \frac{1}{2} \frac{mcv^2}{\sqrt{c^2 - v^2}}}$$

$$E_p = \frac{1}{2}\left(E - \mathcal{L}\right) = \frac{1}{2}\left(\sqrt{\frac{m^2c^4v^2}{c^2 - v^2} + m^2c^4} + V + mc\sqrt{c^2 - v^2} + V\right)$$

$$= V + \frac{1}{2}\left(\frac{mc^3}{\sqrt{c^2 - v^2}} + \frac{mc\,(c^2 - v^2)}{\sqrt{c^2 - v^2}}\right) = V + \frac{1}{2}\left(\frac{2mc^3 - mcv^2}{\sqrt{c^2 - v^2}}\right)$$

$$\Longrightarrow \boxed{E_p = V + \frac{mc\,(2c^2 - v^2)}{2\sqrt{c^2 - v^2}}}$$

1) Limite non relativiste $v \ll c$:

Lagrangien : $\mathcal{L} = -mc\sqrt{c^2-v^2} - V(\vec{r})$

$$= -mc^2\sqrt{1-\frac{v^2}{c^2}} - V = -mc^2 + \frac{1}{2}mv^2 - V$$

$$\boxed{\mathcal{L} \simeq \frac{1}{2}mv^2 - (mc^2 + V)}$$

énergie de masse au repos

Impulsion : $\vec{p} = \frac{mc\vec{v}}{\sqrt{c^2-v^2}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \boxed{m\vec{v} = \vec{p}}$

Energie totale : $E = \sqrt{p^2c^2+m^2c^4}+V$

$$= mc^2\sqrt{1+\frac{p^2}{m^2c^2}}+V$$

$$\simeq mc^2\left(1+\frac{p^2}{2m^2c^2}\right)+V$$

$$\boxed{E = \frac{p^2}{2m^2} + mc^2 + V}$$

Energie cinétique : $T = \frac{1}{2}\frac{mcv^2}{\sqrt{c^2-v^2}} = \frac{1}{2}\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} \simeq \boxed{\frac{1}{2}mv^2 = T}$

Energie potentielle : $E_p = V + \frac{mc(2c^2-v^2)}{2\sqrt{c^2-v^2}}$

$$= V + \frac{m(2c^2-v^2)}{2\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{E_p \simeq V + mc^2}$$

Dans la mesure où une énergie est définie à une constante près, le terme mc^2 ne pose pas problème lorsqu'on compare avec expressions habituelles.