Phase-charge duality in Josephson junction circuits: effect of microwave irradiation*

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I. Ultrasmall Josephson junctions

Some basic notions
Small Josephson junction:

**Josephson relations:**

\[
I = I_c \sin \phi \\
\dot{\phi} = 2eV/\hbar
\]

**Small Josephson junction: two energy scales**

\[
E_C = \frac{(2e)^2}{2C} \\
E_J = \hbar I_c/(2e)
\]

**Hamiltonian**

\[
H = E_C \left(\frac{Q}{2e}\right)^2 - E_J \cos \phi - I_b \phi
\]

**Commutator**

\[
[\phi, Q] = 2ie
\]
Superconductor vs. insulator behavior: consequence of phase-charge duality

\[ H = E_c \left( \frac{Q}{2e} \right)^2 - E_J \cos \phi \]

- Localized charge
- Localized phase

Coulomb blockade

- Zero-current state
- Zero-voltage state

Josephson effect
Effect of environment: RCSJ model

\[ \frac{d^2 \phi}{dt^2} + \sigma \frac{d\phi}{dt} + \sin \phi = i_0 \]

- \( \frac{d^2 \phi}{dt^2} \): displacement current
- \( \sigma \frac{d\phi}{dt} \): current through resistor
- \( \sin \phi \): supercurrent

\[ \omega_p = \left( \frac{2eI_c}{\hbar C} \right)^{1/2} \]

\[ \sigma = G \left( \frac{\hbar}{2eI_c C} \right)^{1/2} \]

\[ i_0 = \frac{I_0}{I_c} \]

Dynamics of fictitious phase particle
Effect of environment: RCSJ model

\[ \omega_p = \left( \frac{2eI_c}{\hbar C} \right)^{1/2} \]
\[ \sigma = G \left( \frac{\hbar}{2eI_cC} \right)^{1/2} \]
\[ i_0 = \frac{I_0}{I_c} \]

Overdamped limit: effect of thermal noise

\[ I(V_B) = I_0 \text{Im} \left[ \frac{I_1 - 2i\beta eV_B/\hbar R_B(\beta E_J)}{I_2 - 2i\beta eV_B/\hbar R_B(\beta E_J)} \right] \]

Steinbach et al., '01

Ivanchenko & Zilbermann '69
Effect of microwaves: Shapiro voltage steps

Dynamics of phase

\[ \frac{d^2 \phi}{dt^2} + \sigma \frac{d\phi}{dt} + \sin \phi = i_0 + i_1 \sin \Omega_1 t. \]

Finite voltage: Josephson oscillations

\[ \langle V \rangle = \left\langle \frac{d\phi}{dt} \right\rangle = f_J \]

Phase-locking: voltage-Shapiro steps at

\[ V_n = n hf_1 / 2e \]

\( \sigma > 1 \) \hspace{2cm} \( \sigma < 1 \)

Kautz '96
Duality: Josephson junction versus phase-slip junction

Cooper pair tunneling

\[ \hat{H} = \frac{\hat{Q}^2}{2C} - E_J \cos \hat{\theta} \]

\[ E_J = \frac{\hbar}{2e} I_c \]

Phase-slip

\[ \hat{H} = \frac{\hat{\theta}^2}{2L} - \Delta_0 \cos \hat{q} \]

\[ \Delta_0 = eV_c \]

\[ [\theta, \mathcal{Q}] = -2ie \]

\[ \hat{\theta} \leftrightarrow \hat{q} \]

Schmid (1983)
Likharev, Averin, Zorin (1985)
Schön, Zaikin (1990)
Ingold and Nazarov (1992)
Weiss (1999)
Beloborodov et al. (2002)
Zazunov et al. (2008)
Arutyunov, Golubev & Zaikin (2008)

J.E. Mooij and Yu.V. Nazarov (2006)
Consequences of duality: IV characteristics & Shapiro steps

\[ V_0 \]

\[ \omega_c = \left( \frac{\pi V_c}{eL} \right)^{1/2} \]

\[ \frac{d^2q}{dt^2} + \rho \frac{dq}{dt} + \sin q = v_0 \]

\[ v_0 = \frac{V_0}{V_c} \]

\[ \rho = R \left( \frac{e}{\pi V_c L} \right)^{1/2} \]

Phase vs. charge particle
Consequences of duality: IV characteristics & Shapiro steps

\[ V_0 \]
\[ R \]
\[ L \]

\[ \omega_c = \left( \pi \frac{V_c}{eL} \right)^{1/2} \]

\[ \frac{d^2 q}{dt^2} + \rho \frac{dq}{dt} + \sin q = v_0 + \delta \]

inductance  resistor  phase slip element

\[ v_0 = \frac{V_0}{V_c} \]
\[ \rho = \frac{R(e/\pi V_c L)^{1/2}}{t MWv} \]

Expected I-V characteristics

Effect of microwaves: current-Shapiro steps

\( \rho > 1 \)
\( \rho < 1 \)
II. Voltage-biased junction in an inductive environment

*Single Josephson junction as a phase-slip element*
CJRL-model: Circuit & Hamiltonian


\[ \hat{H} = \hat{H}_J + \hat{H}_L + \hat{H}_B \]

Josephson junction

\[ \hat{H}_J = \frac{\hat{Q}^2}{2C} - E_J \cos \theta \]

Inductance

\[ \hat{H}_L = \left( \frac{\hbar}{2e} \right)^2 \frac{(\phi_b - \phi_R - \theta)^2}{2L} \]

Bath Hamiltonian

Coupling

Bias phase

\[ \hat{H}'_B = \sum_{i=1}^{\infty} \frac{\hat{\Pi}_i^2}{2} + \frac{1}{2} \hat{\omega}_i^2 \hat{\Xi}_i^2 \]

\[ \hat{\phi}_R = \sum_i \lambda_i \hat{\Xi}_i \]

\[ \frac{\hbar}{2e} \dot{\phi}_b = V_b \]
Single junction: Bloch Bands

Periodic Hamiltonian:
\[ \hat{H}_J = \frac{\hat{Q}^2}{2C} - E_J \cos \theta \quad E_J \gg E_C \]

Tight-Binding Model

Sinusoidal lowest band:
\[ \epsilon_0(q) = -\Delta_0 \cos \pi q/e \]

Phase-slip amplitude:
\[ \Delta_0 \approx \left( E_J^3 E_C \right)^{1/4} \exp \left( -\sqrt{8E_J / E_C} \right) \]
Dynamics in lowest Bloch band

\[ \hat{H} = \frac{\hat{Q}^2}{2C} - E_J \cos \hat{\theta} + \frac{1}{2L} \left( \varphi_b - \varphi_R - \hat{\theta} \right)^2 + H_{bath} \]

Equation of motion for quasicharge

\[ \frac{d\hat{q}}{dt} = i[\hat{H}, \hat{q}] = \frac{d\hat{H}}{d\varphi_b} = \hat{I} \]

Quasicharge = total momentum of JJ that is conserved without external current
Equation of motion for quasicharge

\[ L\ddot{q} + \partial \varepsilon_0(q)/\partial q = V_b + \delta \dot{V} \]

\[ E_J \gg E_C \]

\[ Z(\omega) \rightarrow R \]

\[ L\ddot{q} + R\dot{q} + V_c \sin \pi q/e = V_b + \dot{v} \]

\[ \omega_c = (\pi V_c/eL)^{1/2} \]

\[ \rho = R(e/\pi V_c L)^{1/2} \]

\[ d^2\hat{q}/d\tau'^2 + \rho d\hat{q}/d\tau' + \sin \hat{q} = \bar{V}_b + \dot{v} \]

Exactly dual to usual RCSJ model!

\[ \delta \dot{V} = \dot{v}(t) - \int_0^t Z(t - t') \dot{q}(t') \]

\[ Z(\omega) = \left( \frac{\hbar}{2e} \right)^2 \sum_{i=1}^{\infty} \frac{\chi_i^2}{(\omega + i\eta)^2 - \tilde{\omega}_i^2} \]

\[ \int dt e^{i\omega t} \langle \{ \dot{v}(t), \dot{v}(0) \} \rangle / 2 \]

\[ = \hbar \omega \Re[Z(\omega)] \coth(\hbar \omega / 2k_B T) \]
Equation of motion for quasicharge: quasiclassical approach

\[ L \ddot{q} + R \dot{q} + V_c \sin \frac{\pi q}{e} = V_b + \dot{\nu} \]

Quasiclassical dynamics

Narrow wave packets, thermal noise

\[ L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + V_c \sin(\pi q / e) = V_b + \nu \]

Ignore wave-packet spreading on scale \(1/\omega_c, L/R\)

Classical noise:

\[ \langle \nu(t)\nu(0) \rangle = 2k_B T R \delta(t) \]

\(\omega_c, R/L < T < \Delta_0\)
Overdamped quasicharge dynamics in a phase-slip junction: experiment

Corlevi, Guichard, FH & Haviland (2006); Beloborodov, FH & Pistolesi (2002)

\[ L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + V_c \sin(\frac{\pi q}{e}) = V_b + v_{\text{ther}} \]

\[ \langle V \rangle = V_C \Im \left[ \frac{I_1 - i \beta e l_b R / \pi (\beta e V_C / \pi)}{I_1 - i \beta e l_b R / \pi (\beta e V_C / \pi)} \right] \]

Dual to Steinbach et al.!
III. One-dimensional junction arrays

Josephson junction chain as a phase-slip element
(providing its own inductance)
Phase-biased Josephson junction array

Ignore self-capacitance: \( \tilde{E}_C \gg E_J, E_C \).

Action:
\[
S = \int_0^\beta dt \sum_{n=1}^N \left\{ \frac{\theta_n^2}{2E_C} + E_J[1 - \cos\theta_n(t)] \right\}.
\]

Phase bias leads to constraint:
\[
\sum_{n=1}^N \theta_n(t) = \varphi
\]

Strong Josephson coupling & long array:
\( E_J \gg E_C, N \gg 1 \)

Classical energy of chain: minimize potential energy for large \( N \), imposing the constraint
\[
\theta_n = \phi / N \ll 1 \quad \Rightarrow \quad E_{\text{class}} = \frac{E_J}{2N} \phi^2 \quad \text{(inductor)}
\]
Phase-slips in phase-biased arrays

Phase slip: energy unchanged, but constraint violated

Phase slip combined with small adjustment: constraint satisfied

\[ \theta_n = \sum_{m=1}^{n} \theta_n \]

\[ \phi / N \]

\[ \phi \]

\[ \phi + 2\pi \]

\[ \phi - 2\pi / N \]
Hamiltonian for quantum phase-slip array

Matveev, Larkin, Glazman (2002)

Classical energy for $m$ phase slips:

$$E_J (\phi - 2\pi m)^2 / 2N$$

Effect of capacitance: quantum fluctuations

Phase slip at junction $k$:

$$\theta_k(t) = 4 \arctan \exp[\sqrt{E_J E_C} (t - t')]$$

Phase slip amplitude:

$$\Delta_0 \approx \left( E_J^3 E_C \right)^{1/4} \times \exp\left(-\sqrt{8E_J / E_C}\right)$$

Hamiltonian:

$$\hat{H}_{ar} = \frac{E_J}{2N} (2\pi \hat{m} - \phi)^2 - \frac{N\Delta_0}{2} \sum_m [\ket{m+1}\bra{m} + h.c.]$$

Phase-slip can occur on any junction
Phase-slip array in quasi-charge representation

Guichard & FH (2009)

Introduce « quasicharge representation »

\[ [\hat{q}, \hat{m}] = -\frac{i e}{\pi} \]

Rewrite Hamiltonian

\[ \hat{H}_{ar} = \left( \frac{E_J}{2N} \right) \left( 2\pi \hat{m} - \phi \right)^2 - N \Delta_0 \cos \pi \hat{q}/e \]

Quasicharge dynamics related to current

\[ \dot{\hat{q}} = \left( \frac{2e}{\hbar} \right) d\hat{H}/d\phi \]

\[ L_{ar} \ddot{q} + R \dot{q} + V_{c,ar} \sin \pi \hat{q}/e = V_b + \dot{\nu} \]

\[ L_{ar} = \hbar N / 2e I_c \quad V_{c,ar} = NV_c \]

Tunable parameters!

Again exactly dual to usual RCSJ model!
IV. Phase-slip junction under microwave irradiation

*Current Shapiro steps: towards a current standard?*
Microwaves & Shapiro steps

\[
\frac{d^2 q}{dt^2} + \rho \frac{dq}{dt} + \sin q = v_0 \quad v_0 = \frac{V_b}{V_c}
\]

\[
V_b(t) = V + V_{MW} \sin(\omega_{MW} t)
\]

Overdamped case

\(\rho=2, \ V_{MW}=5V_c, \ \omega_{MW}=1.5\omega_c\)

Underdamped case

\(\rho=0.15 \ V_{MW}=3V_c, \ \omega_{MW}=2\omega_c\)
Phase-slip junctions, Shapiro steps & the metrological triangle

Josephson junction
\[ V = \frac{n h f}{2e} \]

Phase-slip junction
\[ I = m e f \]

Quantum Hall effect
\[ I = \frac{p e^2}{h} V \]

\[ V/V_0 \]

\[ I/e_0 \]
Conclusions

I. Ultrasmall Josephson junctions
Some basic notions:
- Junction in electromagnetic environment
- Voltage-Shapiro steps
- Duality

II. Voltage-biased junction in an inductive environment
Single Josephson junction as a phase-slip element:
- Hamiltonian
- Duality with an external inductance

III. One-dimensional junction arrays
Josephson junction chain as a phase-slip element:
- Quantum phase-slips in a phase-biased array
- Duality with intrinsic inductance

IV. Effect of microwaves
- Current-Shapiro steps: closing the metrological triangle?