Control of chaos in Hamiltonian systems

G. Ciraolo, C. Chandre, R. Lima, M. Vittot
Centre de Physique Théorique – CNRS, Marseille

M. Pettini
Osservatorio Astrofisico di Arcetri, Università di Firenze

Ph. Ghendrih
Association Euratom-CEA, CEA Cadarache

Contact: ciraolo@cpt.univ-mrs.fr

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References: J. Phys. A: Math. Gen. **37**, 3589 (2004)
J. Phys. A: Math. Gen. **37**, 6337 (2004)
Phys. Rev. E **69**, 056213 (2004)
Nonlinearity **18** (2005)

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Our strategy of control 1

original system

$$\frac{d\mathbf{x}}{dt} = X(\mathbf{x}, t)$$

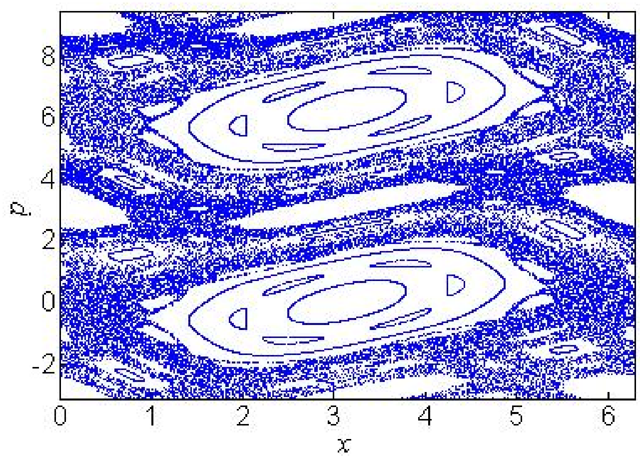
controlled system

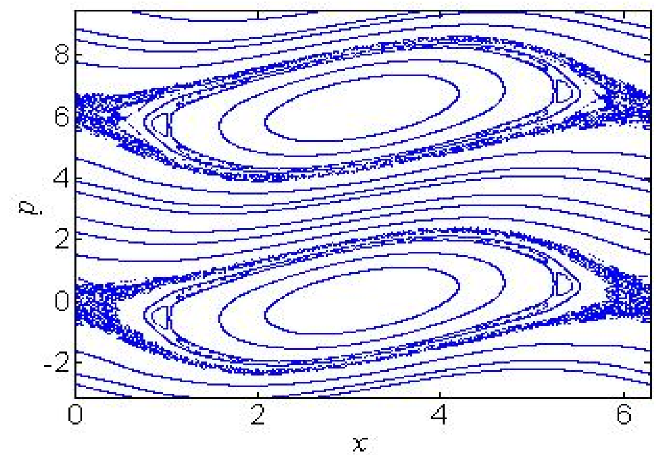
$$\frac{d\mathbf{x}}{dt} = X_c(\mathbf{x}, t)$$

with $X_c = X + f$

control term

chaotic dynamics → regular dynamics





Aim: find a small and apt control f

> small modification of the system / great influence on the dynamics

Our strategy of control 2

Aim: $H = H_0 + \varepsilon V$ chaotic $\longrightarrow H_c = H_0 + \varepsilon V + f$ regular

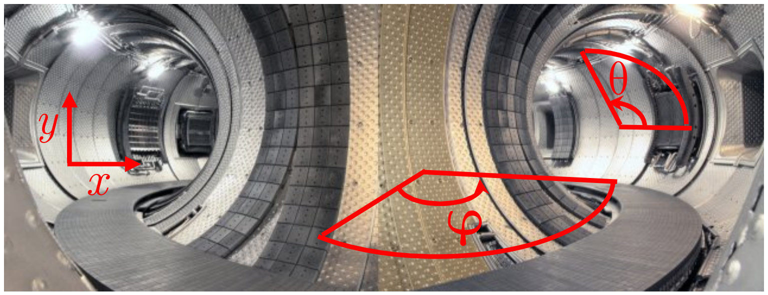
$f = -\varepsilon V$ obvious and useless solution

$\Rightarrow$ tailoring the control term f

Requirements on f :

- small with respect to the perturbation εV
  > here, we require that $f = O(\varepsilon^2)$
- localized in phase space
  > accessible region, fewer energy for the control
- with a certain shape
  > robustness ...
- other requirements ?

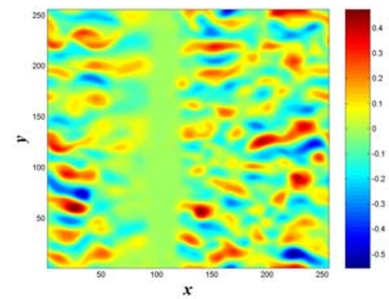
Physical situations



> electrostatic turbulence : $E \times B$ drift motion

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{c}{B^2}\mathbf{E}\times\mathbf{B} = \frac{c}{B}\begin{pmatrix} -\dfrac{\partial V}{\partial y} \\ \dfrac{\partial V}{\partial x} \end{pmatrix}$$

where $\mathbf{E} = -\nabla V(x, y, t)$

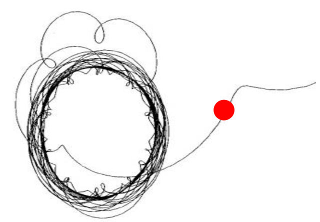


> magnetic field lines

$$\begin{cases} \dfrac{d\psi}{d\varphi} = -\dfrac{\partial H}{\partial \theta} \\ \dfrac{d\theta}{d\varphi} = +\dfrac{\partial H}{\partial \psi} \end{cases}$$

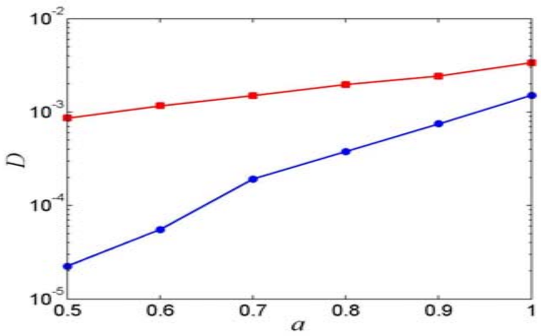
$$H(\psi, \theta, \varphi) = H_0(\psi) + \varepsilon H_1(\psi, \theta, \varphi)$$

> atomic physics : atoms in external fields, traps



> particle accelerators, free electron laser, ...?

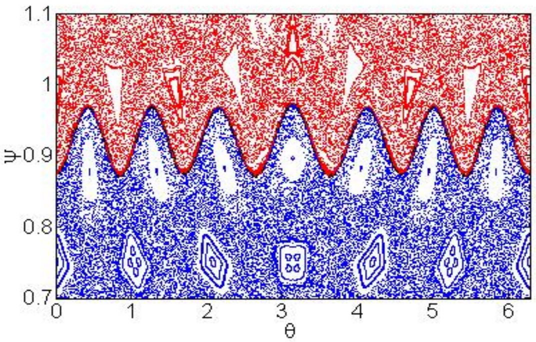
I. *Global control* of Hamiltonian systems :



- method and illustration

- application to *edge plasma turbulence*

II. *Localised control* of Hamiltonian systems



- method and illustration

- application to *magnetic field lines*

Lie formalism

In general

$$H = H_0 + V$$

$\mathcal{L}_{H_0}$ given by $\mathcal{L}_{H_0} V = \{H_0, V\}$

$\{\cdot,\cdot\} =$ Poisson bracket

Γ pseudo inverse of $\mathcal{L}_{H_0}$:

$$\mathcal{L}_{H_0}{}^2\,\Gamma = \mathcal{L}_{H_0}$$

$\mathcal{R}$ projector: $\mathcal{R} = 1 - \mathcal{L}_{H_0}\Gamma$

In action-angle variables:

$$H(\mathbf{A},\boldsymbol{\varphi}) = H_0(\mathbf{A}) + \sum_{\mathbf{k}\in\mathbb{Z}^n} V_{\mathbf{k}}(\mathbf{A})\ \mathrm{e}^{i\,\mathbf{k}\cdot\boldsymbol{\varphi}}$$

where $(\mathbf{A},\boldsymbol{\varphi}) \in \mathbb{R}^n \times \mathbb{T}^n$

$$\mathcal{L}_{H_0} V = \frac{\partial H_0}{\partial \mathbf{A}} \cdot \frac{\partial V}{\partial \boldsymbol{\varphi}} - \frac{\partial V}{\partial \mathbf{A}} \cdot \frac{\partial H_0}{\partial \boldsymbol{\varphi}}$$

$$= \sum_{\mathbf{k}\in\mathbb{Z}^n} i\,\boldsymbol{\omega}\cdot\mathbf{k}\ V_{\mathbf{k}}(\mathbf{A})\ \mathrm{e}^{i\,\mathbf{k}\cdot\boldsymbol{\varphi}}$$

where $\boldsymbol{\omega} = \dfrac{\partial H_0}{\partial \mathbf{A}}$

$$\Gamma V = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^n \\ \boldsymbol{\omega}\cdot\mathbf{k}\neq 0}} \frac{V_{\mathbf{k}}(\mathbf{A})}{i\,\boldsymbol{\omega}\cdot\mathbf{k}}\ \mathrm{e}^{i\,\mathbf{k}\cdot\boldsymbol{\varphi}}$$

$$\mathcal{R} V = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^n \\ \boldsymbol{\omega}\cdot\mathbf{k}= 0}} V_{\mathbf{k}}(\mathbf{A})\ \mathrm{e}^{i\,\mathbf{k}\cdot\boldsymbol{\varphi}}$$

Global control

Proposition 1 : $H_c = H_0 + V + f$ and $H_0 + \mathcal{R}V$ are canonically conjugate

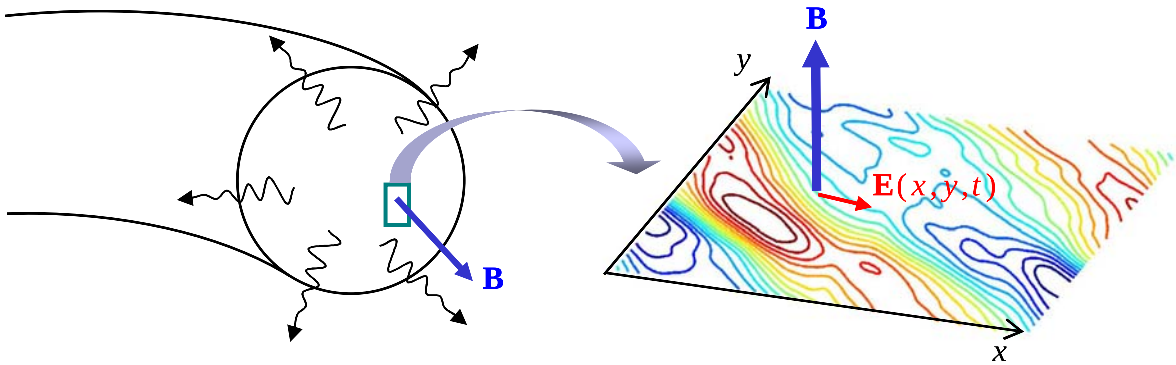
where $$f = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \mathcal{L}_{\Gamma V}{}^{n} (n\mathcal{R} + 1)V$$

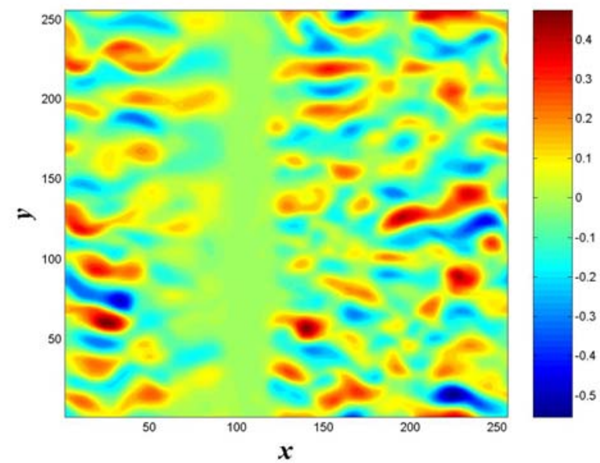
if V is of order $\varepsilon \Rightarrow f$ is of order $O(\varepsilon^2)$

> $f = \sum_{n=2}^{\infty} f_n$ where f_n is of order ε^n

> If $\boldsymbol{\omega} = \dfrac{\partial H_0}{\partial \mathbf{A}}$ is non-resonant and in many other situations, $H_0 + \mathcal{R}V$ is integrable

Control of chaotic transport in a model for $\mathbf{E}\times\mathbf{B}$ *drift in magnetized plasmas*





$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{c}{B^2}\mathbf{E}\times\mathbf{B} = \frac{c}{B}\begin{pmatrix} -\frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial x} \end{pmatrix} \qquad \text{where} \quad \mathbf{E} = -\nabla V(x,y,t)$$

V is the Hamiltonian

$x,\ y$ are conjugate variables

- Algorithm for the computation of the control term f
- Numerical investigation: - of the effect of the control term (in practice f_2)

  - of its robustness

Control of E×B drift

Electrostatic potential $V(x,y,t)$ known on a spatio-temporal grid.

1) Expansion of $V(x,y,t) \approx \sum_k V_k(x,y)\,\mathrm{e}^{i\omega_k t}$

2) Computation of $\Gamma V = \sum_{\omega_k \neq 0} \frac{V_k(x,y)}{i\,\omega_k}\,\mathrm{e}^{i\,\omega_k t}$

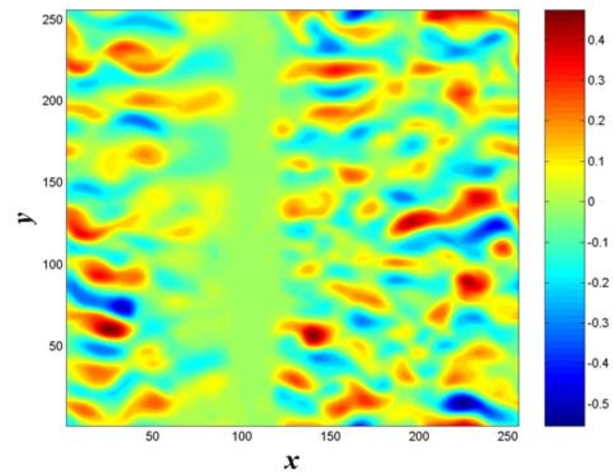
3) Computation of $\boxed{f_2 = -\frac{1}{2}\{\Gamma V, V\}}$

where $\{\,\cdot\,,\,\cdot\,\} = \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\cdot\frac{\partial}{\partial x}$

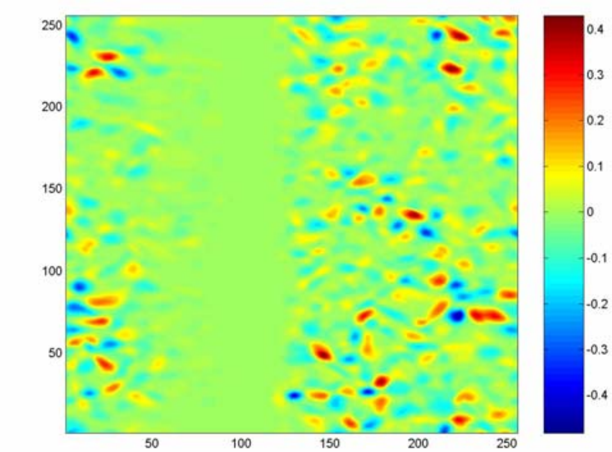
4) Higher orders ? $\boxed{f_n = -\frac{1}{n}\{\Gamma V, f_{n-1}\}}$

(recall: $f = \sum_{n=2}^{\infty} f_n$).

$V(x,y,t=0)$



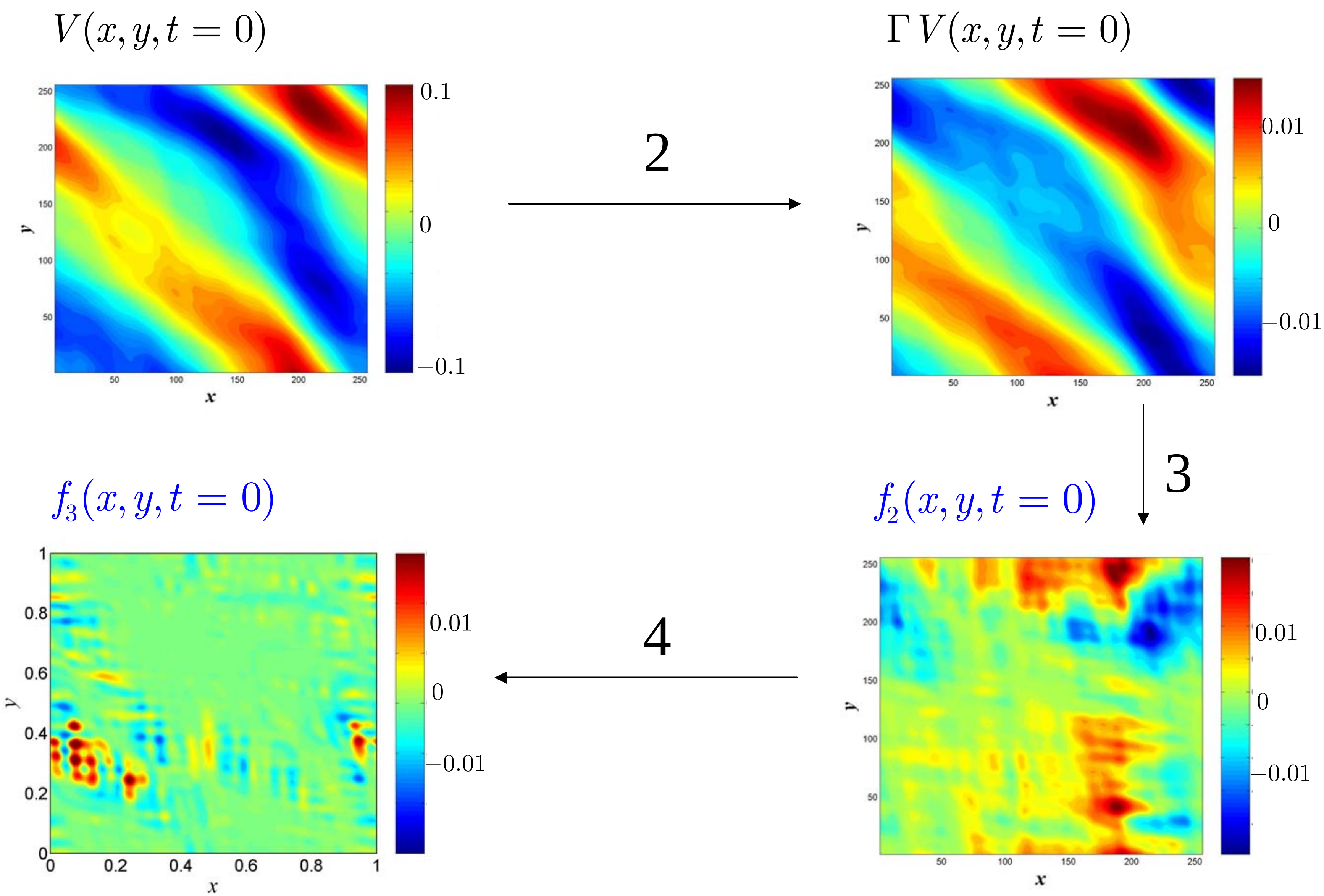
$f_2(x,y,t=0)$



Example:

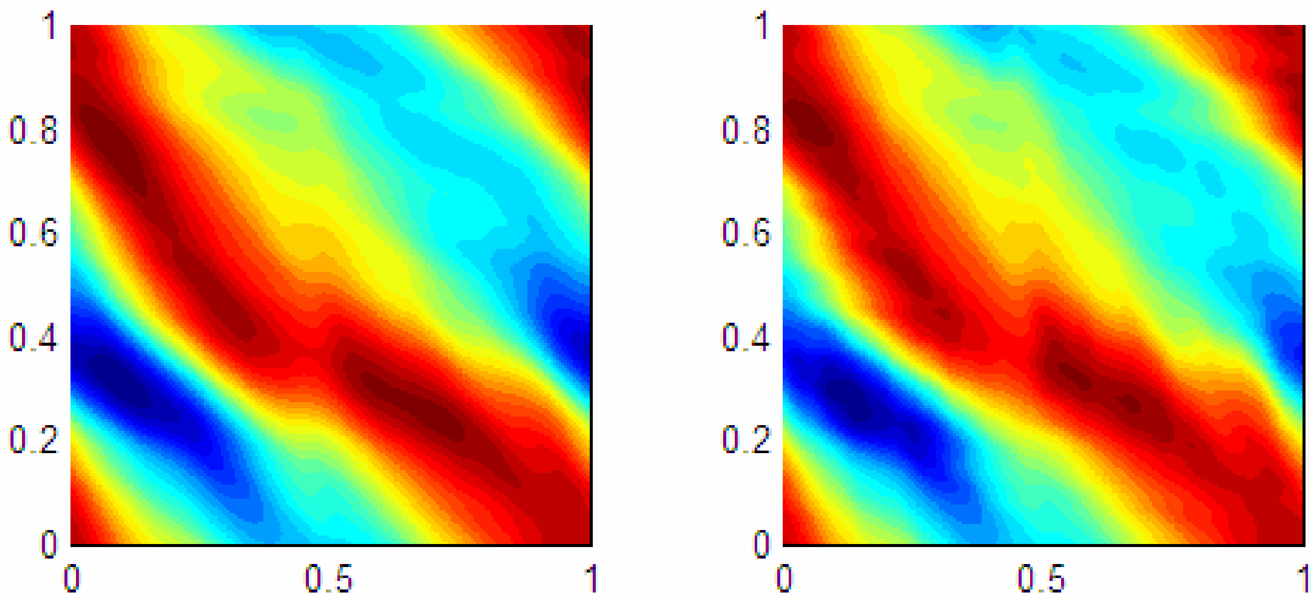
$$V(x,y,t) = \sum_{m,n,k} \frac{a_k}{(n^2+m^2)^{3/2}} \sin\left(2\pi\left(nx+my\right)+\varphi_{nmk}-\omega_k t\right)$$

φ_{nmk} random phases



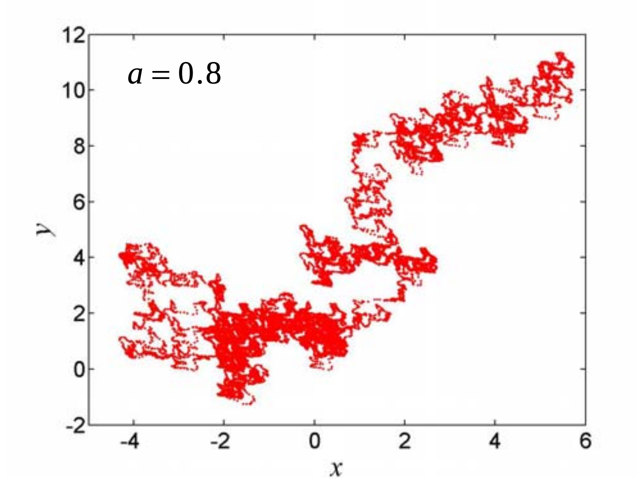
The control term is a small modification of the potential

$V(x,y,t)$ $\qquad\qquad$ $V(x,y,t) + f_2(x,y,t)$

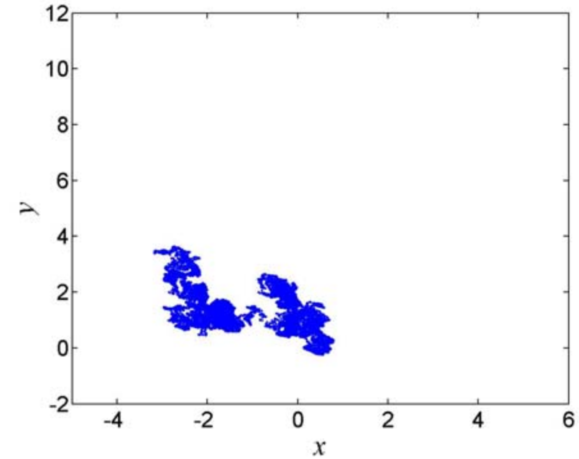


Trajectories of particles

> Poincaré surface
of section

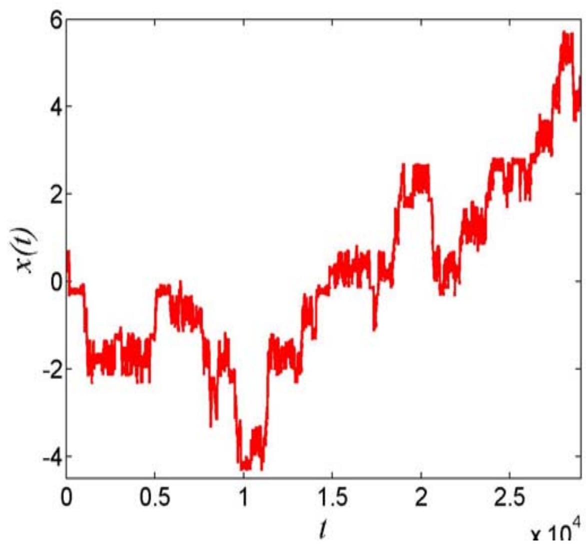


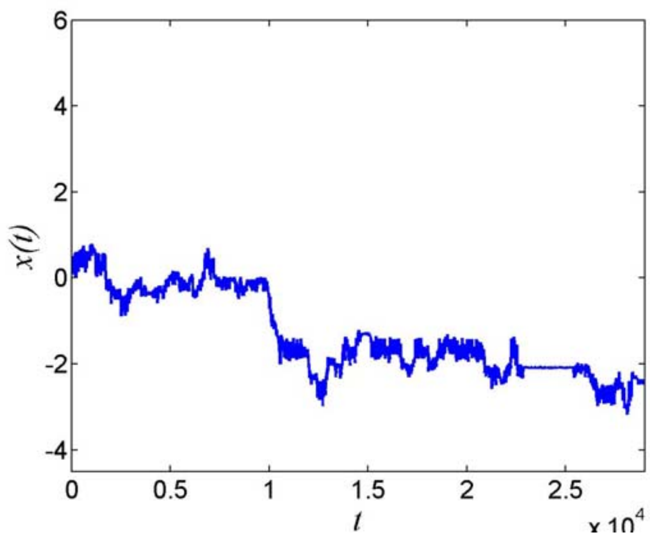
without control



with control f_2

> Projection on
the x axis

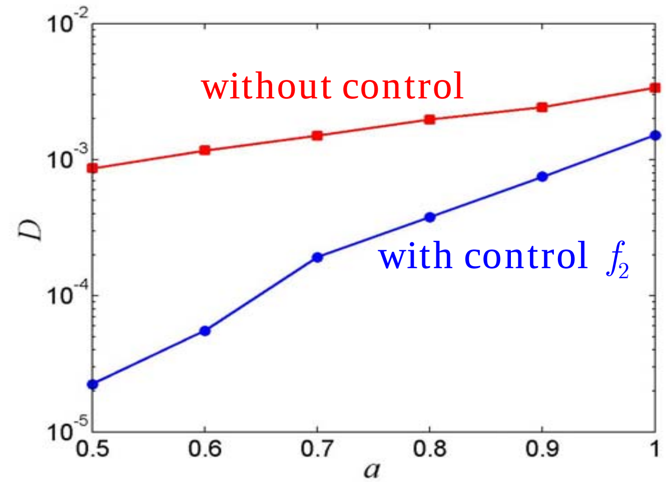




Diffusion of test particles

Diffusion coefficient

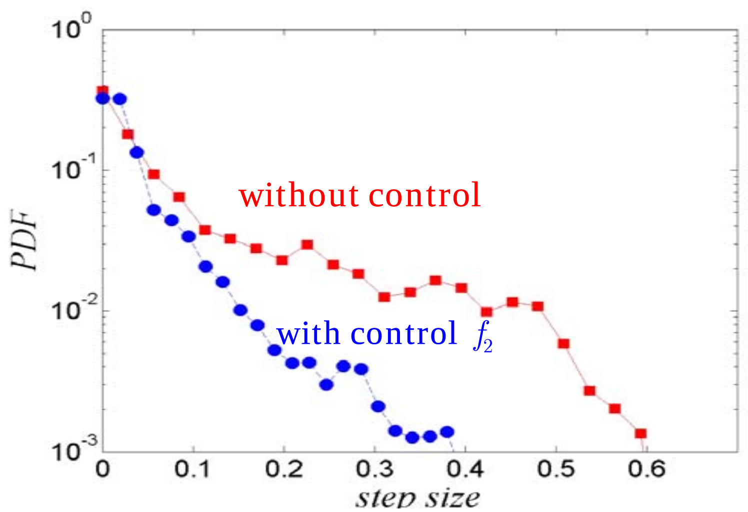
$$D = \lim_{t\to\infty} \frac{1}{Mt} \sum_{i=1}^{M} \left\| \mathbf{x}_i(t) - \mathbf{x}_i(0) \right\|^2$$



> significant reduction of diffusion by f_2

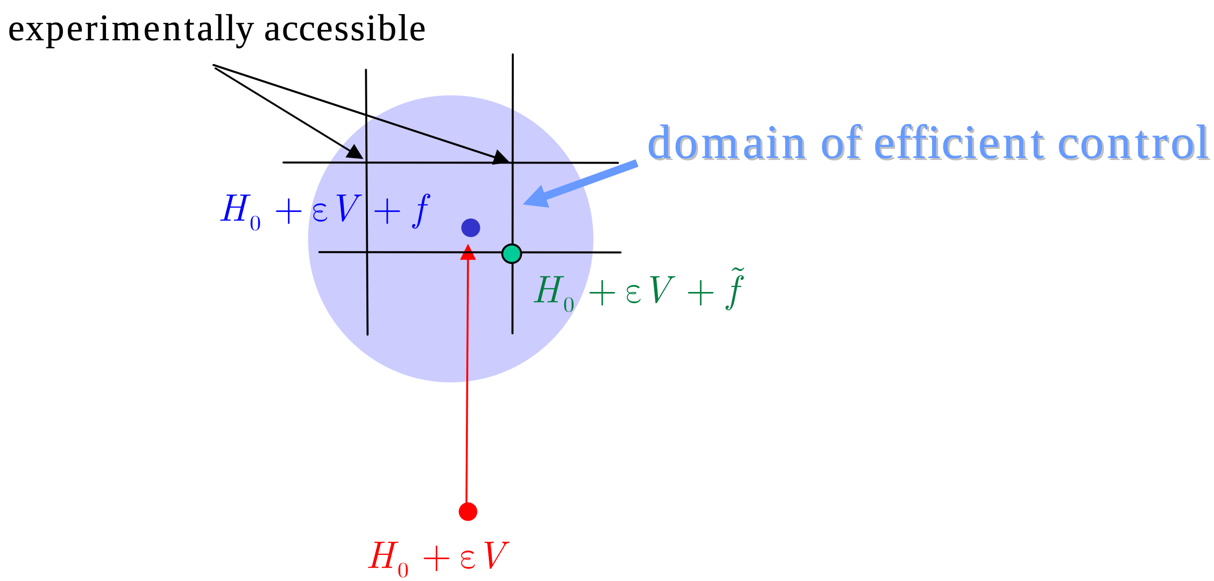
Probability Distribution Function of step sizes

Step size ≡ distance between two successive sign reversals of the velocity



> ***the control quenches the large steps***

Robustness of the control : a crucial requirement



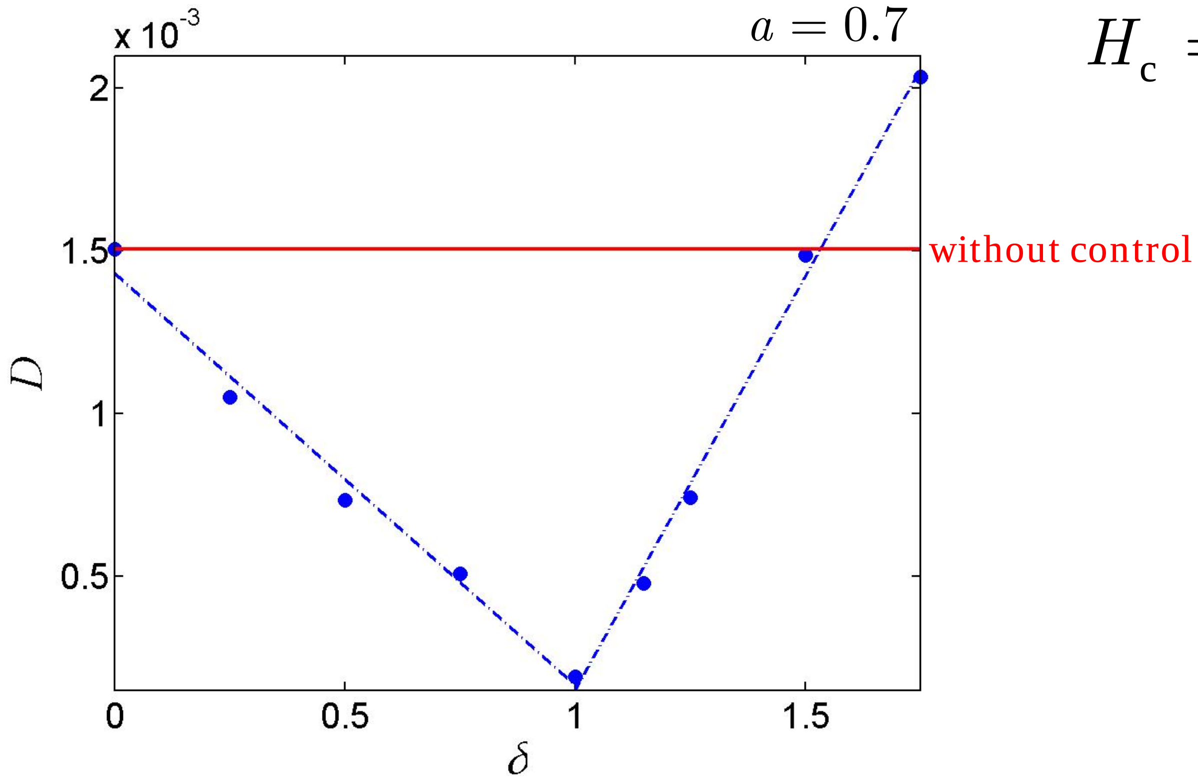
$H_0 + \varepsilon V$ uncontrolled Hamiltonian

$H_0 + \varepsilon V + f$ controlled Hamiltonian, computed theoretically

$H_0 + \varepsilon V + \tilde{f}$ controlled Hamiltonian, implemented experimentally

Robustness 1

> Modification of the amplitude of the control term : $\boxed{f_2 \rightarrow \delta \cdot f_2}$

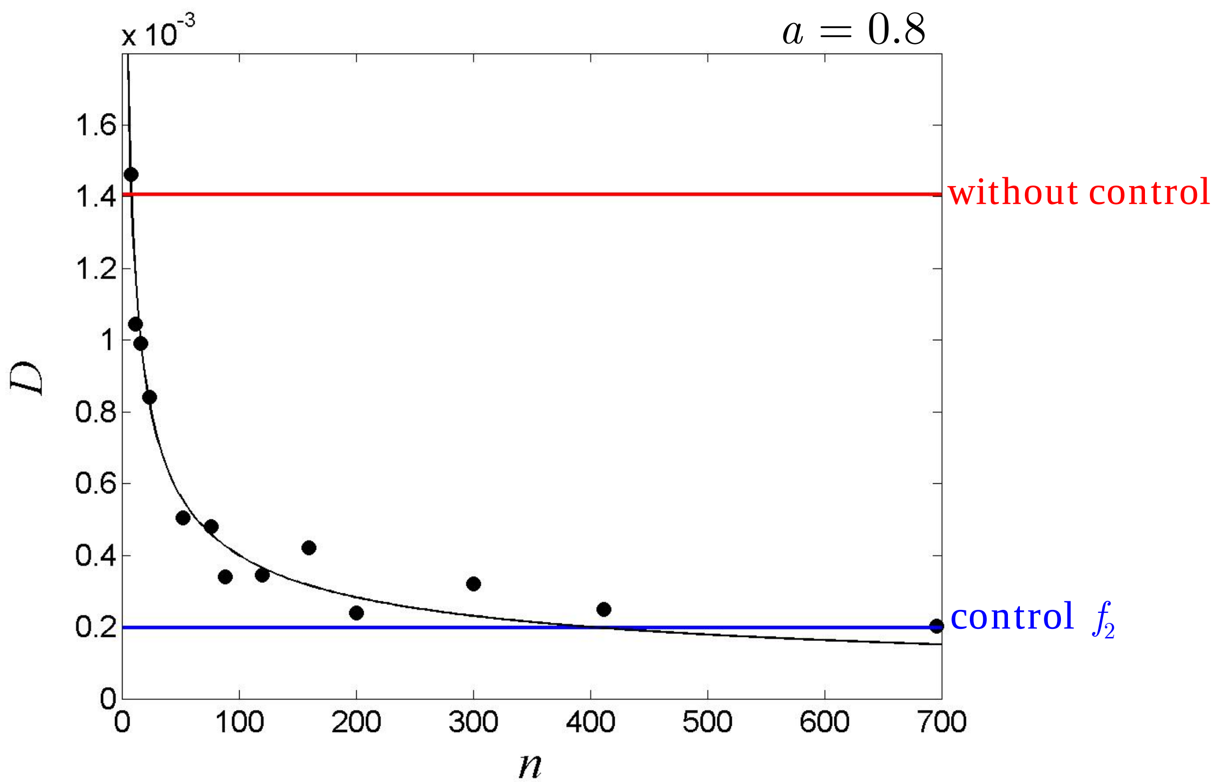


$$H_c = H_0 + \varepsilon V + \delta \cdot f_2$$

- optimal control for $\delta = 1$
- increasing its amplitude does not improve control
- decreasing the amplitude does still give a good control
- reducing energy to control the system ($\delta = 0.5 \Rightarrow$ reduction of 30%)

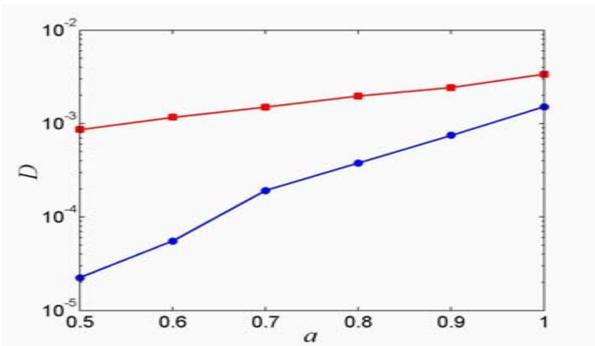
Robustness 2

if the potential is known on a coarse grained grid:
truncation of the Fourier series of the control term f_2



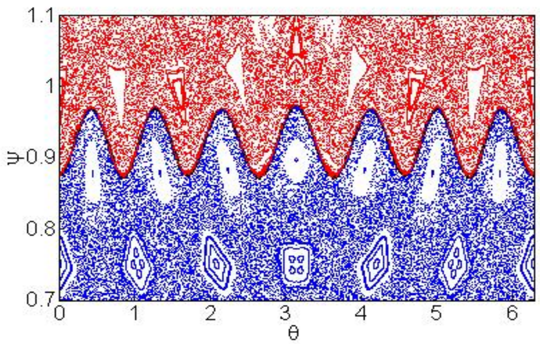
> efficient control with few Fourier modes: 12 modes $\Rightarrow$ reduction of 25%
> simplification of the control term
> reduction of the energy necessary for the control

I. Global control of Hamiltonian systems :



- *method and illustration*
- *application to edge plasma turbulence*

II. *Localised control* of Hamiltonian systems



- method and illustration
- application to *magnetic field lines*

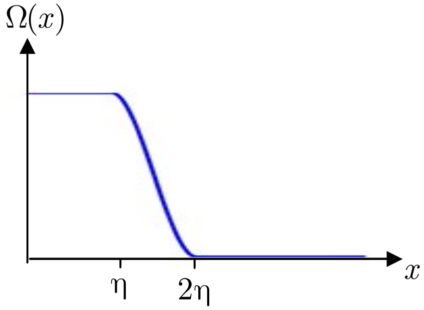
Localised control : channeling chaos by building barriers

Proposition 2 : There exists a control term f of order ε^2 such that

$$H_c(\mathbf{A},\varphi) = H(\mathbf{A},\varphi) + f(\varphi)\,\Omega\left(\left\|\mathbf{A} + \varepsilon\Gamma\partial_\varphi H(\mathbf{0},\varphi)\right\|\right)$$

has an invariant torus at $\mathbf{A} = -\varepsilon\Gamma\partial_\varphi H(\mathbf{0},\varphi)$.

It is given by $f(\varphi) = -H(-\varepsilon\Gamma\partial_\varphi H(\mathbf{0},\varphi),\varphi)$.



> expansion of H around $\mathbf{A} = \mathbf{A}_0$, *e.g.*, $\mathbf{A}_0 = \mathbf{0}$:

Remark: for 2 d.o.f. barrier to diffusion
for n d.o.f. effective barriers

Advantages:

> Explicit expression for the control term: existence and regularity

> Explicit expression for the invariant torus which has been created

> Persistence of the created torus for arbitrarily large values of ε (provided $\| f \| \leq \| V \|$)

Algorithm of construction of a localised control term

on an example: $H(p,x,t) = \frac{p^2}{2} + \varepsilon V(x,t)$

1) translation of the momentum p by ω

$\mathbf{A} = (p, E)$

$\boldsymbol{\varphi} = (x,t)$

$\boldsymbol{\omega} = (\omega, 1)$

$$H(p,E,x,t) = \omega p + E + \varepsilon V(x,t) + \frac{p^2}{2}$$

2) computation of $\Gamma\partial_\varphi V = (\Gamma\partial_x V, \Gamma\partial_t V)$

$$\Gamma\partial_\varphi V = \sum_{k_1,k_2} \frac{1}{\omega k_1 + k_2} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} V_{k_1 k_2}\, \mathrm{e}^{i(k_1 x + k_2 t)}$$

3) computation of $f = -H\left(-\varepsilon\Gamma\partial_x V, -\varepsilon\Gamma\partial_t V, x, t\right)$

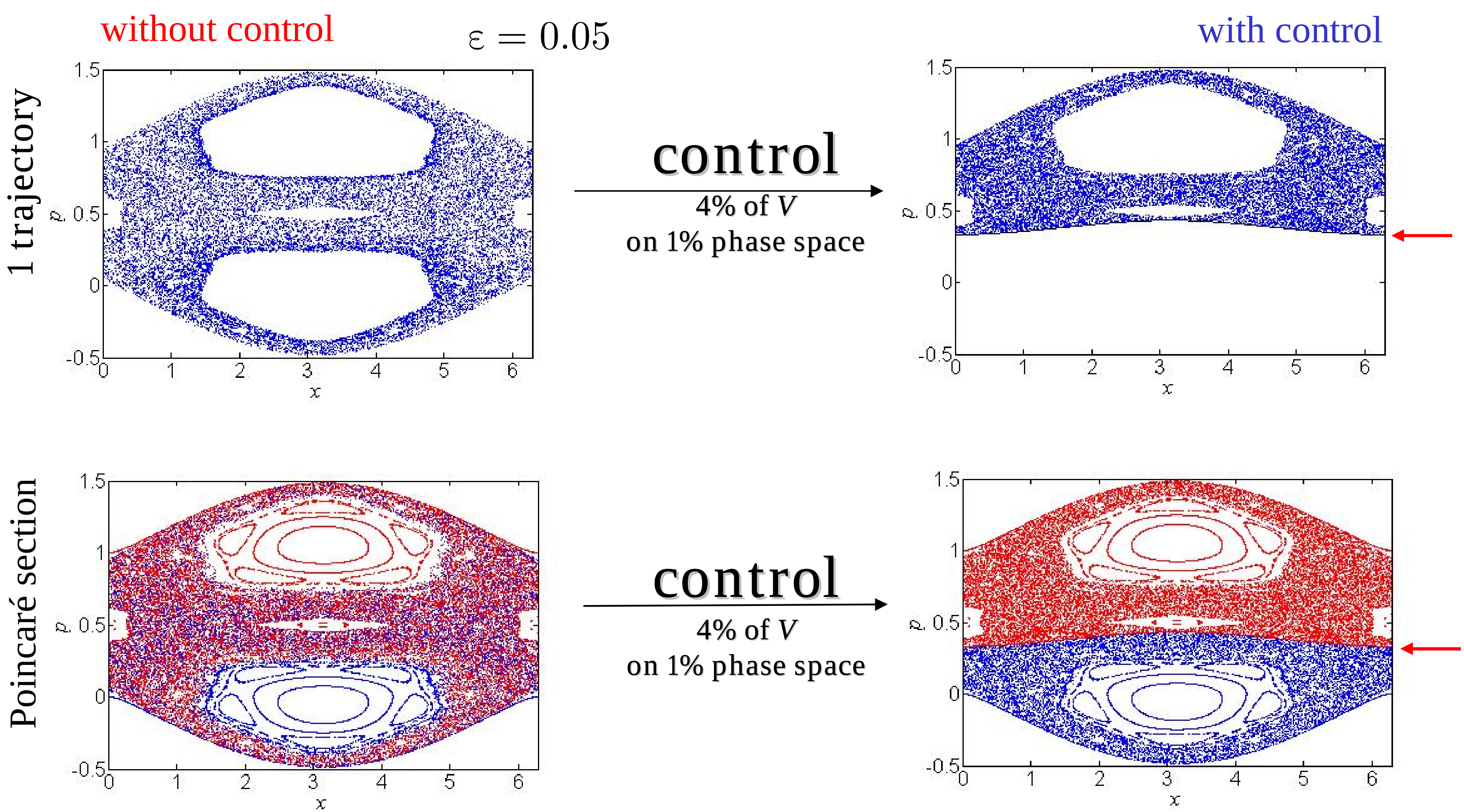
$$f = -\frac{\varepsilon^2}{2}\left(\sum_{k_1,k_2} \frac{k_1}{\omega k_1 + k_2} V_{k_1 k_2}\, \mathrm{e}^{i(k_1 x + k_2 t)}\right)^2$$

4) equation of the invariant torus

$$p_0(x,t) = \omega - \varepsilon \sum_{k_1,k_2} \frac{k_1}{\omega k_1 + k_2} V_{k_1 k_2}\, \mathrm{e}^{i(k_1 x + k_2 t)}$$

Localised control: forced pendulum

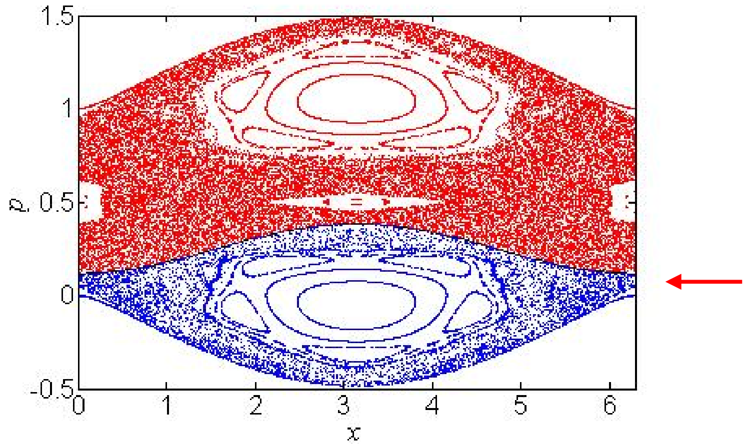
$$H_c = \frac{p^2}{2} + \varepsilon\left(\cos x + \cos(x-t)\right) - \frac{\varepsilon^2}{2}\left(\frac{\cos x}{\omega} + \frac{\cos(x-t)}{\omega - 1}\right)^2 \Omega\left(\left|p - p_0(x,t)\right|\right)$$

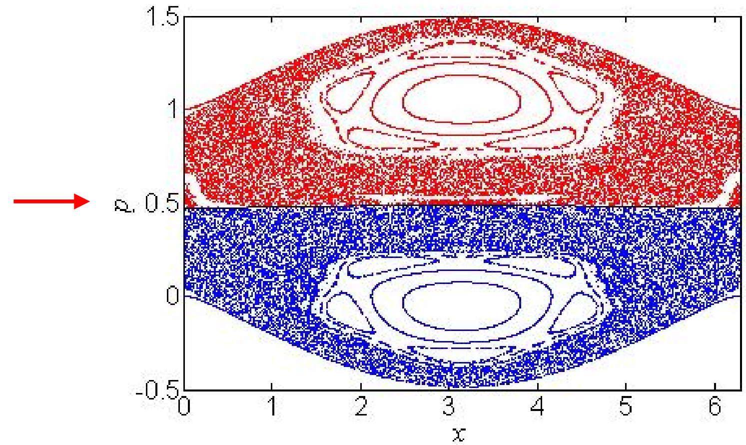


> uncontrolled case : no barrier for $\varepsilon \geq 0.028$

Localised control: channelling chaos by building barriers

• Creation of barriers at different locations

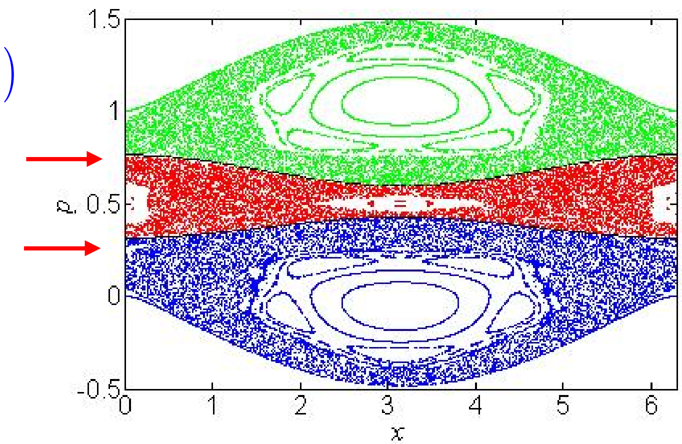




• Creation of set of barriers

$$H_c = \frac{p^2}{2} + \varepsilon\left(\cos x + \cos(x-t)\right) + \varepsilon^2 \sum_{i=1}^{M} f_i(x,t)\,\Omega\left(\left|p - p_{0i}(x,t)\right|\right)$$

$\varepsilon = 0.05$

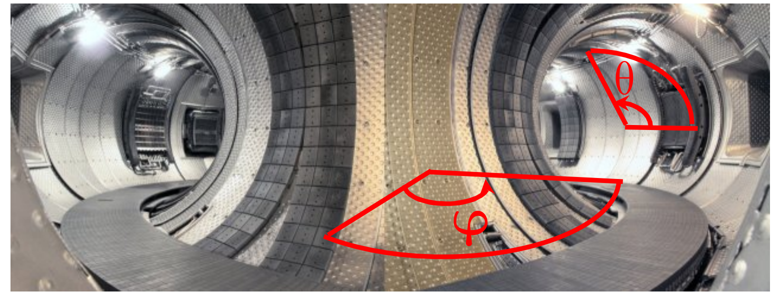


> Trapping of particles

Localised control: magnetic field lines

Magnetic field line dynamics in a toroidal geometry

$$\begin{cases} \dfrac{d\vartheta}{d\varphi} = +\dfrac{\partial H}{\partial \psi} \\ \dfrac{d\psi}{d\varphi} = -\dfrac{\partial H}{\partial \vartheta} \end{cases}$$



Nearly axisymmetric case: $H(\psi,\vartheta,\varphi) = H_0(\psi) + \varepsilon H_1(\psi,\vartheta,\varphi)$

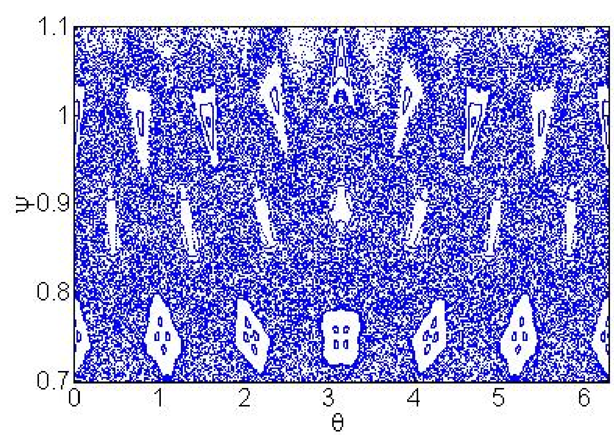
- regular part: $H_0(\psi) = \displaystyle\int \frac{d\psi}{q(\psi)}$

- safety factor: $q(\psi) = \dfrac{4}{(2-\psi)(2-2\psi+\psi^2)}$

- magnetic perturbation:

$$H_1(\psi,\vartheta,\varphi) = \sum_m (-1)^m \frac{\sin\left[(m-m_0)\vartheta_d\right]}{\pi(m-m_0)} \psi^{m/2} \cos(m\vartheta - n\varphi)$$

$\varepsilon = 0.003$

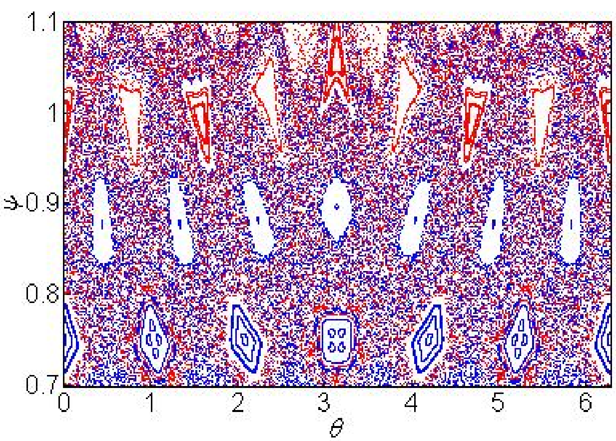


Localised control: magnetic field lines

around $\psi \approx \psi_0$:

$$H_c(\psi,\vartheta,\varphi) = H_0(\psi) + \varepsilon H_1(\psi,\vartheta,\varphi) + \varepsilon^2 f(\vartheta,\varphi)\Omega(\| \psi - \psi_0 + \varepsilon\Gamma\partial_\theta H_1(\psi_0,\theta,\varphi) \|)$$

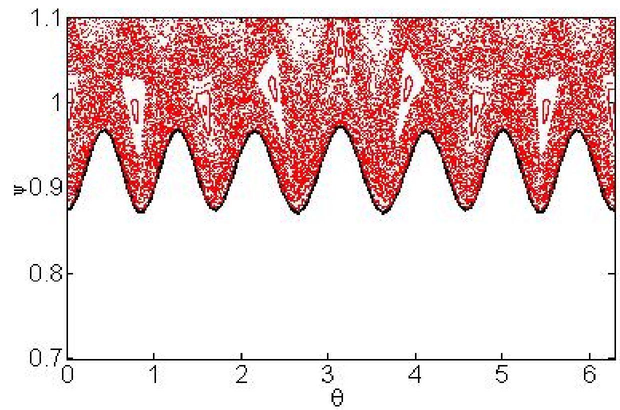
$\varepsilon = 0.003$



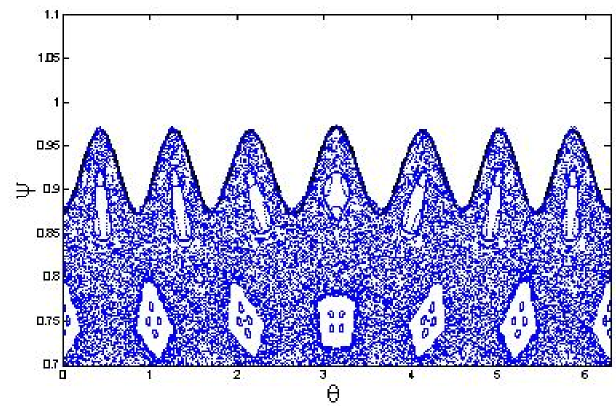
control

5% 0f V
on 7% phase space

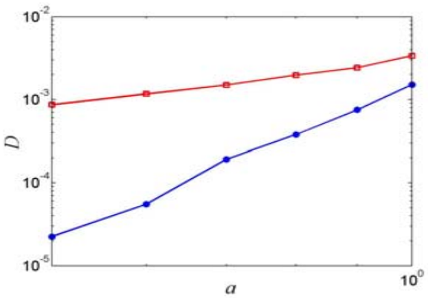
$\varepsilon = 0.003$



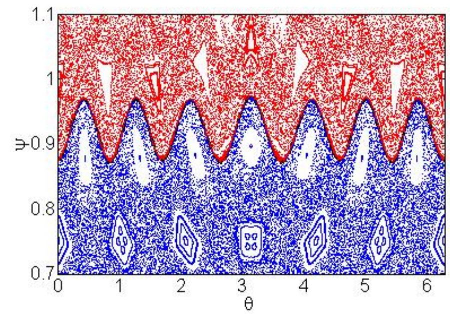
$\varepsilon = 0.003$



Summary and outlook



> Global control of Hamiltonian systems
  - application to electrostatic turbulence
  - robustness



> Localised control of Hamiltonian systems
  - application to magnetic field lines
  - barrier to diffusion

> control of chaos in area-preserving maps [cf. Poster]

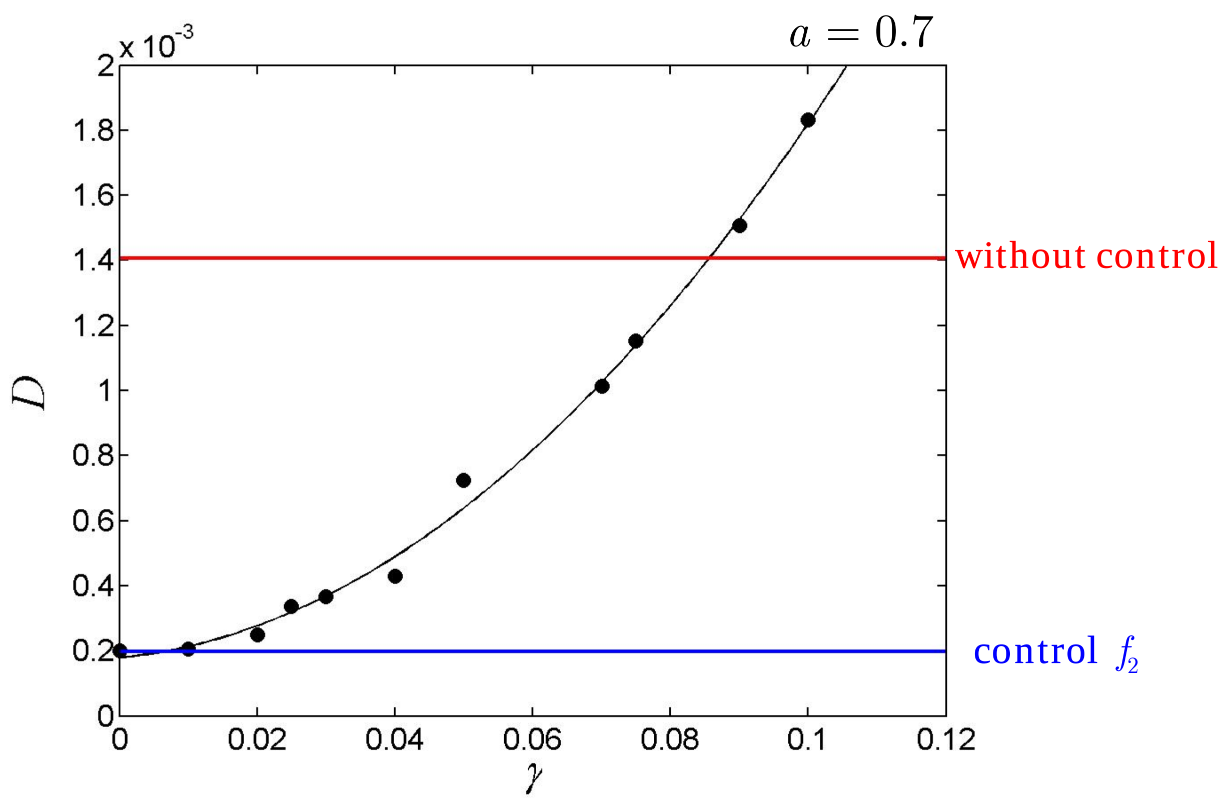
> application on a TWT [cf. Doveil's talk]

> extension to more realistic models in fusion plasmas

> other applications to atomic physics, particle accelerators, free electron laser..?

Robustness 3

> phase error in f_2 $$\tilde{\varphi}_{nm} = \varphi_{nm} + \gamma \cdot \varphi_{nm}^{err}$$



> 50% reduction of the diffusion coefficient for 5 % error in the phases