Wigner time delay distribution in mesoscopic chaotic cavity

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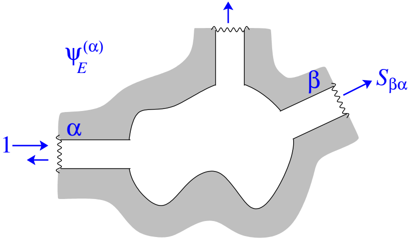
with **Satya N. Majumdar**

→ Coherent transport : $[-\Delta + V(x)]\psi_E(x) = E\psi_E(x)$

Stationary scattering states : $\psi_E^{(\alpha)}(x)$



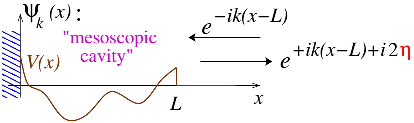
For N open channels :

$N \times N$ scattering matrix $\mathcal{S}(E)$

→ Current (Landauer/Büttiker), shot noise, G_{NS}, etc

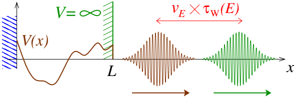
Temporal aspect of quantum scattering : time delay

$$\mathcal{S} = \mathcal{U} \begin{pmatrix} e^{2i\eta_1} & & \\ & \ddots & \\ & & e^{2i\eta_N} \end{pmatrix} \mathcal{U}^\dagger$$



1) **Partial** time delays

$$\tilde{\tau}_a \stackrel{\text{def}}{=} \frac{\mathrm{d}(2\eta_a)}{\mathrm{d}E}$$



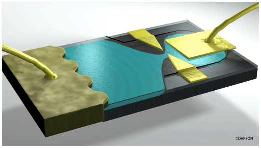
2) Wigner-Smith matrix & **Proper** time delays

$$Q(E) \stackrel{\text{def}}{=} -\mathrm{i}\mathcal{S}(E)^\dagger \frac{\mathrm{d}\mathcal{S}(E)}{\mathrm{d}E} \longrightarrow \text{eigenvalues } \{\tau_a\}$$

3) **Wigner** time delay ($\leftrightarrow$ DoS of open cavity)

$$\boxed{\tau_\mathrm{W}(E) \stackrel{\text{def}}{=} \frac{1}{N}\mathrm{Tr}\{Q(E)\} = -\frac{\mathrm{i}}{N}\frac{\mathrm{d}}{\mathrm{d}E}\ln\det\mathcal{S}(E)} \simeq \frac{2\pi}{N}\mathrm{DoS}$$

AC transport → Admittance $G(\omega)$

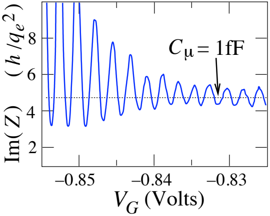


$$\frac{1}{G(\omega)} = \frac{1}{-\mathrm{i}\omega C_\mu} + R_q$$

$\{\tau_a\} \longrightarrow C_\mu$ & R_q (Büttiker '93, etc)

mesoscopic capacitance :

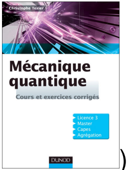
$$C_\mu = e\frac{\mathrm{d}Q}{\mathrm{d}\mu} \quad \Rightarrow \quad \frac{1}{C_\mu} = \underbrace{\frac{1}{C_{\mathrm{geo}}}}_{\text{electrostat.}} + \underbrace{\frac{1}{e^2\rho(\epsilon_F)}}_{\text{Fermi gas}}$$



LPA's experiment (ENS, Paris) :

J. Gabelli, G. Fève, B. Plaçais,...

J. Gabelli *et al*, Science **313**, 499 (2006)

(Problem 10.2, 

)

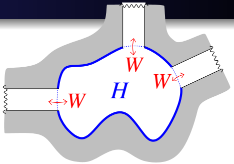
Questions :

Mesoscopic cavity $\Rightarrow$ complex dynamics (chaos)

Q : What are the statistical properties of τ_{W} ?

Chaotic cavities - RMT approach

Several formulations
of Random Matrix Theory :



1. Stochastic approach (SA) : $\mathcal{S} \longrightarrow$ COE, CUE or CSE
   $\rightarrow$ no E-structure !

2. Hamiltonian approach, or « Heidelberg » formulation (HA) :

$$\mathcal{S}(E) = -\mathbf{1} + 2\mathrm{i}W \frac{1}{E - H + \mathrm{i}W^\dagger W} W^\dagger$$

$H = H^\dagger$ & W
2 random matrices

3. « Alternative » stochastic approach (ASA) :
   idea of HA within SA

Brouwer & Büttiker, Europhys. Lett. (1997)

$P(\tau_1, \cdots, \tau_N)$: Brouwer, Frahm & Beenakker, PRL (1997)

Outline

1. Coulomb gas formulation
2. Freezing transition
3. Numerics
4. Concluding remarks

I. Coulomb gas formulation

Laguerre ensemble : (for τ_a's in units of $\tau_{\mathrm{H}} = h/\Delta$)

$$P(\gamma_1, \cdots, \gamma_N) \propto \prod_{i<j} |\gamma_i - \gamma_j|^\beta \prod_k \gamma_k^{\beta N/2} e^{-\frac{\beta}{2}\gamma_k}$$

with $\beta \in \{1, 2, 4\}$

Brouwer, Frahm & Beenakker, '97

$\{\gamma_a = 1/\tau_a\}$ are eigenvalues of Wishart matrices $X^\dagger X$

Coulomb gas method (F. Dyson, '62)

$$P(\gamma_1, \cdots, \gamma_N) \propto e^{-\beta E(\gamma_1, \cdots, \gamma_N)}$$

with $$E(\gamma_1, \cdots, \gamma_N) = \underbrace{\frac{1}{2}\sum_k \gamma_k}_{\sim N\gamma_i} - \underbrace{\frac{N}{2}\sum_k \ln \gamma_k}_{\sim N^2 \ln \gamma_i} - \underbrace{\sum_{i<j} \ln |\gamma_i - \gamma_j|}_{\sim N^2 \ln \gamma_i}$$

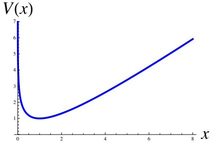
Scaling

$$N\gamma_i \sim N^2 \Rightarrow \gamma_i \stackrel{\text{def}}{=} N x_i \quad \text{with } x_i = \mathcal{O}(N^0)$$

Rescaling $x_i \stackrel{\text{def}}{=} \gamma_i/N = 1/(\tau_i N)$ & $s \stackrel{\text{def}}{=} N\tau_{\mathrm{W}}$

$$P(x_1,\cdots,x_N) \propto e^{-\frac{\beta}{2}N^2\mathscr{E}[\rho]} \quad \text{where} \quad \rho(x) = \frac{1}{N}\sum_i \delta(x-x_i)$$

$$\boxed{\mathscr{E}[\rho] = \int_0^\infty \mathrm{d}x\,(x-\ln x)\,\rho(x) - \int_0^\infty \mathrm{d}x\mathrm{d}x'\,\rho(x)\,\rho(x')\,\ln|x-x'|}$$

Confinement 

& logarithmic repulsion

$$\tau_{\mathrm{W}} = \frac{1}{N}\sum_i \gamma_i^{-1} \longrightarrow \boxed{s \stackrel{\text{def}}{=} N\frac{\tau_{\mathrm{W}}}{\tau_{\mathrm{H}}} = 2\pi\,\frac{\mathrm{DoS}}{\langle\mathrm{DoS}\rangle} = \frac{1}{N}\sum_i x_i^{-1} = \int \frac{\mathrm{d}x}{x}\rho(x)}$$

$\tau_{\mathrm{H}} = h/\Delta$: Heisenberg time

$\Delta = 1/\langle\mathrm{DoS}\rangle$: mean level spacing

Path integral – Saddle point approximation ($N \gg 1$)

$$P_N(s) = \frac{\int \mathcal{D}\rho\, e^{-\frac{\beta}{2}N^2 \mathscr{E}[\rho]}\, \delta\left(1 - \int \mathrm{d}x \rho(x)\right)\, \delta\left(s - \int \frac{\mathrm{d}x}{x}\rho(x)\right)}{\int \mathcal{D}\rho\, e^{-\frac{\beta}{2}N^2 \mathscr{E}[\rho]}\, \delta\left(1 - \int \mathrm{d}x \rho(x)\right)}$$

$\Rightarrow$ Minimize $\mathscr{E}[\rho]$ under two constraints $\longrightarrow$ two Lagrange multipliers

$$\mathscr{F}[\rho] = \mathscr{E}[\rho] + \mu_0 \left(\int \mathrm{d}x\, \rho(x) - 1\right) + \mu_1 \left(\int \mathrm{d}x\, \frac{\rho(x)}{x} - s\right)$$

Optimal distribution $\rho_*(x; s)$ solves $\dfrac{\delta \mathscr{F}[\rho]}{\delta \rho(x)} = 0$

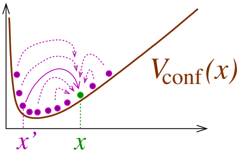
$$P_N(s) \underset{N\to\infty}{\sim} \exp\left\{-\frac{\beta}{2} N^2 \mathscr{E}[\rho_*]\right\}$$

Equilibrium condition

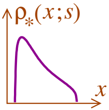
$$\frac{\delta\mathscr{F}[\rho]}{\delta\rho(x)} = 0 \qquad \text{gives}$$

$$\overbrace{\mu_0}^{\text{chemical potential}} + \underbrace{\frac{\mu_1}{x} + x - \ln x}_{V_{\text{conf}}(x)=V(x)+\mu_1/x} \underbrace{-2\int_a^b \mathrm{d}x'\,\rho(x')\,\ln|x-x'|}_{\text{interaction with bulk}} = 0 \qquad \text{for } x\in[a,b]$$

$$\boxed{\underbrace{-1+\frac{1}{x}+\frac{\mu_1}{x^2}}_{F_{\text{conf}}(x)} + \underbrace{2\,⨍_a^b \mathrm{d}x'\,\frac{\rho(x')}{x-x'}}_{F_{\text{bulk}}(x)} = 0} \qquad \text{for } x\in[a,b]$$



$\Rightarrow$ Solution



Tool : Tricomi's theorem

Given a function g, consider the integral equation for ρ

$$\fint_a^b \frac{\mathrm{d}t}{\pi}\,\frac{\rho(t)}{x-t} = g(x) \qquad \text{for } x \in [a,b]$$

Assuming that ρ has a compact support $[a,b]$, then

$$\rho(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}\left\{C + \fint_a^b \frac{\mathrm{d}t}{\pi}\,\frac{\sqrt{(t-a)(b-t)}}{t-x}\,g(t)\right\}$$

Maximum of $P_N(s)$

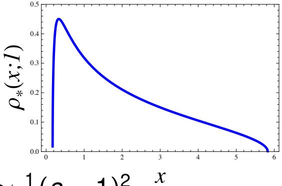
General solution :

$$\rho_*(x;s) = \frac{x+c}{2\pi x^2}\sqrt{(x-a)(b-x)} \quad ; \quad c = \frac{\mu_1}{\sqrt{ab}}$$

Most probable τ_{W} : remove constraint $\Rightarrow$ $\mu_1 = 0$ leading to

$$\begin{cases} a = 3 - 2\sqrt{2} \\ b = 3 + 2\sqrt{2} \end{cases}$$ (Marčenko-Pastur law)

$\Rightarrow s = \int \frac{\rho_*(x)}{x} = 1$ most probable value



For $s \sim 1$ one finds : $\mathscr{E}[\rho_*(x;s)] - \mathscr{E}[\rho_*(x;1)] \simeq \frac{1}{4}(s-1)^2$

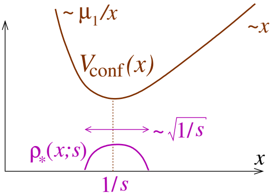
$$P_N(s) \underset{s\sim 1}{\sim} \exp -\frac{\beta N^2}{8}(s-1)^2 \quad \Rightarrow \quad \mathrm{Var}(\tau_{\mathrm{W}}) \simeq \frac{4\tau_{\mathrm{H}}^2}{\beta N^4}$$

i.e. $\mathrm{Var}(\mathrm{DoS}) \simeq \frac{4}{\beta N^2 \Delta^2}$

Lehmann et al '95 and Mezzadri & Simm '12 : $\mathrm{Var}(\tau_{\mathrm{W}}) = \frac{4\tau_{\mathrm{H}}^2}{(N+1)(N\beta-2)N^2}$

Large deviations for $s \to 0$

$$(*) \qquad s = \frac{1}{N}\sum_i x_i^{-1} \to 0 \qquad \Rightarrow \qquad x_i \sim \frac{1}{s} \to \infty \ \forall i$$



$$s \to 0 \Rightarrow \mu_1 \simeq 1/s^2 \to \infty$$

$$\Rightarrow \qquad \mathscr{E}[\rho_*(x;s)] \simeq 1/s$$

Detailed analysis :

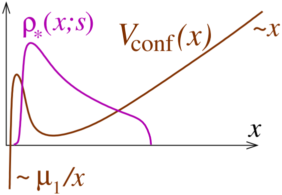
$$\rho_*(x;s) \underset{s\to 0}{\simeq} \frac{1}{\pi}\sqrt{2s-(x\,s-1)^2} \quad \& \quad \mathscr{E}[\rho_*(x;s)] \underset{s\to 0}{\simeq} \frac{1}{s} + \frac{3}{2}\ln s + \frac{1+\ln 2}{2}$$

(like Wigner semi-circle law for GE)

$$\text{i.e.} \quad \boxed{P_N(s) \underset{s\to 0}{\sim} s^{-\frac{3}{4}N^2\beta} \exp -\frac{N^2\beta}{2s}}$$

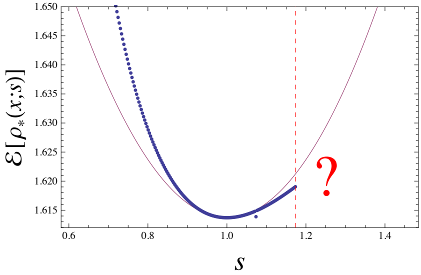
Large deviations for $s \gg 1$?

$$s > 1 \Rightarrow \mu_1 < 0$$



$$\frac{\delta \mathscr{F}[\rho]}{\delta\rho(x)} = 0$$: no real solution for $s > s_c$

$$s_c = \frac{10+6\times 2^{1/3}-11\times 2^{2/3}}{3\,(6-6\times 2^{1/3}+2^{2/3})} = 1.1738...$$



II. Large deviations for $s > 1$: freezing transition

New scenario : splitting of a single charge

$$\rho(x) = \underbrace{\frac{1}{N}\delta(x - x_1)}_{\text{isolated charge}} + \underbrace{\tilde{\rho}(x)}_{\text{continuous}}$$

Saddle point eqs. :

$$\frac{\delta \mathscr{F}[\rho]}{\delta \tilde{\rho}(x)} = 0 \quad \& \quad \frac{\partial \mathscr{F}[\rho]}{\partial x_1} = 0$$

Equilibrium condition :

$$-1 + \frac{1}{x} + \frac{\mu_1}{x^2} + \frac{2}{N}\frac{1}{x - x_1} + 2\,⨍_a^b \mathrm{d}x'\,\frac{\tilde{\rho}(x')}{x - x'} = 0 \qquad \text{for } x \in [a, b]$$

$$-1 + \frac{1}{x_1} + \frac{\mu_1}{x_1^2} + 2\int_a^b \mathrm{d}x'\,\frac{\tilde{\rho}(x')}{x_1 - x'} = 0 \qquad \text{with } x_1 < a$$

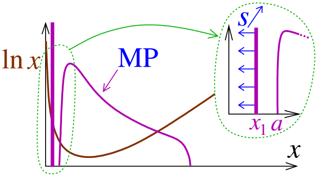
$\Rightarrow$ 4 eqs. for a, b, μ_1 & x_1

Freezing transition for $s > 1$

$$(*) \qquad s = \frac{1}{Nx_1} + \frac{1}{N}\sum_{i>1} x_i^{-1} \to \infty \qquad \Rightarrow \qquad \text{one charge } x_1 \sim \frac{1}{Ns} \to 0$$

- $\mu_1 \simeq -x_1 \to 0^-$
- $\tilde\rho(x;s) \underset{N\to\infty}{\longrightarrow} \rho_*(x;1)$ for $s > 1$

$\tilde\rho(x;s)$ freezes to the MP law $\rho_*(x;1)$



$$\mathscr{E}[\rho_*(x;s)] = \frac{1}{N}(x_1 - \ln x_1) + (\cdots) \underset{s\gg 1}{\simeq} \frac{1}{N}\ln s + \mathrm{Cste}$$

Power law tail

$$\ln P_N(s) \sim -\frac{\beta N}{2}\ln s + \cdots$$

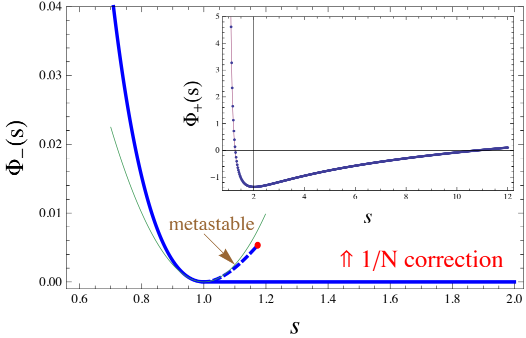
Scattering interpretation : a single time τ_1 dominates $\Rightarrow$ resonance

Large deviation functions

Energy of the gas :

$\mathscr{E}[\rho_*] = \text{cst} + \Phi_-(s)$

$\mathscr{E}[\rho_*] = \text{cst} + \frac{1}{N}\Phi_+(s)$

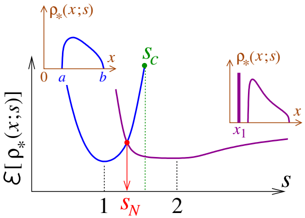


$$P_N(s) \underset{N\to\infty}{\sim} \begin{cases} \exp\left[-\frac{\beta}{2}N^{2}\,\Phi_-(s)\right] & \text{for } s < s_N \\ N^{-\frac{\beta N}{2}} \exp\left[-\frac{\beta}{2}N^{1}\,\Phi_+(s)\right] & \text{for } s > s_N \end{cases}$$

A second order phase transition

New phase exists for $s > 1$

(and not only $s > s_c$)



Crossover at s_N

$$s < s_N \quad \rightarrow \quad \mathscr{E}[\rho_*(x;s)] - \mathscr{E}[\rho_*(x;1)] \underset{s\to1}{\simeq} \frac{1}{4}(s-1)^2$$

$$s > s_N \quad \rightarrow \quad \mathscr{E}[\rho_*(x;s)] - \mathscr{E}[\rho_*(x;1)] \underset{s\to1^+}{\simeq} \frac{1}{N}\frac{1}{s-1}$$

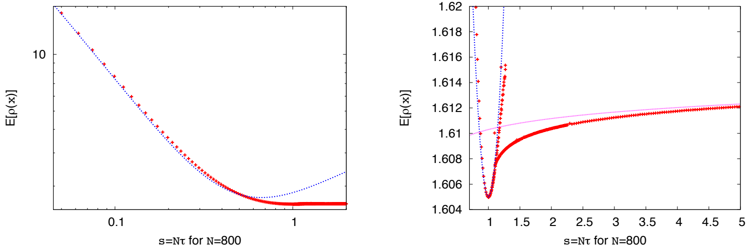
$$\boxed{s_N \simeq 1 + (4/N)^{1/3}}$$

$s_N \underset{N\to\infty}{\longrightarrow} 1^+$: Freezing is here a **2nd** order phase transition

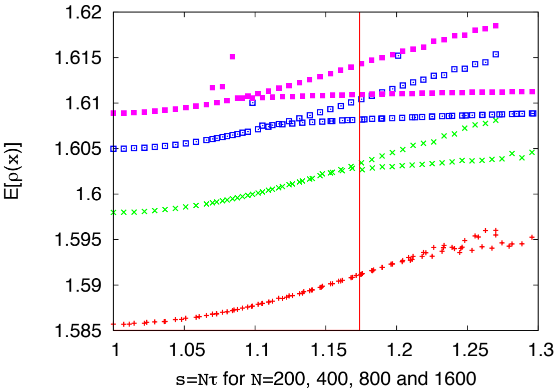
III. Numerics

Monte Carlo simulation of the gas

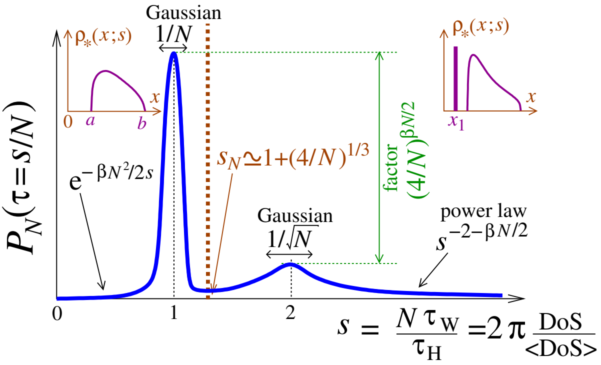
Demonstrates the phase transition :



For s going from $\infty \to 1^+$: force the existence of one isolated charge


1.62
1.615
1.61
1.605
1.6
1.595
1.59
1.585
E[ρ(x)]
1
1.05
1.1
1.15
1.2
1.25
1.3
s=Nτ for N=200, 400, 800 and 1600

IV. Conclusions



Power law $P_N(s) \sim s^{-2-\beta N/2}$: one narrow resonance

CT & S. N. Majumdar, Phys. Rev. Lett. **110**, 250602 (2013)