Hall current and electron polarizability of a two-dimensional electron gas subjected to weak superlattice potentials

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A theory of the Hall effect in two-dimensional electron systems subjected to weak periodic modulation of the background potential is presented. It is shown that the nondissipative Hall current is strongly affected by the static electron polarizability, which is responsible for the nonmonotonic sequence of quantum Hall plateaus in the dependence on the magnetic field strength. This static electron polarizability is quantized whenever the Fermi energy lies within an energy gap.

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I. INTRODUCTION

The Hall resistance increases with increasing magnetic field strength in most studied systems. This property is also usually preserved for the quantum Hall effect. Deviations from this standard dependence are expected for two-dimensional electron gas subject, besides the perpendicular magnetic field, to a periodic potential. Such systems have a complicated energy structure due to the interplay of the two characteristic areas: the area of the unit cell of the periodic potential and the area per unit magnetic-flux quantum. For such two-dimensional systems, a nonmonotonic sequence of quantum Hall plateaus in the dependence on the applied magnetic field was predicted[1,2] soon after the discovery of the integer quantum Hall effect.[3] Recently, the prediction has been experimentally verified for a weakly modulated two-dimensional electron gas in strong magnetic fields.[4] Similar behavior was also predicted for three-dimensional systems.[5,6] To our knowledge, the physical origin of such nonstandard Hall-resistance behavior has not yet been understood. As we show in the present paper, it is due to a local charge polarization effect represented by static electron polarizability.

Hall resistance R_H is standardly measured on long-strip shaped samples, and it is defined as the ratio of the voltage drop between strip edges and the applied current. Recently, it has been argued that for Bloch electrons, the Hall resistance is affected by the local charge polarization induced by the voltage across the strip.[7] The main aim of the presented treatment is to analyze the polarization effect in its purest form. For this reason the following virtual experiment will be studied. Consider a strip with electrically connected ends so that nondissipative current can flow through it. Applying an external electric field across the strip, a voltage drop $U_\perp$ between strip edges and a Hall current J_H are induced. Thus, we can define the Hall conductance as their ratio

$$G_H = \frac{J_H}{U_\perp}. \tag{1}$$

In the quantum Hall regime, the measured Hall resistance just amounts to $1/G_H$.

Properties of the Hall conductance G_H will be analyzed for two-dimensional electron systems in strong magnetic fields. A periodic background potential with a rectangular unit cell and an potential amplitude much smaller than the Landau-level spacing will be considered. In this limit, the periodic potential weakly perturbs the free electron Landau-level structure.[8–10] It guarantees that the periodic potential does not lead to a Landau-level mixing and that it can be treated in the lowest-order perturbation expansion. Then, the most important effect is that each Landau level becomes broadened and its energy dispersion shows a complicated self-similar structure as a function of the magnetic field strength B, often presented in the form of the aesthetically pleasing figure which has become known as the Hofstadter butterfly.[11] As first noticed by Wannier,[12] energy gaps within the internal structure of the broadened Landau level can only appear if the following relation is satisfied:

$$N = \frac{s}{A_0} + \frac{\sigma}{2\pi l_B^2}, \quad l_B^2 = \frac{\hbar}{m^* \omega_c} = \frac{1}{2\pi}\frac{hc}{eB}, \tag{2}$$

where l_B is the so-called magnetic length, ω_c is the cyclotron frequency, and $A_0 = a_x a_y$ denotes the unit cell area. Assuming zero temperature, the number of electrons N equals the integrated density of states $N(\mu)$. Integers σ and s are the so-called gap quantum numbers that specify the dependence of $N(\mu)$ on B and A_0.[7,13] They can also be viewed as bulk invariants that correspond to a response coefficient related to the pressure on the boundary,[14] which is just equal to $N(\mu)$ if the Fermi energy lies within the gap of the bulk spectrum. In the presence of the magnetic field, the pressure is composed of the Lorentz force and the gradient force from the background potential. Values of gap quantum numbers are then determined by the condition that both forces have to be compensated by the boundary force. Consequently, two types of edge states, magnetic and nonmagnetic, appear at the system boundaries,[13] with σ counting the number of magnetic edge-state branches. It was realized[1,2,15,16] that σ is associated with the quantum Hall effect. Thouless *et al.*[1] identified σ as a topological index of the dependence of eigenfunctions on wave vectors in the magnetic Brillouin zone, thereby spawning the topological approach to the quantum Hall effect. In the following, we will refer to σ and s as the topological numbers.

The Hall resistance attains quantized values,

$$R_H^Q = \frac{1}{G_H^Q} = -\frac{h}{e^2}\frac{1}{\sigma} = -\frac{B}{ec\left(N - \frac{s}{A_0}\right)}, \tag{3}$$

whenever the Fermi energy μ is located within the gap of the bulk-energy spectrum. The topological number σ is thus directly measurable. The question arises if there exists a physical quantity, measurable at least in principle, allowing direct measurement of the topological number s. It will be shown that this quantity is the static electron polarizability.

The paper is organized as follows. We recall in Sec. II the results of the eigenvalue problem for the cosine-potential modulation. In contrast to the standard presentation of the single-particle spectrum, we use the extended zone scheme here. The system geometry is an infinitely long strip, and we summarize the basic edge-state properties. The results of this section are not new, but the reader may find useful to find them collected, with a unified perspective and notation. In the following sections, the main attention will be devoted to the properties of electrons with energy within energy subbands, since the properties of edge states (the only states in the energy gaps) are well known. Section III gives expressions for the electronic-wave-function center-of-mass positions in the periodic potential unit cell and computes their linear response to an electric field. Section IV is devoted to the integrated density of states $N(\mu)$. In order to find the link between the properties of a system in which the Fermi energy lies within an energy gap and those for which it is located within an energy subband, the, in general, noninteger quantities $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$ satisfying Eq. (2) for any value of the Fermi energy are defined. We call $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$ the effective topological numbers, as they are the generalization of the topological numbers defined in energy gaps. The following sections are devoted to properties and interpretation of the effective topological numbers. In Sec. V, their relation to the magnetization and internal pressure is discussed, while in Sec. VI, $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$ are shown to be simply related to measurable quantities, Hall conductance and static electron polarizability, respectively. Finally, the main results are summarized in the last section.

II. ENERGY SPECTRUM

A. Model Hamiltonian

For the sake of simplicity, the cosine form of the potential modulation of the two-dimensional electron gas will be considered,

$$V(\vec{r}) = -V_0\cos(K_x x) - V_0\cos(K_y y), \tag{4}$$

where $K_x = 2\pi/a_x$ and $K_y = 2\pi/a_y$ are elementary translations of the reciprocal lattice. Magnetic field $\vec{B}$ is assumed to be applied perpendicularly to the x-y plane, i.e., along the $\hat{z}$ direction. Since we will discuss properties of a strip of finite width along the $\hat{y}$ direction, the Landau gauge $\vec{A} \equiv (-By, 0, 0)$ is chosen for the vector potential $\vec{A}$. The resulting single-particle model Hamiltonian reads

$$H = \frac{p_y^2}{2m^*} + \frac{m^*}{2}\left[\frac{p_x}{m^*} - \omega_c y\right]^2 + V(\vec{r}) + V_c(y), \tag{5}$$

where $\omega_c \equiv eB/m^*c$ denotes the cyclotron frequency with m^* and e being the electron effective mass and absolute value of the electron charge, respectively. The confining potential $V_c(y)$, which determines the strip width, is assumed to be different from zero only at the edge regions.

The unmodulated system, $V(\vec{r}) \equiv 0$, has an energy spectrum composed of Landau levels having the energy $\hbar\omega_c(n+1/2)$ within the region far from the strip boundaries. The exact eigenstates $|n, k_x\rangle$ are with a great accuracy approached by wave functions having the following form

$$\Psi^{(0)}_{n,k_x}(\vec{r}) = \frac{1}{\sqrt{2\pi}} e^{ik_x x}\phi_n(y - l_B^2 k_x), \tag{6}$$

where $l_B^2 k_x$ denotes the mass-center position of the normalized oscillator function $\phi_n(y)$. For slowly varying confining potential, the wave function at edge regions can be assumed unchanged while the Landau-level energy is shifted by the potential energy of the oscillator mass center, $V_c(l_B^2 k_x)$.

To establish the effect of the periodic potential $V(\vec{r})$, we diagonalize the Hamiltonian defined by Eq. (5) in the basis of the unmodulated system eigenfunctions, Eq. (6). The already mentioned limitation of our treatment to a weak periodic modulation implies that the amplitude V_0 has to be much less than the Landau-level spacing $\hbar\omega_c$. Then, the Landau-level mixing by the periodic potential can be neglected[9,10] and only the periodic potential matrix elements diagonal in the Landau-level index n are to be taken into account. Since the modulation in the $\hat{x}$ direction couples Landau states with centers differing by integer multiples of $l_B^2 K_x$, the zero-order eigenfunctions can be written in the following way:

$$\Psi_{n,k_x}(\vec{r}) = \sum_{\lambda=-\infty}^{+\infty} u_n(k_x + \lambda K_x)\Psi^{(0)}_{n,k_x+\lambda K_x}(\vec{r}), \tag{7}$$

where λ is an integer index. Amplitudes $u_n(k_x+\lambda K_x)$ are solutions to the eigenvalue problem defined by the k_x-dependent effective Hamiltonian $H(k_x)$, which has, in this representation, the form of the tridiagonal matrix,

$$-\frac{\tilde{V}_0^{(n)}(a_x)}{2}u_n[k_x + (\lambda-1)K_x] + [H_{\lambda,\lambda}(k_x) - E_n(k_x+\lambda K_x)]$$

$$\times u_n(k_x+\lambda K_x) - \frac{\tilde{V}_0^{(n)}(a_x)}{2}u_n[k_x+(\lambda+1)K_x] = 0, \tag{8}$$

with

$$H_{\lambda,\lambda}(k_x) = \hbar\omega_c(n+1/2) + V_c(l_B^2(k_x+\lambda K_x)) - \tilde{V}_0^{(n)}(a_y)\cos(l_B^2(k_x+\lambda K_x)K_y). \tag{9}$$

The effective potential amplitudes read

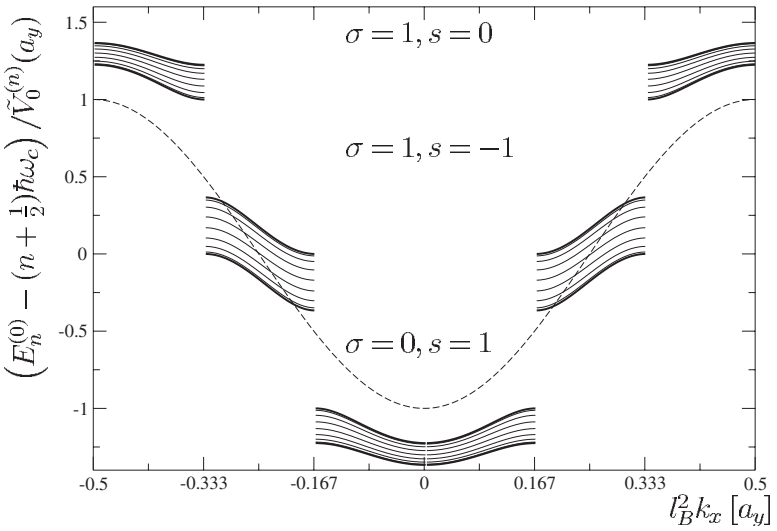

FIG. 1. Energy dispersion as a function of $l_B^2 k_x$ for $q/p=1/3$ and $a_x=a_y$ and for several values of β. The edges of the energy subbands are shown with thicker lines. Expectation values of the cosine-potential modulation along the $\hat{y}$ direction are shown by the dotted line.

$$\tilde{V}_0^{(n)}(a) \equiv V_0 e^{-(\pi/2)(2\pi l_B^2/a^2)} L_n\left(\pi \frac{2\pi l_B^2}{a^2}\right), \tag{10}$$

where $L_n(z)$ denote the Laguerre polynomials.[10] Assuming that the confining potential increases to infinity at some values of mass-center positions, the tridiagonal matrix given by Eq. (8) is of finite dimensions and the eigenvalue problem can be, at least in principle, solved.

B. Bulk-energy structure of broadened Landau levels

The eigenvalue problem becomes substantially simplified if we limit our consideration to the so-called rational magnetic fields, satisfying the following condition:

$$\frac{2\pi l_B^2}{a_x a_y} = \frac{hc/e}{a_x a_y B} \equiv \frac{\Phi_0}{\Phi} = \frac{q}{p}, \tag{11}$$

where q and p are integers. Physically, this commensurability condition means that there are p flux quanta Φ_0 per q unit cells. Then, the matrix element $H_{\lambda,\lambda}(k_x)$ is periodic with period p, $H_{\lambda+p,\lambda+p}(k_x)=H_{\lambda,\lambda}(k_x)$, for $l_B^2 k_x$ in the bulk region where $V_c(y)=0$. The amplitudes $u_n(k_x)$ have periodic moduli as well,

$$|u_n(k_x+pK_x)| = \left|u_n\left(\frac{l_B^2(k_x+pK_x)}{l_B^2}\right)\right| = \left|u_n\left(\frac{l_B^2 k_x + qa_y}{l_B^2}\right)\right| = |u_n(k_x)|, \tag{12}$$

but they can differ by phase factors. Assume that the phase difference between the amplitudes $u_n(k_x)$ and $u_n(k_x+pK_x)$ acquires a certain value β from the interval $(0,2\pi)$. This boundary condition reduces the eigenvalue problem defined by Eq. (8) to an eigenvalue problem given by a $p\times p$ matrix, which can easily be solved. For each β, which we call the branch index, the energy dispersion as a function k_x can be obtained. Broadened Landau levels become composed of p subbands, and according Usov,[9] the eigenenergies are periodic in the wave number k_x as follows:

$$E_{n,\beta}^{(0)}\left(k_x+\frac{pK_x}{q}\right) = E_{n,\beta}^{(0)}\left(\frac{l_B^2 k_x + a_y}{l_B^2}\right) = E_{n,\beta}^{(0)}(k_x). \tag{13}$$

Note that for the considered Hamiltonian, the internal structure of broadened Landau levels does not depend on the level index n. It is only their overall width which scales in accord with the n dependence of the effective potential $\tilde{V}_0^{(n)}(a)$ given by Eq. (10).[10] The obtained dispersions for $q/p=1/3$ and $q/p=3/5$ are shown in Figs. 1 and 2, respectively.

The above treatment is equivalent to the standard application of the Born-Karman conditions along the $\hat{y}$ direction.[9,10] The only difference is that instead of the wave number k_y, we use the branch index β. We also did not introduce the magnetic Brillouin zone, so that the original meaning of the wave number k_x, determining the mass-center positions $l_B^2 k_x$ of the harmonic oscillators entering the base functions, Eqs. (6) and (7), has been preserved. Therefore, we can omit the usually introduced subband index, and the energy dispersions presented as a function of $l_B^2 k_x$ are periodic with the period a_y.

The presented bulk-energy spectra are based on the implicit assumption that a dissipation process of some sort exists, giving rise to a finite coherence length much less than the strip width. Then, the interference between the bulk and the edge states is suppressed, and the bulk spectra do not depend on the form of the confining potential. The probability of finding an electron within an interval of length a_y, far from the edge region, is given by the average over all branches β. Assuming a fully filled Landau level, the probability that an electron within the unit cell area A_0 has an energy belonging to one particular subband is equal to $1/q$, or equivalently, there have to be $1/(2\pi l_B^2 p)$ states within the unit area per one subband. This condition has to be satisfied by the averaging procedures over the branch index β.

C. Edge states

Within edge regions, a nonzero confining potential gives rise to edge states. They have been studied numerically for cosine modulation, Eq. (4), assuming a sharp confining potential modeling hard walls or a smooth confining potential

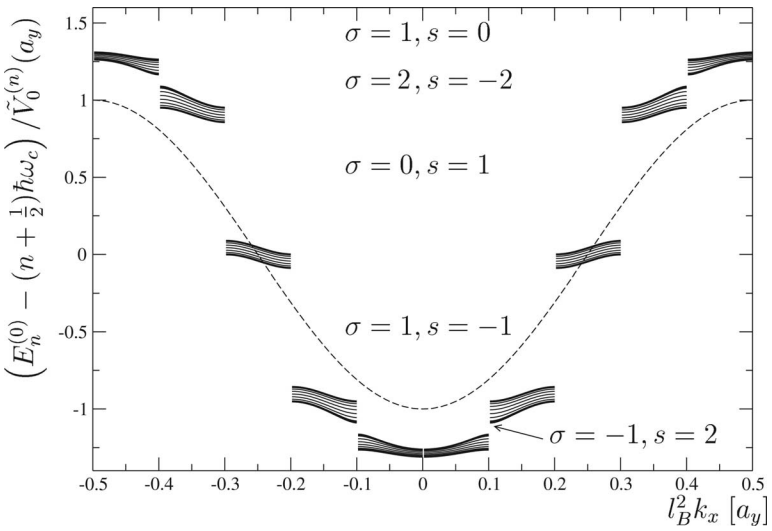

FIG. 2. Energy dispersion as a function of $l_B^2 k_x$ for $q/p=3/5$ and $a_x=a_y$. Values of β are the same as that used for dispersions in Fig. 1. Expectation values of the cosine-potential modulation along the $\hat{y}$ direction are shown by the dotted line.

unmodulated by the periodic potential.[13,17] It has been found that two types of edge states, magnetic and nonmagnetic, exist.

Magnetic edge states are forming branches having opposite velocities at opposite strip edges. It has been shown that the sum of the chiralities of all magnetic edge states at a given energy in a bulk-energy gap equals the topological number σ for the gap. In particular, magnetic edge states must always exist at all energies in a gap for which σ is nonzero. Therefore, the energy branches of these magnetic edge states bridge the energy gap between adjacent subbands.[13] They contribute a finite amount to the magnetization per unit area of the system in the thermodynamic limit, despite the fact that their contribution to the density of states vanishes. Their contribution to the magnetic moment tends to compensate the contribution from the bulk states and maintain the total magnetic moment small. This is the physical reason for the stability of magnetic edge-state branches with respect to varying conditions at the system edges. To preserve the chirality, the effect of magnetic edge states can be included by inserting σ edge-state branches into the bulk-energy gap. They are formed by contributions of bulk branches β, and they can have arbitrary dispersion, i.e., the dependence on $l_B^2 k_x$ at the edge region, since it does not affect system properties in the thermodynamic limit. Only the proper sign of the velocity has to be preserved.

For the considered weak cosine modulation, Eq. (4), it has been rigorously established[1,18] that the topological numbers σ and s which have to satisfy the Diophantine equation, Eq. (2), for any Landau level are uniquely determined by the auxiliary inequality

$$|s| < p/2, \tag{14}$$

and that a gap occurs when this inequality is sufficient to determine s uniquely. That is why we limit our consideration to rational magnetic fields for which p is an odd integer. Then, all p subbands are separated by energy gaps.[19]

The other types of edge states are nonmagnetic states. They appear in pairs of opposite velocities but located at a particular edge region. They show a behavior which is typical for edge states of two-dimensional periodic systems in zero magnetic field.[20] If the system contains hard walls, the occurrence of nonmagnetic edge states in a gap depends on the hard wall position relative to maxima or minima of the periodic potential. They form branches which often cover an energy interval narrower than the full gap. If the hard wall position is gradually shifted over the period a_y, just s nonmagnetic edge states per interval a_x cross the gap.[13] This behavior is general and it does not depend on the form of the confining potential. In real systems, a variation of the confining potential along the $\hat{x}$ direction always appears. It leads to the formation of the nonmagnetic edge states which become localized in all directions[13,20] covering whole gap.

In thermodynamic limit, the system properties cannot depend on the form of the confining potential. Bulk branches forming a particular subband give rise to edge states whose effect can be included by considering a single branch of magnetic or nonmagnetic edge states in the dependence on their chirality. While the velocity of the magnetic edge states is nonzero, the nonmagnetic localized states have zero velocity and cannot contribute to the current.

III. MASS-CENTER POSITIONS AND LINEAR RESPONSE TO THE ELECTRIC FIELD

A. Mass-center positions

To define mass-center positions, we first compute the average electron mass distribution $g_{n,\beta,k_x}(y)$ along the $\hat{y}$ direction for an electron in the state $|n,\beta,k_x\rangle$. Corresponding zero-order wave functions are given by Eq. (7), where the index β has been added, $u_n(k_x) \to u_{n,\beta}(k_x)$, to specify the selected solution of the Harper's equation. Using the periodicity of amplitudes $u_{n,\beta}(k_x)$ given by Eq. (12), and having in mind that the shift of the periodic potential by the lattice constant can only change the wave function phase, we get

$$\frac{g^{(0)}_{n,\beta,k_x}(y)}{m^*} = \sum_{\lambda_p=-\infty}^{+\infty} w^{(0)}_{n,\beta,k_x}(y - \lambda_p a_y), \tag{15}$$

where λ_p is an integer index and the local distribution for the considered rational magnetic fields with $p=2i+1$ can be written as

$$w^{(0)}_{n,\beta,k_x}(y) = \sum_{\lambda=-i}^{+i} |u_{n,\beta}(k_x + \lambda K_x)|^2 |\phi_n(y - l_B^2(k_x + \lambda K_x))|^2. \tag{16}$$

The electron mass distribution along the $\hat{y}$ direction, which is periodic with a period a_y, is thus expressed as the sum of identical local distributions shifted along the $\hat{y}$ direction by $\lambda_p a_y$. The form of the local distribution is such that in the limit of a vanishing amplitude of the periodic potential, the mass-center position of the local distribution reaches the value defined by eigenstate given by Eq. (6). Indeed, for a vanishing potential, $|u_{n,\beta}(k_x)|^2$ becomes a constant independent of k_x, and the resulting mass-center position reaches the desired value $l_B^2 k_x$. The used empty-lattice condition allows us to define mass-center positions for periodic system uniquely.

To get a precise evaluation of the mass-center position of a given local distribution, we use the relation between the velocity and the position operators,

$$y = \frac{p_x}{m^* \omega_c} - \frac{v_x}{\omega_c}. \tag{17}$$

Using the eigenstates to the first order in the modulation potential,[10] we get the standard expression for the velocity expectation values along the $\hat{x}$ direction,

$$v^{(0)}_{n,\beta}(k_x) = \frac{1}{\hbar} \frac{dE^{(0)}_{n,\beta}(k_x)}{dk_x}, \tag{18}$$

while the contribution of the operator p_x reads

$$\left\langle \frac{p_x}{m^* \omega_c} \right\rangle_{n,\beta,k_x} = l_B^2 k_x + l_B^2 \sum_{\lambda=-i}^{+i} \lambda K_x |u_{n,\beta}(k_x + \lambda K_x)|^2. \tag{19}$$

The resulting expression for mass-center positions can be written in the following way:

$$Y^{(0)}_{n,\beta}(k_x) = l_B^2 k_x - \frac{l_B^2}{\hbar} p^{(0)}_{n,\beta}(k_x), \tag{20}$$

where the momentum

$$p^{(0)}_{n,\beta}(k_x) = m^* v^{(0)}_{n,\beta}(k_x) - \hbar \sum_{\lambda=-i}^{+i} \lambda K_x |u_{n,\beta}(k_x + \lambda K_x)|^2 \tag{21}$$

is a periodic function of k_x,

$$p^{(0)}_{n,\beta}(k_x + a_y l_B^{-2}) = p^{(0)}_{n,\beta}(k_x). \tag{22}$$

The mass-center positions $Y^{(0)}_{n,\beta}(k_x)$ defined in such a way have the following interpretation. Consider an electron in the state $|n, \beta, k_x\rangle$, spread over the width Na_y. In each of the unit cells of the width a_y, it has a local distribution whose mass-center position is given by Eq. (20). The first term on the right-hand side of Eq. (20) represents the mass-center positions of oscillator functions, Eq. (6), and corresponds to the unmodulated system, i.e., the vanishing periodic potential. The range of the values of this first term coincides with the strip width. The second term describes the shift due to the force caused by the periodic potential. It is a periodic function of $l_B^2 k_x$ with a period a_y. Note that for $p=1$, or for vanishing potential modulation along the $\hat{x}$ direction, Eq. (20) coincides with existing results.[21]

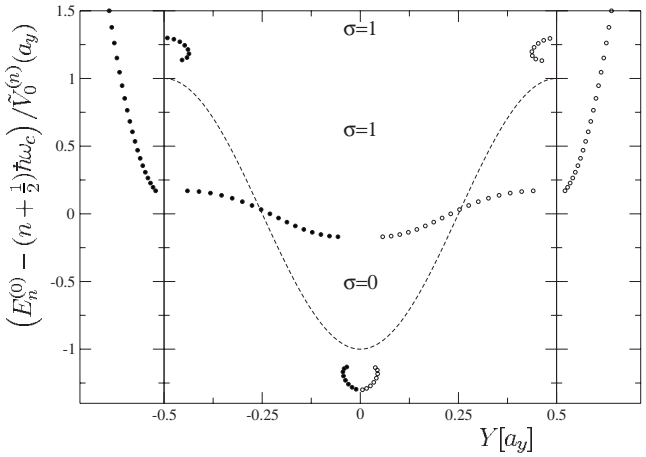

FIG. 3. Center panel: energy dispersion as a function of the mass-center positions Y for $q/p=1/3$ and $a_x=a_y$. The filled (empty) dots represent states with negative (positive) velocities along the $\hat{x}$ direction. The potential modulation along the $\hat{y}$ direction is given by the dotted line. Left and right panels: sketch of the magnetic edge-state branches near the strip edges.

To illustrate the distances of mass-center positions for a given energy, the eigenenergies as a function of the mass-center positions within the interval of the length a_y are presented in Figs. 3 and 4 for $q/p=1/3$ and $q/p=3/5$, respectively. Only curves for a single value of the branch index β are shown, since for any allowed value, they are qualitatively identical. In these figures, also magnetic edge-state branches at the strip edges satisfying the chirality condition are sketched.

Two remarks are in order here. First, the above treatment represents mass-center positions of the bulk electrons. Since

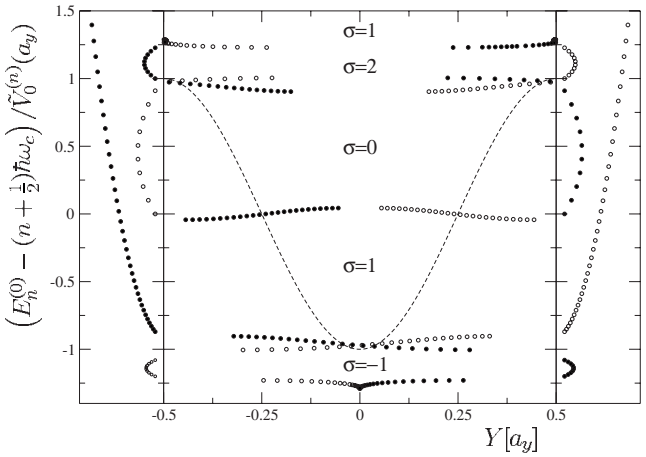

FIG. 4. Center panel: energy dispersion as a function of the mass-center positions Y for $q/p=3/5$ and $a_x=a_y$. The filled (empty) dots represent states with negative (positive) velocities along the $\hat{x}$ direction. The potential modulation along the $\hat{y}$ direction is given by the dotted line. Left and right panels: sketch of the magnetic edge-state branches near the strip edges.

our aim is to characterize properties of macroscopic systems, we implicitly assume the existence of a finite coherence length, much less than the strip width, which prevents interference between bulk and edge states. Thus, the bulk-state properties are supposed to be practically independent of the details of the boundary conditions. The mass-center positions of edge states are assumed to be located close to the physical system edges. In the thermodynamical limit, i.e., the limit of the infinite system area, their mass-center positions can be identified directly with the physical system boundaries with no substantial effect to any measurable physical quantities.

Second, in order to allow for an easy fulfillment of the aforementioned condition in the empty-lattice limit, we have confined our treatment to the odd values of the integer p. Satisfying this condition for even values of p is not trivial, since the chosen form of the potential modulation leads to unstable solutions[9] which are, in fact, very sensitive also to boundary conditions. Then, an additional averaging procedure has to be adopted to obtain results characterizing macroscopic systems, where the interference between bulk and edge states is supposed to be suppressed. This problem will be treated elsewhere.

B. Linear response to an electric field

Consider a homogeneous electric field across the strip, i.e., along the $\hat{y}$ direction, $\mathcal{E}_y$. Assume that it is weak enough so that the condition $e\mathcal{E}_y a_y \ll \tilde{V}_0(a_y)$ is satisfied. As it gives rise only to a weak perturbation of the background potential modulation, we can limit our consideration to the linear response in $\mathcal{E}_y$. The corresponding potential term which has to be added to the model Hamiltonian, Eq. (5), reads

$$\Delta H = e\mathcal{E}_y y. \tag{23}$$

The resulting single electron Hamiltonian can be written in the following way:

$$H' \equiv H + \Delta H = \frac{m^*}{2}\left[\frac{\hbar}{m^*}\left(\frac{p_x}{\hbar} - \Delta k_x\right) - \omega_c y\right]^2 + \frac{\hbar^2}{m^*}\frac{p_x}{\hbar}\Delta k_x - \frac{\hbar^2}{2m^*}(\Delta k_x)^2 + \frac{p_y^2}{2m^*} + V(\vec{r}), \tag{24}$$

where

$$\Delta k_x \equiv \frac{e\mathcal{E}_y}{\hbar\omega_c}. \tag{25}$$

Using the same form of the zero-order eigenfunctions as that given by Eq. (7), only diagonal elements of the k_x-dependent effective Hamiltonian [Eq. (8)] are affected,

$$H'_{\lambda,\lambda}(k'_x) = \hbar\omega_c(n+1/2) + \frac{\hbar^2}{m^*}(k'_x + \lambda K_x)\Delta k_x - \tilde{V}_0(a_y)\cos(l_B^2(k'_x + \lambda K_x)K_y), \tag{26}$$

where

$$k'_x \equiv k_x - \Delta k_x. \tag{27}$$

To establish wave function amplitudes $u_n(k_x)$, the assumption $e\mathcal{E}_y a_y \ll \tilde{V}_0(a_y)$ allows for the neglect of terms linear in λ in components $H'_{\lambda,\lambda}(k'_x)$ of the $p\times p$ Hamiltonian matrix. Therefore, the following approximation can be used:

$$\frac{\hbar^2}{m^*}(k'_x + \lambda K_x)\Delta k_x \approx \frac{\hbar^2}{m^*}k'_x\Delta k_x. \tag{28}$$

This approach fully preserves the interference between coupled states.

Up to the linear order in the electric field, the energy spectrum is given as follows:

$$E_{n,\beta}(k_x) = E^{(0)}_{n,\beta}(k_x) + e\mathcal{E}_y Y^{(0)}_{n,\beta}(k_x), \tag{29}$$

and the velocity expectation values are

$$v_{n,\beta}(k_x) = v^{(0)}_{n,\beta}(k_x) + \frac{1}{\hbar}\frac{dY^{(0)}_{n,\beta}(k_x)}{dk_x}e\mathcal{E}_y. \tag{30}$$

For mass-center positions and momentum, we obtain

$$Y_{n,\beta}(k_x) = Y^{(0)}_{n,\beta}(k'_x) = Y^{(0)}_{n,\beta}(k_x) - \frac{dY^{(0)}_{n,\beta}(k_x)}{dk_x}\Delta k_x, \tag{31}$$

$$p_{n,\beta}(k_x) = p^{(0)}_{n,\beta}(k'_x) = p^{(0)}_{n,\beta}(k_x) - \frac{dp^{(0)}_{n,\beta}(k_x)}{dk_x}\Delta k_x. \tag{32}$$

The changes of the mass-center positions due to the applied electric field are then given by

$$\frac{1}{l_B^2}\frac{dY^{(0)}_{n,\beta}(k_x)}{dk_x}e\mathcal{E}_y = e\mathcal{E}_y - \frac{1}{\hbar}\frac{dp^{(0)}_{n,\beta}(k_x)}{dk_x}e\mathcal{E}_y. \tag{33}$$

This equality represents the differential form of expression (20) for the center-of-mass positions.

IV. INTEGRATED DENSITY OF STATES IN TERMS OF THE EFFECTIVE TOPOLOGICAL NUMBERS

In this section, we obtain an equation for the integrated density of states in the form of Eq. (2). Using the results of the previous section, we define quantities $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$, which are the generalization of the topological numbers σ and s for the cases when the Fermi energy is not located within a bulk-energy gap.

At the considered zero temperature, the electron concentration coincides with the integrated density of states $N(\mu)$ for given Fermi energy μ. Eigenenergies are periodic in k_x with the period $a_y l_B^{-2}$, see Eq. (13), and within each period, there are no more than two crossings of each particular branch β with the energy μ. We can thus introduce left and right k_x positions of occupied state intervals as $k_x^{(L,R)}(\beta,\mu)$ with indices (L) and (R) denoting left and right boundaries.

To get the expression for the integrated density of states $N(\mu)$, we simply integrate Eq. (33) divided by $e\mathcal{E}_y$ over k_x. The constant term gives $N(\mu)$, the mass-center part defines $\tilde{\sigma}$, and the momentum part defines $\tilde{s}$. In analogy with Eq. (2), we get

$$N(\mu) = \frac{\tilde{s}(\mu)}{A_0} + \frac{\tilde{\sigma}(\mu)}{2\pi l_B^2}, \tag{34}$$

with

$$N(\mu) = \frac{n}{2\pi l_B^2} + \frac{1}{N_\beta}\sum_{\beta=1}^{N_\beta} \frac{k_x^{(R)} - k_x^{(L)}}{2\pi a_y}, \tag{35}$$

$$\tilde{s}(\mu) = \frac{1}{N_\beta}\sum_{\beta=1}^{N_\beta} \frac{p_{n,\beta}^{(0)}(k_x^{(R)}) - p_{n,\beta}^{(0)}(k_x^{(L)})}{2\pi\hbar/a_x}, \tag{36}$$

and

$$\tilde{\sigma}(\mu) = n + \frac{1}{N_\beta}\sum_{\beta=1}^{N_\beta} \frac{Y_{n,\beta}^{(0)}(k_x^{(R)}) - Y_{n,\beta}^{(0)}(k_x^{(L)})}{a_y}. \tag{37}$$

For brevity, the dependence of $k_x^{(R)}$ and $k_x^{(L)}$ on the branch index β and on the value of the Fermi energy μ has been omitted. The factor proportional to the Landau-level index n in the expression for the integrated density $N(\mu)$ comes from the contribution of n fully occupied Landau levels which are below the considered level.

The above expressions, which are obtained with the Fermi energy inside an energy subband, can be given a meaning even when the Fermi energy is in gap, by considering the edge states. Indeed, when the energy is in a gap, there are no states in the bulk of the strip, but states are present near the edges of the strip. The magnetic edge states, shown in Figs. 3 and 4, come in pairs with opposite velocities at the two edges, thus giving a macroscopic contribution to the magnetic moment. This contribution can be formally split into contributions per period a_y, which defines the corresponding values for $Y_{n,\beta}^{(0)}(k_x^{(R,L)})$, and consequently yields a contribution to $\tilde{\sigma}(\mu)$. Note that it is just the edge-state contribution which gives the factor n in the expression for $\tilde{\sigma}(\mu)$, as each filled Landau level contributes one magnetic edge-state branch. Similarly, the contribution of nonmagnetic edge states can be written formally as a contribution per period a_y, which yields values for $p_{n,\beta}^{(0)}(k_x^{(R,L)})$ and thus for $\tilde{s}(\mu)$. This procedure ensures the validity of Eq. (2), i.e., the condition for topological numbers which has to be satisfied at gap energies.[1,13,14]

As we are considering the case of rational magnetic fields [see Eq. (11)], Eq. (2) is an equation for integer numbers, a Diophantine equation, which, for the considered case of the weak potential modulation, is often presented in the following form:

$$\nu = \sigma + s\frac{q}{p}, \tag{38}$$

where $\nu = 2\pi l_B^2 N$ is the filling factor, which is 1 for a fully occupied Landau level. It is thus natural to consider the effective topological numbers as a function of ν, rather than a function of the Fermi energy μ. The dependence of $\tilde{\sigma}(\nu)$ and $\tilde{s}(\nu)$ on the filling factor $\nu = 2\pi l_B^2 N$ is presented in Figs. 5 and 6 for the cases $q/p = 1/3$ and $q/p = 3/5$, respectively. For the considered odd values of the integer p, the energy gaps within the bulk region always appear at filling factors ν equal to integer multiples of $1/p$. At these values of ν, the effective topological numbers $\tilde{\sigma}(\nu)$ and $\tilde{s}(\nu)$ become integers just equal to the corresponding values of σ and s, as can be seen in the presented figures.

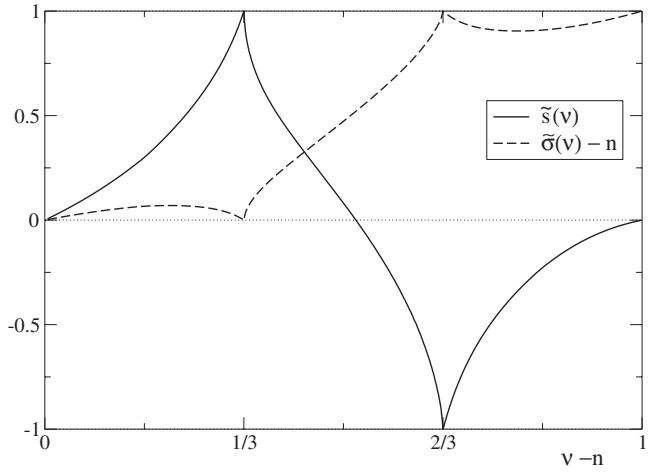

FIG. 5. Dependence of the effective topological numbers, $\tilde{\sigma}-n$ and $\tilde{s}$, for the nth Landau level, on the filling factor ν for $q/p = 1/3$ and $a_x = a_y$.

V. MAGNETIZATION AND INTERNAL PRESSURE

This section is devoted to the relation between quantities characterizing macroscopic properties of the considered system, the magnetization and the internal pressure, and the effective topological numbers $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$. We will show that Eq. (34) is equivalent to that already derived by means of thermodynamics,[7]

$$-\left(\frac{\partial P}{\partial \mu}\right)_{T,V,B} + B\left(\frac{\partial M^{(a)}}{\partial \mu}\right)_{T,V,B} = 0, \tag{39}$$

where P denotes the internal pressure and $M^{(a)}$ is the part of the magnetization which originates in the motion of the electron mass centers. The above equation has been derived as the consequence of the equality of statistical forces acting in the studied system. The proof that it is equivalent to the identity given by Eq. (34) will lead to the conclusion that both equations are of the same physical origin.

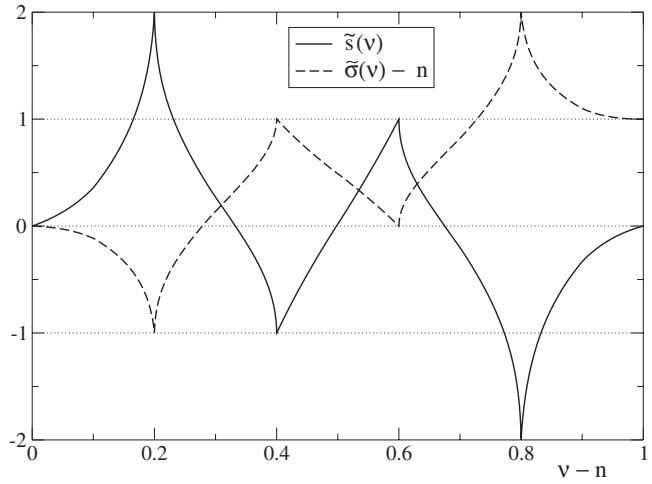

FIG. 6. Dependence of the effective topological numbers, $\tilde{\sigma}-n$ and $\tilde{s}$, for the nth Landau level, on the filling factor ν for $q/p = 3/5$ and $a_x = a_y$.

A. Magnetization

Magnetic moment of the electron within an eigenstate $|n,\beta,k_x\rangle$ is composed of two contributions, both parallel to the applied magnetic field, i.e., perpendicular to the electron system. The first term represents the moment arising due to the relative motion of the electron with respect to its center of mass. In the strong field and weak periodic modulation limits used, the Landau-level mixing can be neglected and the term is just equal to the magnetic moment of the electron in the absence of the modulation. It depends on the Landau-level index n only and it is equal to $-\mu_B^*(2n+1)$, with $\mu_B^* \equiv e\hbar/(2m^*c)$ the effective Bohr magneton. The second term represents the effect of the potential modulation as well as confining potential and originates from the motion of the electron mass centers. The magnetic moment per unit area, magnetization $M(\mu)$, can thus be divided into two parts,

$$M(\mu) \equiv M^{(i)}(\mu) + M^{(a)}(\mu) = \frac{e}{c}\mathrm{Tr}[f_0(H)v_x y], \tag{40}$$

where $f_0(E)$ stands for the Fermi-Dirac distribution function. The magnetic moment $M^{(i)}(\mu)$ with the origin in the relative motion reads

$$M^{(i)}(\mu) = -\mu_B^* \frac{eB}{hc}\sum_{n=0}^{+\infty}(2n+1)\nu_n(\mu), \tag{41}$$

where $\nu_n(\mu)$ is the filling factor of the particular Landau level n.

The mass-center motion contribution $M^{(a)}(\mu)$ is given as follows:

$$M^{(a)}(\mu) = \frac{e}{c}\sum_{n,\beta}\int f_0(E_{n,\beta}^{(0)}(k_x))\frac{v_{n,\beta}^{(0)}(k_x)Y_{n,\beta}^{(0)}(k_x)}{2\pi a_y N_\beta}dk_x, \tag{42}$$

where the k_x integration is taken over the length period $l_B^{-2}a_y$. Inserting Eq. (18) for the velocity expectation values, we get the following zero-temperature result:

$$M^{(a)}(\mu) = \frac{e}{ch}\sum_{n,\beta}\int_0^\mu \frac{Y_{n,\beta}^{(0)}(k_x^{(R)}) - Y_{n,\beta}^{(0)}(k_x^{(L)})}{a_y N_\beta}dE, \tag{43}$$

where $k_x^{(L,R)}$ are boundaries of occupied states of the given branch β below energy E. Comparison with the definition of the effective topological number $\tilde{\sigma}$, Eq. (37), immediately yields

$$\tilde{\sigma}(\mu) = \frac{hc}{e}\frac{\partial M^{(a)}(\mu)}{\partial \mu}. \tag{44}$$

For energies within the gap regions, only magnetic edge states give nonzero contribution and $\tilde{\sigma}$ reduces to the usual topological number σ. The above equation becomes

$$\left(\frac{\partial M^{(a)}(\mu)}{\partial \mu}\right)_{\mu\in\text{ gap}} = \frac{e}{hc}\sigma. \tag{45}$$

This relation was already derived by the use of thermodynamical arguments.[13]

B. Internal pressure

Internal pressure in two-dimensional systems is defined as the derivative of the grand-canonical potential Ω with respect to the area A,

$$P = -\left(\frac{\partial\Omega}{\partial A}\right)_{T,B,\mu} = -\left(\frac{\partial\Omega}{\partial A}\right)_{T,B,\mu,A_0} + P_0, \tag{46}$$

$$P_0(\mu) \equiv -\frac{A_0}{A}\left(\frac{\partial\Omega}{\partial A_0}\right)_{T,B,\mu,A}. \tag{47}$$

The first term on the right side of the above equation, Eq. (46), represents scaling of the internal pressure with A under the constraint of fixed unit cell area A_0. For a system of noninteracting particles, its derivative with respect to the chemical potential equals the particle concentration $N(\mu)$. The second term $P_0(\mu)$ arises due to the variation of A_0 and vanishes for zero potential modulation.

To establish the internal pressure, we apply its quantum mechanical definition. Let us assume that the change of the sample area A is due to the change of the strip width accompanied by the corresponding change of the lattice constant a_y. Since

$$\frac{dH}{da_y} = -\frac{y}{a_y}\frac{dV(\vec{r})}{dy}, \tag{48}$$

we get

$$P(\mu) = \mathrm{Tr}\left[f_0(H)y\frac{dV(\vec{r})}{dy}\right]. \tag{49}$$

Similarly, as in the case of the magnetic moment, the above expression can be divided into two parts. The contribution which represents the effect of the mass-center positions reads

$$P^{(a)}(\mu) = \sum_{n,\beta}\int f_0(E_{n,\beta}(k_x))\frac{Y_{n,\beta}^{(0)}(k_x)V'_{n,\beta}(k_x)}{2\pi a_y N_\beta}dk_x, \tag{50}$$

where

$$V'_{n,\beta}(k_x) = \langle n,\beta,k_x|\frac{dV(\vec{r})}{dy}|n,\beta,k_x\rangle. \tag{51}$$

Since the expectation value of the force along the $\hat{y}$ direction defined by the acceleration operator a_y,

$$m^* a_y = m^*\omega_c v_x - \frac{dV(\vec{r})}{dy}, \tag{52}$$

vanishes for an electron in any eigenstate, the quantity $V'_{n,\beta}(k_x)$ entering Eq. (50) can be replaced by $m^*\omega_c v_{n,\beta}^{(0)}(k_x)$. If we further insert for $Y_{n,\beta}^{(0)}(k_x)$ the expression given by Eq. (20), we obtain, for the internal pressure at zero temperature,

$$P^{(a)}(\mu) = \int_0^\mu N(E)dE + P_0^{(a)}(\mu), \tag{53}$$

where

$$P_0^{(a)}(\mu) = -\sum_{n,\beta} \int_0^{\mu} \frac{p_{n,\beta}^{(0)}(k_x^{(R)}) - p_{n,\beta}^{(0)}(k_x^{(L)})}{2\pi\hbar a_y N_\beta} dE. \quad (54)$$

The derivative of the internal pressure $P^{(a)}(\mu)$ with respect to μ has the following form:

$$\frac{\partial P^{(a)}(\mu)}{\partial \mu} = N(\mu) + \frac{\partial P_0^{(a)}(\mu)}{\partial \mu}. \quad (55)$$

If we define $p_{n,\beta}^{(0)}(k_x^{(L,R)})$ for energies at which there are no bulk states by means described in the above, we obtain the general relation[13,14,22]

$$\left(\frac{\partial P_0^{(a)}(\mu)}{\partial \mu}\right)_{\mu\epsilon \text{ gap}} = \left(\frac{\partial P_0(\mu)}{\partial \mu}\right)_{\mu\epsilon \text{ gap}} = -\frac{s}{A_0}. \quad (56)$$

Thus, the effective topological number $\tilde{s}(\mu)$ defined by Eq. (36) can be expressed as the derivative of the pressure $P_0^{(a)}(\mu)$ with respect to the chemical potential,

$$\tilde{s}(\mu) = -A_0 \frac{\partial P_0^{(a)}(\mu)}{\partial \mu}. \quad (57)$$

Note that in the considered limit of the weak periodic modulation, the topological number s for gaps between broadened Landau levels equals zero. Therefore, the summation over the Landau-level index can be excluded from the internal pressure evaluation by Eq. (54).

To conclude this section, let us note that the product of the magnetic field strength B and the part of the magnetization stemming from the relative electron motion, $M^{(i)}(\mu)$, has the meaning of pressure. It can be easily shown that the corresponding part of the internal pressure $P_0^{(i)}(\mu)$ just equals $-BM^{(i)}(\mu)$. Thus, the effect of the relative electron motion can be fully included into the definition of the internal pressure. Then, the resulting pressure $P(\mu)$ coincides with the pressure $P^{(a)}(\mu)$ defined in this section.

VI. EFFECTIVE TOPOLOGICAL NUMBERS IN TERMS OF MEASURABLE QUANTITIES

In this section, we show that the effective topological numbers $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$ are simply related to, at least in principle, measurable quantities: the Hall conductance for $\tilde{\sigma}(\mu)$ and the static electron polarizability for $\tilde{s}(\mu)$.

A. Hall conductance

Quantum Hall effect for two-dimensional system of Bloch electrons in the presence of a magnetic field has been the subject of many theoretical papers[1,2,13–17] and the technological progress in sample preparation allowed for verification of the theoretical predictions.[4] Nearly no attention has been devoted to the properties of the Hall effect between two adjacent quantized values. Only recently it has been argued[23,24] that the Hall current is related to the Fermi electron contribution to the part of the magnetic moment which stems from the motion of the electron mass centers, i.e., to the magnetic moment $M^{(a)}(\mu)$ defined in Sec. V A. In the following, another procedure leading to the same result will be used.

The applied electric field $\mathcal{E}_y$ changes the velocity of electrons originally having energy $E_{n,\beta}^{(0)}(k_x)$, and for the current density along the $\hat{x}$ direction, we have

$$j_x = -\frac{e}{N_\beta} \sum_{n,\beta} \int f_0(E_{n,\beta}^{(0)}(k_x)) \frac{v_{n,\beta}(k_x)}{2\pi a_y} dk_x. \quad (58)$$

Inserting for $v_{n,\beta}(k_x)$ its linear expansion in $\mathcal{E}_y$, Eq. (30), we get

$$j_x = -e^2 \mathcal{E}_y \sum_{n,\beta} \int \frac{f_0(E_{n,\beta}^{(0)}(k_x))}{2\pi\hbar a_y N_\beta} \frac{dY_{n,\beta}^{(0)}(k_x)}{dk_x} dk_x. \quad (59)$$

Comparison with the expression for $M^{(a)}(\mu)$, Eq. (43), immediately gives the relation[7] we have been looking for,

$$j_x = -ec \frac{\partial M^{(a)}(\mu)}{\partial \mu} \mathcal{E}_y. \quad (60)$$

For the Hall conductance, we get

$$G_H \equiv \frac{j_x}{\mathcal{E}_y} = -ec\left(\frac{\partial M^{(a)}(\mu)}{\partial \mu}\right)_{T,A,B} = -\frac{e^2}{h}\tilde{\sigma}(\mu)$$
$$= -\frac{ec}{B}\left(N(\mu) - \frac{\tilde{s}(\mu)}{A_0}\right). \quad (61)$$

The last equality has been obtained by using identity relating effective topological numbers $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$ with the integrated density of states $N(\mu)$, Eq. (34).

For the Fermi energy located within an energy gap of bulk states, the only contribution of Fermi electrons to the total magnetic moment is due to the electrons in edge states. In the thermodynamic limit, the derivatives of $M(\mu)$ and $M^{(a)}(\mu)$ with respect of the μ coincide, and using Maxwell relations, we get

$$G_H^Q = -ec\left(\frac{\partial M(\mu)}{\partial \mu}\right)_{T,A,B} = -ec\left(\frac{\partial N(\mu)}{\partial B}\right)_{T,A,\mu\epsilon \text{ gap}}$$
$$= -\frac{e^2}{h}\sigma = \frac{1}{R_H^Q}. \quad (62)$$

Note that the above general formula (61) for the Hall conductance G_H is also directly applicable to the experimental data satisfying the condition that the Hall field is much larger than the longitudinal field. In these cases, the measured Hall resistance is approximately determined by $1/G_H$.

B. Static electron polarizability

The polarizability arises from the charge redistribution induced by the so-called local electric field inside the bulk of the system.[25] It is defined as the ratio of the induced dipole moments to the field actually acting on charges within the unit cell area. In the considered transport regime, only the polarizability across the strip induced by the electric field along the $\hat{y}$ direction, $\mathcal{E}_y$, has a physical meaning.

The electronic states we consider are long extended states along the $\hat{x}$ direction and they are localized along the $\hat{y}$ direction around their mass-center positions $Y_{n,\beta}^{(0)}(k_x)$. The

mass-center shift induced by $\mathcal{E}_y$ is given by Eq. (31). Besides electron charges, there are positive charges of ions determining the background potential. The standard assumption that there is only a constant shift of these charges without any local charge redistribution is used. The resulting shift along the $\hat{y}$ direction is $-l_B^2 \Delta k_x$, with Δk_x defined by Eq. (25). The average displacement of the electron and corresponding positive charges along the $\hat{y}$ direction reads

$$\Delta Y_{n,\beta}(k_x) = -\frac{dY_{n,\beta}^{(0)}(k_x)}{dk_x}\Delta k_x + l_B^2 \Delta k_x = \frac{l_B^2}{\hbar}\frac{dp_{n,\beta}^{(0)}(k_x)}{dk_x}\Delta k_x, \tag{63}$$

where the definition of the mass-center position $Y_{n,\beta}^{(0)}(k_x)$, Eq. (20), has been used. Since $p_{n,\beta}^{(0)}(k_x)$ defines the periodic part of the mass-center positions, the displacement has the same periodicity with the period a_y/l_B^2, see Eq. (22). Note that the total charge neutrality of the system has been assumed, i.e., the sum over all charges vanishes.

The dipole moment due to the shift of the electron in the state $|n,\beta,k_x\rangle$ is given as the product of the electron charge and the above-defined displacement, Eq. (63). For the static electron polarizability, we get

$$\alpha(\mu) = \frac{e^2}{m^* \omega_c^2} \sum_{n,\beta} \int \frac{f_0(E_{n,\beta}^{(0)}(k_x))}{2\pi\hbar a_y N_\beta} \frac{dp_{n,\beta}^{(0)}(k_x)}{dk_x} dk_x. \tag{64}$$

Comparison with the expression for the internal pressure $P_0^{(a)}(\mu)$, Eq. (54), and that for the effective topological number $\tilde{s}(\mu)$, Eq. (36), gives

$$\alpha(\mu) = -\frac{e^2}{m^* \omega_c^2}\frac{\partial P_0^{(a)}(\mu)}{\partial \mu} = \frac{e^2}{m^* \omega_c^2}\frac{\tilde{s}(\mu)}{A_0}. \tag{65}$$

When the Fermi energy is located within an energy gap, the polarizability has a quantized value

$$\alpha^Q = \frac{e^2}{m^* \omega_c^2}\frac{s}{A_0}, \tag{66}$$

where s is the topological number for the considered gap. However, contrary to quantum Hall values, the proportionality constant relating α^Q and the gap quantum number s is not a universal quantity, but it is determined by the force constant $m^* \omega_c^2$. The static electron polarizability as a function of the filling factor smoothly interpolates between adjacent quantized values as can be seen from the dependence of $\tilde{s}(\nu)$ presented in Fig. 6. For a completely filled broadened Landau level, s equals zero and there is no polarization effect. It can be nonzero only for noninteger filling factors. These properties are specific for the considered limit of a potential modulation weak in comparison to the Landau-level energy separation $\hbar\omega_c$.

VII. SUMMARY AND CONCLUDING REMARKS

In this paper, we have studied the Hall effect for two-dimensional electron systems subject to a weak periodic potential. The Hall conductance has been obtained as a linear response to the electric field across the strip. It gives rise to two response forces: the Lorentz force related to the current induced along the strip and a gradient force related to the electron polarizability. If there is no potential modulation, the polarizability vanishes and the force due to the electric field across the strip can only be compensated by the Lorentz force. As a result, the Hall resistance is exclusively determined by the electron concentration. In the case of nonzero potential modulation, the electron polarizability affects the resulting value of the induced Hall current and, consequently, the Hall resistance is affected. In the particular case when the system becomes an insulator, like when the filling factor $\nu = 1/3$ for $q/p = 1/3$, there is no current flow, i.e., no Lorentz force appears. Then, the only response to the electric field across the strip is the polarization of the system, as expected. In the general case, the competition of the Lorentz force and the force related to the electron polarizability, which together have to compensate the electric force, is responsible for the nonmonotonic dependence of the Hall resistance on the magnetic field strength.

We have defined effective topological numbers $\tilde{\sigma}(\mu)$ and $\tilde{s}(\mu)$ as response coefficients related to the Lorentz force and the static electron polarizability, respectively. The sum of $\tilde{\sigma}(\mu)$ divided by the area per unit flux quantum and $\tilde{s}(\mu)$ divided by the unit cell area gives the electron concentration, Eq. (34). This relation is equivalent to the steady state condition that the total force acting on electrons has to vanish. Whenever the Fermi energy is located within an energy gap of the bulk-energy spectra, effective topological numbers obey integer values satisfying the necessary condition for the gap appearance, Eq. (2). The Hall resistance obeys quantized values given by σ. Similarly, the static electron polarizability obeys quantized values defined by the integer s, but contrary to the quantum Hall values, the proportionality constant is not a universal quantity. It is determined by the force constant which is a material dependent quantity. Also, the localized states within bulk gaps affect polarizability contrary to the Hall effect.

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