Problem session from the conference: Dynamics in Teichmüller theory

0.1. **J. Athreya.** Philosophical question: examples of generic behavior.

(1) E.g., fix a stratum $\mathcal{H}$. Give an example of $(X,\omega) \in \mathcal{H}$ such that $\mathrm{SL}(X,\omega) = \{\mathrm{id}\}$ and $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega)} = \mathcal{H}$. (Take $g > 2$, following Calta/McMullen classification.) (Hubert–Lanneau–Möller have techniques that work for certain strata.) Try to avoid things of the form $(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}))^n \ltimes (\mathbb{R}^2/\mathbb{Z}^2)^n$. Give explicit construction.
(2) Give examples of Birkhoff generic points for $\{g_t\}$ or $\{h_s\}$, in the sense that

$$\frac{1}{T}\int_0^T \chi_A(g_t x)\,dt = \int_{\mathcal{H}} \chi_A\,d\mu \qquad \text{or} \qquad \frac{1}{T}\int_0^T \chi_A(h_s x)\,dt = \int_{\mathcal{H}} \chi_A\,d\mu.$$

(3) For $1 \leq \alpha \leq 3$, there exists $x \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ such that $\mathrm{H}\dim(\overline{\{g_t x\}_{t\geq 0}}) = \alpha$. Answer same question for strata (i.e., for $1 \leq \alpha \leq \dim \mathcal{H}$).

Program: Let $\Sigma = \Sigma_g$ be a compact surface, $\Gamma = \Gamma_g$ the mapping class group, $\pi = \pi_1(\Sigma)$, $G = \mathrm{SU}(2)$. Teichmüller space can be represented as the connected component of discrete, faithful representations $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$. For $\mathrm{Hom}(\pi, G)/G$, Goldman showed there exists an ergodic Γ-invariant volume form. Do lattice Veech groups act ergodically? Known that there exists a pseudo-Anosov element that does not act ergodically. What about other connected components of $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$, or $\mathrm{Hom}(\pi, G)/G$ for other compact Lie groups?

0.2. **J.-C. Yoccoz.** If you have a Rauzy diagram $\mathcal{D}$ corresponding to some stratum, can construct a surface for each vertex following canonical choices. Each edge will give an identification $Q(M_\pi) \xrightarrow{\simeq} Q(M_{\pi'})$. For any loop in the diagram, have therefore an isomorphism from a Teichmüller space to itself, which represents a mapping class; i.e., there is a map $\pi_1(\mathcal{D}, \pi) \to \mathrm{Mod}(M_\pi)$. Is this map surjective? (Do not require that loops follow arrows of $\mathcal{D}$.)

0.3. **P. Hubert.** Infinite square-tiled surfaces, perhaps with points having infinite angles (e.g., infinite staircase).

(1) Such a surface has a linear flow, which is "periodic" in rational directions (either cylinder or infinite band). When is the linear flow of slope $\alpha \notin \mathbb{Q}$ ergodic? Are there other good conditions under which one can say something about the dynamics?
(2) What can be said about the Veech groups of these objects? Staircase is a lattice example, with Veech group $\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$. Can we find many lattice examples? (In these cases, answering the first question is easier.) What kinds of finitely generated, non-lattice, and not generated by a single parabolic, groups can arise?
(3) Another lattice example, not a covering of the torus: infinite ladder, with edges labelled by integers in reverse order, has Veech group $\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$. What are the dynamics of the linear flow on this surface?
(4) Perhaps a more interesting class: stable abelian differentials.

0.4. **A. Zorich.** Suppose you have an abstrat compact manifold M^n, as nice as you wish, and a closed C^∞ 1-form ω on M with Morse-type singularities (i.e., since locally $\omega = \mathrm{d}f$, require that f is a Morse function). Define a codimension 1 foliation L^{n-1} on M by ω. Let $c_1, \ldots, c_n$ be a basis of cycles on M, and let the periods be

$$a_1 = \int_{c_1} \omega, \quad \ldots, \quad a_n = \int_{c_n} \omega.$$

Compute the dimension of $\langle a_1, \ldots, a_n \rangle_{\mathbb{Q}}$. If this dimension is 1, leaves are compact submanifolds. If dimension is 2, get union of cylinders with handles, which resembles a covering of a compact manifold, but is only quasi-periodic.

Can you embed a K–Z type covering as a leaf of such a foliation?

Can we define dynamics on a family of such manifolds? This would produce a renormalization procedure for the original billiard.

0.5. **M. Möller.** Finite square-tiled surfaces, with corresponding Teichmüller curve $C = \mathbb{H}/\mathrm{SL}(X, \omega)$, and C_2 a finite index covering (to eliminate torsion, other problems). Consider the action of $\pi_1(C_2)$ on $H_1(X, \mathbb{C})$, which has an invariant subspace $\langle \omega, \overline{\omega} \rangle_{\mathbb{C}}$.

(1) What can we say about the action of $\pi_1(C)$ on the orthogonal space $V = \langle \omega, \overline{\omega} \rangle_{\mathbb{C}}^{\perp}$? Give examples where V is irreducible, reducible as a vector space with $\pi_1(C)$-action (a.k.a. representation space, local system, flat bundle). What kind of components can occur? Reasonable invariant: what is the algebraic monodromy group, i.e., the smallest Lie group containing the image of $\pi_1(C)$ in $\mathrm{Sp}(V)$?
(2) Calculate Lyapunov exponents of these examples.
(3) Suppose have covering $Y \to X$, with X the regular pentagon (or any primitive Veech surface in genus 2) and Y another Veech surface. Consider C_Y and the action of $\pi_1(C_Y)$ on $H_1(Y, \mathbb{C})$. Then there is one subspace $\mathbb{L} = \langle \omega, \overline{\omega} \rangle$ with Lyapunov exponent, and the second Lyapunov exponent $\mathbb{L}^\sigma$ is 1/3. Since Y has higher genus, there are other exponents. Does the second Lyapunov exponent come from the primitive surface, with all remaining exponents less than 1/3, or some other behavior?

0.6. **Y. Cheung.**

0.6.1. *Question of Boshernitzan.* Let n be the number of squares in a square-tiled surface. Fix a direction, say with slope the golden mean. Known that flow is uniquely ergodic. How long do you have to wait before any trajectory becomes ε-dense? Give an answer that depends only on ε and n. Zorich suggests that surfaces in hyperelliptic components will pose the biggest problem, because second Lyapunov exponent is large.

0.6.2. *Question from 2003.* Known generically that $\#\{\gamma \text{ s.c.} : |\gamma| < T\} \sim CT^2$. Give an example where exact quadratic asymptotics do not exist, i.e.,

$$\lim_{t\to\infty} \frac{1}{T} \#\{\gamma \text{ s.c.} : |\gamma| < T\} \qquad \text{does not exist.}$$

0.6.3. Are there examples P such that $0 < \mathrm{H}\dim NE(P) < 1/2$?