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GOLDEN OLDIE

Republication of: Quantum theory of weak gravitational fields

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QUANTUM THEORY OF WEAK GRAVITATIONAL FIELDS[1]

By M. Bronstein.

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§ 1. General remarks

It is known that the deviations of a space-time continuum from "Euclideanness" can be characterized by the components of the Riemann - Christoffel tensor. When these deviations are small, this fourth-rank tensor field can be derived from a symmetric second-rank tensor field as follows:

$$(\mu\rho\nu\sigma) = \frac{1}{2}\left(\frac{\partial^2 h_{\mu\nu}}{\partial x_\rho \partial x_\sigma} + \frac{\partial^2 h_{\rho\sigma}}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 h_{\mu\sigma}}{\partial x_\rho \partial x_\nu} - \frac{\partial^2 h_{\rho\nu}}{\partial x_\mu \partial x_\sigma}\right), \tag{1}$$

where $h_{\mu\nu}$ is the small deviation of the fundamental metric tensor from its Minkowskian value $\Delta_{\mu\nu}$ ($\Delta_{00} = 1,\ \Delta_{11} = \Delta_{22} = \Delta_{33} = -1;\ \Delta_{\mu\nu} = 0,\ \text{if } \mu \neq \nu$). In these circumstances, we consider the world as "Euclidean" with the metric tensor $\Delta_{\mu\nu}$, and $(\mu\rho\nu\sigma)$ as the components of a fourth-rank tensor field embedded in this flat world. Thereby $h_{\mu\nu}$ play the role of the "potentials", whose values can be fixed by four additional "gauge conditions"

$$[\alpha\alpha, \beta] = 0 \qquad (\beta = 0,\ 1,\ 2,\ 3). \tag{2}$$

(Here and in what follows the summation convention applies only to Greek indices, also in the sense that e.g. $A_{\alpha\alpha}$ denotes $A_{00} - A_{11} - A_{22} - A_{33}$; it allows us not to worry about the difference between co- and contravariant components; $[\alpha\beta, \gamma]$ is the usual notation for the three-index Christoffel symbol).

In empty space, the equations of gravitation read

$$(\mu\rho\nu\rho) = 0. \tag{3}$$

[1]A more detailed summary of this work appears simultaneously in "Journal of Experimental and Theoretical Physics" (russian).

Under the “gauge conditions” (2), they are fully equivalent to the usual wave equations for potentials

$$\Box h_{\mu\nu} = 0. \tag{4}$$

In what follows, we consider a quantum-mechanical continuous system, for which the classical equations of motion can be written in the form (4); along with the additional conditions (2), which, as we shall show, are compatible with the S c h r ö d i n g e r equation for the quantum-mechanical system under consideration, this system is identical to the gravitational field in empty space. This treatment is, to some extent, analogous to F e r m i 's quantization of electrodynamics: F e r m i 's depends on a non-gauge-invariant Lagrangian; our quantization of the gravitational field also relies on quantities that are not (even approximately) relativistic invariants.

§ 2. Hamiltonian form and plane waves

Let us consider a dynamical continuum with ten fields $h_{\mu\nu}$ $(\mu \leq \nu)$, which play the role of mechanical coordinates with the Lagrangian density

$$[\alpha\alpha,\beta]\,[\beta\gamma,\gamma] - [\alpha\beta,\gamma]\,[\alpha\gamma,\beta] + \frac{1}{2}\,[\alpha\alpha,\beta]\,[\gamma\gamma,\beta]\,.$$

Rather complicated calculations, which we omit here for brevity, show that the corresponding Hamiltonian density is

$$2\left(p_{00} + \sum_l \frac{\partial h_{0l}}{\partial x_l}\right)^2 - \frac{1}{4}\left(3p_{00} - \sum_l p_{ll} + 2\sum_l \frac{\partial h_{0l}}{\partial x_l}\right)^2 -$$

$$-\frac{1}{2}\sum_l\left(-p_{0l} + \frac{\partial h_{00}}{\partial x_l} + \sum_m \frac{\partial h_{ml}}{\partial x_m}\right)^2 +$$

$$+\frac{1}{4}\sum_n\left(2p_{nn} + p_{00} - \sum_l p_{ll} + 2\frac{\partial h_{0n}}{\partial x_n}\right)^2 +$$

$$+\frac{1}{2}\sum_{l<m}\left(p_{lm} + \frac{\partial h_{0m}}{\partial x_l} + \frac{\partial h_{0l}}{\partial x_m}\right)^2 + \frac{1}{8}\sum_l\left(\frac{\partial h_{00}}{\partial x_l}\right)^2 -$$

$$-\frac{1}{4}\sum_{lm}\left(\frac{\partial h_{0l}}{\partial x_m} - \frac{\partial h_{0m}}{\partial x_l}\right)^2 + \frac{1}{4}\sum_{lm}\frac{\partial h_{00}}{\partial x_l}\frac{\partial h_{mm}}{\partial x_l} - \frac{1}{2}\left(\sum_l \frac{\partial h_{0l}}{\partial x_l}\right)^2 -$$

$$-\frac{1}{8}\sum_{m}\left(\frac{\partial}{\partial x_m}\sum_{l}h_{ll}\right)^2+\frac{1}{4}\sum_{lmn}\left(\frac{\partial h_{ln}}{\partial x_m}\right)^2+$$
$$+\frac{1}{2}\sum_{lmn}\left(\frac{\partial h_{lm}}{\partial x_m}\frac{\partial h_{ln}}{\partial x_n}-\frac{\partial h_{lm}}{\partial x_n}\frac{\partial h_{ln}}{\partial x_m}\right),$$

where $p_{\alpha\beta}$ are the momenta conjugate to the coordinates $h_{\alpha\beta}$ and that the corresponding equations of motion are (4). Here and in what follows, Latin indices take the values $(1, 2, 3)$ only. For simplicity we have also set the speed of light to 1 and the Newtonian gravitational constant to $1/16\pi$. (The reader can easily verify the value of gravitational constant if he compares our formulas e.g. with the formulas 58.1 and 59.4 of E d d i n g t o n's book "Relativitätstheorie in mathematischer Behandlung", Berlin, Springer 1925.)

Now let us introduce the Fourier expansion,

$$h_{\alpha\beta}=\frac{1}{(2\pi)^{3/2}}\int d\mathfrak{k}\left[h_{\alpha\beta,\mathfrak{k}}e^{-i(\omega t-\mathfrak{k}\mathfrak{r})}+h^{+}_{\alpha\beta,\mathfrak{k}}e^{i(\omega t-\mathfrak{k}\mathfrak{r})}\right]$$

(where $\omega=|\mathfrak{k}|$). The Hamiltonian can the be calculated to be

$$H=\int d\mathfrak{k}\omega^2\left\{\frac{1}{2}\left(h^{+}_{00,\mathfrak{k}}+\sum_{l}h^{+}_{ll,\mathfrak{k}}\right)\left(h_{00,\mathfrak{k}}+\sum_{l}h_{ll,\mathfrak{k}}\right)+\right.$$
$$\left.+\sum_{l\neq m}\left(h_{lm,\mathfrak{k}}h^{+}_{lm,\mathfrak{k}}-h^{+}_{ll,\mathfrak{k}}h_{mm,\mathfrak{k}}\right)-2\sum_{l}h_{0l,\mathfrak{k}}h^{+}_{0l,\mathfrak{k}}\right\}. \tag{5}$$

(At first sight, the factor-ordering seems to be arbitrary, but it will be clear from what follows that this is the only one that will avoid quantum "zero-point energy".)

After Fourier expansion, the conditions (2) take the following form

$$\left.\begin{array}{c}\frac{1}{2}\omega\left(h_{00,\mathfrak{k}}+\sum_l h_{ll,\mathfrak{k}}\right)+\sum_l \mathfrak{k}_l h_{0l,\mathfrak{k}}=0,\\ \omega h_{0l,\mathfrak{k}}+\sum_m \mathfrak{k}_m h_{ml,\mathfrak{k}}+\frac{1}{2}\mathfrak{k}_l\left(h_{00,\mathfrak{k}}-\sum_m h_{mm,\mathfrak{k}}\right)=0\\ (l=1,\ 2,\ 3).\end{array}\right\} \tag{6}$$

Hence the number of independent $h_{\mu\nu,\mathfrak{k}}$ (for a given $\mathfrak{k}$) is equal to $10-4=6$. However one may easily show that for many problems where e.g., energy transfer by gravitational waves is discussed, the number of independent polarizations is even fewer. Let us consider e.g., the case $\mathfrak{k}\|z$. Then the conditions (6) become

$$h_{11,\mathfrak{k}} + h_{22,\mathfrak{k}} = h_{00,\mathfrak{k}} + 2h_{03,\mathfrak{k}} + h_{33,\mathfrak{k}} = h_{01,\mathfrak{k}} + h_{31,\mathfrak{k}} = \\ = h_{02,\mathfrak{k}} + h_{32,\mathfrak{k}} = 0. \tag{6'}$$

Taking into account the relation (6′) one may write the energy content of the gravitational wave in the form

$$2d\mathfrak{k}\omega^2 \left(h^+_{12,\mathfrak{k}} h_{12,\mathfrak{k}} + h^+_{11,\mathfrak{k}} h_{11,\mathfrak{k}}\right).$$

(Using the conditions (6′) the integrand in (5) can be easily brought to this form, if all $h_{\mu\nu,\mathfrak{k}}$ and $h^+_{\mu\nu,\mathfrak{k}}$ commute; later we shall see, that even when the h and h^+ are quantized and do not commute, a similar result holds.) Thus, without changing the energy content of the gravitational wave, we can subject the 10 amplitudes $h_{\mu\nu,\mathfrak{k}}$ to four additional conditions besides (6). For example, we can choose (also when $\mathfrak{k}$ is not parallel to z) the following (albeit relativistically non-invariant) additional conditions:

$$h_{00,\mathfrak{k}} = h_{01,\mathfrak{k}} = h_{02,\mathfrak{k}} = h_{03,\mathfrak{k}} = 0.$$

Thus we see among other things that for the energy transfer only the transverse gravitational waves are significant, namely those with two independent polarizations. However, the situation is quite different if one considers not energy transfer but e.g., gravitational interaction between two massive bodies: we shall see below that for this interaction only the longitudinal $h_{\mu\nu}$ waves dominate.

§3. Commutation relations and eigenvalues of the energy.

The Heisenberg – Pauli commutation relations read[2]:

$$\left.\begin{array}{l} [h_{\alpha\beta}(\mathfrak{r}), h_{\alpha'\beta'}(\mathfrak{r}')] = 0, \\ [p_{\alpha\beta}(\mathfrak{r}), p_{\alpha'\beta'}(\mathfrak{r}')] = 0, \\ [p_{\alpha\beta}(\mathfrak{r}), h_{\alpha'\beta'}(\mathfrak{r}')] = \frac{h}{i}\delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta(\mathfrak{r}-\mathfrak{r}') \qquad (\alpha \leqslant \beta, \quad \alpha' \leqslant \beta'). \end{array}\right\} \tag{7}$$

[2]Here h means $h/2\pi$.

After Fourier expansion, we obtain

$$\left.\begin{aligned}
&[h_{\alpha\beta,\mathfrak{k}}, h_{\alpha'\beta',\mathfrak{k}'}] = 0, \qquad \left[h^+_{\alpha\beta,\mathfrak{k}}, h^+_{\alpha'\beta',\mathfrak{k}'}\right] = 0,\\
&\left[h^+_{00,\mathfrak{k}}, h_{00,\mathfrak{k}'}\right] = \left[h^+_{00,\mathfrak{k}}, h_{ll,\mathfrak{k}'}\right] = \left[h^+_{ll,\mathfrak{k}}, h_{ll,\mathfrak{k}'}\right] = -\frac{h}{2\omega}\delta(\mathfrak{k}-\mathfrak{k}'),\\
&\left[h^+_{00,\mathfrak{k}}, h_{0l,\mathfrak{k}'}\right] = \left[h^+_{ll,\mathfrak{k}}, h_{0m,\mathfrak{k}'}\right] = \left[h^+_{0l,\mathfrak{k}}, h_{mn,\mathfrak{k}'}\right] = 0,\\
&\left[h^+_{00,\mathfrak{k}}, h_{lm,\mathfrak{k}'}\right] = \left[h^+_{nn,\mathfrak{k}}, h_{lm,\mathfrak{k}'}\right] = 0 \qquad (l \neq m),\\
&\left[h^+_{ll,\mathfrak{k}}, h_{mm,\mathfrak{k}'}\right] = \frac{h}{2\omega}\delta(\mathfrak{k}-\mathfrak{k}') \qquad (l \neq m),\\
&\left[h^+_{0l,\mathfrak{k}}, h_{0m,\mathfrak{k}'}\right] = \frac{h}{2\omega}\delta_{lm}\delta(\mathfrak{k}-\mathfrak{k}'),\\
&\left[h^+_{lm,\mathfrak{k}}, h_{pq,\mathfrak{k}'}\right] = -\frac{h}{2\omega}\delta_{lp}\delta_{mq}\delta(\mathfrak{k}-\mathfrak{k}') \quad (l < m, \quad p < q).
\end{aligned}\right\} \tag{8}$$

We introduce the operators

$$A = \frac{1}{2}\omega\left(h_{00,\mathfrak{k}} + \sum_l h_{ll,\mathfrak{k}}\right) + \sum_l \mathfrak{k}_l h_{0l,\mathfrak{k}},$$

$$B_l = \omega h_{0l,\mathfrak{k}} + \sum_m \mathfrak{k}_m h_{ml,\mathfrak{k}} + \frac{1}{2}\mathfrak{k}_l\left(h_{00,\mathfrak{k}} - \sum_m h_{mm,\mathfrak{k}}\right).$$

Calculation shows that due to (8) all eight operators A, B_l, A^+, B_l^+ ($l = 1, 2, 3$) commute with each other. But they do not commute with the Hamiltonian (5), i.e. they are not integrals of motion. However in the particular case, when

$$A = B_l = A^+ = B_l^+ = 0,$$

one can easily show, that the Poisson brackets of any of these operators and the Hamiltonian, i.e. the rates of change of these operators, are all zero. It means that the conditions (6) (and their conjugates) are compatible with each other and with the quantum-mechanical equations of motion.

The commutation relations (8), the Hamiltonian operator (5) and the additional conditions (6) (together with the ansatz for the interaction between gravitational field and matter, which will be introduced later) form the foundation of the quantum theory of gravity proposed here. Note that the quantum-mechanical Hamiltonian operator can never be uniquely specified by the correspondence principle: it is always possible to change the Hamiltonian by introducing additional terms that go to zero when $h \to 0$

(e.g., "spin terms" in the theory of the electron); in general even relativistic requirements are not sufficient to fix these "spin terms" uniquely. Nevertheless, we believe that adding such terms is not necessary here.

Now to calculate the eigenvalues of the energy! As in the quantum electrodynamics this is accomplished by introducing the new variables ξ that satisfy the commutation relations

$$\left[\xi, \xi^+\right] = 1.$$

The eigenvalues of $\xi\xi^\dagger$ are well-known to equal $n+1$, and those of $\xi^+\xi$ equal n, where n is a positive integer or zero.

Here it is not possible to introduce these ξ-variables in a symmetric manner. One possible solution of the problem looks like this,

$$\frac{1}{2}\left(h_{00,\mathfrak{k}} + \sum_l h_{ll,\mathfrak{k}}\right) = \sqrt{\frac{h}{2\omega d\mathfrak{k}}}\xi_{00,\mathfrak{k}}e^{i\omega t},$$

$$\begin{aligned}
h_{11,\mathfrak{k}} &= \sqrt{\frac{h}{2\omega d\mathfrak{k}}}\left(\frac{\xi^+_{11,\mathfrak{k}}}{\sqrt{3}}e^{-i\omega t} + \frac{\xi_{22,\mathfrak{k}}}{\sqrt{3}}e^{i\omega t} + \xi_{33,\mathfrak{k}}e^{i\omega t}\right), \\
h_{22,\mathfrak{k}} &= \sqrt{\frac{h}{2\omega d\mathfrak{k}}}\left(\frac{\xi^+_{11,\mathfrak{k}}}{\sqrt{3}}e^{-i\omega t} + \frac{\xi_{22,\mathfrak{k}}}{\sqrt{3}}e^{i\omega t} - \xi_{33,\mathfrak{k}}e^{i\omega t}\right), \\
h_{33,\mathfrak{k}} &= \sqrt{\frac{h}{2\omega d\mathfrak{k}}}\left(\frac{\xi^+_{11,\mathfrak{k}}}{\sqrt{3}}e^{-i\omega t} - \frac{2}{\sqrt{3}}\xi_{22,\mathfrak{k}}e^{i\omega t}\right), \\
h_{lm,\mathfrak{k}} &= \sqrt{\frac{h}{2\omega d\mathfrak{k}}}\xi_{lm,\mathfrak{k}}e^{i\omega t} \qquad (l \neq m), \\
h_{0l,\mathfrak{k}} &= \sqrt{\frac{h}{2\omega d\mathfrak{k}}}\xi^+_{0l,\mathfrak{k}}e^{-i\omega t}.
\end{aligned}$$

The Hamiltonian in the new variables transforms into

$$\left.\begin{aligned}
H = \sum_{\mathfrak{k}} h\omega\, (\xi^+_{00,\mathfrak{k}}\xi_{00,\mathfrak{k}} + \xi_{12,\mathfrak{k}}\xi^+_{12,\mathfrak{k}} + \xi_{23,\mathfrak{k}}\xi^+_{23,\mathfrak{k}} + \xi_{13,\mathfrak{k}}\xi^+_{13,\mathfrak{k}} - \\
-\xi^+_{01,\mathfrak{k}}\xi_{01,\mathfrak{k}} - \xi^+_{02,\mathfrak{k}}\xi_{02,\mathfrak{k}} - \xi^+_{03,\mathfrak{k}}\xi_{03,\mathfrak{k}} - \xi_{11,\mathfrak{k}}\xi^+_{11,\mathfrak{k}} + \\
+\xi^+_{22,\mathfrak{k}}\xi_{22,\mathfrak{k}} + \xi^+_{33,\mathfrak{k}}\xi_{33,\mathfrak{k}}) \,.
\end{aligned}\right\} \qquad (5')$$

Therefore the eigenvalues of the energy (for each value of $\mathfrak{k}$) are

$$h\omega\left(n_{00} + n_{12} + n_{23} + n_{31} - n_{01} - n_{02} - n_{03} - n_{11} + n_{22} + n_{33} + 2\right),$$

where $n_{00},\ n_{12},...$ are ten quantum numbers ($n = 0,\ 1,\ 2,...$).

The conditions (6) make this expression positive definite. To see this, we again consider the case $\mathfrak{k}||z$. From (6) we then obtain the following conditions:

$$\xi_{00,\mathfrak{k}}e^{i\omega t} + \xi^{\dagger}_{03,\mathfrak{k}}e^{-i\omega t} = 0, \qquad \xi^{\dagger}_{01,\mathfrak{k}}e^{-i\omega t} + \xi_{13,\mathfrak{k}}e^{i\omega t} = 0,$$

$$\xi^{\dagger}_{02,\mathfrak{k}}e^{-i\omega t} + \xi_{23,\mathfrak{k}}e^{i\omega t} = 0, \qquad \xi^{\dagger}_{11,\mathfrak{k}}e^{-i\omega t} + \xi_{22,\mathfrak{k}}e^{i\omega t} = 0.$$

From this it follows that

$$n_{01} = n_{31} + 1, \qquad n_{02} = n_{23} + 1, \qquad n_{22} = n_{11} + 1, \qquad n_{03} = n_{00} + 1.$$

The eigenvalues of the energy for this $\mathfrak{k}$ become

$$h\omega\left(\xi^{\dagger}_{12,\mathfrak{k}}\xi_{12,\mathfrak{k}} + \xi^{\dagger}_{33,\mathfrak{k}}\xi_{33,\mathfrak{k}}\right) = h\omega\,(n_{12} + n_{33})\,.$$

We see, consequently, that the energy of the gravitational field consists of positive gravitational quanta, of two polarizations for each wave vector $\mathfrak{k}$. In analogy to the classical case, also here only the transverse gravitational excitations matter: e.g., for $\mathfrak{k}||z$, these are h_{12} and $1/2\,(h_{11} - h_{22})$.

No "zero-point energy terms" arise in the process, due to the suitably chosen factor-ordering in the expression (5).

§ 4. Let us undertake a little gedanken experiment!

In order to understand somewhat better the physical content of the quantum theory of the gravitational field, let us consider the measurement of one of the field quantities appearing here, for example, of the three-index Christoffel symbol [00, 1]. The classical Einstein equations of motion read in our case (all $h_{\mu\nu} \ll 1$):

$$\frac{d^2x}{dt^2} = \frac{\partial h_{01}}{\partial t} - \frac{1}{2}\frac{\partial h_{00}}{\partial x} = [00, 1]. \tag{9}$$

Following Bohr and Rosenfeld [3] let us consider the measurement of a space-time average of [00, 1] in volume V and time interval T. Take a test body of volume V. Let its mass be ρV. The above equation of motion, which is valid only when the speed of the test body is small compared to the speed of light, makes the following measurement possible: let us measure

[3] Bohr N. and Rosenfeld L., Dansk. Vidensk. Selskab., Math.-fys. Meddel. **12**, 8. 1933.

the momentum of the test body at the beginning and at the end of a time interval T; then by definition the required average is

$$\frac{(p_x)_{t+T} - (p_x)_t}{\rho VT}.$$

Therefore the measurement of $[00, 1]$ is associated with an uncertainty of order

$$\Delta[00, 1] \approx \Delta p_x/\rho VT, \tag{10}$$

where Δp_x is the uncertainty in momentum. Let the duration of the momentum measurement be Δt (of course, $\Delta t \ll T$); Δx be the uncertainty in the coordinate associated with measuring the momentum. The uncertainty Δp_x consists of two terms: the usual quantum-mechanical $h/\Delta x$ and one associated with the gravitational field produced by the test body itself because of its recoil due to the measurement. Because of E i n s t e i n 's equation of gravitation $\Box h_{01} = \rho v_x$ the uncertainty in h_{01}, which appears as a consequence of the undetermined recoil speed $\Delta x/\Delta t$, must be of the order $\rho \frac{\Delta x}{\Delta t} \cdot \Delta t^2$. One can see from (9), that the corresponding uncertainty in $[00, 1]$ is of the order $\rho\Delta x$, and therefore during every measurement of momentum there appears an additional uncertainty in momentum connected with the gravitational field, which is of order $\rho\Delta x \cdot \rho V\Delta t$. In order to simplify the comparison with the usual units of measurement we suspend our convention $c = 1, G = 1/16\pi$ (till the end of this section). For the momentum we obtain

$$\Delta p_x \approx \frac{h}{\Delta x} + G\rho^2 V\Delta x\Delta t.$$

It can be shown (similarly to the arguments of B o h r and R o s e n f e l d) that the second term can be made arbitrarily small compared to the first. But in order to make the best measurement of $[00, 1]$ it seems more appropriate to bring Δp_x to its minimum, i.e. to make both terms of the same order. Thus one should choose Δx of the order

$$\Delta x \approx \frac{1}{\rho}\left(\frac{h}{GV\Delta t}\right)^{1/2}.$$

For $\Delta[00, 1]$ we obtain

$$\Delta[00, 1] \gtrsim \frac{h^{1/2}G^{1/2}\Delta t^{1/2}}{V^{1/2}T}. \tag{11}$$

Thus an absolutely precise measurement of the gravitational field would be possible only if an arbitrarily rapid measurement of momentum were possible. But two circumstances make the latter impossible: firstly, according to the definition of the measurement, $\Delta x \ll V^{1/3}$ should hold, and it leads to

$$\Delta t \gg \frac{h}{\rho^2 G V^{5/3}}.$$

Secondly, according to the theory of relativity, Δx can never be greater than $c\Delta t$, and it leads to

$$\Delta t \gtrsim \frac{h^{1/3}}{c^{2/3}\rho^{2/3}V^{1/3}G^{1/3}}.$$

It follows from (11) that $\Delta[00,1]$ can never be made smaller than

$$\frac{h}{\rho T V^{4/3}} \qquad \text{or} \qquad \frac{h^{2/3}G^{1/3}}{c^{1/3}\rho^{1/3}V^{2/3}T}.$$

Of these two bounds the first is the only significant one for the case of a light test body ($\rho V \lesssim h^{1/2}c^{1/2}G^{1/2}$, i.e. smaller than approximately 0.01 mg). For a heavier test body the second one is the most significant. It is clear that for the most precise measurement possible of $[00,1]$ heavy test bodies should be used, so only the second bound is of theoretical importance. We finally have

$$\Delta[00,1] \gtrsim \frac{h^{2/3}G^{1/3}}{c^{1/3}\rho^{1/3}V^{2/3}T}. \tag{12}$$

Thus it is clear that in the region where all $h_{\mu\nu}$ are small compared to 1 (this is just the meaning of the word "weak" in the title of this paper), accuracy of gravity measurements can be increased arbitrarily: as in this domain of phenomena the approximate linearized equations (1) hold, consequently the superposition principle is also valid, and it is therefore always possible to create a test body of arbitrary large ρ. From this we conclude that it is possible, just as, e.g., this paper attempts to do, to construct a completely self-consistent quantum theory of gravity within the framework of special relativity (i.e. when the space-time continuum is "Euclidean"). However, within the domain of General Relativity theory, where deviations from "Euclideanness" can be arbitrary large, the situation is quite different. Indeed the gravitational radius of the test body used for the measurement ($G\rho V/c^2$) cannot be larger than its linear dimensions $\left(V^{1/3}\right)$; this means that the upper bound on its density is ($\rho \lesssim c^2/GV^{2/3}$). Consequently, in this domain the possibilities of measurement are even more restricted than those due to the quantum-mechanical commutation relations. Without a deep revision of classical notions it seems hardly possible to extend the quantum theory of gravity also to this domain.

§5. Interaction with matter

The Ansatz for the energy of interaction between gravitational field and matter that is consistent with the correspondence principle can be obtained from the general relativistic form of the Dirac wave equation established by V. Fock.[4] When all $h_{\mu\nu}$ are small compared to 1, this equation can be written, for vanishing electromagnetic field, in the following form:

$$\frac{h}{i}\sum_{k=0}^{3} e_k a_k \left(\frac{\partial}{\partial x_k} - \frac{1}{2}\sum_{l=0}^{3} e_l h_{kl}\frac{\partial}{\partial x_l}\right)\psi +$$
$$+\left(\frac{1}{8}\frac{h}{i}\sum_{k=0}^{3}\frac{\partial h_{00}}{\partial x_k}\alpha_k - m\beta\right)\psi = 0,$$

where

$$e_0 = 1, \qquad e_1 = e_2 = e_3 = -1$$

and

$$\alpha_0 = 1, \qquad \alpha_1 = \rho_1\sigma_1, \qquad \alpha_2 = \rho_1\sigma_2, \qquad \alpha_3 = \rho_1\sigma_3, \qquad \beta = \rho_3$$

(the Dirac matrices). If we introduce two two-component functions χ and φ instead of the four-component ψ-function:

$$\psi_1 = \chi_1 e^{-imt/h}, \qquad \psi_2 = \chi_2 e^{-imt/h}, \qquad \psi_3 = \varphi_1 e^{-imt/h}, \qquad \psi_4 = \varphi_2 e^{-imt/h},$$

then for decreasing particle velocity, χ goes to zero, and φ tends to its non-relativistic wave function. From the Schrödinger equation for these φ_x one can see that the interaction energy between the particle and the gravitational field takes the form

$$V = \frac{m}{2}h_{00} + \frac{h}{i}\sum_k h_{0k}\frac{\partial}{\partial x_k} - \frac{h^2}{2m}\sum_{kl} h_{kl}\frac{\partial^2}{\partial x_k \partial x_l} - \frac{h^2}{4m}\sum_{kl}\frac{\partial h_{kl}}{\partial x_l}\frac{\partial}{\partial x_k} +$$

[4] V. Fock, ZS. f. Phys., **57**, 261, 1929.

$$+\frac{h}{4i}\sum_{l}\frac{\partial h_{0l}}{\partial x_l}+\frac{h^2}{4mi}\sum_{jklm}\sigma_j e_{jlm}\frac{\partial h_{km}}{\partial x_l}\frac{\partial}{\partial x_k}+\frac{h}{4}\sum_{jlm}\sigma_j e_{jlm}\frac{\partial h_{0m}}{\partial x_l},$$

where e_{jlm} is the skew-symmetric unit tensor (i.e. $e_{123}=1$ and e_{jlm} is anti-symmetric with respect to each pair of its indices). When the wavelength of the gravitational perturbations is sufficiently large, this expression simplifies to

$$V=\frac{m}{2}h_{00}+\frac{h}{i}\sum_{k}h_{0k}\frac{\partial}{\partial x_k}-\frac{h^2}{2m}\sum_{kl}h_{kl}\frac{\partial^2}{\partial x_k\partial x_l}. \qquad (13)$$

We shall use the relation (13) below. Note that even simple considerations based on the correspondence principle, without going through the Dirac-Fock equation, also lead to the relation(13) for the interaction energy.

§ 6. Energy transfer by gravitational waves

One of the simplest applications of the quantum theory of gravity sketched above is in the calculation of the energy radiation into gravitational waves emitted by material systems. Here we make use of the conditions (6) and (6′) together with $h_{00,\mathfrak{k}}=h_{01,\mathfrak{k}}=h_{02,\mathfrak{k}}=h_{03,\mathfrak{k}}=0$. In the ξ-variables (see above, §3) for the case $\mathfrak{k}\|z$ it leads to

$$V=\frac{m}{8\pi}\sqrt{\frac{hd\mathfrak{k}}{\pi\omega}}\left\{\left[\xi\left(\dot{x}_1^2-\dot{x}_2^2\right)+2\eta\dot{x}_1\dot{x}_2\right]e^{i\mathfrak{k}\mathfrak{r}}+\right.$$
$$\left.+\left[\xi^{+}\left(\dot{x}_1^2-\dot{x}_2^2\right)+2\eta^{+}\dot{x}_1\dot{x}_2\right]e^{-i\mathfrak{k}\mathfrak{r}}\right\}, \qquad (14)$$

where for brevity $\dot{x}_k$ is written instead of $\frac{h}{mi}\frac{\partial}{\partial x_k}$, and ξ and η are written instead of ξ_{33} and ξ_{12}. Let us denote the initial or the final state of the emitting particle by k or l, the initial or the final state of the gravitational excitations by k' or l'. For the transition probability per unit time quantum mechanics gives the well-known expression

$$\frac{2\pi}{h}\delta\left(E_l+E_{l'}-E_k-E_{k'}\right)\left|\left(kk'\left|V\right|ll'\right)\right|^2.$$

Using the known values of oscillator matrix elements we conclude that the probability of spontaneous emission of a gravitation quantum per unit time, with a given $\mathfrak{k}$ ($\mathfrak{k}||z$) and with ξ - polarization is

$$\frac{m^2}{32\pi^2}\frac{d\mathfrak{k}}{\omega}\left|\left(k\left|\dot{x}_1^2-\dot{x}_2^2\right|l\right)\right|^2\delta\left(E_l-E_k+h\omega\right),$$

and with η-polarization

$$\frac{m^2}{32\pi^2}\frac{d\mathfrak{k}}{\omega}\left|\left(k\left|2\dot{x}_1\dot{x}_2\right|l\right)\right|^2\delta\left(E_l-E_k+h\omega\right).$$

The probability of the transition $k\to l$ with the simultaneous emission of a gravitation quantum into a cone $d\Omega(\|z)$ per unit time is therefore

$$\frac{d\Omega}{8\pi^2}\frac{m^2\omega}{h}\left\{\left|\left(k\left|\frac{\dot{x}_1^2-\dot{x}_2^2}{2}\right|l\right)\right|^2+\left|\left(k\left|\dot{x}_1\dot{x}_2\right|l\right)\right|^2\right\}.$$

It is not difficult to generalize this expression to arbitrary direction (not only $\|z$) and then to integrate over all directions. The calculation leads to the following result: the total probability (per unit time) of the transition $k\to l$ with simultaneous emission of a gravitation quantum of the frequency $\omega=\left(E_k-E_l\right)/h$ is

$$\frac{m^2\omega}{10\pi h}\left\{\sum_{pq}\left|\left(k\left|\dot{x}_p\dot{x}_q\right|l\right)\right|^2-\frac{1}{3}\left|\left(k\left|\sum_p\dot{x}_p^2\right|l\right)\right|^2\right\}.$$

It is not difficult to generalize this expression also to an arbitrary system of matter particles. For the energy that such a system loses to gravitational waves radiated per unit time during the transition $k\to l$, we obtain the expression

$$\frac{\omega^2}{10\pi}\left\{\sum_{pq}\left|\left(k\left|\sum m\dot{x}_p\dot{x}_q\right|l\right)\right|^2-\frac{1}{3}\left|\left(k\left|\sum_p\sum m\dot{x}_p^2\right|l\right)\right|^2\right\}\tag{15}$$

(the summation symbol $\sum$ without indices means summation over different particles). This formula is the quantum-theoretic generalization of E i n s t e i n 's known result.

Indeed, E i n s t e i n's expression for the energy radiated per unit time in the form of gravitational waves (with $G=1/16\pi$) is[5]

$$\frac{1}{80\pi}\left\{\sum_{pq}\left(\frac{d^3}{dt^3}\sum mx_px_q\right)^2-\frac{1}{3}\left(\sum_p\frac{d^3}{dt^3}\sum mx_p^2\right)^2\right\}.$$

[5]In E i n s t e i n 's work (Berl. Ber., 1918, p. 154) there is $1/160\pi$ instead of $1/80\pi$ due to a calculation error. More recent calculations by E d d i n g t o n (see his textbook or Proc. Roy. Soc. **102**, 281, 1922) lead to the correct coefficient.

If $I_{pq} \equiv \sum m x_p x_q$ can be represented through a Fourier series of the form

$$I_{pq} = \sum_{k=-\infty}^{+\infty} I_{pq}^{(k)} e^{ik\omega_0 t},$$

then this classical expression for the emission of energy at frequency $\omega = k\omega_0$ becomes

$$\frac{\omega^6}{40\pi} \left\{ \sum_{pq} \left| I_{pq}^{(k)} \right|^2 - \frac{1}{3} \left| \sum_p I_{pp}^{(k)} \right|^2 \right\}. \tag{16}$$

On the other hand we have,

$$\left(k \left| \sum |m\dot{x}_p \dot{x}_q| \right| l \right) = \sum \sum_j \left(k \left| \sqrt{m} \dot{x}_p \right| j \right) \left(j \left| \sqrt{m} \dot{x}_q \right| l \right) =$$

$$= -\sum \sum_j m \left(k \left| x_p \right| j \right) \left(j \left| x_q \right| l \right) \omega_{kj} \omega_{jl}.$$

At lower frequencies and higher quantum numbers, as one can easily calculate e.g. for the case of a rotator, we can put ω^2 instead of $2\omega_{kj}\omega_{jl}$ ($\omega =$ the emitted frequency), and it gives approximately

$$\left(k \left| \sum m\dot{x}_p \dot{x}_q \right| l \right) = -\frac{\omega^2}{2} \left(k \left| I_{pq} \right| l \right).$$

Then the expression (15) becomes

$$\frac{\omega^6}{40\pi} \left\{ \sum_{pq} \left| \left(k \left| I_{pq} \right| l \right) \right|^2 - \frac{1}{3} \left| \sum_p \left(k \left| I_{pp} \right| l \right) \right|^2 \right\}.$$

If we replace Fourier amplitudes for matrix elements, this expression goes over into Einstein's (16) classical one. Thus in the limit $h \to 0$ the quantum theory of gravity agrees with classical Einstein theory.

§ 7. Derivation of Newton's law of gravitation

Dirac showed[6] that various interactions between charges can be always interpreted as realized through a mediation by an intermediate agent, namely

[6]Dirac P.A.M.- Proc. Roy. Soc. **136**, 453, 1932.

by the quantized field. Here we will show that is also the case for gravitational phenomena. At first glance, this looks a little paradoxical because both expressions for the interaction between field and matter are almost exactly the same [$e\Phi$ is the main term of interaction in the electromagnetic case; if we write $m/2$ instead of the charge e, and replace the scalar potential Φ with the scalar potential h_{00}, we obtain the main term in (13)]; yet, the very same scheme should in one case explain the repulsion of particles of the same type (Coulomb force), but in the other case the attraction (Newtonian forces). The solution of the paradox lies in the fact that in quantum electrodynamics, the commutation relations for the potential Φ are

$$\left[\Phi_{\mathfrak{k}}^{+}, \Phi_{\mathfrak{k}'}\right] = \frac{h}{2\omega}\delta(\mathfrak{k}-\mathfrak{k}'),$$

while in our case another commutation relation applies, namely (cf. (8))

$$\left[h_{00,\mathfrak{k}}^{+}, h_{00,\mathfrak{k}'}\right] = -\frac{h}{2\omega}\delta(\mathfrak{k}-\mathfrak{k}').$$

Neither commutation relations are introduced ad hoc, but originated quite naturally from the general quantum-mechanical formalism. As we shall see this suffices to obtain the correct sign of the gravitational interactions. Thus, the fundamental difference between Coulomb and Newtonian forces is explained from quantum mechanics.

Following the idea of Dirac, Fock and Podolsky [7] derived Coulomb's law. Our calculation proceeds exactly parallel to theirs. We start from the equations

$$\left(\frac{1}{2m_1}\mathfrak{p}_1{}^2 + \frac{m_1}{2}h_{00}(\mathfrak{r}_1)\right)\psi + \frac{h}{i}\frac{\partial\psi}{\partial t_1} = 0,$$
$$\left(\frac{1}{2m_2}\mathfrak{p}_2{}^2 + \frac{m_2}{2}h_{00}(\mathfrak{r}_2)\right)\psi + \frac{h}{i}\frac{\partial\psi}{\partial t_2} = 0.$$

At $t_1 = t_2 = t$ we have

$$\left(\frac{1}{2m_1}\mathfrak{p}_1{}^2 + \frac{1}{2m_2}\mathfrak{p}_2{}^2 + \frac{h}{i}\frac{\partial}{\partial t}\right)\psi = -\left(\frac{m_1}{2}h_{00}(\mathfrak{r}_1) + \frac{m_2}{2}h_{00}(\mathfrak{r}_2)\right)\psi.$$

[7] V. Fock and B. Podolsky, Sow. Phys. **1**, 801, 1932 (Part II).

The expansion of the solution in the powers of m_1 and m_2 is required.

Due to the above-mentioned difference between the commutation relations, instead of the formulas (39) and (40) of Fock and Podolsky we obtain the following formulas:

$$-\left(\frac{1}{2m_1}\mathfrak{p}_1{}^2+\frac{1}{2m_2}\mathfrak{p}_2{}^2+\frac{h}{i}\frac{\partial}{\partial t}\right)\varphi_2 \sim$$
$$\sim \frac{m_1}{2(2\pi)^{3/2}}\int h_{00,\mathfrak{k}}\varphi_1(\mathfrak{p}_1-h\mathfrak{k},\mathfrak{p}_2)e^{-i\omega t}d\mathfrak{k}+$$
$$+\frac{m_2}{2(2\pi)^{3/2}}\int h_{00,\mathfrak{k}}\varphi_1(\mathfrak{p}_1,\mathfrak{p}_2-h\mathfrak{k})e^{-i\omega t}d\mathfrak{k};$$
$$-\left(\frac{1}{2m_1}\mathfrak{p}_1{}^2+\frac{1}{2m_2}\mathfrak{p}_2{}^2+\frac{h}{i}\frac{\partial}{\partial t}\right)\varphi_1 \sim$$
$$\sim \frac{m_1}{2(2\pi)^{3/2}}\int h^+_{00,\mathfrak{k}}\varphi_0(\mathfrak{p}_1+h\mathfrak{k},\mathfrak{p}_2)e^{i\omega t}d\mathfrak{k}+$$
$$+\frac{m_2}{2(2\pi)^{3/2}}\int h^+_{00,\mathfrak{k}}\varphi_0(\mathfrak{p}_1,\mathfrak{p}_2+h\mathfrak{k})e^{i\omega t}d\mathfrak{k}.$$

Then, [8]

$$\varphi_1(\mathfrak{p}_1,\mathfrak{p}_2)=$$
$$-\frac{m_1}{2(2\pi)^{3/2}h^3}h^+_{00,\frac{\mathfrak{p}_1{}^0-\mathfrak{p}_1}{h}}\frac{\delta(\mathfrak{p}_2-\mathfrak{p}_2{}^0)\delta_{j0}}{W-W_0+|\mathfrak{p}_1{}^0-\mathfrak{p}_1|}e^{i\frac{|\mathfrak{p}_1{}^0-\mathfrak{p}_1|-W_0}{h}t}-$$
$$-\frac{m_2}{2(2\pi)^{3/2}h^3}h^+_{00,\frac{\mathfrak{p}_2{}^0-\mathfrak{p}_2}{h}}\frac{\delta(\mathfrak{p}_1-\mathfrak{p}_1{}^0)\delta_{j0}}{W-W_0+|\mathfrak{p}_2{}^0-\mathfrak{p}_2|}e^{i\frac{|\mathfrak{p}_2{}^0-\mathfrak{p}_2|-W_0}{h}t}.$$

Let us put

$$h^+_{00,\mathfrak{k}}h_{00,\mathfrak{k}'}\sim 0 \qquad \text{and} \qquad h_{00,\mathfrak{k}}h^+_{00,\mathfrak{k}'}\sim\frac{h}{2\omega}\delta(\mathfrak{k}-\mathfrak{k}').$$

After removing the infinite self-interaction term we finally obtain

$$\left(\frac{1}{2m_1}\mathfrak{p}_1{}^2+\frac{1}{2m_2}\mathfrak{p}_2{}^2+\frac{h}{i}\frac{\partial}{\partial t}\right)\sim$$
$$\sim\frac{m_1m_2}{4(2\pi)^{3/2}h}\frac{\delta(\mathfrak{p}_1-\mathfrak{p}_1{}^0+\mathfrak{p}_2-\mathfrak{p}_2{}^0)\delta_{j0}}{|\mathfrak{p}_1-\mathfrak{p}_1{}^0|^2}e^{-\frac{i}{h}W_0t}.$$

[8]For the notations see V. Fock and B. Podolsky, loc. cit.

The sign of the right-hand side is different than in the F o c k - P o d o l s k y formula (42). When we go back to the configuration space we accordingly obtain the S c h r ö d i n g e r equation with the potential energy

$$-\frac{m_1 m_2}{16\pi|\mathfrak{r}_1 - \mathfrak{r}_2|},$$

and thus we have recovered Newtonian gravitation as a necessary consequence of the quantum theory of gravity.

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