Averaged number of visits

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December 17, 2008

Abstract

We introduce a new indicator for dynamical systems, the averaged number of visits, to estimate the frequency of visits in small regions, when a map is iterated up to the inverse of the measure of this region. We compute this quantity analytically and numerically for various systems and we show that it depends on the ergodic properties of the systems and on their topological properties like the presence of periodic points.

In a previous article in *Chaos* [1], we studied the statistics of the distribution of the number of visits in small sets for dynamical systems with weak ergodic properties: irrational rotations, shear maps, standard map. We now pursue this investigation by introducing a new statistical indicator which counts the averaged number of visits (ANV) in small sets when the orbits of the sample points are iterated up to the inverse of the measure of the set, or even further. This expected value could be different from one (and this is already a refinement of Kac's theorem), and it depends on topological local properties (like the presence of periodic points) and on the ergodic properties of the map. We also produce sets which do not have any limiting distribution at all for the number of visits. We will present a complete mathematical description of the ANV in the case of ϕ-mixing systems with a particular emphasis on the returns around periodic points; the ANV will depend smoothly on the period. We will then consider the examples studied in [1], for their importance in the understanding of transport and diffusion phenomena. We will show that the ANV behaves in a different qualitative manner if compared with the ϕ-mixing situations. In particular the lack of ergodicity for the shear map will be reflected in a large standard deviation for the ANV. For the standard map in regular and mixed regions, the ANV and its variance will also manifest a clear departure from the chaotic regimes typical of ϕ-mixing measures. This suggests that it is necessary to pass to the higher moments of the process in order to deal with systems with weak mixing properties and with systems whose phase space consists or orbits with qualitative different behaviors.

1 Introduction

One of the consequences of the ergodicity of a dynamical system is that the frequency of visits of a subset of positive measure is exactly the measure of the set. To be more precise, let $(\mathcal{X}, \mathcal{B}, \mu, T)$

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be a measure-preserving dynamical system, where $\mathcal{X}$ is the space, $\mathcal{B}$ the σ-algebra on it and T a measurable map with preserve the probability measure μ. If we moreover assume that μ is ergodic, then for a given $A \in \mathcal{B}$, $\mu(A) > 0$, we have:

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A \left(T^j(x)\right) = \mu(A) \,,$$

for μ-a.e. x (χ_A which denotes the characteristic function of the set A). We stress that this frequency does not depend on the topological nature of the set A (provided it is of positive measure), neither on other geometrical or statistical properties of the map T (hyperbolicity, mixing, etc.). It would be useful to have another indicator of the frequency of visit which reflects the statistical properties of T more accurately. Something is known on the distribution of the following random variable, called the number of visits of $A_n \in \mathcal{B}$ up to the time $\left[\frac{t}{\mu(A_n)}\right]$ of the trajectory originating from x:

$$\xi_n^E(x,t) \equiv \sum_{j=1}^{\left[\frac{t}{\mu(A_n)}\right]} \chi_{A_n}\left[T^j(x)\right] \,, \qquad \forall x \in \mathcal{X} \,, \tag{1}$$

where A_n is a sequence of measurable sets for which $\mu(A_n) \to 0+$. More precisely: for a wide class of dynamical systems (see below for the precise definitions) and for suitable choices of the sets $\{A_n\}_{n\in\mathbb{N}}$, the existence of the following limit distribution $F_k^E(t)$ as been proven:

$$F_k^E(t) \equiv \lim_{n\to+\infty} \mu\left(\left\{x \in \mathcal{X} \;\middle|\; \xi_n^E(x,t) = k\right\}\right) \,. \tag{2}$$

Similarly, if the point x starts out in A_n then $\xi_n^R = \xi_n^E\,|_{A_n}$ counts the returns to A_n on the orbitsegment of length $[\frac{t}{\mu(A_n)}]$. In this case, and under the same conditions as above, one obtains:

$$F_k^R(t) \equiv \lim_{n\to+\infty} \mu_n\left(\left\{x \in A_n \;\middle|\; \xi_n^R(x,t) = k\right\}\right) \,, \tag{3}$$

where $\mu_n = \frac{\mu|_{A_n}}{\mu(A_n)}$ is the conditional measure on A_n. In the following we will refer to the quantities $F_k^E(t)$ as the distribution of the number of visits for entry times and to $F_k^R(t)$ as the distribution of the number of visits for return times (into the sets A_n). In the following we also put $\xi_A^{E,R}(x,t)$ to indicate the number of visits of any measurable set A of positive measure:

$$\xi_A^{E,R}(x,t) \equiv \sum_{j=1}^{\left[\frac{t}{\mu(A)}\right]} \chi_A\left[T^j(x)\right] \,. \tag{4}$$

The existence of the limit (2) and (3) means that, for $* = E, R$, the random variables $\xi_n^*(\cdot,t)$ converge in distribution toward a random variable whose distribution is given by the $F_k^*(t)$. We associate with them the expectations:

$$\mathcal{M}^*(t) \equiv \sum_{k=1}^{+\infty} k F_k^*(t) \qquad * = E, R \,, \tag{5}$$

and, whenever $\mathcal{M}^*(t) < \infty$, the relative variances:

$$\mathcal{V}^*(t) \equiv \sum_{k=0}^{+\infty} \left[k - \mathcal{M}^*(t)\right]^2 F_k^*(t) \qquad * = E, R \,. \tag{6}$$

For physical reasons (which will become clear in a moment) we also consider the average values at $t = 1$:

$$M^* \equiv \mathcal{M}^*(1) \equiv \sum_{k=1}^{+\infty} k F_k^*(1) \qquad * = E, R \,, \tag{7}$$

and, correspondingly , $V^E \equiv \mathcal{V}^E(1)$, $V^R \equiv \mathcal{V}^R(1)$. The statistical and physical interpretations of (7) are quite interesting and they are the main object of this paper. M^* denotes **the averaged number of visits (ANV) to the sets A_n when the sample initial points are iterated up to time $\mu(A_n)^{-1}$ in the limit** $\mu(A_n) \to 0+$. We will also estimate the deviation with respect to this mean value by computing its relative variance. In the last Section of the paper (numerical computations) we will use the abbreviation ANV in a more general setting to indicate the expectations $\mathcal{M}^*(t)$, for any positive t.

In practice it is easier to compute first the expectations of $\xi_n^*(\cdot, t)$ and then to take the limit for $n \to +\infty$ afterwards. Suppose therefore that, for each t, the following limits exist:

$$\overline{\mathcal{M}^E}(t) \equiv \lim_{n\to+\infty} \mathbb{E}_\mu \left[\xi_n^E(x,t)\right] = \lim_{n\to+\infty} \int_{\mathcal{X}} \xi_n^E(x,t)\, d\mu(x) \,, \tag{8}$$

$$\overline{\mathcal{M}^R}(t) \equiv \lim_{n\to+\infty} \mathbb{E}_{\mu_n} \left[\xi_n^R(x,t)\right] = \lim_{n\to+\infty} \int_{A_n} \xi_n^R(x,t)\, d\mu_n(x) \,, \tag{9}$$

where again $\overline{M^E} \equiv \overline{\mathcal{M}^E}(1)$ and $\overline{M^R} \equiv \overline{\mathcal{M}^R}(1)$. In general $\mathcal{M}^*(t) \le \overline{\mathcal{M}^*}(t)$ and the equality holds whenever the random variables $\xi_n^*(\cdot, t)$ are uniformly integrable. We will see that a sufficient condition for the uniform integrability of these processes is that the measure μ is $\phi-$mixing (see Sect. 2.2 for a precise definition of this property). The quantities (5) and (8,9) will be computed analytically and numerically for various dynamical systems. We will see that the ANV will depend crucially on two properties of the system:
(i) the “location” of the set A_n; for instance for different points x, at which the return sets A_n are centered, one typically gets different values for the ANV; in particular, periodic points have a limiting statistics which differs from generic points (generic w.r.t. the invariant measure);
(ii) the ANV depends on the ergodic properties of the measure μ; for strongly mixing systems this value is usually 1 at generic points but, for instance, is less than 1 for rotations.
In these respects, the ANV will deserve the role of the frequency of visits indicator that we addressed at the beginning of this Introduction.

Let us now discuss the characteristics of this indicator.

a) Kac’s theorem states that for an ergodic measure μ and a measurable set A of positive measure the expectation of the first return $\tau_A(x) \equiv \inf\{n \in \mathbb{Z}^+ \mid T^n(x) \in A\}$ to the set A satisfies

$$\int_A \tau_A(x)\, d\mu_A(x) = \frac{1}{\mu(A)} \,.$$

Hence we can say that on average $x \in A$ returns to A for the first time after time $[\frac{1}{\mu(A)}]$, or, equivalently we could say that the averaged number of visits to A is 1 on a time interval of length $[\frac{1}{\mu(A)}]$. However the ANV is a finer measure of the return time since it involves the limit of $\mu(A_n) \to 0$ as $n \to \infty$ and the value 1 is only recovered if the measure has some strong mixing properties and then only for generic points (periodic points yield different values). Moreover and contrary to Kac’s theorem, we will explicitly compute the variance as a measure of the deviation with respect to the expected value. The ANV is a local refinement of Kac’s result.

b) Let us now consider more precisely the quantities which are upper bounds for the expectations M^*. As pointed out, upper bounds $\overline{M^*}$ of the expectations M^* are much easier to compute analytically and numerically. They are also of intrinsic interest. For instance, if we take, as usual, a decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets centered at $z = \bigcap_n A_n$, then we could see $\overline{M^R}$ as a function of $z \in \mathcal{X}$, $\overline{M^R}(z)$, whose expression is:

$$\overline{M^R}(z) = \lim_{n\to+\infty} \sum_{k=1}^{\left[\frac{1}{\mu(A_n(z))}\right]} \frac{\mu\left[A_n(z) \cap T^k(A_n(z))\right]}{\mu(A_n(z))} \,,$$

(provided the limit exists). Since $\overline{M^R}(z)$ is not a constant (as it could be different for generic and periodic points), it would be interesting to characterise its level sets $\mathcal{X}_\lambda = \left\{z \in \mathcal{X} : \ \overline{M^R}(z) = \lambda\right\}$

and determine its geometric properties (e.g. its Hausdorff dimension). This is particularly appealing since we will show in Sect. 2.4 that there are points (let's call them exceptional) which do not have a limiting return time distribution at all.

c) We will see that systems with different ergodic properties will give the same dependence on t of the expectations $\mathcal{M}^*(t)$. Those differences will be captured by studying higher moments of the random variables $\xi_A^*(\cdot, t)$. Here we restrict to their variances. Strong mixing properties imply that the variances as well as the expectations are linear in t (when the sequence of sets A_n is taken around generic points), since in this case the distribution of the number of visits is Poissonian. We will give some examples of systems which are far from mixing and for which the expectations are linear in t and the variances are either linear in t (shear flow in Sect. 3.2), or obey a power law behavior in t (standard map in Sect. 3.3). In the former case the departure from the mixing behavior will be manifested by the extremely large standard deviation when compared with the expectation computed around periodic points of small period. In the latter case the non-linearity of the variance is probably due to the presence of orbits whose total measure is small, but whose long term dynamics is very regular. This is achieved by relatively rare orbits which jump between different chaotic configurations. We think that these fine local features of the orbits, which are closely related to the transport properties of the system, will be better depicted by introducing the moments:

$$\mathbb{E}_{\mu_A}\left(\left|\xi_A^R(x,t) - \mathcal{M}^R(t)\right|^q\right), \tag{10}$$

where μ_A is the conditional measure on A. We guess that in the limit $\mu(A) \to 0+$ and for large t, the expectation (10) will scale as $t^{f(q)}$, where the scaling exponent $f(q)$ will reflect the underlying characteristics of the dynamics. In particular when the system is not sufficiently mixing we believe that the function $f(q)$ becomes non linear.

As pointed out above the ANV could enlighten the transport features of a systems. In fact the ANV has a few precursors that have been used to understand the anomalous transport for maps of physical relevance: standard map, web map, billiards (see [2] for a complete and detailed account of these applications). The quantity that has been widely studied is the so-called *Poincaré recurrence distribution* which is the density function $p^R(t)$ of the probability distribution $1 - F_0^R(t)$ (we assume that the target set A has small measure): $1 - F_0^R(t) = \int_0^t p^R(s)ds$. As we will see in the next Section (see Remark after Proposition 3), $p^R(t)$ is the probability density function of the distribution of the first return time into a set A of vanishing measure, and this quantity has been extensively studied by several authors, see for instance [3, 4, 5, 6, 7, 8] and references therein. For systems with strong mixing properties, $p^R(t)$ decays exponentially in t, while it obeys a power law decay in situations where regular and chaotic motions coexist or the map is not ergodic (e.g. the shear flow which has rigorously been studied in [9]). In the former case, typical of the standard map, it has been argued that the power law decay of $p^R(t)$ "is a result of a non-uniformity of phase space, and can be conjectured as a universal phenomenon for typical Hamiltonian systems with a stochastic sea and islands" ([2], p. 472). In [10, 11], following the approach of [12], the Poincaré recurrence was studied by introducing a (multi)-fractal time scale which allowed to fit the power-tail of the distribution.

By considering the successive return times, and not only the first one, we are naturally lead to investigate the distributions $F_k^R(t)$ of the number of visits. This was achieved in [1] where we show that the functional dependence on t of such a distribution is again sensitive to the ergodic properties of the map: it is Poissonian for mixing transformations and it follows a power law decay for the shear and the standard maps.

The ANV and its variance are the first two moments of the limiting process $\xi_A^R(\cdot, t)$ (as the measure of A goes to 0) and its distribution is given by the functions $F_k^R(t)$. To summarize, the rich statistical properties of these two moments appear at two level:
(i) *locally* at the level of *topological dynamics*, since they are able to discriminate among different target sets A around generic, periodic or exceptional points and this for systems with different kinds of ergodic behavior;
(ii)*globally* at the level of *measurable dynamics*, since the dependence of $\mathcal{M}^R(t)$ and $\mathcal{V}^R(t)$ on the

parameter t reflects the ergodic properties of the systems and is sensitive to the different kinds of motions, chaotic or regular, as we will show on concrete examples in Sec. 3.

Of course these two approaches could be refined by passing to higher moments. This will be the topic of a forthcoming paper.

In the next Section we will provide new analytic results on the distributions (2,3). In the last chapter, we will compute the expectations (7) and (8,9) and the relative variances in some interesting cases where rigorous estimations of the distributions of the number of visits are still missing.

2 Analytic results

2.1 Relations between entry and return time distributions

We begin by establishing a general relation, for measurable dynamical systems, between the two distributions $F_k^E(t)$ and $F_k^R(t)$. For this we introduce the first entrance time of the point $x \in \mathcal{X}$ into the measurable set A by:

$$\tau_A^E(x) \equiv \min\left(\left\{k \geq 1 \;\middle|\; T^k(x) \in A\right\} \cup \{+\infty\}\right) , \tag{11}$$

and analogously the k-th entrance time which is defined recursively:

$$\tau_{k,A}^E(x) \equiv \begin{cases} \tau_A^E(x) & , \text{if } k = 1 \\ \tau_{k-1,A}^E(x) + \tau_A^E\left(T^{\tau_{k-1,A}^E(x)}(x)\right) & , \text{if } k \geq 2 \end{cases} . \tag{12}$$

This definition can be generalized to any $k \geq 0$ by consistently put $\tau_{0,A}^E(x) = 0$. If we restrict the choice of the initial point x to the set A, we get respectively the first return time $\tau_A^R(x)$ and the k-th return time $\tau_{k,A}^R(x)$ of the point x into A. The first return time is μ-a.e. finite by Poincaré's recurrence theorem (since μ is invariant). We now introduce the intermediate distributions for $k \geq 0$:

$$F_{k,A}^E(t) \equiv \mu\left(\left\{x \in \mathcal{X} \;\middle|\; \xi_A^E(x,t) = k\right\}\right) , \tag{13}$$

$$F_{k,A}^R(t) \equiv \mu_A\left(\left\{x \in A \;\middle|\; \xi_A^R(x,t) = k\right\}\right) , \tag{14}$$

where μ_A is the conditional measure on the set A. We also introduce the intermediate distributions for the k-th entrance and return time and for any $k \geq 0$:

$$G_{k,A}^E(t) \equiv \mu\left(\left\{x \in \mathcal{X} \;\middle|\; \tau_{k,A}^E \mu(A) \leq t\right\}\right) , \tag{15}$$

$$G_{k,A}^R(t) = \mu_A\left(\left\{x \in A \;\middle|\; \tau_{k,A}^R \mu(A) \leq t\right\}\right) , \tag{16}$$

with $G_{0,A}^*(t) = 1$, *=E,R, and where the normalization of the k-th time by the measure of the set follows from Kac's theorem (see the Introduction: we are still supposing that μ is ergodic).

It is straightforward to check that the invariance of μ and the definition (12) of the successive return times allows us to establish the following equations among the preceding intermediate distributions for $k \geq 0$:

$$F_{k,A}^*(t) = G_{k,A}^*(t) - G_{k+1,A}^*(t) , \qquad * = E, R . \tag{17}$$

A recent result [13] shows that the asymptotic distribution of the successive entrance and return times (15, 16) are related under suitable conditions. This result generalizes a previous contribution by [14], which holds for $k = 1$ and links the distributions of the first entrance and return times.

Theorem 1 ([13]) *Let $\{A_n\}_{n\geq 1}$ be a sequence of measurable subset of $\mathcal{X}$ of positive measure such that $\mu(A_n) \to 0+$ as $n \to +\infty$. If two of the three sequences $\left\{G_{k-1,A_n}^R(t)\right\}_{n\geq 1}$, $\left\{G_{k,A_n}^R(t)\right\}_{n\geq 1}$, $\left\{G_{k,A_n}^E(t)\right\}_{n\geq 1}$, $k \geq 0$, converge weakly, then the third one converges too and the asymptotic distributions satisfy the identity:*

$$\int_0^t \left[G_{k-1}^R(s) - G_k^R(s)\right] \, ds = G_k^E(t) , \qquad \forall\, t \geq 0 . \tag{18}$$

As an immediate corollary to this theorem we get a similar relation for the distributions of the number of visits for entrance an return times.

Proposition 2 *Under the assumption of the previous theorem and with the additional requirement that the distributions* $\left\{G^R_{k+1,A_n}(t)\right\}_{n\geq 1}$*, and* $\left\{G^E_{k+1,A_n}(t)\right\}_{n\geq 1}$ *converge weakly, we have:*

$$\int_0^t \left[F^R_{k-1}(s) - F^R_k(s)\right]\, ds = F^E_k(t)\,, \qquad \forall\, t \geq 0\,. \tag{19}$$

Proof. The additional assumptions of our proposition give an equality like (18) with k replaced by $k+1$. Subtracting both, term by term, we get:

$$\int_0^t \left\{\left[G^R_{k-1}(s) - G^R_k(s)\right] - \left[G^R_k(s) - G^R_{k+1}(s)\right]\right\}\, ds = G^E_k(t) - G^E_{k+1}(t) \qquad \forall\, t \geq 0\,. \tag{20}$$

Now observe that the following limits exist thanks to (17) and the existence of the limiting distributions prescribed by our hypotheses:

$$F^E_k(t) \equiv \lim_{n\to+\infty} F^E_{k,A_n}(t) = \lim_{n\to+\infty} G^E_{k,A_n}(t) - \lim_{n\to+\infty} G^E_{k+1,A_n}(t)$$

which is the r.h.s. of (19). The same argument applies to the other two differences in the integral (20), producing the term $F^R_{k-1}(s) - F^R_k(s)$, i.e. the l.h.s. of (19), and therefore proving the proposition.

Remark: Formula (19) immediately implies that the limiting distributions $F^E_k(t)$ are continuous as a function of t, while nothing is *a priori* known about the regularity of the $F^R_k(t)$.
Also note that the Poisson distribution $\frac{t^k e^{-t}}{k!}$ is a fixed point for (19).

2.2 Example: ϕ-mixing systems

Formula (19) can be tested on an important class of dynamical systems, the ϕ-mixing systems; we recall here the definition. Let $\mathcal{A}$ be a finite measurable partition of $\mathcal{X}$ and denote by $\mathcal{A}^n \equiv \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ its n-th join which is also a measurable partition of $\mathcal{X}$ for every $n \geq 1$. We call the atoms of $\mathcal{A}^n$ n-cylinders; we also put $\mathcal{A}^* \equiv \cup_{n\geq 1}\mathcal{A}^n$ for the collection of all cylinders in $\mathcal{X}$ and put $|A| \equiv n$ if $A \in \mathcal{A}^n$. We will assume that the partition $\mathcal{A}$ is generating (i.e. the atoms of $\mathcal{A}^\infty$ are single points in $\mathcal{X}$).

Let $\mathcal{B}$ is the σ-algebra generated by $\mathcal{A}^*$. We then call the dynamical system $(\mathcal{X}, \mathcal{B}, \mu, T)$ *ϕ-mixing* if there exists a function ϕ which decreases monotonically to zero such that:

$$\left|\mu\left(U \cap T^{-m-n}(V)\right) - \mu(U)\mu(V)\right| \leq \phi(m)\mu(U)\mu(V)\,,$$

for all $m, n \geq 0$, where U is a union of n-cylinders and $V \in \mathcal{B}$. This property implies in particular that cylinder sets have exponentially decreasing measure: $\mu(A) \leq C\lambda^n$, for all $A \in \mathcal{A}^n$, $n \geq 1$, and for some $\lambda \in]0,1[$ and some positive constant C [4, 5].

For ϕ-mixing systems the asymptotic distributions for the number of visits for entry and return times are known in two situations. For $x \in \mathcal{X}$ denote by $A_n(x)$ the n-cylinder which contains x. Clearly, since $\mathcal{A}$ is generating we get that $A_{n+1}(x) \subset A_n(x)$ and $x = \bigcap_n A_n(x)$. For μ-a.e. x it has been shown that:

$$F^E_k(t) = F^R_k(t) = \frac{t^k e^{-t}}{k!}\,, \tag{21}$$

that is F^*_k is Poisson distributed. (Note that the mean and the variance for the Poisson distribution (21) are equal to t which implies in particular that the averaged number of visits M^* as well as the variances V^* are equal to 1.) This result has been proved in [6] and in [7] for the particular case of Axiom-A systems endowed with Gibbs measure.

Now let x be a periodic point of (least) period m and suppose that the following limit exists:

$$p \equiv \lim_{n\to+\infty} \frac{\mu\left(A_{n+m}(x)\right)}{\mu\left(A_n(x)\right)}\,, \tag{22}$$

and it is strictly smaller than 1. In [15] it is argued that $p = \lim_{l\to\infty} \frac{1}{l} \log(\mu(A_{lm}(x))$ and this limit is strictly less than one. For longer prime periods the value of p gets closer to 0: this is particularly evident when p can be explicitly computed as in formula (27).
Then the distributions for the entry and return number of visits converge to the compound Poisson distribution of the following type [15](we give at the end of this Section an heuristic explanation of these results):

$$F_k^E(t) = \begin{cases} e^{-t(1-p)} & \text{, if } k = 0 \\ e^{-t(1-p)} \sum_{j=1}^{k} p^{k-j}(1-p)^j \frac{t^j(1-p)^j}{j!} \begin{pmatrix} k-1 \\ j-1 \end{pmatrix} & \text{, if } k \geq 1 \end{cases} , \tag{23}$$

$$F_k^R(t) = e^{-t(1-p)} \sum_{j=0}^{k} p^{k-j}(1-p)^{j+1} \frac{t^j(1-p)^j}{j!} \begin{pmatrix} k \\ j \end{pmatrix} , \qquad \forall k \geq 0 . \tag{24}$$

For the compound Poisson distribution the expectation and the variance of the random variable with distribution (23) (entry times) are:

$$\mathcal{M}^E(t) = t \, , \quad \mathcal{V}^E(t) = \frac{1+p}{1-p} \cdot t \, . \tag{25}$$

For a random variable with distribution (24) (return times) the expectation and the variance are:

$$\mathcal{M}^R(t) = t + \frac{p}{1-p} \, , \quad \mathcal{V}^R(t) = \frac{1+p}{1-p} \cdot t + \frac{p}{(1-p)^2} \, . \tag{26}$$

These expressions can be deduced from the generating functions of the two distributions (23) and (24), which are [15]:

- for $F_k^E(t)$:
$$g^E(z) = e^{-t(1-p)\frac{z-1}{1-pz}} ,$$
- for $F_k^R(t)$:
$$g^R(z) = \frac{1-p}{1-pz} e^{-t(1-p)\frac{z-1}{1-pz}} .$$

The averaged number of visits for entry and return times follows from the preceding expressions computed at $t = 1$, precisely: for entry times $M^E = 1$, $V^E = \frac{1+p}{1-p}$; for return times $M^R = 1 + \frac{p}{1-p}$, $V^R = \frac{1+p}{1-p} + \frac{p}{(1-p)^2}$.
It is easy to check that for return times $\sqrt{V^R} \leq \sqrt{\frac{5}{4}} M^R$, for $0 \leq p < 1$ and therefore the ANV for return is a pretty good statistical indicator. This is not the case for the expectation $M^E (= 1)$ whose variance could jump to infinity whenever p approaches 1. This could easily happen, for example, for a two-state Bernoulli shift with a state, say 0, carrying a weight p very close to 1 and by computing the expectation V^E around the fixed point 0^∞, see also Sect. 2.4 below. Later on we will give another result, Proposition 4, which shows the interest of focusing on the expectations for return times. For this reason the numerical investigations of the last chapter will be done mostly in this direction.

The limit (22), defining p, can be established in some circumstances, for example when μ is the Gibbs measure for a Hölder continuous potential f on an Axiom-A space; then:

$$p = \exp \left\{ \sum_{j=0}^{m-1} f\left[T^j(x)\right] \right\} , \tag{27}$$

where x is a periodic point of minimal period m.

We now present a heuristic argument (which we already used in [1]) that allows to understand the validity of formulas (23) and (24). It was shown in Sect. 2.1 that the distribution of the number of return times $F^R_{k,A}(t)$ satisfies $F^R_{k,A}(t) = G^R_{k,A}(t) - G^R_{k+1,A}(t)$, where $G^R_{k,A}(t) = \mu_A \left(\left\{ x \in A \;\middle|\; \tau^R_{k,A}\mu(A) \le t \right\} \right)$. We then also observe that the difference between two consecutive return times satisfies a similar relation, since by (12) $\tau^R_{k+1,A} - \tau^R_{k,A} = \tau^R_A \circ T^{\tau^R_{k,A}}$ and the measure μ_A is invariant with respect to the induced transformation on A. Moreover $\tau^R_{k,A} = \tau^R_A + (\tau^R_{2,A} - \tau^R_{1,A}) + \cdots + (\tau^R_{k,A} - \tau^R_{k-1,A})$. Now, if we assume that the random variables $\tau^R_A, (\tau^R_{2,A} - \tau^R_{1,A}), \ldots, (\tau^R_{k,A} - \tau^R_{k-1,A})$ (properly multiplied by the measure of A) were i.i.d. with the same distribution function $G^R_{1,A}(t)$, then, as is well known, the distribution function $G^R_{k,A}(t)$ of their sum $\mu(A)\tau^R_{k,A}$ is the convolution product $C^k_A(t) = G^R_{1,A}(t) * G^R_{1,A}(t) * \cdots * G^R_{1,A}(t)$ (k-times). In the case of ϕ-mixing measure, the preceding random variables become independent whenever A is chosen as a nested sequence of cylinders $A_n(x)$ centred at a periodic point x and then letting n go to infinity. Then, if ϕ decays fast enough, $F^R_k(t) = C^k(t) - C^{k+1}(t)$ is the limiting distribution of the first return time, where $C^k(t) = G^R_1(t) * G^R_1(t) ... * G^R_1(t)$ (k-times) and $G^R_1(t) = \lim_{n\to+\infty} \mu_n \left(\left\{ x \in A_n \;\middle|\; \tau^R_{A_n}(x)\mu(A_n) \le t \right\} \right)$. In the case of a Gibbs measure for a Hölder continuous potential f on an Axiom-A space, Hirata [7] showed that at a periodic point of minimal period m, $G^R_1(t) = 1 - (1-p)e^{-t(1-p)}$, where p is given by formula (27). Convoluting this function k times yields formula (24) for $F^R_k(t)$ (see also [16]). If we replace $G^E_1(t)$ by $G^R_1(t)$ and use the identity $G^E_1(t) = \int_0^t (1 - G^R_1(s))ds$ [14] (i.e. $G^E_1(t) = 1 - e^{-t(1-p)}$) then we obtain, by the same argument, equation (23), the limiting distribution of the entry number of visits. Obviously the independence of the successive return times is attained only asymptotically. A rigorous proof for general ϕ-mixing measures is given in [15], where we also give the rates of convergence (which is needed in Sect. 2.3 to get the uniform integrability of our processes).

2.3 Results about mean values

Let us now return to the general properties of the averaged number of visits. Proposition 2 suggests a link between the expectations of the two distributions F^E_k and F^R_k. Hence:

Proposition 3 *In addition to the assumptions of Proposition 2 let us assume that:*

$$\lim_{k\to+\infty} \int_0^1 kF^R_k(s)\, ds = 0 \,.$$

Then:

$$M^E \equiv \sum_{k=1}^{+\infty} kF^E_k(1) = \sum_{k=0}^{+\infty} \int_0^1 F^R_k(s)\, ds \,.$$

Proof. Rewriting (19) yields as:

$$0 = F^E_k(t) + \int_0^t F^R_k(s)\, ds - \int_0^t F^R_{k-1}(s)\, ds \,, \qquad \forall\, t \ge 0 \,. \tag{28}$$

Fix a $K \in \mathbb{Z}^+$, multiply the last identity by k and take the sum of over k. We obtain:

$$0 = \sum_{k=1}^{K} kF^E_k(t) + \sum_{k=1}^{K} k \int_0^t F^R_k(s)\, ds - \sum_{k=1}^{K} k \int_0^t F^R_{k-1}(s)\, ds \,, \tag{29}$$

which becomes:

$$0 = \sum_{k=1}^{K} kF^E_k(t) - \sum_{k=0}^{K-1} \int_0^t F^R_k(s)\, ds + K \int_0^t F^R_K(s)\, ds \,.$$

Taking the limit $K \to +\infty$ proves the proposition.

Remark: The distribution $F^R_0(t)$ in (28) is simply the limiting statistics of the first return into the sets $\{A_n\}_n$ when $n \to +\infty$, namely:

$$F^R_0(t) = \lim_{n\to+\infty} \mu_n \left(\left\{ x \in A_n \;\middle|\; \tau^R_{A_n}(x)\mu(A_n) > t \right\} \right) \,.$$

Moreover, a statement analogous to Proposition 3 is valid for $\mathcal{M}^E(t)$, $t \in \mathbb{R}^+$.

Let us now consider a sequence of sets $\{A_n\}_n$ and suppose that the processes $\left(\xi_n^E(\cdot, 1); \mu\right)$ and $\left(\xi_n^R(\cdot, 1); \mu_n\right)$ converge in distribution to the random variables ψ^E and ψ^R, respectively with distributions $F_k^E(1)$ and $F_k^R(1)$. As pointed out in the Introduction one has [17]:

$$M^E \equiv \mathbb{E}(\psi^E) \equiv \sum_{k=1}^{+\infty} k F_k^E(1) \leq \liminf_{n\to+\infty} \mathbb{E}_\mu \left[\xi_n^E(x,1)\right] \ , \tag{30}$$

$$M^R \equiv \mathbb{E}(\psi^R) \equiv \sum_{k=1}^{+\infty} k F_k^R(1) \leq \liminf_{n\to+\infty} \mathbb{E}_{\mu_n} \left[\xi_n^R(x,1)\right] \tag{31}$$

(note that the expectations on the l.h.s. are defined by their distributions). The equalities in (30, 31) hold whenever the processes are uniformly integrable, which means [17]:

$$\lim_{\alpha\to+\infty} \sup_{n\geq 0} \int_{|\xi_n^E(x,1)|\geq\alpha} \left|\xi_n^E(x,1)\right| \, d\mu(x) \ = \ 0 \, ,$$

and similarly for $\left(\xi_n^R(\cdot, 1); \mu_n\right)$.

Before exploring this condition we can say something about the limits on the r.h.s. of (30, 31). For entry times we will get a simple result under the only condition of stationarity of the measure. Instead for return times we need much stronger assumptions like ϕ-mixing.

Proposition 4 *a) If $(\mathcal{X}, \mathcal{B}, \mu, T)$ is a measure preserving dynamical system then:*

$$\overline{M^E} = \lim_{n\to+\infty} \mathbb{E}_\mu \left[\xi_n^E(x,1)\right] = 1 \, ,$$

whenever $\mu(A_n) > 0$, for all $n \geq 1$.

b) If $(\mathcal{X}, \mathcal{B}, \mu, T)$ is measure preserving and ϕ-mixing, then:

$$\limsup_{n\to+\infty} \mathbb{E}_{\mu_n} \left[\xi_n^R(x,1)\right] \leq 1 + C\left[1+\phi(0)\right] \frac{\lambda}{1-\lambda} \, ,$$

where $\lambda \in]0,1[$ and $C > 0$ are given by the uniform rate of decay of cylinders: $\mu(A_n) \leq C\lambda^n$.

c) If $(\mathcal{X}, \mathcal{B}, \mu, T)$ is measure preserving and ϕ-mixing and $A_n = A_n(z)$ are the n-cylinders at z, then for μ-a.e. z:

$$\limsup_{n\to+\infty} \mathbb{E}_{\mu_n} \left[\xi_n^R(x,1)\right] \leq 1 \, .$$

Proof. Let us put $\upsilon_n = \left[\frac{1}{\mu(A_n)}\right]$. Then

a) By definition (1) we have:

$$\mathbb{E}_\mu \left[\xi_n^E(x,1)\right] = \sum_{k=1}^{\upsilon_n} \int_{\mathcal{X}} \chi_{A_n}\left[T^k(x)\right] \, d\mu(x) = \mu(A_n)\upsilon_n \, , \tag{32}$$

from which the result follows in the limit $n \to +\infty$.

b) Analogously we obtain

$$\mathbb{E}_{\mu_n} \left[\xi_n^R(x,1)\right] = \sum_{k=1}^{\upsilon_n} \frac{\mu\left[A_n \cap T^{-k}(A_n)\right]}{\mu(A_n)} \, .$$

Recall that $\mu(A_n) \leq C\lambda^n$, $\lambda < 1$, $C > 0$ and split the preceding sum into two parts:

$$\mathbb{E}_{\mu_n} \left[\xi_n^R(x,1)\right] = \sum_{k=1}^{n-1} \frac{\mu\left[A_n \cap T^{-k}(A_n)\right]}{\mu(A_n)} + \sum_{k=n}^{\upsilon_n} \frac{\mu\left[A_n \cap T^{-k}(A_n)\right]}{\mu(A_n)} \, . \tag{33}$$

If $A_n \equiv A_n(z)$ for some $z \in \mathcal{X}$, then for $k = 1, 2, \cdots, n-1$ we can write:

$$A_n(z) = A_k(z) \cap T^{-k}\left\{A_{n-k}\left[T^k(z)\right]\right\} \ .$$

For the first sum on the r.h.s. of (33) we therefore get

$$\sum_{k=1}^{n-1} \frac{\mu\left[A_n \cap T^{-k}(A_n)\right]}{\mu(A_n)} = \sum_{k=1}^{n-1} \frac{\mu\left\{A_k(z) \cap T^{-k}\left[A_{n-k}\left(T^k(z)\right) \cap A_n(z)\right]\right\}}{\mu\left[A_n(z)\right]} \leq$$

$$\leq \left[1+\phi(0)\right] \sum_{k=1}^{n-1} \frac{\mu\left[A_k(z)\right] \mu\left\{A_{n-k}\left[T^k(z)\right] \cap A_n(z)\right\}}{\mu\left[A_n(z)\right]} \leq$$

$$\leq \left[1+\phi(0)\right] \sum_{k=1}^{+\infty} \frac{\mu\left[A_k(z)\right] \mu\left[A_n(z)\right]}{\mu\left[A_n(z)\right]} \leq \left[1+\phi(0)\right] C \sum_{k=1}^{+\infty} \lambda^k \leq C\left[1+\phi(0)\right] \frac{\lambda}{1-\lambda} \ , \qquad (34)$$

where in first inequality we used the ϕ-mixing condition with $m = 0$. The second term on the r.h.s. of (33) is estimated as follows:

$$\sum_{k=n}^{\upsilon_n} \frac{\mu(A_n \cap T^{-k}(A_n))}{\mu(A_n)} = \sum_{j=0}^{\upsilon_n - n} \frac{\mu(A_n \cap T^{-n-j}(A_n))}{\mu(A_n)} \leq$$

$$\leq \sum_{j=0}^{\upsilon_n - n} \frac{(1+\phi(j))\mu(A_n)^2}{\mu(A_n)} \leq \mu(A_n) \sum_{j=0}^{\upsilon_n} (1+\phi(j)) \ \to \ 1$$

as $n \to \infty$ since ϕ is monotonically decreasing to zero. This gives the desired result.

c) Since a ϕ-mixing system has positive entropy, we can use the result from [18] which says that:

$$\liminf_{n\to+\infty} \frac{\min\left\{\tau_{A_n(z)}(x) \ | \ x \in A_n(z)\right\}}{n} \geq 1 \ , \qquad \text{for } \mu\text{-a.e. } z \in \mathcal{X} \ . \qquad (35)$$

Therefore in this situation the first term on the r.h.s. of equality (33) is zero, and we only have to estimate the second term which was done in (b).

Remark: Because of (30) and a), we see that $M^E < +\infty$. However the existence of M^R is guaranteed for ϕ-mixing systems (from (31) and b)), in which case it is finite.
The preceding proposition works as well if we consider the processes $\xi_n^*(\cdot, t)$ instead of $\xi_n^*(\cdot, 1)$. Indeed for entry times we get:

$$\lim_{n\to+\infty} \mathbb{E}_\mu\left[\xi_n^E(x,t)\right] = t \ ,$$

and for return times:

$$\limsup_{n\to+\infty} \mathbb{E}_{\mu_n}\left[\xi_n^R(x,t)\right] \leq t + C\left[1+\phi(0)\right] \frac{\lambda}{1-\lambda} \ .$$

In the same way, for all $t \in \mathbb{R}^+$, $\mathcal{M}^E(t) < +\infty$, while $\mathcal{M}^R(t) < +\infty$ whenever the system is ϕ-mixing.

This shows that the expectation cannot grow more than linearly in t around generic points, and in an affine way around any other point. We already saw that for ϕ-mixing systems the expectation for entry times was t for cylinder sets shrinking to zero around generic and periodic points. This could suggest that the processes $\xi_n^E(\cdot, t)$ is really uniformly integrable. In order to prove this, we first need to state a result proved in [15], which gives the error in the convergence to the asymptotic distributions $F_k^E(t)$ and $F_k^R(t)$. We quote the result around periodic points, that around generic points being obtained by putting $p = 0$.

Theorem 5 ([15]) *Let $(\mathcal{X}, \mathcal{B}, \mu, T)$ be a ϕ-mixing system with finite partition $\mathcal{A}$, z a periodic point with minimal period m, p the quantity defined in (22) and:*

$$q_n \equiv \sup_{\ell \geq n} \left| \frac{\mu\left[A_{\ell+n}(x)\right]}{\mu\left[A_\ell(x)\right]} - p \right| .$$

Let $\left(\xi_n^E(\cdot, t); \mu\right)$ and $\left(\xi_n^R(\cdot, t); \mu_n\right)$ be the processes defined in (1) relative to $A_n(z)$ (which is the n-cylinder of $\mathcal{A}^n$ containing z) and define the function

$$\Psi_p(t,k) = \begin{cases} \frac{t^k(1-p)^k\sqrt{k}}{k!} e^{2k+3t} & \text{if } t(1-p) > \frac{1}{2}pk \\ (2p)^k \exp\left(t\frac{1+2p}{1-4p}\right) & \text{if } t(1-p) \leq \frac{1}{2}pk \end{cases}$$

Then there exists a constant $C^E \in \mathbb{R}^+$ so that, for every $\delta \in \mathbb{R}^+$ and every $t \in \mathbb{R}^+$:

$$\left.\begin{array}{r} \left|\mu(\xi_n^E(x,t) = k) - F_k^E(t)\right| \\ \frac{1}{n}\left|\mu_n(\xi_n^R(x,t) = k) - F_k^R(t)\right| \end{array}\right\} \leq$$

$$\leq C^E \mu(A_n) \frac{(t(1-p))^{k-1}}{k!} e^{3k} + C^E \left(p^{\frac{n}{m}} + \frac{\delta}{t}\mu(A_n) + q_n + \phi(\delta)\right) \Psi_p(t,k) ,$$

where the distribution F_k^* have been defined in (23,24).

We are now ready to state our next result.

Proposition 6 *Under the assumptions of the preceding theorem the processes $\left(\xi_n^E(\cdot, 1); \mu\right)$ and $\left(\xi_n^R(\cdot, 1); \mu_n\right)$ are uniformly integrable (provided $p < \frac{1}{2}$), and consequently the equalities in (30) and (31) hold.*

Proof. We do the proof for a given positive t (we will successively put $t = 1$) and for $\left(\xi_n^E(\cdot, t); \mu\right)$, the other being similar (recall that $\mathcal{M}^R(t)$ is finite for ϕ-mixing systems, i.e. ξ_n^R is integrable for every t). Let $t > 0$. Since $\mu(\left(x \in \mathcal{X} \;\middle|\; \xi_n^E(x,t) \geq \alpha\right) \to 0$ as $\alpha \to \infty$ it is sufficient to show that $\int_{\xi_n^E(x,t) \geq \alpha} \xi_n^E(x,t)\, d\mu(x) \to 0$ as $\alpha \to \infty$ in order to get uniform integrability of ξ_n^E [19]. Since $p > 0$ we can assume that $\alpha \in \mathbb{N}$ is large enough so that $\alpha \geq \frac{2t(1-p)}{p}$. Using the previous theorem we then obtain:

$$\int_{\xi_n^E(x,t) \geq \alpha} \xi_n^E(x,t)\, d\mu(x) = \sum_{j=\alpha}^{+\infty} j\mu\left(\left\{x \in \mathcal{X} \;\middle|\; \xi_n^E(x,t) = j\right\}\right) \leq$$

$$\leq \sum_{j=\alpha}^{+\infty} jF_j^E(t) + \sum_{j=\alpha}^{+\infty} jC^E\mu(A_n)\frac{(t(1-p))^{j-1}}{j!}e^{3j} + \sum_{j=\alpha}^{+\infty} jC^E\epsilon(2p)^j e^{t\frac{1+2p}{1-4p}} \leq$$

$$\leq \sum_{j=\alpha}^{+\infty} jF_j^E(t) + c_1 \sum_{j=\alpha}^{+\infty} \frac{(t(1-p)e^3)^{j-1}}{(j-1)!} + c_2 \sum_{j=\alpha}^{+\infty} j(2p)^j ,$$

where $\epsilon \equiv p^{\frac{n}{m}} + \frac{\delta}{t}\mu(A_n) + q_n + \phi(\delta)$ and the two constant $c_1 \geq C^E\mu(A_n)e^3$ and $c_2 \geq C^E\epsilon e^{t\frac{1+2p}{1-4p}}$ depend on t, but not on n. The first sum, which is independent of n, goes to zero as $\alpha \to +\infty$ as $\mathcal{M}^E(t) < \infty$ (see Remark to Proposition 4). The second sum is the rest of an exponential series, and so it goes to 0 for $\alpha \to +\infty$; the last one vanishes for the same reason whenever $p < \frac{1}{2}$. This concludes the proof.

2.4 Example: non existence of limiting distributions

In this Section we give an example to show that in almost every ϕ-mixing dynamical system there are points that do not have a limiting return time distribution; this extends to return times the analogous result for entry times previously proved in [15].

For simplicity's sake let $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ be the full two shift over the symbols $0, 1$ and denote by σ the shift transformation (i.e. $\sigma(x)_j = x_{j+1}, \forall j$ for $x \in \mathcal{X}$). We put on $\mathcal{X}$ the Bernoulli measure

μ with different weights w and $1-w$ ($0 < w < \frac{1}{2}$). Clearly μ is invariant under σ. The map σ has two fixed point: $y = 0^\infty$ and $z = 1^\infty$. According to Theorem 5, the return times to cylinder neighborhoods of y, z are compound Poissonian with the p-weights $p_1 = w$ (at y) and $p_2 = 1-w$ (at z). We now look at points that alternately visit y and z in such a way that it will not allow them to have a limiting return times distribution. We will find cylinder sets $A_{n_1} \supset A_{n_2} \supset A_{n_3} \supset \cdots$ for some numbers $n_1 < n_2 < \cdots$ and then put $x = \bigcap_j A_{n_j}$. Fix an integer $K \in \mathbb{N}$ and $T > 0$ and let $\varepsilon < \frac{1}{3}|p_1 - p_2|$ be positive. By Theorem 5 there exists n_1 so that the cylinder $A_{n_1}(y) = A_{n_1}(0^{n_1})$ satisfies:

$$\left|\mu_{n_1}\left(\left\{x \in \mathcal{X} \mid \xi^R_{n_1}(x,t) = k\right\}\right) - F^R_k(t)\right| < \varepsilon$$

for $t \leq T$ and $k = 1, \ldots, K$ with the p-value equal to p_1. Now let $n_2 > n_1$ be so that the cylinder $A_{n_2}(0^{n_1}1^{n_2-n_1}) \subset A_{n_1}$ satisfies:

$$\left|\mu_{n_2}\left(\left\{x \in \mathcal{X} \mid \xi^R_{n_2}(x,t) = k\right\}\right) - F^R_k(t)\right| < \varepsilon$$

for $t \leq T$ and $k = 1, \ldots, K$ for the p-value p_2. This can be done because the limiting distribution is invariant under the shift σ (i.e. the limiting distribution of the cylinder $A_{n_2}(0^{n_1}1^{n_2-n_1})$ as $n_2 \to \infty$ is equal to the limiting distribution of the cylinder $A_n(1^\infty)$ as $n \to \infty$). Continuing in this way we find an increasing sequence of integers $n_1, n_2, n_3, \ldots$ so that the distribution of $\xi^R_{n_j}$ alternates within an error of ε between the distribution F^R_k for the p-value p_1 (odd j) and F^R_k (even j) for the p-value p_2 for all $t \leq T$ and orders $r \leq K$. Hence the return time distribution of the point

$$x = \bigcap_j A_{n_j}(0^{n_1}1^{n_2-n_1} \cdots *^{n_j - n_{j-1} - \cdots - n_1})$$

($*$ is 0 if j is odd and 1 if j is even) oscillates between two distinct distributions and therefore cannot converge to a limiting distribution of its own. Around this point the ANV computed with formula (5) will therefore be meaningless. Instead it makes sense to compute it *via* (8). In fact part b) of Proposition 4 holds for any sequence of cylinders whose measure goes to zero. We thus get that around x the lim sup of the expectations will be bounded, in the limit of large n, by $1 + C\left[1 + \phi(0)\right]\frac{\lambda}{1-\lambda}$, which is also in agreement with Kac's theorem.

3 Numerical results

We first summarize what we got up to now. For systems with strong mixing properties (ϕ-mixing), the ANV for entry times was 1 when the sequence of cylinders defining the processes was taken around generic or periodic points. What made the difference between these two kinds of points was the variance. The situation is different for the ANV for return times. In this case our quantity (and the relative variances) take different values according to the point around which the cylinders converge. In particular the ANV is larger than 1 around periodic points.

We recall that the class of ϕ-mixing systems includes irreducible and aperiodic finite state Markov chains and Gibbs states with summable variations (see for example [20]). Smooth systems which share such a behavior (eventually after a suitable coding), are Anosov diffeomorphisms, hyperbolic attractors, uniformly expanding maps, hyperbolic Julia sets. The lack of hyperbolicity seems not to affect the compound Poissonian distribution of return and entry times in some interesting circumstances. We computed numerically the ANV for the Hénon map (presence of homoclinic tangencies) and for the Lozi map (presence of discontinuities) and found good agreement with the behavior of ϕ-mixing systems [16] (in [1] we showed numerically that for the Hénon map the distribution of the number of visits follows pretty well the Poissonian law). Probably such behavior is robust and persist even for measures with weaker mixing properties: the events given by the successive recurrence into a set A could become independent when the measure of A goes to zero as we described at the end of Sect. 2.2, thus producing a Poissonian statistics for the distribution of the number of visits. This matter deserves to be better investigated.

To consider some systems which are not mixing (or ergodic) at all or for which the ergodic properties are still unknown let us focus on the following examples: (1) the irrational rotations of the circle, (2) a shear flow on the cylinder and (3) the standard map. We would like to point out that

in those examples there are only a few analytical results. The irrational rotations are chosen since they probably represent the easiest ergodic dynamical system; nevertheless the recurrence properties are difficult to establish, as we will see in a moment. The shear flow and the standard map are studied for their importance in the understanding of transport phenomena. The shear flow is not ergodic, but it manifests a local mixing property which gives non trivial recurrence statistics [9]. The standard map is considered for the coexistence of different types of motion, integrable on the quasi-periodic tori, ergodic (at least around homoclinic points), intermittent. This last issue is what implies stronger deviations in long time asymptotics and is due to the presence of islands, in the chaotic sea, which confines the motion for long times in the so-called sticky domains. The study of recurrence phenomena could help in the understanding and the classification of these different features of the dynamics [2]. As we said in the Introduction, in our previous works [9, 1] we applied to the standard map the statistics of the first return times and of the number of visits with a particular emphasis on the regions which are at the boundary of regular and chaotic motion.
Coming back to the ANV, we will focus mainly on the expectation and the variance of the random variable $\xi_A^R(\cdot, t)$, since they offer a richer variety of behaviors, as Proposition 4 suggests, and we compute them throughout the quantities $\overline{\mathcal{M}^R}(t)$ and $\overline{\mathcal{V}^R}(t)$ (note that by the definition (9), $\overline{\mathcal{M}^R}(t)$ is a non-decreasing function of t).

3.1 Irrational rotations

Let us take the irrational rotation of the circle:

$$R_\alpha(x) = x + \alpha \qquad \bmod 1 ,$$

with $\alpha \in]0,1[\setminus\mathbb{Q}$. Let $\alpha = [0, a_1, a_2, a_3, \ldots]$ be its continued fraction expansion and $\frac{p_n}{q_n} = [0, a_1, \ldots, a_n]$ its n-th approximants which satisfy the recurrence relations:

$$p_k = a_k p_{k-1} + p_{k-2} , \qquad q_k = a_k q_{k-1} + q_{k-2} ,$$

with $p_{-2} = 0$, $p_{-1} = 1$ and $q_{-2} = 1$, $q_{-1} = 0$. The descending sequence of sets $\{A_n(z)\}_{n\geq 1}$ at a point z on the circle are defined in the following way: $A_n(z)$ is the closed interval with endpoints $R_\alpha^{q_{n-1}}(z)$ and $R_\alpha^{q_n}(z)$. Using these sets, Coelho and De Faria [8] proved the existence of two limiting distributions for the statistics of the first return time (cf. Remark of Proposition 3):

$$F_0^R(t) = \lim_{n\to+\infty} \mu_n \left(\left\{ x \in A_n(z) \;\middle|\; \tau^R_{A_n(z)}(x)\mu\left[A_n(z)\right] > t \right\} \right) ,$$

where μ is the Lebesgue measure. We consider here the most interesting of these two laws, which is obtained in this way. The coefficient a_n in the continued fraction expansion of α is equal to $a_n = \left[1/H^{n-1}(\alpha)\right]$, where H denotes the Gauss transformation. By setting $b_j = \left[1/H^{j-1}(\beta)\right]$, with $\beta \in 0, 1$ and $j \geq 1$, we can construct the following quantities:

$$\Gamma^n(\alpha, \beta) = (H^n(\alpha), [0, a_n, a_{n-1}, \ldots, a_1, b_1, b_2, \ldots]) , \quad n \geq 1 .$$

Coelho and De Faria proved that the distribution function $\mu\left(\{x \mid \tau_{A_n}(x)\mu(A_n) \leq t\}\right)$ converges if $\Gamma^n(\alpha, \beta) \to (\theta, \omega)$, with $\theta \in]0, 1]$ and $\omega \in [0, 1[$. Moreover the limit distribution $F_0^R(t)$ is a piecewise linear function. Up to now nothing is known analytically about the existence of the distribution of the number of visits $F_k^E(t)$ and $F_k^R(t)$. In [1] we performed a numerically analysis in the particular case of the golden ratio $\alpha = \frac{\sqrt{5}-1}{2}$, $\alpha = [0, 1, 1, \ldots]$, and found that the function $F_k^R(t)$ is a piecewise constant function which for each k assumes three different values. It is easy to see that in this case Γ^n converges to $(\theta, \omega) = (\alpha, \alpha)$.

In figure 1 we plot $\overline{\mathcal{M}^R}(t)$ for the set $A_{20}\left(\frac{1}{2}\right)$: it is a piecewise constant function close to the identity, but definitely less than it (in particular $\overline{M^R} < 1$). Figure 2 shows the variance for this set: we observe a piecewise constant function, with a possibly oscillating modulation. For entry times we verified that, as prescribed by Proposition 4, $\overline{\mathcal{M}^E}(t) = t$; instead the variance, in figure 3, is a continuous oscillating function, which suggests that the second moment of ξ^E is uniformly summable in t, provided $F_k^E(t)$ is continuous for any k (cf. Proposition 2). All these results are very stable w.r.t. the change of sets, both in size and in position (the latter case follows by the translation invariance of the map).

Figure 1: Irrational rotations: $\overline{\mathcal{M}^R}(t)$ for $A_{20}\left(\frac{1}{2}\right)$: dotted line is the identity (a magnification from the time range $[0, 100]$)

Figure 2: Irrational rotations: $\overline{\mathcal{V}^R}(t)$ for $A_{20}\left(\frac{1}{2}\right)$ (a magnification from the time range $[0, 100]$)

3.2 Shear flow

In the papers [9, 1, 16] we studied the following skew map on the cylinder $\mathcal{C} \equiv \mathbb{T} \times [0, 1]$:

$$R: \begin{cases} x' = x + y \mod 1 \\ y' = y \end{cases} .$$

This map has interesting properties:
(I) By perturbing R we obtain, according to KAM theory, a transformation that is integrable only for a subset of $\mathcal{C}$ whose Lebesgue measure approaches 1 as the amplitude of the perturbation vanishes.
(II) This map is the shear of the linearized dynamics at the fixed point of the following class of area preserving map on $\mathbb{T}^2$:

$$\begin{cases} x' = x + h(x) + y \mod 1 \\ y' = h(x) + y \mod 1 \end{cases} ,$$

where h is a smooth symmetric function vanishing at zero with its first derivative. These maps are not uniformly hyperbolic and, as a consequence, the systems spend a substantial fraction of time near the fixed point, which makes the motion rather regular as in the sticky regions of the standard map [21].
(III) R describes the flow on a square billiard, when restricted to $\mathbb{T}^2$.
(IV) Our shear map is a particular case of a large class of skew-systems whose importance has been recently addressed in the book [22] about the understanding of transport phenomena. Recurrence is a key tool for that and we furnish here new and quantitative techniques to implement this approach.

For this map, we rigorously computed the statistics of the first return time in regions around the fixed points and we found a polynomial decay, which is quite different from the exponential decay e^{-t} exhibited by systems with strong mixing properties. Moreover we computed, almost numerically, the distribution of the number of visits $F_k^R(t)$ in some sets and we showed the following behaviour: for small values of t there is a *plateau*, which is followed by a spike, and, for $t \gg 1$, the distribution decays like $t^{-\beta}$, with $\beta \approx 2$. Again, this behavior is different from the Poissonian distributions, typical of mixing systems. We also observed a dependence on k: like for rotations on the circle, $F_k^R(t)$ seems to increase with k. When $k = 0$ we recover the Poincaré recurrence distribution $p^R(t)$ (see the Introduction), since $-\frac{dF_0^R(t)}{dt} = p^R(t)$. As we said above, in [9] we rigorously proved that for a large class of domains: $F_0^R(t) = \frac{1}{2t^2}$, $t \geq 1$. Therefore the Poincaré recurrence distribution scales like $\frac{1}{t^3}$. It is interesting that the same tail has been found for the Sinai billiard [2]: as we noted before, our shear map, when restricted to the two-dimensional torus, describes the flow on a square billiard. For the Cassini billiard the scaling exponent was found to be -3.15 [23].

Before continuing, we need to make an important observation which will play a role in the last Section. The length $\left[\frac{t}{\mu(A)}\right]$ of the observation window for the random variables $\xi_A^*(x, t)$ $(* = E, R)$ was basically motivated by Kac's theorem, which states that for ergodic systems $\langle \tau_A \rangle \equiv \mathbb{E}_{\mu_A}[\tau_A(x)] = \mu(A)^{-1}$. For systems that are not ergodic the rescaling factor $\mu(A)^{-1}$ should consequently be replaced

Figure 3: Irrational rotations: $\overline{\mathcal{V}^E}(t)$ for $A_{20}\left(\frac{1}{2}\right)$ (a magnification from the time range $[0, 100]$)

Figure 4: Shear flow: $\overline{\mathcal{M}^R}(t)$ around a generic point (cell size is 10^{-3})

Figure 5: Shear flow: $\overline{\mathcal{V}^R}(t)$ around a generic point up time $[0,60]$ (cell size is 10^{-3}); the line is approximately $6.25t$; for $t \sim 60$ numerical instabilities appear

by the expectation $\langle \tau_A \rangle$. We thus define the non-ergodic cousin of (4) by:

$$\hat{\xi}^*_A(x,t) \equiv \sum_{j=1}^{[t\langle \tau_A \rangle]} \chi_A \left[T^j(x) \right] , \qquad * = E, R . \tag{36}$$

For the shear map, which is non-ergodic, we computed the distributions of the number of visits by taking formula (36); we will follow the same procedure in Sect. 3.3 and in all the situations for which we do not know if the invariant measure is ergodic. It is important to observe that the statement a) of the Proposition 4 does not longer hold. In fact, formula (32) in the proof is replaced by:

$$\mathbb{E}_\mu \left[\hat{\xi}^E_n(x,1) \right] = \sum_{k=1}^{[\langle \tau_{A_n} \rangle]} \int_{\mathcal{X}} \chi_{A_n} \left[T^k(x) \right] \, d\mu(x) = \mu(A_n) \cdot [\langle \tau_{A_n} \rangle] ,$$

so:

$$\overline{M^E} = \lim_{n \to +\infty} \mu(A_n) \cdot [\langle \tau_{A_n} \rangle] ,$$

whenever the limit exists. In particular, for the shear map, we know, from [1], that the mean return time in a square $A_\epsilon \equiv [0,\epsilon] \times [y_0,\epsilon]$ (for each $y_0 \in [0, 1-\epsilon]$) is $\langle \tau_{A_\epsilon} \rangle = 1/\epsilon$. This mean return time could also be computed by applying formula (13) in [24] which gives the average first return time for homeomorphisms which are not necessarily ergodic. So, we easily notice that:

$$\overline{M^E} = \lim_{\epsilon \to 0+} \mu(A_\epsilon) \cdot [\langle \tau_{A_\epsilon} \rangle] = \lim_{\epsilon \to 0+} \epsilon^2 \left[\frac{1}{\epsilon} \right] = 0 .$$

This is not surprising, since the invariance of the cylinders imposes:

$$F^E_{k,A_\epsilon}(t) \le \epsilon ,$$

for all k's, which implies the vanishing of the limiting functions; in this case, the process $\xi^E_{A_\epsilon}(\cdot, 1)$ does not converge to any random variable.

Figure 4 shows that the ANV of returns is the identity, when it is computed around generic points. Around the fixed points and around periodic points of small periods, we found that it becomes an affine function $t + a$, where the coefficient a is roughly equal to 1. The variance $\overline{\mathcal{V}^R}(t)$ is also close to an affine function of t, with a slope which is always bigger than 1 (it is almost 6 around generic points) and which decreases with the period around periodic points (it is almost 1000 around the fixed points, and 80 for period 5). Although the limiting distributions for the shear map are far from being Poissonian, the qualitative averaged behaviours given by the first two moments seem to share the same features as for ϕ-mixing systems (cf. (26)). Nevertheless there is an important difference, as we have already anticipated in the Introduction, which appears when we consider the distribution of the number of visits around periodic points of small period. The ANV ($t = 1$) is of order 1, but the variance has a huge value, and this important deviation reflects the lack of strong statistical properties of the shear map, in particular the absence of mixing and the persistence of asymptotic correlations (recall that in Sect. 2.2 we showed that for ϕ-mixing systems the quantities M^R and V^R are of the same order). We also would like to point out that the computation of the variance is affected by statistical errors and numerical instabilities when the time t increases; for this reason the range of values of t for the variance is distinctly smaller than that for the expectation.

3.3 Standard map

In [1] we computed numerically the distribution of the number of returns $F_k^R(t)$ for the standard map:

$$\begin{cases} y' = y - \frac{\eta}{2\pi}\sin(2\pi x) \bmod 1 \\ x' = x + y' \bmod 1 \end{cases}$$

for the coupling parameter $\eta = 3$, in such a way that the boundary between the integrable orbits and the chaotic region is as sharp as possible compared to the size of the sets used to compute the distributions. Because of the lack of ergodicity we have to use the functions $\hat{\xi}_A^*(x,t)$ $(* = E, R)$, which was defined in (36). Based on [1] we consider three cases.

i) *Regular regions.* For the statistics of return times, we found an asymptotic power law decay like:
$$F_k^R(t) \simeq \frac{\alpha}{t^\beta}$$
with $\alpha > 0$ and $\beta \in [1.90, 2.05]$ and we observed a slight dependence of β on the order k. From Figures 6 and 7, which contain mean and variance of some sets embedded in the regular zone, we can deduce that the dependence of $F_k^R(t)$ on k, included that of β, is strong enough to guarantee the existence of the respective limiting moments. The expectation is affine in t with possibly different slopes (for the straight line with slope 1.26 the coefficient for $t = 0$ is very small and not detectable in the picture), while the variance shows a power law behaviour that depends on the choice of the target set, but which roughly behaves, for large t, as $\overline{\mathcal{V}^R}(t) \sim at^\gamma$, with a a small factor and $\gamma \simeq 2$. We note that for $t = 0$ the variance exhibits values which, in our figure, range from 1 to 40 which is surely due to the presence of periodic points (the larger the period the smaller the coefficient for $t = 0$, see also the discussion in the shear map Section). The fact that the slope of the expectation is larger than 1, when compared to that of the ϕ-mixing regime, means that the majority of the orbits starting from a set A in the regular region tend to return more often to this set (or to spend much more time in it), instead of visiting with the same or comparable frequency all the other accessible regions of the phase space.

ii) *Chaotic regions.* In the chaotic sea of the standard map, one would expect a Poisson distribution for $F_k^R(t)$ which would be typical for mixing systems. Instead we pointed out in [1] a departure from the expected Poisson law for sufficiently large values of t, showing a polynomial decay with an exponent near 2. When $k = 0$ we recover the Poincaré recurrence distribution $p^R(t) = -\frac{dF_0^R(t)}{dt}$. Therefore in our case the Poincaré recurrence distribution scales like e^{-t} for short times and has a long time tail of the form $p^R(t) \simeq \frac{1}{t^3}$. This should be compared with previous results on the standard map in the stochastic sea for $\eta = 6.476939$ [25]. These authors also noted a departure from the exponential regime for large t and observed a power law tail with exponent -3.5. A possible explanation for this phenomenon comes from the fact that orbits originating from points far away from the regular region usually approach it, so that the distribution of the number of visits will be influenced by that typical of integrable regions, which is just polynomial, as we saw above. About the difference between the two scaling exponents we share a comment from [25], p. 4915: "This variety of power law exponents...related to different ranges of stochasticity parameters shows that there is not yet a self-consistent theory predicting universality of these exponents".
The expectation of the ANV for returns is very close to the identity (figure 8), the same as for strongly mixing systems. However the variance departs from the identity by showing slopes slightly larger than 1.

iii) *Mixed regions.* We finally consider domains that straddle the boundary between the regular and the chaotic regions. In [1] we argued, with analytic arguments, that the distribution of the number of visits should be given by a linear superposition of a power law (the contribution of the integrable orbits) and a Poisson distribution (from the chaotic sea). With these assumptions we were able to recover the distributions of the number of visits computed numerically. Concerning the ANV, the expectation is affine in t with a slope usually bigger than 1 and which reduces

Figure 6: Standard map: two examples of $\overline{\mathcal{M}^R}(t)$ in the regular zones; the upper line, relative to the first zone is like $0.75 + 1.45t$, while the other one is approximately $1.26t$ (dotted line is t)

Figure 7: Standard map: $\overline{\mathcal{V}^R}(t)$ in the same sets of Figure 6; the upper line is approximately $40 + 1.22t^{1.74}$, while the other one is near to $1 + 0.11t^{1.99}$ (dotted line is $\frac{1}{4}t^2$)

to 1 when the portion of chaotic region prevails in the target set. The presence of the constant term for $t = 0$ shows the influence of the regular region through the presence of periodic orbits. The variance has a power law behaviour of the type $\overline{\mathcal{V}^R}(t) \sim at^{\gamma}$, with $a \sim 10^1 \div 10^3$ and $\gamma \simeq 2$ (see figures 10 and 11). We cannot exclude the presence of constant terms when $t = 0$ for the variance too, but they are very small and indistinguishable from numerical errors. This means we observed strong numerical instabilities when t increases, which forced us to keep to the range of values of t where the numerical computation were reliable.

4 Conclusions and perspectives

The interesting variety of behaviours we observed in the different regions of the standard map are surely deserving of further careful investigations, especially by looking at the higher moments of our processes. At the level of the first two moments we clearly observed a departure from the behaviour of the ϕ-mixing systems: first of all, the slopes of the expectations are larger than 1 (compare with Proposition 4) and the variance shows a power law trend like t^2 when we consider orbits in weakly chaotic (regular and mixed) regions. For the shear map the lack of ergodicity appeared in the very large values of the variance around periodic points of small period. For irrational rotations the fascinating behavior that we discovered requires a deep theoretical analysis. Our approach of using fine properties of the recurrence to understand the statistical properties of dynamical systems, should be applied to other situations where recurrence is closely related to transport phenomena of physical interest: billiards, hamiltonian systems with vortices, diffusion in symplectic maps, intermittent maps and the class of linked twist maps assumed in [22] as a paradigm of local mixing. Our next objective will be the analysis of the moments $\mathbb{E}_{\mu_A}\left(\left|\xi_A^R(x,t) - \mathcal{M}^R(t)\right|^q\right) \sim t^{f(q)}$, and in particular of the scaling function $f(q)$; this is in some sense reminiscent of the local scaling in turbulence where the non-linearity of suitable scaling exponents could discriminate between laminar and turbulent motion. It is well known that in this case the non-linearity of the scaling exponent is due to rare events with big fluctuations. The same could happen when we look at the distribution of local returns in systems where different kinds of motion coexist. The presence of orbits with an exceptional behavior could influence the distribution of the number of visits and give asymptotic laws far from the Poissonian regime typical of chaotic dynamics.

Acknowledgments

E.L. and S.V. have been supported by the GDRE GREFI-MEFI (*Gruppo di ricerca europeo franco-italiano in fisica matematica*). E.L. thanks the *Ambasciata d'Italia a Parigi* for financial support during the preparation of this work. N.H. was supported by a grant from the NSF (DMS-0301910) and by a grant from the Université de Toulon et du Var, France. We also thanks D. Baro, L. Rossi, G. Servizi and G. Turchetti for useful discussions.
We finally thank the referees whose useful comments and suggestions helped us to improve the paper.

Figure 8: Standard map: $\overline{\mathcal{M}^R}(t)$ in the chaotic zone around a generic point

Figure 9: Standard map: two examples of $\overline{\mathcal{V}^R}(t)$ in the chaotic zone around generic points; the upper line, relative to the first is like $1.31t$, while the other one is approximately $1.17t$ (dotted line is t)

Figure 10: Standard map: two examples of $\overline{\mathcal{M}^R}(t)$ in the mixed zone; the upper line refers to the first set and is approximately $0.22 + 15.4t$; the lower line for the second set is close to $4.57 + 4.76t$ (dotted line is t)

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Figure 11: Standard map: $\overline{\mathcal{V}^R}(t)$ in the same sets of Figure 10; the upper line is like $732t^2$, while the other one is approximately $26t^2$ (dotted line is $30t^2$)

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