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Mellin transforms of correlation integrals and generalized dimension of strange sets

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The Mellin transform of the correlation integral is introduced and proved to be equal to the energy integral whose divergence abscissa is a lower bound to the Hausdorff dimension. For some Julia sets exact results are obtained. For the linear Cantor sets on the real axis it is shown that the energy integral is meromorphic, and the real pole, determining the divergence abscissa, has a sequence of satellite poles equally spaced on a line parallel to the imaginary axis, which explain the oscillations observed in numerical calculations of the correlation integral. The order-d generalized energy integrals are introduced as Mellin transforms of the order-d correlation integrals and for the Cantor sets they are proved to have the same singularities as the ordinary energy integrals. Letting r_d be the residue of the real pole corresponding to the divergence abscissa it is proved that $\lim_{d\to\infty}(-d^{-1}\ln r_d)$ is the second Renyi entropy. Some numerical results obtained for the energy integrals are discussed.

I. INTRODUCTION

After the recent rise of interest in the chaotic behavior of deterministic systems, several investigations were devoted to understanding the geometrical properties of strange attractors (or repellers).[1] The generalized dimensions and the entropies were usually considered and numerical methods to compute them have been developed.[2] In the mathematical literature the theory of dimensions is developed using the "energy integrals"[3,4] while the "correlation integrals" are introduced in the physical literature. In this work we show that the Mellin transform, which proved to be fruitful in the analysis of the Julia sets,[5] is the bridge between correlation and energy integrals.

The analytic structure of the energy integrals is investigated and for a class of Cantor sets it is rigorously described. For the Cantor sets with a single scale λ or s scales $\lambda_1,\lambda_2,\ldots,\lambda_s$ the energy integral is meromorphic and the pole on the real axis corresponding to the correlation dimension is followed by equally spaced satellites sitting on a line parallel to the imaginary axis. As a consequence one can prove that the correlation integral, defined according to Grassberger and Procaccia,[2] has infinitely many oscillations. Numerical evidence for such a phenomenon was given.[6]

For any integer $d\geq 2$ we define the "order-d energy integrals" as the Mellin transform of the "order-d correlation integrals" considered by Grassberger and Procaccia[7,8] and Takens.[9] For the Cantor set their analytic structure is the same as for $d=2$, and for $d\to\infty$ it is shown that the residue at the pole corresponding to the correlation dimension determines the entropy K_2 (second Renyi entropy).[7]

For two simple Julia sets[10] of the quadratic map $z'=z^2-p$ (the circle for $p=0$ and the segment for $p\neq 0$), the energy integral for the balanced measure proves to be a meromorphic function with all the poles on the real axis. The satellites of the dominant pole, which make the correlation integral oscillate, seem to be related to fractal sets.[5]

For other dynamical systems numerical results can be obtained using the physical (invariant, ergodic) measure defined by the time averages. The numerical detection of the dominant pole and its residue is easily carried out using rational interpolations to the energy integrals, and, for the Hénon map, the comparison with the results obtained from the correlation integrals is favorable.

The plan of the work is the following: in Sec. II we introduce the Mellin transforms and quote the basic properties of the correlation integrals, in Sec. III we analyze the analytic structure of the Cantor sets, in Sec. IV the generalized order-d energy integrals are defined as Mellin transforms of the corresponding order-d correlation integrals and explicit results are derived for the Cantor sets. Finally, in Sec. V the order-d energy integrals for the Julia sets are analyzed and numerical results are discussed for the attractor of the Hénon map.[11]

II. ENERGY AND CORRELATION INTEGRALS

We recall the definitions of the Hausdorff and fractal dimensions D and F of a compact set and quote a theorem on the energy integrals which allows bounds to be obtained. The definitions of D and F are purely topological: Let $\{B_i\}$ be a covering of a compact set $E\in\mathbb{R}^n$ with

$$\epsilon_i=\mathrm{diam}(B_i)\leq\epsilon \tag{2.1}$$

and let $H(\epsilon;\alpha)$ be

$$H(\epsilon;\alpha)=\inf_{\{B\}} \sum_i \epsilon_i^\alpha, \quad \alpha>0 \tag{2.2}$$

where the infimum is on all possible coverings. The limit for $\epsilon \to 0$ diverges for $\alpha \le D$

$$H(\alpha)=\lim_{\epsilon\to 0} H(\epsilon;\alpha)=\begin{cases} 0, & \alpha>D \\ \infty, & \alpha<D. \end{cases} \tag{2.3}$$

The numerical computation of D is manifestly impossible; moreover, if we choose a covering with spheres of radius ϵ and let $N(\epsilon)$ be the least number for a covering of E, then if we define the fractal dimension

$$F=\lim\sup_{\epsilon\to 0} \left[-\frac{\ln N(\epsilon)}{\ln \epsilon} \right], \tag{2.4}$$

it can be easily shown that, since $H(\epsilon;\alpha) \le N(\epsilon)\epsilon^\alpha$ and $N(\epsilon)<\epsilon^{-(\rho+F)}$ for any ρ, provided that ϵ is small enough, taking the limit $\epsilon \to 0$, we must have

$$D \le F . \tag{2.5}$$

The fractal dimension or capacity F can be numerically computed even though the computing time increases rapidly with n. For the one-dimensional expanding sets and for the hyperbolic Julia sets it has been proved that $F=D$.[12]

Lower bounds to D can be obtained if we introduce a probability measure. Let μ be a normalized measure with support in E and define the energy integral $\Phi(\alpha)$ by

$$\Phi(\alpha;\mu)=\int_{E\times E} \frac{d\mu(x)d\mu(y)}{||x-y||^\alpha}, \tag{2.6}$$

where $||\cdot||$ denotes any norm in $\mathbb{R}^n$. This is an obvious generalization of the electrostatic energy. Another useful expression of the energy integral involves the Fourier transform $\sigma(p)$ of the measure and for $E\subset R$ explicitly reads (see Appendix A)

$$\Phi(\alpha)=2\Gamma(1-\alpha)\sin\left[\frac{\pi\alpha}{2}\right]\int_{-\infty}^{\infty} |p|^{\alpha-1}|\sigma(p)|^2 dp . \tag{2.7}$$

The equilibrium measure $\mu_{\text{eq}}(\alpha)$ is defined for any α as the measure corresponding to the infimum of $\Phi(\alpha;\mu)$. Then the following result is true (see corollary 6.5 of Ref. 3):

$$D=\inf_\alpha \{\alpha:\Phi[\alpha;\mu_{\text{eq}}(\alpha)]=\infty\} . \tag{2.8}$$

As a consequency D is the least value of α for which the energy integral computed with the equilibrium measures diverges. If we do not know the equilibrium measure and we replace in (2.8) μ_{eq} with any other measure μ, the new quantity

$$\nu(\mu)=\inf_\alpha \{\alpha:\Phi(\alpha;\mu)=\infty\} \tag{2.9}$$

bounds D from below

$$\nu(\mu) \le D . \tag{2.10}$$

We remark that since all the norms in $\mathbb{R}^n$ are equivalent, one can easily show that $\nu(\mu)$ is independent of the norm, while the analytic structure in α can depend on it. For the maps in $\mathbb{R}^n$ with $n \ge 2$ the Euclidean norm is used as customary in the physical literature.

We can easily prove that $\Phi(\alpha;\mu)$ is the Mellin transform of the correlation integral $C(l;\mu)$ defined by

$$C(l;\mu)=\int_{E\times E} \vartheta(l-||x-y||)d\mu(x)d\mu(y) . \tag{2.11}$$

Indeed letting Δ denote the diameter of E and dropping the reference to μ in C and Φ we have

$$\begin{aligned}\int_0^\Delta l^{-\alpha} dC(l) &= \int_0^\Delta l^{-\alpha}\int_{E\times E} \delta(l-||x-y||)d\mu(x)d\mu(y) \\ &= \int_{E\times E} ||x-y||^{-\alpha} d\mu(x)d\mu(y)=\Phi(\alpha) .\end{aligned} \tag{2.12}$$

According to (2.11) we recall that $C(l)$ is a nondecreasing function of l and $dC(l)$ is a Stieltjes measure. The correlation dimension ν introduced by Grassberger and Procaccia is the limit (when it exists)

$$\nu=\lim_{l\to 0} \frac{\ln C(l)}{\ln l} . \tag{2.13}$$

If $C(l)=l^\nu f(l)$ with $\nu>0$, $\ln f(l)=o(\ln l)$, then the above limit exists. If we further assume that $f(l)=[\ln(\Delta/l)]^m g(l)$ where $m \ge 0$ with $g(l), g'(l)$ bounded in $[0,\epsilon]$ and $f(l)$, $f'(l)$ bounded in $[\epsilon,\Delta]$ for some $\epsilon>0$, then the lowest value of α for which $\Phi(\alpha)$ diverges is also ν.

For some models, where the energy integral is meromorphic with a pole in ν (divergence abscissa), we can take the Mellin antitransform of $\Phi(\alpha)$ and so we determine the correlation integral up to a set of zero Lebesgue measure; it was found to be of the form quoted above.

The energy integral $\Phi(\alpha)$ can be taken as the starting point to obtain bounds to the Hausdorff dimension. This is achieved by analyzing the singularities of $\Phi(\alpha)$: according to (2.6) this function is holomorphic in $\text{Re}\,\alpha<0$ and there it is bounded by $|\Phi(\alpha)| \le \Delta^{-\text{Re}\,\alpha}$. Among the singularities of $\Phi(\alpha)$ located on the real positive axis, the one nearest to the origin is specially relevant because it defines the divergence abscissa of $\Phi(\alpha)$ and the corresponding lower bound to the Hausdorff dimension.

In all the models explicitly solved, see Sec. III, $\Phi(\alpha)$ was found to be meromorphic and the structure of $C(l)$ could be determined. Here we simply recall that the measures $dC(l)$ whose Mellin transforms have isolated poles are well known. For a real simple pole we have

$$\Phi(\alpha)=\frac{\Delta^{-\alpha}}{\nu-\alpha} \to C(l)=\frac{1}{\nu}\left[\frac{1}{\Delta}\right]^\nu,$$

for a real multiple pole

$$\Phi(\alpha)=\frac{\Delta^{-\alpha}}{(\nu-\alpha)^m} \to C(l)=\left[\frac{l}{\Delta}\right]^\nu \frac{1}{\nu^m}\sum_{k=0}^{m-1}\frac{\nu^k}{k!}\left[\ln\frac{\Delta}{l}\right]^k,$$

for a pair of the complex conjugate poles

$$\Phi(\alpha)=\frac{\Delta^{-\alpha}}{\nu+i\eta-\alpha}+\text{c.c.}\to C(l)$$
$$=\left[\frac{l}{\Delta}\right]^{\nu} B\cos\left[n\ln\frac{\Delta}{l}+\beta\right],$$

where $B=(\nu^2+\eta^2)^{-1/2}$ and $\tan\beta=-\eta/\nu$. Complex multiple poles lead to analog expressions for $C(l)$; given any superposition of poles the corresponding $C(l)$ can also be immediately written.

For the regular sets with continuous measures we have examined, $\Phi(\alpha)$ has only real poles while for a fractal like the Cantor set $\Phi(\alpha)$ has poles on lines parallel to the imaginary axis: in the former case $f(l)$ is analytic in $[0,\Delta]$, in the second $f(l)$ is a periodic function of $\ln(\Delta/l)$ and $C(l)$ exhibits damped oscillations.

III. THE CANTOR SETS

A. The Cantor set with one scale

We first consider a Cantor set $E\subset[0,1]$ with a single ratio of dissection $\lambda=1/q$ with $q>2$. Let T denote the map $[0,1]\to[0,q/2]$

$$T(x)=\begin{cases} qx, & 0\le x<\frac{1}{2} \\ q(1-x), & \frac{1}{2}\le x\le 1, \end{cases} \tag{3.1}$$

and by $T_1^{-1}(x),T_2^{-1}(x)$ the two inverses on $[0,1]\subset[0,q/2]$

$$T_1^{-1}(x)=\lambda x,\quad T_2^{-1}(x)=1-\lambda x\ . \tag{3.2}$$

Since T^{-1} maps [0,1] into disjoint intervals of [0,1] and it is contractive with contraction rate λ, the Cantor set E is given by the preimages of $I=[0,1]$,

$$E=\lim_{n\to\infty} T^{-n}(I)=\bigcap_{n=0}^{\infty}\ \bigcup_{k=1,2,\ldots,2^n} T_k^{-n}(I)\ . \tag{3.3}$$

The Cantor set can also be generated by the maps

$$\hat{T}_k^{-1}(x)=\epsilon_k(1-\lambda)+\lambda x\ , \tag{3.4}$$

with $\epsilon_0=0$, $\epsilon_1=1$, then E is immediately found as the closure of the set of preimages of zero which read

$$x=\epsilon_1(1-\lambda)+\epsilon_2\lambda(1-\lambda)+\cdots+\epsilon_n\lambda^{n-1}(1-\lambda)\ . \tag{3.5}$$

A Stieltjes invariant measure $d\mu$ is obtained, according to Zygmund,[13] as the weak limit of the following sequence of measures $d\mu_n(x)$. Letting

$$T^{-n}([0,1])=\bigcup_{k=1}^{2^n} I_k\ ,$$

where I_k are ordered disjoint intervals of equal length λ^n, we consider a continuous nondecreasing function $\hat{\mu}_n(x)$ constant for $x\notin I_k$, linear for $x\in I_k$ with slope $(2\lambda)^{-n}$ and values $(k-1)2^{-n},k2^{-n}$ at the ends of I_k. For any interval $(x_1,x_2)\subset I_k$ we define

$$\mu_n(x_1,x_2)\equiv\int_{x_1}^{x_2} d\mu_n(x)$$
$$=\hat{\mu}_n(x_2)-\hat{\mu}_n(x_1)=-(2\lambda)^{-n}(x_2-x_1)\ , \tag{3.6}$$

and observing that the preimages of (x_1,x_2) are disjoint

$$\mu_{n+1}(T^{-1}(x_1,x_2))\equiv\mu_{n+1}(T_1^{-1}(x_1,x_2))+\mu_{n+1}(T_2^{-1}(x_1,x_2))$$
$$=2\lambda(2\lambda)^{-(n+1)}(x_2-x_1)=\mu_n(x_1,x_2)\ . \tag{3.7}$$

Moreover, one obviously has

$$\mu_{n+1}(T_1^{-1}(x_1,x_2))=\mu_{n+1}(T_2^{-1}(x_1,x_2))$$
$$=\tfrac{1}{2}\mu_n(x_1,x_2)\ , \tag{3.8}$$

and in the limit $n\to\infty$ one obtains a measure μ which is invariant and balanced[5] (if for a generic subset A of [0,1] a suitable covering is considered):

$$\mu(A)=\mu(T^{-1}A),\quad \mu(T_1^{-1}A)=\mu(T_2^{-1}A)=\tfrac{1}{2}\mu(A)\ . \tag{3.9}$$

The Fourier transform $\sigma(p)$ of the measure is known and reads[13]

$$|\sigma(p)|^2=\frac{1}{2\pi}\left|\int_{-\infty}^{+\infty} e^{ipx}d\mu(x)\right|^2$$
$$=\frac{1}{2\pi}\prod_{k=1}^{\infty}\cos^2\left[\frac{p}{2}\lambda^k(1-\lambda)\right]. \tag{3.10}$$

In order to determine the analytic structure of $\Phi(\alpha)$ one can exploit the invariance and balance properties of the measure which imply

$$\int_0^1 f(x)d\mu(x)=\tfrac{1}{2}\int_0^1 f(T_1^{-1}(x))d\mu(x)$$
$$+\tfrac{1}{2}\int_0^1 f(T_2^{-1}(x))d\mu(x)\ . \tag{3.11}$$

Using this invariance on $\Phi(\alpha)$ we obtain

$$\Phi(\alpha)=\tfrac{1}{4}\sum_{j,k=1}^{2}\int_0^1 |T_j^{-1}(x)-T_k^{-1}(y)|^{-\alpha}d\mu(x)d\mu(y)$$
$$=\frac{\lambda^{-\alpha}}{2}\int_0^1 |x-y|^{-\alpha}d\mu(x)d\mu(y)$$
$$+\tfrac{1}{2}\int_0^1 |1-\lambda x-\lambda y|^{-\alpha}d\mu(x)d\mu(y)$$
$$=\frac{\lambda^{-\alpha}}{2}\Phi(\alpha)+\varepsilon(\alpha)\ ,$$

where $\varepsilon(\alpha)$ is an entire function of α because $1-\lambda x-\lambda y\ge 1-2\lambda>0$. As a consequence

$$\Phi(\alpha)=\left[1-\frac{\lambda^{-\alpha}}{2}\right]^{-1}\varepsilon(\alpha)\ , \qquad (3.12)$$

which shows that $\Phi(\alpha)$ is meromorphic with poles at

$$\alpha=-\frac{\ln 2}{\ln\lambda}-i\frac{2\pi n}{\ln\lambda}=\frac{\ln 2}{\ln q}+i\frac{2\pi n}{\ln q},\quad n=0,\pm 1,\ldots\ . \qquad (3.13)$$

The real pole gives the Hausdorff dimension $D=\ln 2/\ln q$ (since it is equal to the fractal dimension) and is followed by a sequence of satellites. The residue of the first poles (for $q\geq 4$) can be proved to be nonzero (see Appendix B) and the correlation integral exhibits oscillations

$$C(l)=l^{\ln 2/\ln q}\sum_{n=0}^{\infty}B_n\cos\left[2\pi n\frac{\ln l}{\ln q}+\beta_n\right]\ . \qquad (3.14)$$

B. The Cantor set with two scales

We consider the mapping

$$T(x)=\begin{cases}\dfrac{x}{\lambda_1}, & 0\leq x<\dfrac{\lambda_1}{\lambda_1+\lambda_2}\\[2ex] \dfrac{1-x}{\lambda_2}, & \dfrac{\lambda_1}{\lambda_1+\lambda_2}\leq x<1\end{cases} \qquad (3.15)$$

with $\lambda_1>0$, $\lambda_2>0$, $\lambda_1+\lambda_2<1$ and its inverses

$$T_1^{-1}(x)=\lambda_1 x,\quad T_2^{-1}(x)=1-\lambda_2 x\ . \qquad (3.16)$$

The Cantor set is the limit set of the preimages of any closed subinterval of [0,1], and we consider an invariant probability measure μ such that

$$\int_0^1 f(x)d\mu(x)=p_1\int_0^1 f(T_1^{-1}(x))d\mu(x)+p_2\int_0^1 f(T_2^{-1}(x))d\mu(x)\ , \qquad (3.17)$$

with $p_1+p_2=1$. The existence of such a measure has been proved in Ref. 14. Using the invariance of the measure on $\Phi(\alpha)$ we obtain

$$\Phi(\alpha)=(p_1^2\lambda_1^{-\alpha}+p_2^2\lambda_2^{-\alpha})\Phi(\alpha)+2p_1p_2\int_{E\times E}|1-\lambda_1 x-\lambda_2 y|^{-\alpha}d\mu(x)d\mu(y), \qquad (3.18)$$

and since $\lambda_1 x+\lambda_2 y\leq\lambda_1+\lambda_2<1$ the last integral in (3.18) is an entire function of α and the singularities of $\Phi(\alpha)$ are determined by

$$G(p_1,p_2,\alpha)=p_1^2\lambda_1^{-\alpha}+p_2^2\lambda_2^{-\alpha}=1 \qquad (3.19)$$

under the constraint $p_1+p_2=1$.

For real positive α Eq. (3.19) has only one solution: indeed, G is a monotone increasing function of α with $G\to+\infty$ for $\alpha\to+\infty$ and $G(p_1,p_2,0)=p_1^2+p_2^2<1$. Let ν be the real positive solution of (3.19): since $\nu=\nu(p_1,p_2)$ we maximize it with respect to p_1 and p_2. Letting D be the maximum of ν and observing that it is achieved when

$$\frac{\partial}{\partial p_1}G(p_1,1-p_1,D)=0\ ,$$

we obtain

$$p_1=\lambda_1^D,\quad p_2=\lambda_2^D,\quad \lambda_1^D+\lambda_2^D=1\ . \qquad (3.20)$$

As a consequence, all the singularities of $\Phi(\alpha)$ are given by

$$\lambda_1^{2D-\alpha}+\lambda_2^{2D-\alpha}=1\ . \qquad (3.21)$$

The zeros of (3.21) are necessarily isolated, because the left-hand side (lhs) is an entire function of α. Therefore $\Phi(\alpha)$ is meromorphic. Furthermore, one proves easily that the zeros remain inside a finite strip parallel to the imaginary axis.

It is important to note that the Cantor sets with two or more scales (see below) satisfy the open set condition of the theorem (8.6) in Ref. 3, so the value of D which maximizes (3.19) is exactly the Hausdorff dimension of the set E. Then it is not surprising that if we use the balanced measure with probabilities $p_1=\lambda_1^D$ and $p_2=\lambda_2^D$ to compute the energy integral, its (real) abscissa of divergence is just the Hausdorff dimension of the set. It can be shown that this measure is the maximal pressure measure for the function $-D\ln|T'(z)|$, i.e., the unique ergodic measure which maximizes the expression $\sup\{h(\mu)-D\int\ln|T'(z)|d\mu\}=0$ where $h(\mu)$ is the Kolmogorov entropy of μ and the supremum is taken on the set of T-invariant probability measures on the Cantor set. This follows because the Cantor set is a mixing repeller[15] and the Kolmogorov entropy and the Lyapunov exponent $\lambda(\mu_b)$ of a balanced measure μ_b with weights p_1,p_2 read[14]

$$h(\mu_b)=-p_1\ln p_1-p_2\ln p_2\ ,$$
$$\lambda(\mu_b)=\int_E\ln|T'(z)|d\mu_b=-p_1\ln\lambda_1-p_2\ln\lambda_2\ .$$

C. The Cantor set with s scales

We consider a mapping $T(x)$ defined in [0,1] with s inverses defined by

$$T_j^{-1}(x)=a_j+\lambda_j x,\quad j=1,2,\ldots,s \qquad (3.22)$$

with $0<a_1<a_1+\lambda_1<\cdots<a_s<a_s+\lambda_s<1$ which implies $\lambda_1+\lambda_2+\cdots+\lambda_s<1$. Letting the Cantor set E be defined by $E=\lim_{n\to\infty}T^{-n}(I)$ for $I\subset[0,1]$ and μ be a measure such that

$$\int f(x)d\mu(x)=\sum_{j=1}^{s}p_j\int f(T_j^{-1}x)d\mu(x)\ , \qquad (3.23)$$

with $\sum_j p_j=1$ it is easy to show that the singularities of $\Phi(\alpha)$ are given by

$$\sum\lambda_i^{2D-\alpha}=1\ , \qquad (3.24)$$

where $\alpha=D$ is the unique real positive solution of (3.24).

IV. THE GENERALIZED ENERGY INTEGRALS

The energy integral $\Phi(\alpha)$ is defined as the mean of the distance $||x-y||$ of two points of a set E raised to the power $-\alpha$; the generalized energy integral of order d has the same definition where the distance of two points is replaced by the distance $||x-y||_d$ of two pieces of orbit formed by d consecutive points.

$$||x-y||_d=\left[\sum_{j=0}^{d-1}||T^jx-T^jy||^2\right]^{1/2} . \tag{4.1}$$

Labeling the order-d energy integral by $\Phi_d(\alpha)$ we write

$$\Phi_d(\alpha)=\int_{E\times E}||x-y||_d{}^{-\alpha}d\mu(x)d\mu(y) . \tag{4.2}$$

Just as in Sec. II it is obvious to check that letting $C_d(l)$ be the generalized correlation integral

$$C_d(l)=\int_{E\times E}\vartheta(l-||x-y||_d)d\mu(x)d\mu(y) ,$$

then $dC_d(l)$ is a positive Stieltjes measure and $\Phi_d(\alpha)$ is its Mellin transform

$$\Phi_d(\alpha)=\int_0^\Delta l^{-\alpha}dC_d(l) . \tag{4.3}$$

According to Grassberger and Procaccia,[6] if $C(l)$ vanishes like l^ν then $C_d(l)$ also does, and if $C_d(l)=l^\nu f_d(l)$ for the same ν independent of d with $f_d(l)$ regular at $l=0$, then $f_d(0)\simeq e^{-dK_2}$ for $d\to+\infty$ where K_2 is the second Renyi entropy, defined in Sec. (4.1) of Ref. 1.

If $f_d(l)$ is analytic for $l=0$ then the smallest real pole of $\Phi_d(\alpha)$ is at $\alpha=\nu$ and the residue behaves as νe^{-dK_2} for $d\to\infty$. The structure of $\Phi_d(\alpha)$ can be analyzed in the case of the Cantor sets: the singularities of $\Phi_d(\alpha)$ are the same as for $\Phi(\alpha)$ and consequently $f_d(l)$ has not a finite limit as $l\to 0$ even though it is bounded. However, the residue at the smallest real pole should still behave as νe^{-dK_2} for $d\to+\infty$.

For the Cantor set with a single scale using the invariance with respect to the balanced measure, we obtain

$$\Phi_d(\alpha)=\tfrac{1}{2}\int(\lambda\,|x-y|^2+|x-y|^2+|Tx-Ty|^2+\cdots+|T^{d-2}x-T^{d-2}y|^2)^{-\alpha/2}d\mu(x)d\mu(y)$$
$$+\tfrac{1}{2}\int[\,|1-\lambda(x+y)|^2+|x-y|^2+|Tx-Ty|^2+\cdots+|T^{d-2}x-T^{d-2}y|^2]^{-\alpha/2}d\mu(x)d\mu(y) ,$$

where the last integral defines an integer function of α since $l-\lambda(x+y)$ is strictly positive.

Using the invariance with respect to the measure k times we have

$$\Phi_d(\alpha)=\frac{1}{2^k}\int[(\lambda^{2k}+\lambda^{2(k-1)}+\cdots+\lambda^2+1)\,|x-y|^2$$
$$+|Tx-Ty|^2+\cdots+|T^{d-1-k}x-T^{d-1-k}y|^2]^{-\alpha/2}d\mu(x)d\mu(y)+\varepsilon_k(\alpha) ,$$

where $\varepsilon_k(\alpha)$ is an entire function of α, and finally for $k=d-1$

$$\Phi_d(\alpha)=\frac{1}{2^{d-1}}\left[\sum_{k=0}^{d-1}\lambda^{2k}\right]^{-\alpha/2}\Phi(\alpha)+\varepsilon(\alpha) , \tag{4.4}$$

where again $\varepsilon(\alpha)$ is an entire function.

As a consequence if r is the residue of $\Phi(\alpha)$ at the pole $\alpha=D$, the corresponding pole of $\Phi_d(\alpha)$ has a residue r_d which reads

$$r_d=2r\left[\frac{1-\lambda^{2d}}{1-\lambda^2}\right]^{-D/2}e^{-d\ln 2} . \tag{4.5}$$

The law of convergence of $(-\ln r_d)/d$ is in this case explicit.

In the case of a Cantor set with two scales, still using $d-1$ times the invariance of the measure we obtain

$$\Phi_d(\alpha)=\Phi(\alpha)\sum_{k_1,\ldots,k_{d-1}=1}^{2}p_{k_1}^2\cdots p_{k_{d-1}}^2(\lambda_{k_1}^2\cdots\lambda_{k_{d-1}}^2+\lambda_2^2\cdots\lambda_{k_{d-1}}^2+\cdots+\lambda_{k_{d-1}}^2+1)^{-\alpha/2}+\varepsilon(\alpha) .$$

Letting ν be the real positive pole of $\Phi(\alpha)$ and r be its residue, the corresponding residue r_d of the same pole of $\Phi_d(\alpha)$ is given by

$$r_d=r\sum_{k_1,k_2,\ldots,k_{d-1}=1}^{2}(p_{k_1}p_{k_2}\cdots p_{k_{d-1}})^2[\lambda_{k_1}^2\cdots\lambda_{k_{d-1}}^2+\lambda_2^2\cdots\lambda_{k_{d-1}}^2+\cdots+\lambda_{k_{d-1}}^2+1]^{-\nu/2} . \tag{4.6}$$

From (4.6) we obtain an upper bound by replacing the square brackets with 1 and a lower bound by replacing every term in the square brackets with 1.

As a consequence we write if $r\ge 0$ (when $r<0$ the bounds are intercharged)

$$rd^{-\nu/2}\sum_{k_1,k_2,\ldots,k_{d-1}=1}^{2}(p_{k_1}p_{k_2}\cdots p_{k_{d-1}})^2\le r_d\le r\sum_{k_1,k_2,\ldots,k_{d-1}=1}^{2}(p_{k_1}p_{k_2}\cdots p_{k_{d-1}})^2 , \tag{4.7}$$

TABLE I. First singularity of the Padé interpolations to $\Phi(-1/k)$ for the Hénon map started at the point $\mathbf{x}_0=(0.1,0.1)$.

N	[3/2]	[3/3]	[4/3]	[4/4]	[5/4]	[5/5]
$2^{11}=2048$	1.203	1.211	1.228	1.215	1.227	1.222
$2^{13}=8192$	1.196	1.200	1.200	1.200	1.200	1.201
$2^{15}=32\,768$	1.196	1.200	1.199	1.200	1.201	1.200

and observing that the sums are equal to $(p_1^2+p_2^2)^{d-1}$ the limit $-(1/d)\ln r_d$ for $d\to+\infty$ can be computed and reads

$$K_2=-\ln(p_1^2+p_2^2)\ . \tag{4.8}$$

The procedure is easily generalized to the Cantor sets with s scales and the result for the second Renyi entropy reads

$$K_2=-\ln(p_1^2+p_2^2+\cdots+p_s^2)\ . \tag{4.9}$$

We have shown[16] that for hyperbolic polynomial totally disconnected Julia sets of degree n, the generalized energy integral allows one to compute the entropy (in this case second Renyi entropy = Kolmogorov entropy = $\ln n$) with respect to the balanced measure (Brolin measure) if the ordinary energy integral has a pole at the divergence abscissa with residue different from zero.

In Ref. 14 it was shown that the Cantor set with s scales is isomorphic to the one-sided $(p_1,p_2,\dots,p_s)$ shift, where $p_1,p_2,\dots,p_s$ are the weights of the balanced measure. So it is easy to compute the Kolmogorov entropy $K_1(\mu)$, which has the expression

$$K_1(\mu)=-\sum_{i=1}^{s}p_i\ln(p_i)\ . \tag{4.10}$$

If we now consider the second Renyi entropy, first of all it is easy to prove that $K_2\le K_1$, and then, adapting the theorem (4.1) in Ref. 17, one can show that K_2 is isomorphism invariant.

It remains to compute the K_2 entropy of the one-sided $(p_1,p_2,\dots,p_s)$ shift. But, following again Ref. 17 (Sec. 6 of Chap. IV), it is easy to show that K_2 is given by (4.9). The topological entropy of the s-Cantor set is $\ln s$ and this value is the Kolmogorov entropy and the second Renyi entropy of the balanced measure with equal weights $p_i=1/s$, $i=1,2,\dots,s$.

V. NUMERICAL RESULTS

Even though the results obtained for the Cantor sets can suggest the analytic structure to be expected for $\Phi(\alpha)$ when dealing with attractors (or repellers) with a simple scaling law, if one really wants to compute the Haussdorff dimension (or a lower bound to it) for a given system numerical methods have to be used. The location of the smallest real pole of $\Phi(\alpha)$ or $\Phi_d(\alpha)$ can be found by interpolating with rational approximation the function computed at a sequence of negative values of α. The best choice was almost invariably found to be $\alpha=-1/n$ for $n=1,2,\dots$. Since $\Phi(\alpha)$ behaves as $\Delta^{-\alpha}$ it is convenient to interpolate $\hat{\Phi}(\alpha)=\Delta^{\alpha}\Phi(\alpha)$.

Indeed, as a stability check of the position of the smallest real pole, one should interpolate $\tau^{\alpha}\Phi(\alpha)$ and vary τ around $\tau=\Delta$. The results should not depend on τ and consequently the variation with τ (the computational time is absolutely negligible) can allow to estimate the error.

The rational interpolations of the $[n/n]$ type were first checked on two explicit examples given by the Julia sets for the quadratic map $z'=z^2$ (circle) for which

$$\Phi(\alpha)=\frac{\Gamma(1-\alpha)}{\Gamma^2[1-(\alpha/2)]}\ , \tag{5.1}$$

and the $z'=z^2-2$ (interval $[-2,2]$) for which

$$\Phi(\alpha)=\left[\frac{\Gamma(1-\alpha)}{\Gamma^2[1-(\alpha/2)]}\right]^2, \tag{5.2}$$

as shown in Appendix A. In the first case $\hat{\Phi}(\alpha)=2^{\alpha}\Phi(\alpha)$ is Stieltjes and the method works remarkably well; in the second, due to the double poles, the convergence is much slower.

The method was tested on the Hénon map $\mathbf{x}'=T\mathbf{x}$ defined by

$$\begin{cases}x'=y+1-1.4x^2,\\ y'=0.3x.\end{cases} \tag{5.3}$$

To carry on the computations we assumed that the Hénon map admits the "physical measure" $\mu_{\rm ph}$ (Sinai-Bowen-Ruelle measure),[1] that is an invariant ergodic measure with support on the attractor $E\subset\mathbb{R}^n$ and such that

$$\int_E f\,d\mu_{\rm ph}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^n x_0),\quad f\in\mathscr{C}(\mathbb{R}^n)$$

where x_0 is any point in the basin of attraction of E (up to a set of zero Lebesgue measure).

We recall that a class of very similar maps (Lozi-Misiurewicz) has physical measure. So we have

TABLE II. The same for $\Phi(-1/k)\Delta^{-1/k}$, $\Delta=2.58$.

N	[3/2]	[3/3]	[4/3]	[4/4]	[5/4]	[5/5]
$2^{11}=2048$	1.229	1.213	1.220	1.218	1.220	1.223
$2^{13}=8192$	1.220	1.195	1.199	1.187	1.199	1.199
$2^{15}=32\,768$	1.220	1.194	1.198	1.195	1.200	1.195

$$\Phi(\alpha)=\lim_{N\to\infty}\left[\frac{1}{N^2}\sum_{i,j=1}^{N}||T^ix-T^jx||^{-\alpha}\right]. \tag{5.4}$$

After computing an orbit on the attractor of length N the $\Phi(\alpha)$ is evaluated at $\alpha=-1/n$ for $n\leq 10$. For a fixed value of N the smallest real pole $\alpha=\nu$ obtained for several $[m/m]$ and $[m+1/m]$ rational interpolations of Padé type with $m=3,4,5$ is quite stable. The results do not change if one scales $\Phi(\alpha)$ according to $\tau^\alpha\Phi(\alpha)$ as one should expect since the singularity structure is not altered.

This transformation is useful to separate the spurious singularities due to numerical noise (such as poles with large real part and nonnegligible residue) from the real singularities: the former are extremely sensitive to τ while the latter are stable. The optimal choice seems to be τ close to Δ, the diameter of the set. The accuracy increases with N but with $N=2^{11}=2048$ it is already of 1%. In Tables I and II the results obtained for various approximants are reported for the Hénon attractor.

A more detailed investigation of the singularity structure of $\Phi(\alpha)$ could be made if the coefficients of $\Phi(\alpha)$ were exactly known. In Tables III and IV we exhibit the poles and residues of the first Padé interpolations of the sequences $\Phi(-1/n)\Delta^{-1/n}$ and $\Phi(-n)\Delta^{-n}$ for the Julia set $z'=z^2$ where $\Delta=2$.

In the first case one observes a very fast convergence to the first singularities while in the second case the convergence is much slower, an accuracy $\simeq 10^{-4}$ on the first pole being reached with the [8/8] rather than [3/3] approximant. Nevertheless, there are sets like the Cantor and the real Julia sets for which the moments of the measure $d\mu$ can be exactly computed by recurrence and consequently the $\Phi(2n)$ are also exactly known (see Appendix C). In this case accurate results can be obtained if one computes sufficiently high approximants using a suitably large number of significant digits. When the singularities are double poles as in 5.2 the method still works but the convergence slows down.

TABLE III. Poles and residues of the Padé interpolations to $-\Phi(-1/k)\Delta^{-1/k}$ for the Julia set $z'=z^2$, $\Delta=2$.

n	Poles	Residues
	[3/2]	
1	3.724	0.700
2	1.000 8	0.639
	[3/3]	
1	11.22	1.66
2	3.190	0.439
3	1.000 06	0.637
	[4/3]	
1	7.924	0.989
2	3.072	0.372
3	1.000 007	0.637
	[4/4]	
1	21.75	2.31
2	6.072	0.553
3	3.017	0.333
4	1.000 000 4	0.637
	[5/5]	
1	37.42	3.05
2	10.16	0.734
3	5.200	0.319
4	3.000 8	0.319
5	1.000 000 001	0.637

TABLE IV. Poles and residues of the Padé interpolations to $-\Phi(-k)\Delta^{-k}$ for the Julia set $z'=z^2$, $\Delta=2$.

n	Poles	Residues
	[3/2]	
1	5.880	1.195
2	1.043	0.705
	[3/3]	
1	22.74	2.69
2	4.293	0.757
3	1.016	0.665
	[4/3]	
1	15.02	1.70
2	3.812	0.619
3	1.008	0.652
	[4/4]	
1	48.13	3.76
2	10.07	1.02
3	3.425	0.496
4	1.003	0.643
	[5/5]	
1	87.23	4.96
2	19.28	1.31
3	7.082	0.616
4	3.141	0.390
5	1.000 4	0.638

APPENDIX A

In this Appendix we sketch a derivation of formula (2.7) which relates the energy integral $\Phi(\alpha)$ to the Fourier transform of the measure. We define a kernel $K(\mathbf{p})$ according to

$$K(\mathbf{p})=(2\pi)^{-n/2}\int_{\mathbf{R}^n}e^{i\mathbf{p}\cdot\mathbf{x}}||\mathbf{x}||^{-\alpha}d\mathbf{x}, \tag{A1}$$

and the Fourier transform of $d\mu(\mathbf{x})$ as

$$\sigma(\mathbf{p})=(2\pi)^{-n/2}\int_{\mathbf{R}^n}e^{i\mathbf{p}\cdot\mathbf{x}}d\mu(\mathbf{x}), \tag{A2}$$

where the support of the measure is a subset E of R^n. It is not hard to prove that the energy integral is given by[4]

$$\Phi(\alpha)=(2\pi)^{n/2}\int_{\mathbf{R}^n}K(\mathbf{p})\,|\,\sigma(\mathbf{p})\,|^2d\mathbf{p}. \tag{A3}$$

If $n=1$ then

$$K(p)=\frac{2}{\sqrt{2\pi}}\int_0^\infty x^{-\alpha}\cos(px)\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\Gamma(1-\alpha)\sin\left[\frac{\pi\alpha}{2}\right]|p|^{\alpha-1} \qquad \text{(A4)}$$

in agreement with (2.7).

Using these results we compute $\Phi(\alpha)$ for the Julia set of the quadratic map $z'=z^2-2$ for which

$$d\mu(x)=\frac{dx}{\pi\sqrt{4-x^2}},\quad -2\le x\le 2\,. \qquad \text{(A5)}$$

Indeed, inserting in (2.7),

$$\sigma(p)=\frac{\sqrt{2\pi}}{\pi^2}\int_0^2\frac{\cos(px)}{\sqrt{4-x^2}}dx=\frac{1}{\sqrt{2\pi}}J_0(2p)\,, \qquad \text{(A6)}$$

where J_0 denotes the Bessel function, we find

$$\Phi(\alpha)=\frac{2}{\pi}\Gamma(1-\alpha)\sin\left[\frac{\pi\alpha}{2}\right]\int_0^\infty p^{\alpha-1}J_0^2(2p)\,dp$$

$$=\frac{1}{\pi}\sin\left[\frac{\pi\alpha}{2}\right]\frac{\Gamma^2(1-\alpha)\Gamma(\alpha/2)}{\Gamma^3[1-(\alpha/2)]}$$

$$=\left[\frac{\Gamma(1-\alpha)}{\Gamma^2[1-(\alpha/2)]}\right]^2, \qquad \text{(A7)}$$

where formula 6.574.2 (p. 692) of Gradshteyn[18] and the product formula for Γ functions $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$ have been used.

Finally we sketch the computation of $\Phi(\alpha)$ for the circle with uniform measure, that is, the Julia set of the map $z'=z^2$.

$$\Phi(\alpha)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}|e^{i\theta_1}-e^{i\theta_2}|^{-\alpha}d\theta_1 d\theta_2$$

$$=\frac{2^{-\alpha}}{4\pi^2}\int_{-\pi}^{\pi}\left|\sin\left[\frac{\theta_1-\theta_2}{2}\right]\right|^{-\alpha}d\theta_1 d\theta_2\,. \qquad \text{(A8)}$$

Changing the variables according to $\theta_1=v+u$, $\theta_2=v-u$ we obtain

$$\Phi(\alpha)=\frac{2^{-\alpha}}{\pi^2}\int_{-\pi}^{\pi}(\pi-|u|)|\sin u|^{-\alpha}du$$

$$=\frac{2^{-\alpha+1}}{\pi^2}\int_0^\pi u\sin^{-\alpha}u\,du=\frac{\Gamma(1-\alpha)}{\Gamma^2[1-(\alpha/2)]} \qquad \text{(A9)}$$

according to formula 3.821.1 (p. 446) of Gradshteyn.

For comparison with (A7) we compute $\Phi(\alpha)$ for a uniform measure on $[-2,2]$. The map T under which such a set is invariant is given by (3.1) with $q=2$. The uniform measure is invariant and ergodic with respect to T. In this case

$$\Phi(\alpha)=\frac{1}{4^2}\int_{-2}^{2}dx\int_{-2}^{2}dy\,|x-y|^{-\alpha}=2\frac{4^{-\alpha}}{(2-\alpha)(1-\alpha)}\,.$$

The divergence abscissa is still 1 as for (A7) since the set is the same, but the analytic structure is very different.

APPENDIX B

The residue of $\Phi(\alpha)$ at a pole

$$\alpha_n=\frac{\ln 2}{\ln q}+in\frac{2\pi}{\ln q}\,, \qquad \text{(B1)}$$

is given by

$$r_n=\frac{2\lambda^{\alpha_n}}{\ln\lambda}\varepsilon(\alpha_n)=-\frac{1}{\ln q}\int|q-x-y|^{-\alpha_n}d\mu(x)d\mu(y)\,. \qquad \text{(B2)}$$

As a consequence

$$\mathrm{Im}\,r_n=-\frac{1}{\ln q}\int(q-x-y)^{-\ln 2/\ln q}\times\sin\left[2\pi n\frac{\ln(q-x-y)}{\ln q}\right]d\mu(x)d\mu(y)\,, \qquad \text{(B3)}$$

and the residue is manifestly nonvanishing if

$$2\pi n\frac{\ln(q-2)}{\ln q}\ge\pi(2n-1)\,, \qquad \text{(B4)}$$

since the argument of the sine varies between $(2n-1)\pi$ and $2\pi n$. Condition (B4) is equivalent to

$$n\le-\frac{1}{2}\frac{\ln q}{\ln\left[1-\frac{2}{q}\right]}\,. \qquad \text{(B5)}$$

Since $-\ln(1-x)$ is a convex function we have $-\ln(1-x)\le 2x\ln 2$ for $0\le x\le\frac{1}{2}$, and we have

$$-\frac{1}{2}\frac{\ln q}{\ln\left[1-\frac{2}{q}\right]}\ge\frac{q\ln q}{8}\,, \qquad \text{(B6)}$$

and condition (B4) is surely satisfied for

$$n\le\frac{q\ln q}{8}\,. \qquad \text{(B7)}$$

For $q=4$ only the first residue is proved to be nonvanishing and the number increases with q. For the ternary Cantor only numerical evidence for the residues to be nonzero can be given.

APPENDIX C

When the moments μ_k of the measure $d\mu$ are known the sequence $\Phi(2n)$ can be computed recursively. Letting

$$\mu_k=\int_E x^k d\mu(x)\,, \qquad \text{(C1)}$$

when $E\subset R$ is a one-dimensional set, one has

$$\Phi(2n)=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\mu_k\mu_{2n-k}\,. \qquad \text{(C2)}$$

For the Cantor set with one scale the balance property of the measure one allows to write a recurrence for the μ_k.

Using (3.11) we have

$$\mu_k=\frac{1}{2}\int_0^1\left[\frac{x}{q}\right]^k d\mu(x)+\frac{1}{2}\int_0^1\left[1-\frac{x}{q}\right]^k d\mu(x)\ , \qquad (C3)$$

which leads to the recurrence

$$\mu_k=\frac{1}{2}\left[1-\frac{1}{2q^k}-\frac{(-1)^k}{2q^k}\right]^{-1}\sum_{l=0}^{k-1}\left[\frac{-1}{q}\right]^l\begin{bmatrix}k\\ l\end{bmatrix}\mu_l \qquad (C4)$$

initialized by $\mu_0=1$, $\mu_1=\frac{1}{2}$. For the Julia set of $z'=z^2-q$ with $q\geq 2$ the generating function of the moments with respect to the balanced measure is

$$G(z)=\int_{-a}^{a}\frac{d\mu(x)}{1-xz}=\sum_{k=0}^{\infty}\mu_k z^k\ , \qquad (C5)$$

where $a=\frac{1}{2}[1+(1+4g)^{1/2}]$ and satisfies the functional relation

$$G(z)=\frac{1}{1-qz^2}G\left[\frac{z^2}{1-qz^2}\right]. \qquad (C6)$$

Replacing (C5) into (C6) we find $\mu_{2k+1}=0$ and

$$\mu_{2k}=\sum_{l=0}^{[\frac{k}{2}]}q^{k-2l}\begin{bmatrix}k\\ k-2l\end{bmatrix}\mu_{2l}$$

initialized by $\mu_0=1$, $\mu_2=q$.

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