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LETTER TO THE EDITOR

Weak chaos and Poincaré recurrences for area preserving maps

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Abstract
The spectrum $F(t)$ of Poincaré recurrence times for the standard map exhibits two distinct limits: an integrable weak-coupling limit with an inverse power law and a chaotic strong-coupling limit with exponential decay. In the domain where chaotic regions coexist with integrable structures, the spectrum $F(t)$ exhibits a superposition of exponential and power law decay. Such a law can be proved to occur in a model of area-preserving map at the boundary of the mixing and integrable components.

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(Some figures in this article are in colour only in the electronic version)

The dynamical behaviour of an area-preserving map M is well understood in two limit cases, integrable and uniformly mixing, and when small perturbations are introduced. The intermediate region, where regular and chaotic structures coexist, hierarchically organized on an infinite number of scales, is still open to investigation [1]. The numerical exploration of the global structure is performed by associating, with each orbit $\mathcal{O}$, a dynamical or thermodynamical variable f: rotation number, Lyapounov exponent, fractal dimension, entropy and reversibility error. This provides an intuitive picture of the global behaviour of M and its effectiveness depends on the sensitivity of f to the nature of the orbit. The fine structure of an orbit $\mathcal{O}$ can be detected in some cases by computing a spectrum rather than the evolution of a dynamical variable [2, 3]. The decay of correlations and the return times spectrum are well-known examples. The decay of correlations in the thin stochastic layer around a chain of islands, for instance, appears to follow a power law [4–7], which implies anomalous transport [8, 9]. The presence of a slow diffusion in these regions has also been detected with the frequency map analysis [10, 11]. The dynamical behaviour of weakly

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chaotic regions is still an open problem, even from the numerical viewpoint. A qualitative picture of the local structure on different scales is provided by renormalization theory, but a satisfactory interpretation of the statistical properties is not yet available. We suggest that the spectrum of Poincaré recurrences is a useful tool to investigate an area-preserving map due to its universality in the invariant regions with mixing properties and in the domains foliated by invariant curves. When both components are present, the properties of the spectrum depend, in a precise way, on their extension and their proportion in the neighbourhood of the chosen point. This is confirmed by numerical analysis. The stochastic layer surrounding a chain of islands is not a channel uniformly hyperbolic with a finite number of islands foliated by invariant curves, to which our predictions would rigorously apply. Nevertheless in the case of the standard map, a similar naive picture allows us to interpret most of the observed properties of the spectrum and its changes when we move in phase space and vary the size of the initial domain.

The statistics of Poincaré recurrences has been analysed both numerically [12] and theoretically for area-preserving maps. For strongly mixing systems, the return times spectrum $F(t)$ is exponential and bounds on the convergence rate have been given [13]. For irrational translations on the 1D torus, Slater's theorem [14] states that for any interval there are at most three values for the return times at any point of the interval. For Diophantine numbers and a special choice of the interval, the return times are only two and a limit spectrum exists, when the length of the interval goes to zero [15]. For a skew integrable map, which foliates the 2D torus into invariant 1D tori, the spectrum exists and follows a power law [16]. As a consequence, in any integrable region the spectrum is universal, a power law with exponent -2, as we can prove for the standard map in the weak-coupling limit and check numerically within any island where no chaoticity is visible. In a model map having a finite number of integrable and chaotic regions, the spectrum is determined.

Given a dynamical system $(\mathcal{A}, M, \mu)$ where μ is the invariant measure on the space $\mathcal{A}$, we define the recurrence time $\tau_A(\mathbf{x})$, where A is any measurable set in $\mathcal{A}$, as

$$\tau_A(\mathbf{x}) = \inf_{k\geqslant 1}\{\mathbf{x} \in A, T^k(\mathbf{x}) \in A\}. \tag{1}$$

The average recurrence time $\langle\tau_A\rangle$ is defined by

$$\langle\tau_A\rangle = \int \tau_A(\mathbf{x})\,\mathrm{d}\mu_A(\mathbf{x}) \qquad \mu_A(B) = \frac{\mu(A\cap B)}{\mu(A)} \tag{2}$$

where μ_A is the conditional measure of the set. If the system is ergodic, the average recurrence time is given by Kac's lemma according to

$$\langle\tau_A\rangle = \frac{1}{\mu(A)}. \tag{3}$$

The statistical distribution is conveniently defined after scaling the return time with respect to the average time $\langle\tau_A\rangle$. We introduce the spectrum $F(t)$ and a density $\rho(t)$, for $t \in \mathbb{R}_+$, defined by

$$F(t) \equiv \int_t^{+\infty} \rho(s)\,\mathrm{d}s = \lim_{\mu(A)\to 0} F_A(t) \tag{4}$$

when the limit exists and where

$$F_A(t) = \mu_A(A_{>t}) \qquad A_{>t} \equiv \left\{x \in A : \frac{\tau_A(x)}{\langle\tau_A\rangle} > t\right\}. \tag{5}$$

The convergence of the distribution $F_A(t)$ (which is in general a piecewise continuous function since $\tau_A(x)$ takes integer values), to a continuous function, is a remarkable property of recurrences of dynamical systems. In particular, for strongly mixing systems, $F(t) = \rho(t) = \mathrm{e}^{-t}$, for μ almost every point $\mathbf{x}$ around which A shrinks. This holds if $\mathbf{x}$ is not a periodic point whereas, if $\mathbf{x}$ is a periodic point, we have instead $F(t) = \mathrm{e}^{-\alpha t}$, where α depends on the period [17–19].

In this letter, we announce the existence of another limiting distribution function, different from the exponential one, that is a rather new result. In any domain of the plane foliated by the invariant curves of an area-preserving map M, the asymptotic spectrum of return times is $F(t) = c/t^2$. This result is obtained for the skew integrable map S of the cylinder $\mathbb{T}^1 \times [a, b]$ defined by $x' = x + y \bmod 1$, $y' = y$, to which M is diffeomorphic. For the fixed point $(0, 0)$ letting A_ϵ be the square whose opposite vertex is (ϵ, ϵ), we have rigorously proved that $F_{A_\epsilon}(t) = 1$ for $0 \leqslant t < \epsilon$ and $F_{A_\epsilon}(t) = \frac{1}{2}$ for $\epsilon \leqslant t < 1$ and $F_{A_\epsilon}(t) = (1-\epsilon)^2/(2t(t-\epsilon))$ for $t \geqslant 1 + k\epsilon$, where k is any positive integer. In the limit $\epsilon \to 0$ and $k \to \infty$ while keeping t finite the following result holds

$$F(t) = \begin{cases} 1 & t = 0 \\ 1/2 & 0 < t < 1 \\ 1/2t^2 & t \geqslant 1. \end{cases} \tag{6}$$

For any other point (x, y), denoting with A_ϵ the square whose opposite vertices are $\left(x \pm \frac{1}{2}\epsilon, y \pm \frac{1}{2}\epsilon\right)$, the geometric construction needed to evaluate $F_{A_\epsilon}(t)$ was carried out with a computer assisted procedure [20]. Since the inaccuracies are only due to the accumulation of round off errors, the limit $\epsilon \to 0$ could be investigated (the statistical error of the standard computational method is absent). The asymptotic spectrum is still $F(t) = c/t^\alpha$ where $\alpha = 2$ with an error less than 10^{-3} for all the tested values of y: rational numbers, 'quadratic irrationals' and randomly generated numbers in $[0, 1]$.

The reason why a continuous spectrum exists is that, even though the system is not ergodic, it has a 'local-mixing' property due to filamentation. Indeed consider the dynamical system $(M, \mathcal{A}_\epsilon, \mu_\epsilon)$, where $\mathcal{A}_\epsilon = \cup_{n=0}^{\infty} M^n(A_\epsilon)$ is the cylinder $\mathbb{T}^1 \times \left[y - \frac{1}{2}\epsilon, y + \frac{1}{2}\epsilon\right]$ and μ_ϵ is the Lebesgue measure μ_L times ϵ^{-1}. It can be easily shown that $\mu_\epsilon(M^n(A_\epsilon) \times A_\epsilon) - \mu_\epsilon^2(A_\epsilon)$ is zero for $n \to \infty$ and decays to zero as n^{-1}. The average return time is

$$\left\langle \tau_{A_\epsilon} \right\rangle = \frac{1}{\mu_\epsilon(A_\epsilon)} = \frac{\epsilon}{\mu_L(A_\epsilon)} = \frac{1}{\epsilon}. \tag{7}$$

The spectrum $F(t)$ can be computed for a dynamical system consisting of two (or more) invariant components. Supposing the set D, where it is defined, splits into two invariant subsets with a common boundary: D_p and D_m where the map M is integrable and mixing respectively, and that the measure μ is the suitably normalized Lebesgue measure, it can be proved [16] that, given a set A having nonempty intersections A_p, A_m with D_p, D_m, the spectrum of return times to A is given by

$$F_A(t) = (1-p) F_{A_m}\left(t \frac{\langle \tau_A \rangle}{\left\langle \tau_{A_m} \right\rangle}\right) + p F_{A_p}\left(t \frac{\langle \tau_A \rangle}{\left\langle \tau_{A_p} \right\rangle}\right) \tag{8}$$

where $p = \mu_L(A_p)/\mu_L(A)$. Since the motion in the region D_p is not mixing but rather, for initial conditions in A_p, it takes place in the subset $\overline{D_p} = \cup_{n=0}^{\infty} M^n(A_p) \subseteq D_p$, which is again foliated by invariant curves, we expect the following behaviour for the average return time $\left\langle \tau_{A_p} \right\rangle = \mu_L(\overline{D_p})/\mu_L(A_p)$ in analogy with (7). Using the Kac theorem in the mixing region $\left\langle \tau_{A_m} \right\rangle = \mu_L(D_m)/\mu_L(A_m)$ and the average return time in A given by $\langle \tau_A \rangle = p\left\langle \tau_{A_p} \right\rangle + (1-p)\left\langle \tau_{A_m} \right\rangle$, the ratio of return times becomes $\langle \tau_A \rangle / \left\langle \tau_{A_p} \right\rangle = p/P$ and $\langle \tau_A \rangle / \left\langle \tau_{A_m} \right\rangle = (1-p)/(1-P)$ where $P = \mu_L(\overline{D}_p)/\mu_L(D)$. When $\mu_L(A)$ approaches zero, the functions $F_{A_p}(t)$, $F_{A_m}(t)$ can be replaced by their limits given by e^{-t} and by equation (6) respectively so that we have

$$F_A(t) \simeq (1-p) \exp\left(-t \frac{1-p}{1-P}\right) + \frac{p}{2}\left(\frac{P}{pt}\right)^2 \tag{9}$$

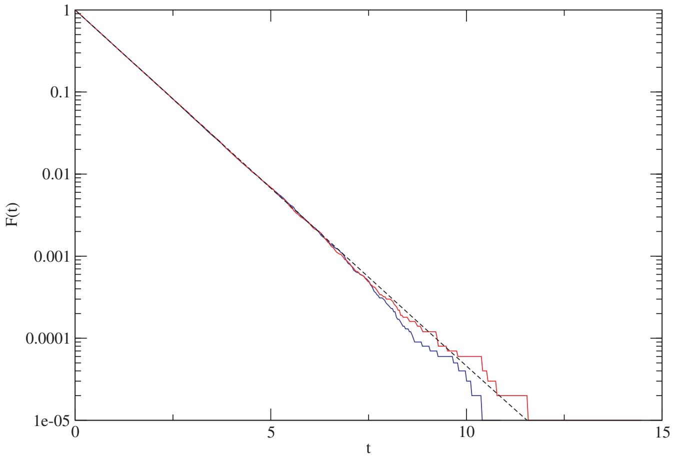

Figure 1. Return times spectrum $F(t)$ of the standard map $M_{\rm I}$ with $\lambda = 10$ (upper continuous (red) line) and of the perturbed cat map $M_{\rm II}$ with $\lambda = 0.1$ (lower continuous (blue) line) obtained with $N = 10^5$ initial points chosen in a square of side $\epsilon = 0.01$ and centre (0.45, 0.05) in a semi-log scale. The black broken line corresponds to the exact spectrum $F(t) = {\rm e}^{-t}$.

where the second term is replaced by p if $t = 0$ and by $p/2$ if $0 < t < P/p$. Equation (9) has been written by sending first $\mu(A)$ to zero, waiting to take the limit $P \to 0$ (see below), just to show finite size effects. By using the exact formula for F_{A_p}, written before (6), one can show that its contribution $\theta(t)$, which is zero except for $t = 0$, where its value is 1. In fact in the limit $\mu(A) \to 0$ the weight p may be kept constant, while P approaches zero because letting $\mu(A) = \epsilon^2$ we have $P \propto \epsilon p^{1/2}$. The limit spectrum becomes therefore $F(t) = (1 - p)\,{\rm e}^{-(1-p)t} + p\theta(t)$: choosing a point at the boundary between D_p and D_m the exponential decay rate is $1 - p$ and varies from 1 to 0. The same effect is obtained by translating a set A from the D_m to D_p: the exponential decay rate decreases to zero when A leaves D_m and the power law decay (6) is recovered. We assume a similar description holds when the region D_p is a layer of weak chaos surrounding an island. It has been observed that the spectrum there follows a power law and we suppose, for simplicity, the exponent is still -2. Letting $\mu_L(A) = \epsilon^2$ so that $\mu_L(A_p) = \epsilon^2 p$, we can only bound the measure of the cylinder $\overline{D}_p$ according to $c(p)\epsilon \leqslant \mu_L(\overline{D}_p) \leqslant \mu_L(D_p)$. As a consequence P may have a finite limit when $\epsilon \to 0$ and the limit spectrum $F(t)$ may exhibit an algebraic component. Letting $P \ll 1$, as we expect for the standard map when $\lambda \gg 1$, since the measure of the chaotic sea is of order 1, the decay of $F(t)$ is exponential, but, for t large enough, the power law decay prevails.

In order to compare the previous results with the numerical spectra of physically relevant models, we considered the following area-preserving maps

$$M_I : \begin{cases} y' = y - \frac{\lambda}{2\pi}\sin(2\pi x) \bmod 1 \\ x' = x + y' \bmod 1 \end{cases} \qquad M_{II} : \begin{cases} y' = y + x - \frac{\lambda}{2\pi}\sin(2\pi x) \bmod 1 \\ x' = x + y' \bmod 1. \end{cases} \tag{10}$$

The first one is the standard Chirikov map, which becomes the skew integrable map S for $\lambda = 0$, the second is a perturbation of the Arnold cat map, to which it reduces for $\lambda = 0$. The maps M_I for $\lambda \gg 1$ and M_{II} for $0 \leqslant \lambda < 1$ have a similar behaviour and their recurrence times spectra agree with $F(t) = {\rm e}^{-t}$, see figure 1, within the numerical errors due to statistical uncertainty (finite number of points) and the finite size of the initial set A_ϵ. The map M_{II} for $|\lambda| < 1$ is an Anosov system and $F(t) = {\rm e}^{-t}$ is a rigorous result [21].

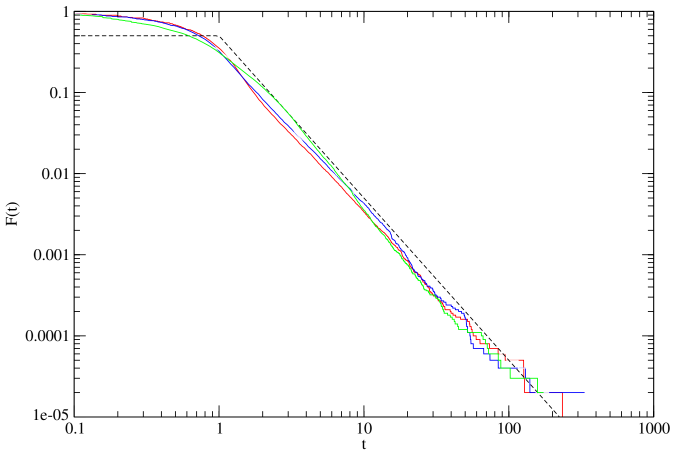

Figure 2. Return times spectra for the standard map M_I: the $N = 10^5$ initial points are taken in a box of side $\epsilon = 0.005$ centred at the point $(0.01, y)$ belonging to the orbit with golden mean rotation number. The curves on a log–log scale correspond to $\lambda = 0.2$ with $y = 0.6133$ (continuous grey (red) line), $\lambda = 0.5$ with $y = 0.6074$ (continuous dark (blue) line) and $\lambda = 0.9$ with $y = 0.6008$ (continuous light grey (green) line). The black broken line is the theoretical return times spectrum given by equation (6).

For $\lambda \ll 1$, the standard map M_I is quasi-integrable and the return times spectrum has a power law behaviour. Indeed for any domain $A \subset \mathbb{T}^2$, letting A_{tori} be the intersection of A with all the invariant tori in $\mathbb{T}^2$, the KAM theorem states that $\mu(A_{\text{tori}}) \to \mu(A)$ as $\lambda \to 0$. Moreover, the existing tori can be smoothly interpolated. As a consequence, in the limit $\lambda \to 0$, the same spectrum is found as for the skew integrable map corresponding to $\lambda = 0$. Consider the strip around the orbit $y(x)$, having a Diophantine winding number ω, delimited by orbits $y_\pm(x)$ having winding numbers $\omega \pm \frac{1}{2}\epsilon$, respectively. For λ and ϵ sufficiently small, all the orbits within the strip are smoothly conjugated with the orbits of the skew integrable map $X' = X + Y \bmod 1$, $Y' = Y$ within the cylinder $\mathbb{T}^1 \times \left(|Y - \omega| < \frac{1}{2}\right)$ (the conjugation is given by the perturbative expansion in λ truncated at a suitable order). The spectrum $F_{A_\epsilon}(t)$ for the skew integrable map S with respect to the square A_ϵ with vertices $\left(\pm\frac{1}{2}\epsilon, \omega \pm \frac{1}{2}\epsilon\right)$ and the spectrum of the standard map M_I with respect to its image $(\Phi(A_\epsilon)$ where $M \circ \Phi \simeq \Phi \circ S)$ are the same, up a to small controlled error. In figure 2, we show the return times spectra $F_{A_\epsilon}(t)$ for a square of side ϵ and centre $(2\epsilon, y)$ belonging to the orbit with golden mean rotation number, for different values of λ. The decay follows closely the power law defined by equation (6). Choosing initial points with a different rotation number, the spectrum does not change appreciably and the exponent of the power law decay is compatible with -2.

The dynamics of the standard map, in the intermediate region $(1 < \lambda < 5)$ after the break up of the last KAM curve, and before the area of islands has come very close to zero, is characterized by the presence of a major region of stable orbits (a single island bifurcating into two at $\lambda \sim 4.3$) surrounded by a layer of variable thickness, where Cantori are present and limit the communication of the homoclinic tangle with the exterior region. The iteration of a domain A_ϵ in the chaotic sea, intersects this region affecting the return times spectra. The net result is the same as if A_ϵ had a small intersection with a region of regular orbits and consequently the behaviour of the return times spectrum may be described by (9). As shown by figure 3, the spectrum of the return times, where A_ϵ is a box in the middle of the chaotic sea (centre $(0.45, 0.05)$ and side $\epsilon = 0.01$), shows an exponential behaviour e^{-t} with a tail decaying as t^{-2} according to (9) with $p = P = 0.04$ for $\lambda = 2$ and $p = P = 0.016$ for

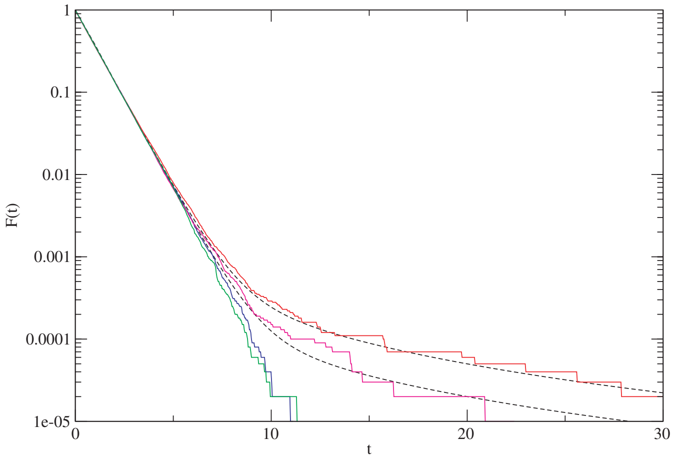

Figure 3. Spectrum of return times for the map M_{I} with $\lambda = 2$. The initial $N = 10^5$ points are chosen in a square centred at (0.45, 0.05) of side $\epsilon = 0.01$. The spectra are shown for $\lambda = 2$ (upper continuous (red) line), $\lambda = 3$ (bottom continuous dark (blue) line), $\lambda = 4$ (medium continuous (purple) line), $\lambda = 5$ (bottom continuous grey (green) curve). A fit to the spectrum, obtained with formula (9) with $P = p = 0.04$ for $\lambda = 2$ and $P = p = 0.016$ for $\lambda = 4$ respectively, is shown by the black broken lines.

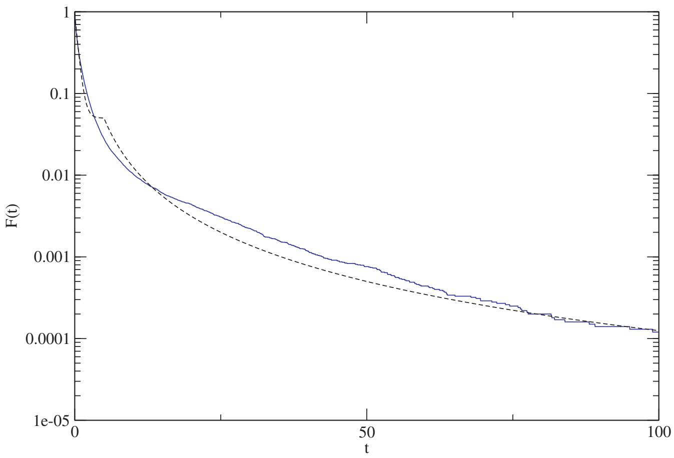

Figure 4. Spectrum of return times for the map M_{I} with $\lambda = 0.8$ (continuous (blue) line). The initial $N = 10^5$ points are chosen in a square centred at (0.45, 0.05) of side $\epsilon = 0.01$. A fit to the spectrum, obtained with formula (9) with $P = 0.5$ and $p = 0.1$, is shown by the black broken line.

$\lambda = 4$. For $\lambda = 3$ the chaotic layer is thinner and the spectrum is fit with $p = P = 0.003$; for $\lambda = 5$ the layer is so thin and its area so small that the algebraic tail in the spectrum is hardly detectable. Decreasing λ, below the critical value P increases and the weight p of the power law component in the spectrum also increases as shown in figure 4, where for $\lambda = 0.8$ the decay is described by $p = 0.1$ and $P = 0.5$. We would like to underline that the spectrum changes somehow with the size of the box (the quality of the fit changes too), whereas it is not sensitive to a change of the initial point, provided it does not approach the stochastic layer around the region of invariant curves. We have verified that, choosing A_ϵ within the region of invariant curves surrounding the origin, within a primary or secondary chain of islands, $F_{A_\epsilon}(t)$

is nicely described by (6). The change of the spectrum entering an integrable region from a chaotic sea is nicely described by $F(t) \sim e^{-(1-p)t}$ as one can see for $\lambda = 3$, since the layer around the major island is thin (choose for instance a box of side $\epsilon = 0.02$ and centre $(x, 0.1)$ moving x from 0.12 to 0.17).

To conclude we underline that the return times spectra exhibit two distinct types of behaviour in the integrable and strongly chaotic regimes. In the transition regions, the spectrum changes exhibiting a combination of the previous laws, as already observed in a model of stationary flow with hexagonal symmetry, where the transport is anomalous [9]. The spectrum, derived for a system with two invariant components, provides an effective way to quantify the relative size of the regular versus mixing domains in the regions of the phase space where both coexist. The analogy we propose with the two-component model, where such a law can be proved to hold, suggests a possible research pathway for a better understanding of mixed regions exhibiting weak chaos, anomalous transport and power law decay of correlations.

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